

**Ph.D. PRELIMINARY QUALIFYING EXAMINATION, JULY 2015**

(Course Work)

Mathematics

Paper II—PERFECT GRAPHS AND GRAPH COLOURING

Time : Three Hours

Maximum : 70 Marks

*Answer either Part A or Part B of each question.**Each question carries 17 ½ marks.***Unit 1**

1. Define an orientation of a graph  $G$ . Also draw all non-isomorphic orientations of a pentagon.
2. Define  $\alpha$  – perfect property of a graph  $G$ . Also show that the complement of an interval graph satisfies the transitive orientation property.

*Or*

3. Let  $G = (V, E)$  be an undirected graph. A subset  $A \subseteq V$  is called an edge cover of  $G$  if for every edge  $xy \in E$ , either  $x \in A$  or  $y \in A$  or both. Prove that  $A$  is a minimum edge cover if and only if  $V - A$  is a maximum stable set.
4. Draw the windmill graph and its interval representation. Also discuss any one of the applications of interval graphs.

**Unit 2**

5. Define a partitionable graph. Also show that a partitionable graph  $G$  contains exactly  $n$  maximum cliques and  $n$  maximum stable sets
6. Let  $G$  be an undirected graph such that every odd cycle has two chords. Then show that  $G$  is perfect.

*Or*

7. Prove that every triangulated graph  $G = (V, E)$  has a simplicial vertex. Moreover, if  $G$  is not a clique, then it has two non-adjacent simplicial vertices.
8. If  $G$  is a  $p$ -critical graph on  $n$  vertices, then show that  $n = \alpha(G) \omega(G) + 1$  and for all vertices  $x$  of  $G$ ,  $\alpha(G) = k(G - x)$  and  $\omega(G) = \chi(G - x)$ .

## Unit 3

9. Show that an undirected graph  $G = (V, E)$  may have at most one colour class which spans all of  $V$ .
10. Prove that every transitive orientation of a comparability graph  $G$  is orientable from some  $G$ -decomposition of  $G$ .

Or

11. How many non-isomorphic graphs are there with the following degree sequences :
- (i)  $[3, 3, 2, 2, 1, 1]$  .
- (ii)  $[5, 5, 4, 3, 3, 3, 2, 1]$  .
12. Let  $[e_1, e_2, \dots, e_k]$  be a scheme for an undirected graph  $G$ , and let  $\pi$  be a permutation of the numbers  $\{1, 2, \dots, k\}$ . Then show that  $[e_{\pi(1)}, e_{\pi(2)}, e_{\pi(3)}, \dots, e_{\pi(k)}]$  is also a scheme for  $G$ .

## Unit 4

13. Define  $\chi(G)$  of a graph. Prove that if the odd cycles in  $G$  are pair wise intersecting, then  $\chi(G) \leq 5$ .
14. If  $G$  is a colour-critical graph, then the graph  $G'$  generated from it by applying Mycielskie's construction is also colour-critical.

Or

15. Prove that the minimum number of edges in a connected  $n$ -vertex graph with chromatic number  $k$  is  $kC_2 + n - k$ .
16. Explain Zykov's construction procedure for colour-critical graphs.

**PH.D. PRELIMINARY QUALIFYING EXAMINATION, JULY 2015**

(Course Work)

Mathematics

Paper II—GRAPH THEORY

Time : Three Hours

Maximum : 70 Marks

*Answer either Part A or Part B of each question.  
Each question carries 17.5 marks.*

**Unit 1**

1. Prove that a graph  $G : (V, E)$ , with order  $n$  is self-complementary if and only if  $n = 4k+1$ , where  $k$  is an integer.
2. Prove that a self complementary graph has a cut vertex if and only if it has a pendant vertex.

*Or*

3. Prove that every loop less graph  $G$  has a spanning  $k$ -partite sub-graph  $H$  such that

$$e(H) \geq \left(1 - \frac{1}{K}\right)e(G).$$

4. Prove that the union of the edge sets of distinct  $u, v$  paths contains a cycle.

**Unit 2**

5. Let  $G$  be a simple graph, which is not complete and not  $k$ - connected. Prove that  $G$  has a separating set of size  $k-1$ .
6. Prove that the Peterson graph is 3- connected.

*Or*

7. For a connected graph  $G$ , with at least 3 vertices, prove that the following statements are equivalent :

- (i)  $G$  is 2- edge connected.
- (ii) Every edge of  $G$  appears in a cycle.
- (iii)  $G$  has a closed trail containing any specified pair of vertices.

8. Prove that every minimally 2- connected graph has a vertex of degree 2.

**Unit 3**

9. Define the ring sum of 2 circuits in a graph  $G$ , and prove that it is either a circuit or an edge disjoint union of circuits.
10. Prove that the dimension of the circuit space  $W_G$  is equal to the nullity  $\mu$  of the graph, and the number of circuit vectors (including 0) in  $W_G$  is  $2^\mu$

*Or*

11. Prove that in a vector space of a graph, the circuit subspace and the cut set subspace are orthogonal to each other. Also illustrate this with an example.
12. In the vector space  $W_G$  (associated with a graph  $G$ ), do the vectors associated with the spanning trees of  $G$  form a vector space over  $GF(2)$ . Explain.

**Unit 4**

13. Define the incidence matrix of a graph, and show that the reduced incidence matrix of a tree is non-singular.
14. Let  $B$  be the circuit matrix of a connected matrix  $G$  with order  $p$  and size  $q$ . Then prove that the rank of  $B = q - p + 1$ .

*Or*

15. Let  $X$  be the adjacency matrix of a simple connected graph. Then prove that the distance between 2 vertices  $v_i$  and  $v_j$  ( $i \neq j$ ) is  $k$  if and only if  $k$  is the smallest integer for which the  $(i, j)^{\text{th}}$  entry in  $X^k$  is non-zero.
16. Show that every circuit has an even number of edges in common with any cut set.

**Ph.D. PRELIMINARY QUALIFYING EXAMINATION  
JANUARY 2015**

(Course Work)

Mathematics

Paper I—RESEARCH METHODOLOGY

Time : Three Hours

Maximum : 70 Marks

*Write all questions.  
All questions carry equal marks.*

**Unit I**

1. A. (a) Write down the  $\text{\LaTeX}$  source code to produce the following

$$\left| \bigvee_{i \in I, j \in J} A_{ij}' \right| = \sum_{i \in I} \sum_{j \in J} |A_{ij}'|.$$

- (b) Write down the  $\text{\LaTeX}$  source code to produce the following

$$\left( \begin{array}{cc|c} x_{11} & x_{12} & \\ x_{21} & x_{22} & \\ \hline & & 0 \\ y & & \\ x & & \end{array} \right).$$

Or

- B. (a) Write down the  $\text{\LaTeX}$  source code to produce the following

$$\begin{aligned} y &= a + b + c + d \text{ if } x = 0 \\ &= e \text{ if } x = 1. \end{aligned}$$

- (b) Write down the  $\text{\LaTeX}$  source code to produce the following

Item	Quantity		Price in Rs.
Rice	300	400	42.50
Sugar	100	90	41.30
Wheat	500	500	32.00

Turn over

## Unit II

2. A. (a) How will you consecutively number theorems, lemma, results and axioms in the output using L<sup>A</sup>T<sub>E</sub>X? Explain using an example.
- (b) Explain the use of cite command for cross reference using example.
- B. (a) Explain the use of the index environment using examples.
- (b) Explain the use of BibTeX to produce bibliography.

## Unit III

3. A. (a) Prove the Riemann-Lebesgue lemma. State its generalized form for any bounded measurable function.
- (b) Prove Cantor-Lebesgue theorem: If a trigonometric series  $\frac{a_0}{2} + \sum_{n \in \mathbb{N}} a_n \cos nx + \sum_{n \in \mathbb{N}} b_n \sin nx$  converges on a set of positive Lebesgue measure, then prove that  $a_n, b_n \rightarrow 0$ .
- B. (a) If  $0 \leq a \leq b \leq \pi$  and  $n \in \mathbb{N} \cup \{0\}$ , show that

$$\int_a^b \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin \frac{t}{2}} dt \leq 4\pi.$$

(b) Show that  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$ .

4. A. (a) If  $f$  is a continuous piecewise smooth  $2\pi$ -periodic function, then prove that its Fourier series converges absolutely and uniformly to  $f$ .
- (b) Give an example of a divergent Fourier series.
- B. (a) Describe the fejer theory.

(b) Show that  $1 - \frac{1}{4} + \frac{1}{9} - \dots = \frac{\pi^2}{12}$ .

## Ph.D PRELIMINARY QUALIFYING EXAMINATION, JANUARY 2015

(Course Work)

Mathematics

Paper II—(Elective)—SEMIGROUPS, GRAPHS AND FUZZY GRAPHS

Time : Three Hours

Maximum : 70 Marks

*Answer either A or B of each question. Each question carries equal marks.*

- I. A (a) Let  $S$  be a semigroup. Prove that  $S$  is a rectangular band if and only if  $S$  is isomorphic to the direct product of a left zero semigroup and a right zero semigroup.
- (b) Let  $(S, \cdot)$  be a commutative semigroup of idempotents. Prove that the relation  $\leq$  on  $S$  defined by :
- $$a \leq b \text{ if and only if } a \cdot b = a$$
- is a partial order on  $S$  and  $S$  is a lower semilattice with respect to the above order.
- (c) Prove that the set  $\mathcal{T}_X$  of all functions from  $X$  into itself is a semigroup with identity element under the binary operation composition of functions. If  $X = \{a, b, c\}$ , then find all idempotents in  $\mathcal{T}_X$ .
- B (a) Prove that in a periodic semigroup  $\mathcal{D} = \mathcal{J}$ .
- (b) Let  $G$  be a group and let  $g \in G$ . Find the  $\mathcal{L}$ -class,  $\mathcal{R}$ -class and  $\mathcal{D}$ -class of  $g$ .
- (c) If  $e$  is an idempotent in a semigroup  $S$ , then prove that the  $\mathcal{H}$ -class of  $e$  is a subgroup of  $S$ . Also prove that no  $\mathcal{H}$ -class contains more than one idempotent.
- II. A (a) Let  $H$  and  $K$  be two group  $\mathcal{H}$ -classes in the same regular  $\mathcal{D}$ -class. Prove that  $H$  and  $K$  are isomorphic.
- (b) Let  $a, b$  be elements in a regular semigroup  $S$ . Prove that  $(a, b) \in \mathcal{L}$  if and only if there exist  $a' \in V(a)$  and  $b' \in V(b)$  such that  $a' a = b' b$ , where  $V(a)$  denotes the set of inverse of  $a$ .
- (c) Let  $S$  be a regular semigroup and let  $e, f$  be idempotents in  $S$ . Prove that the sandwich set  $S(e, f)$  is nonempty.
- B (a) Prove that a semigroup  $S$  is zero simple if and only if  $SaS = S$  for every  $a$  in  $S - \{0\}$ .
- (b) Prove that any finite 0-simple semigroup is completely 0-simple.
- (c) Define completely regular semigroups. Give a characterization for completely regular semigroups.
- III. A (a) Prove that a Cayley graph is vertex transitive.
- (b) Show that the Petersen graph is not a Cayley graph.
- (c) Define edge transitive graphs. Give an example of an edge transitive graph.

Turn over

- B (a) Let  $G$  be a group and let  $e$  be the identity element of  $G$ . If  $G$  acts regularly on the vertices of the graph  $X$ , then prove that  $X$  is a Cayley graph.
- (b) Give an example of a non-Hamiltonian directed Cayley graph.
- (c) Prove that a connected  $s$ -arc transitive graph with girth  $2s - 2$  is distance transitive with diameter  $s - 1$ .
- IV. A (a) Let  $\rho$  and  $\varpi$  be fuzzy relations on a fuzzy subset  $\mu$  of  $S$ . If  $\rho$  and  $\varpi$  are symmetric, then prove that  $\rho \circ \varpi$  is symmetric if and only if  $\rho \circ \varpi = \varpi \circ \rho$ .
- (b) If  $G$  is a fuzzy forest, then prove that the edges of  $F$  are the bridges of  $G$ .
- (c) Let  $(\mu, \rho)$  be a cycle. Prove that  $(\mu, \rho)$  is a fuzzy cycle if and only if  $(\mu, \rho)$  is a fuzzy tree.
- B (a) If  $w$  is a common vertex of at least two bridges, then prove that  $w$  is a cut vertex.
- (b) Prove that  $G = (\mu, \rho)$  is a fuzzy tree if and only if the following are equivalent for all (vertices)  $u, v$  :
- (i)  $(u, v)$  is a bridge.
  - (ii)  $\rho^\infty(u, v) = \rho(u, v)$ , where  $\rho^\infty$  denotes the strength of connectedness between  $u$  and  $v$ .
- (c) Define cycle rank. Give an example of a non-isomorphic fuzzy graphs with the same cycle rank.

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(Pages : 2)

Name.....

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**Ph.D. PRELIMINARY QUALIFYING EXAMINATION, JANUARY 2015**

(Course Work)

Mathematics

Paper II—CECH CLOSURE SPACES AND PERMUTATION GROUPS

Time : Three Hours

Maximum : 70 Marks

*Answer either Part A or Part B of each question.*

*Each question carries 17.5 marks.*

1. A (a) Define a closure space. What is closure preserving family of subsets. Prove that every finite family is closure preserving? Is every locally finite family closure preserving? Justify your answer.  
(b) Define regularly closed and regularly open subsets of a closure space. Give examples and non-examples for each.  
B Define a net in a closure space. Give example. Show that  $x$  is a cluster point of a subset  $Y$  in a closure space  $P$ , if and only if it is a limit point in  $P$  of a net with range  $Y - \{x\}$ .
2. A (a) Define a continuous map from a closure space  $P$  into a closure space  $L$ . State and prove a necessary and sufficient condition for a map  $f : P \rightarrow L$  to be continuous at a point  $x$  in  $P$ .  
(b) What is a topological modification  $v$  of a closure operator  $u$  on a set  $P$ . Show that  $v$  is a topological modification of  $u$  if and only if  $v$  is a closure operator for set  $P$  and each map  $f$  of  $(P, u)$  into a topological space  $Q$  is continuous if and only if  $f$  is continuous as a map of  $(P, v)$  into  $Q$ .
3. A (a) State and prove Cauchy-Frobenius lemma.  
(b) Show that the stabilizer  $G_\alpha$ ,  $\alpha \in \Omega$  form a single conjugacy class of subgroups of  $G$ .  
B (a) Define the automorphism group of a relational structure.  
(b) If the group  $G$  acts transitively on the two sets  $\Omega$  and  $\mu$ , then state and prove a condition for the two actions to be equivalent.
4. A Define and illustrate with examples and non-examples :
  - (a) Stable fixed point.
  - (b) Asymptotically stable fixed point.
  - (c) Neutrally stable fixed point.
  - (d) Chartic function.

Turn over

B (a) Give a simple population model with complicated dynamics.

(b) If  $h$  is topological conjugacy between  $f : X \rightarrow X$  and  $g : Y \rightarrow Y$  and  $x \in X$  is an eventual fixed point of  $f$ , then prove that  $h(x)$  is an eventual fixed point of  $g$ .

## Ph.D. PRELIMINARY QUALIFYING EXAMINATION, JANUARY 2015

(Course Work)

Mathematics

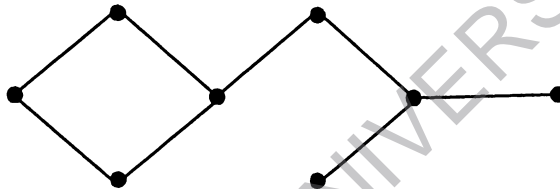
Paper II–Elective—DOMINATION IN GRAPHS

Time : Three Hours

Maximum : 70 Marks

Answer **either A or B** of each question.  
Each question carries **equal** marks.

- I. A (a) Prove that every connected graph  $G$  of order  $n \geq 2$  has a dominating set  $S$  whose complement  $V - S$  is also a dominating set.
- (b) Find the domination number ( $\gamma(G)$ ), upper domination number ( $\Gamma(G)$ ) and independent domination number of the following graph.



- (c) Give an application of domination theory in real life situations.

- B (a) Let  $G$  be a connected claw-free and net-free graph. Prove that  $\gamma(G) \leq \left\lceil \frac{n}{3} \right\rceil$ .

- (b) Prove that for any graph  $G$ ,  $\rho(G) \leq \gamma(G)$ , where  $\rho(G)$  is the packing number of  $G$ .

- (c) Let  $G$  be a graph with no isolated vertices. Prove that

$$\gamma(G) \leq \frac{n(1 + 1n(\delta(G) + 1))}{\delta(G) + 1}.$$

- II. A (a) Let  $G$  be a graph with  $n$  vertices and  $m$  edges. Prove that

$$n - m \leq \gamma(G) \leq n + 1 - \sqrt{1 + 2m}.$$

- (b) Let  $G$  be a graph with no isolated vertices. prove that  $\gamma(G) \leq \beta_2(G)$ , where  $\beta_2(G)$  is the 2-dependence number of  $G$ .

- (c) Find a graph  $G$  with 7 vertices, 10 edges and having  $\gamma(G) = 3$ .
- B (a) Explain hereditary superhereditary properties with the help of an example.
- (b) Prove that every maximal independent set in graph  $G$  is a minimal dominating set.
- (c) Prove that a dominating set  $S$  is a minimal dominating set if and only if it is dominating and irredundant.
- III. A (a) Prove that for any graph  $G$ ,  $\Theta(G) = \Gamma(G)$ , where  $\Theta(G)$  is the upper perfect neighborhood number of  $G$ .
- (b) Prove that, for any tree  $T$  with  $n \geq 2$ , there exists a vertex  $v \in V$  such that  $\gamma(T - v) = \gamma(T)$ .
- (c) Prove that a graph  $G$  is in CER if and only if  $G$  is a galaxy, where
- $$\text{CER} = \{G : \gamma(G - e) \neq \gamma(G) \text{ for all } e \in E(G)\}.$$
- B (a) Give an example of graph in CEA with order 8, where
- $$\text{CEA} = \{G : \gamma(G + e) \neq \gamma(G) \text{ for all } e \in E(G)\}.$$
- (b) Give a characterization of graphs in UVR, where
- $$\text{UVR} = \{G : \gamma(G - v) = \gamma(G) \text{ for all } v \in G\}.$$
- (c) Let  $D$  be symmetric digraph. Prove that  $S \subset V$  is a kernel if and only if  $S$  is a maximal independent set.
- IV. A (a) If a graph  $G$  does not contain the star  $K_{1, k+1}$ ,  $k \geq 2$  as an induced graph, then prove that  $i(G) \leq (k-1)\gamma(G) - (k-2)$ , where  $i(G)$  is the independent domination number of  $G$ .
- (b) If  $G$  is a connected graph with  $n \geq 3$  vertices, then prove that
- $$\gamma_t(G) \leq \frac{2n}{3}, \text{ where } \gamma_t(G) \text{ is the total domination number of } G.$$
- B (a) If  $G$  is a connected graph with  $n \geq 3$  vertices, then prove that
- $$\gamma_c(G) = n - \epsilon_T(G) \leq n - 2, \text{ where } \gamma_c(G) \text{ is the connected domination number of } G.$$
- (b) Give an example of a dominating cycle.
- (c) Let  $G$  be a graph with  $\delta(G) \geq k$ . Prove that  $k$ -domination number  $\gamma_k(G) = \min \{r_k(H)\}$  where the minimum is taken over all spanning  $k$ -bipartite subgraphs  $H$  of  $G$ .