D 33001 **(Pages : 3)** Name.......................................

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 01—ALGEBRA—I

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

*Answer **all** questions.*
Each question carries 2 marks.

1. Find the order of the element $(2,3)$ in $Z_6 \times Z_{15}$.

2. Compute the factor group $\mathbb{Z}/\{0\}$.

3. Let $G = S_3$ act on the set $\{1,2,3\}$ by $\sigma x = \sigma(x)$ for $\sigma \in S_3$ and $x \in X$. Find the isotropy group G_x for $x = 2$.

4. Find the number of orbits in $\{1, 2, 3, 4, 5, 6, 7, 8\}$ under the cyclic subgroup $<(1, 3, 5, 6)>$ of S_8.

5. How many polynomials are there of degree less than or equal to 2 in $Z_5[x]$.

6. Prove that a field contains no proper non-trivial ideals.

7. Show that a finite extension field E of a field F is an algebraic extension of F.

8. Find the degree and a basis for $Q\left(\sqrt{2}, \sqrt[3]{2}\right)$ over Q.

(8 × 2 = 16 marks)

Part B

*Answer any **four** questions.*
Each question carries 4 marks.

9. State and prove fundamental homomorphism theorem.

10. Define the centre of a group. Find the centre of the group S_3.

Turn over

11. The edges of an equilateral triangle can be painted with four different colours of paint. Also assume that only one colour is used on each edge and the different colours must be used on different edges. Find the number of distinguishable ways the triangle can be painted.

12. Prove that the multiplicative group of all non-zero elements of a finite field is cyclic.

13. Let $\varphi : R \to R'$ be a ring homomorphism with kernel H. Prove that the additive cosets of H form a ring R/H.

14. State and prove a necessary and sufficient condition for a field F to be algebraically closed.

(4 × 4 = 16 marks)

Part C

Answer **either** *part A* **or** *part B each of the four questions.*
Each question carries 12 *marks.*

15. A a) Let m, n be relatively prime. Show that $Z_m \times Z_n$ is cyclic if and only if m and n are relatively prime.

b) Show that M is a maximal normal subgroup of a group G if and only if G/M is simple.

B Let H be a subgroup of a group G. Show that the following are equivalent :

(a) $ghg^{-1} \in H, \forall g \in G, h \in H$.

(b) $ghg^{-1} = H, \forall g \in G$.

(c) $gH = Hg, \forall g \in G$

16. A Let X be a G-set and let $x \in X$. Prove that :

i) $|Gx| = (G : G_x)$

ii) If $|G|$ is finite, then $|Gx|$ is a divisor of $|G|$.

B State and prove Burnside's formula.

17. A a) State and prove an irreducibility criteria for a polynomial $f(x) \in Z[x]$ over Q.

b) Show that the cyclotomic polynomial $\varphi_p(x)$ is irreducible over Q for any prime p.

B a) Prove factor theorem for polynomials over a field F.

b) Let F be the ring of all functions mapping $\mathbb{R}$ into $\mathbb{R}$. Let N be the subring of all functions f such that $f(2)=0$. Is N an ideal in ? Why ? or why not ?

18. A a) State and prove Kronecker's theorem.

b) Let F be a field of prime characteristic p with algebraic closure $\overline{F}$. Prove that $x^{p^n}-x$ has atmost p^n distinct zeroes in $\overline{F}$.

c) Prove that if α, β are constructible numbers then $\alpha\beta, \alpha/\beta$ are constructible.

B a) Let E be an extension field of F with $\alpha \in E$ algebraic over F. Show that there exists an unique irreducible polynomial for α. Also prove that if $f(\alpha)=0$ for some $f(x) \in F[x]$ with $f(\alpha)=0$ then $p(x)$ divides $f(x)$.

b) Suppose F_i is a field for $i=1, 2 \ldots r$ and F_{i+1} is a finite extension of F_i. Prove that F_r is a finite extension of F_1 and $[F_r : F_1]=[F_r : F_{r-1}][F_{r-1} : F_{r-2}] \ldots\ldots [F_2 : F_1]$.

c) Find all the primitive fifth roots of unity in Z_{11}.

(4 × 12 = 48 marks)

302902

D 33006 (Pages : 3) Name.......................................

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 01—ALGEBRA – I

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Find the order of the element $(3,10,9)$ in $Z_4 \times Z_{12} \times Z_{15}$.

2. Compute the factor group $\frac{Z_2 \times Z_4}{\langle(0,1)\rangle}$.

3. State First Isomorphism Theorem.

4. Establish that any group of order 15 is cyclic.

5. Find all composition series of $S_3 \times Z_2$.

6. Show that both $\{1\}$ and $\{2,3\}$ are bases for Z_6.

7. Show that if $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is in Z $[x]$ with $a_0 \neq 0$, and if $f(x)$ has a zero in Q, then it has a zero m in Z.

8. Give an example to show that a factor ring of a ring with divisors of zero may be an integral domain.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Show that if H and N are subgroups of a group G, and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

10. Find all abelian groups, upto isomorphism, of order 1049.

Turn over

11. Show that every groups of order p^2, p a prime, is abelian.

12. Let H and K be normal subgroups of a group G with $K \leq H$. Show that $G/H \simeq (G/K)/(H/K)$.

13. Show that a subgroup of a solvable group is solvable.

14. Let a group G be generated by $A = \{a_i : i \varepsilon I\}$ and let G' be any group. Show that if a_i' for $i \varepsilon I$ are any elements in G', not necessarily distinct, then there is at most one homomorphism $\phi : G \to G'$ such that $\phi(a_i) = a_i'$. Further show that if G is free on A, then there is exactly one such homomorphism.

15. Let F be a field. Show that a non-zero polynomial $f(x) \varepsilon F[x]$ of degree n can have at most n zeros in F.

16. Demonstrate that $x^3 + 3x^2 - 8$ is irreducible over Q.

17. Let R be a ring with unity. Show that :
    (i) if R has characteristic $n > 1$, then R contains a subring isomorphism to Z_n.
    (ii) if R has characteristic 0, then R contains a subring isomorphic to Z.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) Let H be a subgroup of a group G. Show that $g h g^{-1} \in H$ for all $g \in G$ and $h \in H$ iff $g H g^{-1} = H$ for all $g \in G$.

    (b) Let G be a group. Show that the set of all commutators $a b a^{-1} b^{-1}$ for $a, b \varepsilon G$ generates a normal subgroup C of G. Further show that if N is a normal subgroup of G, then G/N is abelian iff $C \leq N$.

19. (a) Show that if G is a finite group and a prime p divides $|G|$, then the number of Sylow p-subgroups is congruent to 1 modulo p and divides $|G|$.

    (b) Show that A_4 has no subgroup of order 6.

20. (a) Let p and q be distinct primes with $p < q$ and let G be a group of order pq. Show that G is not simple. Further show that if q is not congruent to 1 modulo p, then G is abelian and cyclic.

(b) Show that no group of order 36 is simple.

21. (a) Let F be a subfield of a field E, let α be any element of E, and let x be an indeterminate. Show that the map $\phi_\alpha : F[x] \to E$ defined by :

$\phi_\alpha(a_0 + a_1x + \ldots + a_nx^n) = a_0 + a_1\alpha + \ldots + a_n\alpha^n$ for $(a_0 + a_1x + \ldots + a_nx^n) \varepsilon F[x]$ is a homomorphism of $F[x]$ into E. Also, show that $\phi_\alpha(x) = \alpha$, and ϕ_α maps F isomorphically by the identity map.

(b) Show that $x^3 + 3x + 2 \varepsilon Z_5[x]$ is irreducible over Z_5.

22. (a) Let $p \in Z$ be a prime. Suppose that $f(x) = a_nx^n + \ldots + a_0$ is in Z [x], and $a_n \not\equiv 0 (\text{mod}\, p)$, but $a_i \equiv 0 (\text{mod}\, p)$ for all $i < n$, with $a_0 \not\equiv 0 (\text{mod}\, p^2)$. Show that $f(x)$ is irreducible over Q.

(b) Let F be a field. Show that an ideal $\langle p(x) \rangle \neq \{0\}$ is maximal iff $p(x)$ is irreducible over F.

(3 × 8 = 24 marks)

D 33004 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 04—DISCRETE MATHEMATICS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Prove that a bipartite graph contains no odd cycles.
2. Define a Hamiltonian graph. Give an example of a Hamiltonian graph with 6 vertices.
3. Define (i) Lattice ; and (ii) Lattice diagram. Illustrate with an example.
4. Write the following Boolean function in its disjunctive normal form :

   $$f(a, b, c) = (a + b + c)(a' + b + c')(a + b' + c')(a' + b' + c)(a + b + c').$$

5. Define dfa.
6. Define language accepted by an nfa.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 *marks.*

7. Prove that in a connected graph G with at least three vertices, any *two* longest paths have a vertex in common.
8. Prove that in any group of n persons $(n \geq 2)$, there are at least two with the same number of friends.
9. Define a chain in a poset. Prove that the intersection of two chains is a chain.

Turn over

10. Let $(X, +, \cdot, ')$ be a finite Boolean algebra. Then prove that every non-zero element of X contains at least one atom.

11. Let $(X, +, \cdot, ')$ be a Boolean algebra. Then prove the Laws of Tautology and Associate Laws.

12. Prove that $(L_1L_2)^R = L_2^R \, L_1^R$.

13. Find a dfa that recognize the set of all strings on $\sum = \{a, b\}$ starting with the prefix ba.

14. Show that the language $L = \{awa : w \in \{a, b\}^*\}$ is regular.

(5 × 4 = 20 marks)

Part C

Answer **either** *A* or *B of each questions.*
Each question carries 16 *marks.*

15. A. (a) Prove that the following statements are equivalent :

(i) G has exactly one cycle ;

(ii) G is connected and $n = m$;

(iii) For some edge e of G, $G - e$ is a tree ; and

(iv) G is connected and the set of edges of G that are not cut edges forms a cycle.

(b) Describe the common features of K_5 and $K_{3,3}$.

B. (a) For a simple graph G with n vertices, $n \geq 2$, which is complete, prove that $k(G) = n - 1$.

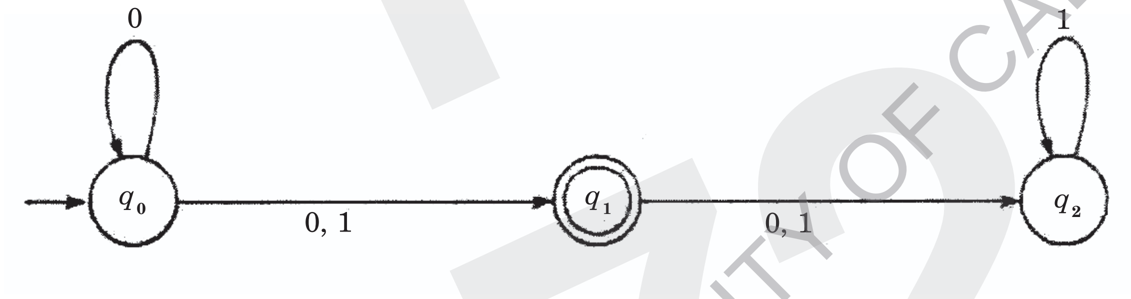
(b) Prove that K_5 and $K_{3,3}$ are non-planar.

16. A. Let $(X, \leq)$ be a poset and A be a non-empty, finite subset of X. Then prove that A has at least one maximal element. Also, prove that A has a maximum element if and only if it has a unique maximal element.

B. Let X be a finite set and $\leq$ be a partial order on X. Define a binary relation R on X by xRy if and only if y covers x. Then prove that $\leq$ is generated by R.

17. A. Let $M = (Q, \Sigma, \delta, q_0, F)$ be a deterministic finite accepter, and let G_M be its associated transition graph. Then prove that for every $q_i, q_j \in Q$, and $w \in \Sigma^+$, $\delta^*(q_i, w) = q_j$ if and only if there is in G_M a walk with label w from q_i to q_j.

B. Convert the nfa in the following table into an equivalent deterministic machine.



(3 × 16 = 48 marks)

D 33002 **(Pages : 4)** **Name...............................**

Reg. No.............................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 02—LINEAR ALGEBRA

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Define vector space.
2. Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for R^3 consisting of $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 1, 1), \alpha_3 = (1, 0, 0)$. What are the co-ordinates of the vector (a, b, c) in the ordered basis B.
3. Let F be a field and T be a linear operator on F^2 defined by $T(x_1, x_2) = (x_1, 0)$. Find $[T]_B$ where B is the standard basis for F^2.
4. Let V be a vector space over the field F. If $S = V$ and S is a subset of V then prove that S^0 is the zerosubspace of V^*.
5. Prove that similar matrices have same characteristic polynomials.
6. Define invariant subspace and give an example for invariant subspace.
7. Let $T : R^2 \to R^2$ defined by $T(x, y) = (x, 0)$. Then prove that T is a projection.
8. Define an innerproduct space.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let V be a vector space which is spanned by a finite set of vectors $\beta_1, \beta_2, ... \beta_m$. Then prove that any independent set of vectors in V is finite and contains no more than m elements.

10. Let V be a finite dimensional vector space over the field F. For each vector α in V define $L_\alpha(f) = f(\alpha)$ for f in V* then prove that the mapping $\alpha \to L_\alpha$ is an isomorphism of V into V**.

11. Let T be a linear operator on a finite dimensional vector space V and let c be a scalar. Then prove that the following are equivalent :

    (i) c is a characteristic value of T.

    (ii) The operator $(T - cl)$ is singular.

    (iii) $\det(T - cl) = 0$.

12. Let T be a diagonalizable linear operator on the n-dimensional vector space V, and let W be a subspace which is invariant under T. Prove that the restriction operator T_W is diagonalizable.

13. Prove that an orthogonal set of nonzero vectors is linearly independent.

14. Let $E_1, ..., E_k$ be linear operators on the vector space V such that $E_1 + ... + E_k = I$. Prove that if $E_i E_j = 0$ for $i \neq j$, then $E_i^2 = E_i$ for each i.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 *marks.*

15. A (a) Let V be an n-dimensional vector space over the field F and let B and B′ are ordered basis for V. Prove that there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that $[\propto]_B = P[\propto]_{B'}$ and $[\propto]_{B'} = P^{-1}[\propto]_B$ for every $\propto$ in V.

    (b) Show that if A is an $m \times n$ matrix with entries in the field F, then row rank (A) = column rank (A).

B (a) If W_1 and W_2 are finite dimensional subspaces V, then $W_1 + W_2$ is finite dimensional and

$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

(b) Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has four elements.

16. A Let V be an n-dimensional vector space over the field F and let W be an m-dimensional vector space over F. Then the space $L(V, W)$ is finite dimensional and has dimension mn.

B Let V, W and Z be finite dimensional vector spaces over the field F, let T be a linear transformation from V into W and U a linear transformation from W into Z. If B, B' and B'' are ordered bases for the spaces V, W and Z, respectively, if A is the matrix of T relative to the pair B, B' and B is the matrix of U relative to the pair B', B'', then prove that the matrix of the composition UT relative to the pair B, B'' is the product matrix $C = BA$.

17. A Let T be a linear operator on a finite-dimensional space V. Let $c_1, c_2, \ldots, c_k$ be the distinct characteristic values of T and let W_i be the null space of $(T - c_i l)$. Then prove that the following are equivalent :

(i) T diagonalizable.

(ii) The characteristic polynomial for T is $f = (x - c_1)^{d1} \ldots (x - c_k)^{dk}$ and $\dim W_i = d_i, i = 1, 2, \ldots k$.

(iii) $\dim W_i + \ldots + \dim W_k = \dim V$.

B State and prove Cayley Hamilton theorem.

18. A Let V be a finite dimensional vector space. Let $W_1, \ldots, W_k$ be subspaces of V and $W = W_1 + \ldots + W_k$. Then prove that the following are equivalent :

(a) $W_1, \ldots, W_k$ are independent.

(b) For each $j, 2 \le j \le k$, we have $W_j \cap (W_1 + \ldots + W_{j-1}) = \{0\}$.

(c) If B_i is an ordered basis for $W_i, 1 \le i \le k$, then the sequence $B = (B_1, \ldots, B_k)$ is an ordered basis for W.

Turn over

B (a) Let W be a subspace of an inner product space V and let β be a vector in V :

(i) The vector α in W is a best approximation to β by vectors in W if and only if β-α is orthogonal to every vector in W.

(ii) if a best approximation to β by vectors in W exists, it is unique.

(iii) If W is finite dimensional and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ is any orthonormal basis for W, then the vector $\alpha = \sum_k \frac{(\beta|\alpha_k)}{\|\alpha_k\|^2}\alpha_k$ is the best approximation to β by vectors in W.

(b) Let V be an inner product space, W a finite dimensional subspace and E the orthogonal projection of V on W. Then prove that the mapping $\beta \to \beta - E\beta$ is the orthogonal projection of V on $W^{\perp}$.

(4 × 12 = 48 marks)

D 33007 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 02—LINEAR ALGEBRA

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Verify whether the set of all polynomials of degree 2 over $\mathbb{R}$ is a vector space over $\mathbb{R}$.

2. Verify whether the set of all matrices of the form $\begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}$ with $x, y \in \mathbb{R}$ is a subspace of the space of all 2×2 matrices over $\mathbb{R}$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (2x + y, x + 2y)$. Verify whether T is a linear transformation.

4. Give an example of a non-zero linear functional on $\mathbb{R}^2$.

5. Find a characteristic vector of the linear operator on $\mathbb{R}^2$ given by $T(x, y) = (x + y, x + y)$.

6. Let T be a linear operator on $\mathbb{R}^3$ given by $T(x, y, z) = (x + y, x + y, z)$. Find a one dimensional T-invariant subspace of $\mathbb{R}^3$.

7. Verify whether $T(x, y) = (x + y, 0)$ is a projection operator on $\mathbb{R}^2$.

8. Let V be an inner product space with inner product $(\mid)$. Show that if $(\alpha \mid \beta) = 0$ for all $\beta \in V$ then $\alpha = 0$.

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Verify whether $(1,2,3) \in \mathbb{R}^3$ is in the span of the set {(1, 1, 0), (0, 0, 1)}.

10. Let V be a vector space over a field F and let $\alpha, \beta \in V$. Show that $W = \{c\alpha + d\beta : c, d \in F\}$ is a subspace of V.

Turn over

11. Show that {(1, 2 , 3), (1, 3, 2), (2, 1, 3)} is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

12. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of a vector space V and $T : V \to V$ be a one to one linear transformation. Show that $\{T(e_1), T(e_2), \ldots, T(e_n)\}$ is also a basis of V.

13. Let f, g be linear functionals on a vector space V and let N (f) = N (g) where N (f) and N (g) are null spaces of f and g respectively. Show that $f = \alpha g$ for some scalar α.

14. Let T be a linear operator on a vector space V and let $\alpha \in V$. Show that if $T(\alpha) = c\alpha$ for some scalar c then $(T^2 + T)(\alpha) = (c^2 + c)\alpha$.

15. Find the minimal polynomial of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

16. Let W_1, W_2 be subspaces of a vector space V and let $V = W_1 \oplus W_2$. Find projections E_1 and E_2 onto W_1 and W_2 respectively such that $E_1 + E_2 = I$ where I is the identity transformation.

17. Let W ={(x, x) : $x \in \mathbb{R}$} be a subspace of $\mathbb{R}^2$ where $\mathbb{R}^2$ has the usual inner product. Find a best approximation for (1,2) in W.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. Let V be a vector space over a field F and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a subset of V.

Let $W = \{x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n : x_i \in F\}$. Prove that :

(a) W is a subspace of V.

(b) $S \subseteq W$.

(c) If U is any subspace of V such that $S \subseteq U$ then $W \subseteq U$.

19. Let V be a finite dimensional vector space and let dim V = n. Prove that :

(a) any set of $n + 1$ vectors in V is linearly dependent.

(b) if S is a subset of $n - 1$ vectors of V then span of S is not equal to V.

20. Let V be a finite dimensional vector space and V* be the dual of V. For each $\alpha \in V$ define L_α on V^* by :

    $L_\alpha(f) = f(\alpha)$ for all $f \in V^*$. Show that

    (a) L_α is a linear functional on V^*.

    (b) The map $\alpha \mapsto L_\alpha$ is an isomorphism from V onto V**.

21. (a) Define characteristic polynomial and minimal polynomial of an $n \times n$ matrix.

    (b) Let $f(x)$ be the characteristic polynomial and $p(x)$ be the minimal polynomial of an $n \times n$ matrix A. Prove that :

    (i) if $p(c) = 0$ for a scalar c then c is a characteristic value of A.

    (ii) if $f(c) = 0$ then $p(c) = 0$.

22. (a) Define orthogonal basis in an inner product space.

    (b) Show that every finite dimensional inner product space has an orthonormal basis.

    (c) Give an orthonormal basis for $\mathbb{R}^2$ with usual inner product.

(3 × 8 = 24 marks)

D 33010 **(Pages : 3)** **Name..**

Reg. No......................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 05—NUMBER THEORY

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Suppose that Q and Q′ are odd and positive. Prove that $\left(\frac{P}{Q}\right)\left(\frac{P}{Q'}\right)=\left(\frac{P}{QQ'}\right)$.

2. Find the smallest integer x for which $\phi(x)=6$.

3. Convert the continued fraction $\langle 0,1,1,100\rangle$ into a rational number.

4. Expand $\sqrt{3}$ as an infinite simple continued fraction.

5. Define the von Mangoldt function $\Lambda(n)$. Also define the functions $\psi(x)$ and $\vartheta(x)$.

6. What is a Dirichlet series ? Give an example.

7. In the 27-letter alphabet (with blank = 26), use the affine enciphering transformation with key $a=13, b=9$ to encipher the message "HELP ME".

8. Define the term 'trapdoor function' as used in public key cryptosystem.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question carries 3 marks.

9. Find all primes p such that $\left(\frac{10}{p}\right) = 1$.

10. Define the divisor function $d(n)$. For each positive integer n, prove that $d(n) = \prod_{p^{\alpha}|n} (\alpha + 1)$.

11. If f and g are totally multiplicative functions such that $f(p) = g(p)$ for all primes p, then prove that $f = g$.

12. Prove that any finite simple continued fraction represents a rational number.

13. Let x be any real number greater than 1. Prove that the n^{th} convergent of $1/x$ is the reciprocal of the $(n-1)^{\text{th}}$ convergent of x.

14. Suppose that $x \geq 2$. Prove that

$$\int_1^x \frac{\psi(u)}{u^2}\, du = \log x + O(1).$$

15. Let σ_a be the abscissa of absolute convergence of the Dirichlet series $A(s) = \sum_{n=1}^{\infty} a_n/n^s$. Prove that $A(s)$ is a continuous function on the open interval $(\sigma_a, +\infty)$.

16. Briefly describe digraph transformations.

17. Find the inverse of the matrix $\begin{pmatrix} 15 & 17 \\ 4 & 9 \end{pmatrix}$ mod 26.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) State and prove Gauss Lemma.

(b) Determine whether $x^2 \equiv 10 \pmod{89}$ has a solution.

19. (a) Let p be a prime. Prove that the largest exponent e such that $p^e \mid n!$ is $e = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]$.

(b) Find a positive integer n such that $\mu(n) + \mu(n+1) + \mu(n+2) = 3$.

20. (a) Prove that the value of any infinite simple continued fraction $\langle a_0, a_1, a_2, \ldots\rangle$ is irrational.

(b) Let ξ be an irrational number and let a/b be a rational number with $b > 0$. If $|\xi b - a| < |\xi k_n - h_n|$ for some $n \geq 0$, then prove that $b \geq k_{n+1}$.

21. (a) If x is a real number, $x > 1$, then prove that there exists atleast one prime number in the open interval $(x, 2x)$.

(b) Suppose that $x \geq 2$. Prove that $\sum_{p \leq x} \frac{1}{p} = \log \log x + b + O(1/\log x)$,

where b is a constant.

22. (a) The message "KVW ? TA!KJB?FVR " (The blanks after ? and R are part of the message) was intercepted. It is known that a linear enciphering transformation is being used with a 30-letter alphabet, in which A-Z have numerical equivalents 0 – 25, blank = 26, ? = 27, ! = 28, . = 29. It is also known that the first six letters of the plaintext are "C.I.A." . Find the deciphering matrix and the full plaintext message.

(b) Briefly describe the procedure to send a signature in RSA cryptosystem.

(3 × 8 = 24 marks)

D 33005 **(Pages : 4)** **Name..**

Reg. No......................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 05—NUMBER THEORY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. If f is a non-zero multiplicative function then prove that $f(1) = 1$.

2. For $n \geq 1$, show that $\log n = \sum_{d|n} \Lambda(d)$.

3. For $x \geq 1$, show that $\sum_{n > x} \frac{1}{n^s} = o\left(x^{1-s}\right)$ if $s > 1$.

4. Evaluate the Legendre's symbol $\left(\frac{7}{11}\right)$.

5. Describe briefly about RSA cryptosystems.

6. How do classical and public Key cryptosystem differ ?

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Prove that if $2^n + 1$ is prime, then n is a power of 2.

8. For $x \geq 2$, show that $\vartheta(x) = \pi(x)\log(x) - \int_x^2 \frac{\pi(t)}{t}\,dt$.

Turn over

9. Show that two lattice points (a, b) and (m, n) are mutually visible if and only if $a - m$ and $b - n$ are relatively prime.

10. Let p be an odd prime. Then for all n prove that $(n \mid p) \equiv n^{\frac{p-1}{2}} \pmod{p}$.

11. If f is an arithmetical function with $f(1) \neq 0$, show that there is a unique arithmetical function f^{-1} such that $f * f^{-1} = f^{-1} * f = I$, where I is the identity function.

12. Determine whether 888 is a quadratics residue or nonresidue of the prime 1999.

13. Briefly describe about digraph transformation.

14. Explain enciphering and deciphering transformation.

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each questions.*
Each question carries 16 *marks.*

15. (A) (a) Given f with $f(1) = 1$. Then prove that :

i) f is multiplicative if and only if $\left(p_1^{a_1} \dots p_2^{a_r}\right) = f\left(p_1^{a_1}\right) \dots f\left(p_1^{a_r}\right)$ for all primes p_i and all integers $a_i \geq 1$.

ii) If f is multiplicative, then f is completely multiplicative if and only if $f\left(p^a\right) = f(p)^a$ for all primes p and all integers $a \geq 1$.

(b) State and prove Quadratic Reciprocity Law.

(B) (a) For all $x \geq 1$, show that $\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$.

(b) Show that the set of lattice points visible from the origin has density $\frac{6}{\pi^2}$.

(c) Prove that the power function $N_\alpha(n) = n^\alpha$, where α is a fixed real or complex number is completely multiplicative.

16. (A) (a) State and prove Euler's summation formula.

(b) If $x \geq 1$, then prove that :

(i) $$\sum_{n \leq x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right).$$

(ii) $$\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O\left(x^{\alpha}\right) \text{ if } \alpha \geq 0.$$

(B) (a) Let p_n denote the n^{th} prime. Show that the following relations are logically equivalent :

(i) $$\lim_{x \to \infty} \frac{\pi(x) \log(x)}{x} = 1.$$

(ii) $$\lim_{x \to \infty} \frac{\pi(x) \log \pi(x)}{x} = 1.$$

(iii) $$\lim_{x \to \infty} \frac{p_n}{n \log n} = 1.$$

(b) State and prove Gauss' lemma.

17. (A) (a) You know that your adversary is using a cryptosystem with a 27-letter alphabet, in which the letters A-Z have numerical equivalents 0-25, and blank = 26. Each digraph then corresponds to an integer between 0 and 728 according to the rule that, if the two letters in the digraph have numerical equivalents x and y, then the digraph has numerical equivalent $27x + y$. Suppose that a study of a large sample of ciphertext reveals that the most frequently occurring digraphs are (in order) "ZA", : "IA", and "IW". Suppose that the most common digraphs in the English language (for text written in our 27-letter alphabet) are "E " (i.e., "E blank"), "S ", "T ". You know that the cryptosystem uses an affine enciphering transformation modulo 729. Find the deciphering key, and read the message "NDXBHO". Also find the enciphering key.

(b) You intercept the message

"FBRTLWUGAJQINZTHHXTEPHBNXSW"

which you know was encoded using a linear enciphering transformation of trigraphs in the 26-letter alphabet A-Z with numerical equivalents 0-25. You also know that the last three trigraphs are the sender's signature "JAMESBOND." Find the deciphering matrix and read the message.

(B) (a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/n\mathbb{Z})$ and set $D = ad - bc$. Then prove that the following are equivalent :

(i) $\text{g.c.d.}(D, N) = 1$;

(ii) A has an inverse matrix ;

(iii) if x and y are not both 0 in $\mathbb{Z}/n\mathbb{Z}$, then $A\begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$;

(iv) A gives a 1-to-1 correspondence of $(\mathbb{Z}/n\mathbb{Z})^2$ with itself.

(b) You are trying to cryptanalyze an affine enciphering transformation of single-letter message units in a 37-letter alphabet. This alphabet includes the numerals 0-9, which are labeled by themselves (i.e., by the integers 0-9). The letters A-Z have numerical equivalents 10-35, respectively, and blank = 36. You intercept the ciphertext "OH7F86BB46R3627O266BB9" (here the O's are the letter "oh", not the numeral zero). You know that the plaintext ends with the signature "007" (zero zero seven). What is the message ?

(3 × 16 = 48 marks)

D 33008 **(Pages : 3)** **Name..**

Reg. No......................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 03—REAL ANALYSIS—I

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *the questions.*
Each question carries 1 *mark.*

1. Define closure of a subset E denoted by $\bar{E}$ in a metric space. Prove that if E is closed then $E = \bar{E}$.
2. Prove that compact subsets of metric spaces are closed.
3. Prove that the set of all real numbers is uncountable.
4. Prove that continuous image of a compact metric space is compact.
5. Let E be a non-compact set in R^1. Then prove that there exists a continuous function on E which is not bounded.
6. Let f be defined on $[a, b]$. If f is differentiable at a point x in $[a, b]$, then prove that f is continuous at x.
7. State Taylor's theorem.
8. If $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then prove that f is continuous on E.

(8 × 1 = 8 marks)

Turn over

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Define limit point of a set E in a metric space. If p is a limit point of a set E, then prove that every neighbourhood of p contains infinitely many points of E.

10. Prove that the intersection of an infinite collection of open sets need not be open. Is the intersection of a finite collection of open sets open ? Justify your claim.

11. Define uniformly continuous mapping on a metric space. If f is a continuous mapping of a compact metric space X into a metric space Y, then prove that f is uniformly continuous on X.

12. Define discontinuities of the second kind. Prove that monotonic functions have no discontinuities of the second kind.

13. Prove that x^n is differentiable and that its derivative is nx^{n-1}.

14. Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$ and if $f'(x)$ exists, then prove that $f'(x) = 0$.

15. If f is continuous on $[a, b]$, then prove that $f \in R(\alpha)$ on $[a, b]$.

16. If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$, then prove that $fg \in R(\alpha)$.

17. Prove that the sequence of functions $\{f_n\}$ defined on E converges uniformly on E if and only if for every $\in > 0$ there exists an integer N such that $m \geq N, n \geq N, x \in E$ implies $|f_n(x) - f_m(x)| \leq \in$.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 *marks.*

18. (a) State and prove Heine-Borel theorem.

(b) Prove that every bounded infinite subset of R^k has a limit point in R^k.

19. (a) Let f be a continuous real function on a metric space X, then prove that the zero space of f is closed.

    (b) Define uniformly continuous function. Prove that a uniformly continuous function of a uniformly continuous function is uniformly continuous.

20. (a) Let f be defined for all real x and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all real x and y. Prove that f is a constant.

    (b) If f is differentiable on $[a, b]$, then prove that f' cannot have any simple discontinuity.

21. (a) Define the Riemann integral of a function f over $[a, b]$.

    (b) If f maps $[a, b]$ into R^k and if $f \in R(\alpha)$ for some monotonically increasing function α on $[a, b]$, then prove that $|f| \in R(\alpha)$, and $\left|\int_a^b f \, d\alpha\right| \leq \int_a^b |f| \, d\alpha$.

22. Suppose K is compact and :

    (a) $\{f_n\}$ is a sequence of continuous functions on K,

    (b) $\{f_n\}$ converges pointwise to a continuous function f on K,

    (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \ldots$

    Then prove that $f_n \to f$ uniformly on K.

(3 × 8 = 24 marks)

D 33003 **(Pages : 3)** **Name.......................................**

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION, NOVEMBER 2022

(CCSS)

Mathematics

MAT 1C 03—REAL ANALYSIS—I

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Let E a non-empty subset of the real numbers which is bounded above and $y = \sup E$. Then prove that $y \in \bar{E}$.
2. Define Cantor set and prove that it is perfect.
3. Define uniform continuity. Give an example of a function which is not uniformly continuous.
4. If f and g are differentiable functions defined on $[a, b]$. Then prove that fg is differentiable.
5. Define Riemann-Stieltjes integral.
6. Show that limit can not be inter changeable in a double sequence.
7. State Cauchy's criteria for uniform convergence of series of functions.
8. Write Weierstrass condition for uniform convergence of sequence of functions.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Prove that closed subsets of a compact set are compact.
10. State and prove a characterisation theorem for connected subsets of the real line.

Turn over

11. If f is a continuous mapping from a metric space X into a metric space Y, and if E a connected subset of X . Then show that $f(\mathrm{E})$ is connected in Y

12. Define uniform convergence. Whether the sequence of functions $f_n(x) = x^n, \forall x \in [0, 1]$ uniformly convergent or not. Justify.

13. Suppose $f \in \mathrm{R}(\alpha)$ on $[a, b]$ and $\varepsilon > 0 \ \exists$ a partition P of $[a, b]$ such that $\mathrm{U}(\mathrm{P}, f, \alpha) - \mathrm{L}(\mathrm{P}, f, \alpha) < \varepsilon$.

    Then prove that for any $t_i \in [x_{i-1}, x_i] \left| \sum f(t_i)\Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon.$

14. Prove that C (X) the class of all complex valued bounded continuous functions form a complete metric space under supermum norm.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 marks.

Unit 1

15. A (a) State and prove Heine Borel theorem.

    (b) Suppose $\mathrm{Y} \subset \mathrm{X}$. A subset E of Y is open relative to Y if and only if $\mathrm{E} = \mathrm{Y} \cap \mathrm{G}$ for some open set G of X.

    B (a) Prove that every bounded infinite set in R^k has a limit point.

    (b) Suppose $\mathrm{K} \subset \mathrm{Y} \subset \mathrm{X}$. Then prove that K is compact relative to Y if and only if K is compact relative to X.

Unit 2

16. A (a) Prove that a mapping f from a metric space X into a metric space Y is continuous iff $f^{-1}(\mathrm{V})$ is open in X for every open set V in Y.

    (b) State and prove Intermediate value theorem for continuous functions.

B (a) Let f be a continuous mapping from a compact metric space X into a compact metric space Y. Then prove that f is uniformly continuous in X.

(b) State and prove the generalised Mean value theorem.

UNIT 3

17. A (a) Assumes α increase monotonically and $\alpha' \in R$ on $[a, b]$. Let f be a bounded real function on $[a, b]$, then prove that $f \in R(\alpha)$ on $[a, b]$ if and only if

$$f\alpha' \in R \text{ on } [a, b] \text{ and } \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)\,dx.$$

(b) Evaluate $\int_0^2 x^2\, dx^2$.

B State and prove bilinearity property of Riemann Stieltjes integral.

UNIT 4

18. A (a) Let $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ and such that $f_n(x_0)$ converges for some $x_0 \in [a, b]$ and f_n' converges uniformly on $[a, b]$ then prove that $\{f_n\}$ convergence uniformly on $[a, b]$ to a function f and $f'(x) = \lim_{n \to \infty} f_n'(x)$.

(b) State and prove the theorem to establish $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$.

B (a) State and prove Stone -Weierstrass theorem.

(b) Show that there exist a real continuous function on the real line which is nowhere differentiable.

(4 × 12 = 48 marks)

D 33009 **(Pages : 4)** **Name.......................................**

Reg. No..................................

FIRST SEMESTER P.G. DEGREE EXAMINATION NOVEMBER 2022

(CCSS)

Mathematics

MAT1 C 04—SCIENTIFIC PROGRAMMING USING PYTHON

(2022 Admissions)

Time : Three Hours Maximum : 50 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Write the output of the following Python code :

```
my_list = [x*x for x in range(5)]
print(sum(my_list))
```

2. Write the output of the following Python code :

```
m = 10
print(9*m == m**2-m)
```

3. Find the output of the following Python code :

```
n = 75
for i in range(2,n):
if n‘%,i == 0:
print(i)
break
```

4. Write the output of the following Python code :

```
a, b= 13, 27
while b:
a, b = b, a%b
print(a)
```

Turn over

5. Write the output of the following Python code :

```
import numpy as np
list1, list2 = [1,3,5], [2,4,6]
list3, list4 = np.array(list1), np.array(list2)
alpha = len(list1+list2)
beta = len(list3+list4)
print("alpha = {}".format(alpha))
print("beta = {}" .format(beta))
```

6. Write the output of the following Python code.

```
from sympy import Symbol
a = Symbol('x')
print(a.name)
```

7. Find the output of the following Python code :

```
from sympy import symbols, factor
x, y = symbols('x y')
expr = x**2 – y**2
print(factor(expr))
```

8. Write the output of the following Python code :

```
from sympy import Symbol, integrate
x = Symbol('x')
f = x**4 – 5
integrate(f,(x,0,1))
```

(8 × 1 = 8 marks)

Part B

Answer any **six** *questions.*
Each question carries 3 *marks.*

9. Demonstrate with example : else and elif.

10. Write a Python program to find the union and intersection of two sets.

11. Explain different methods to import a module in Python programs.

12. Differentiate list and numpy arrays.

13. Write a Python program to find the factorial of a given number.

14. Write a Python program to print the Fibonacci numbers less than 1000.

15. Write a Python program to plot the following data :

Year ...	2010	2011	2012	2013	2014	2015	2016
Avg. Temp ...	28	27	28	29	30	30	31

16. Explain the method of including the title, axes labels and legends in a plot.

17. Explain Lagrange interpolation.

(6 × 3 = 18 marks)

Part C

Answer any **three** *questions.*
Each question carries 8 marks.

18. (a) Write a Python program to check whether given three lengths are the sides of a triangle.

(b) Explain various types of variables available in Python.

19. Consider the function $f(x) = \frac{2x+1}{x-3}$. Write a python program to evaluate the following :

(a) The derivative of f.

(b) Slope of the tangent at $x = 1$.

(c) Equation of the tangent line at $x = 1$.

20. (a) Explain the terms Sum and Prod in sympy module.

(b) Write a Python program to find the partial sums of the following sequence :

$$1 - \frac{x^2}{1!} + \frac{x^4}{4!} - \ldots$$

Turn over

21. Write a Python program to find the standard deviation of the following data :

X	...	20	23	26	29	32
f	...	7	12	13	11	8

22. (a) Write the algorithm for Gauss elimination method for solving a system of linear equations.

(b) Write the Python program for solving a system of linear equations using Gauss elimination method.

($3 \times 8 = 24$ marks)