D 71864 (Pages : 3) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CCSS)

Mathematics

MAT 3E 02—OPERATIONS RESEARCH

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Define a convex function. Prove that $f(x) = x^2, x \in R$ is a convex function.
2. For the problem,

   Maximize $-6x_1 + 2x_2 - 4x_3 + 5x_4$

   subject to $4x_1 - x_2 + 2x_3 + 3x_4 \le 1$

   $x_2 + 4x_3 - 2x_4 \le 2$

   $x_1, x_2\ x_3 \ge 0$

   determine the basic feasible solutions.
3. What are artificial variables in an LP problem and why do we need them ?
4. Prove that the dual of the dual is the primal in LP problems.
5. Define the following terminologies in transportation problems :—
   (i) Transportation array.
   (ii) Loops in transportation array.
6. Briefly describe a method to obtain an initial feasible solution for a transportation problem.
7. What is integer linear programming ? Give an example of a mixed integer linear programming problem.
8. Write Kuhn-Tucker conditions for the problem :

   Minimize $x_1^2 - x_2^3 - x_2\ x_3^2$

   subject to $x_1^2 + x_2 + x_3 = 3$

   $x_1^2 - 5x_2^2 + x_3 \le 0$

   $x_1, x_2\ x_3 \ge 0.$

Turn over

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 16 marks.

9. (A) (a) Let $f(X)$ be a convex differentiable function defined in a convex domain $K \subseteq E_n$. Prove that $f(X_0), X_0 \in K$, is a global minimum iff

$$(X - X_0)^1 \nabla f(X_0) \geq 0 \text{ for all X in K.}$$

(b) Solve the following problem by two-phase simplex method :

Minimize $2x_1 - 3x_2 + 6x_3$

subject to $3x_1 - 4x_2 - 6x_3 \leq 2,$

$2x_1 + x_2 + 2x_3 \geq 11,$

$x_1 + 3x_2 - 2x_3 \leq 5$

$x_1, x_2\ x_3 \geq 0.$

(B) (a) Prove that a vertex of the set of feasible solutions of an LP problem is a basic feasible solution.

(b) Solve the following problem by simplex method :

Maximize $5x_1 + 3x_2 + x_3$

subject to $2x_1 + x_2 + x_3 = 3,$

$-x_1 + 2x_3 = 4$

$x_1, x_2\ x_3 \geq 0.$

10. (A) (a) Prove that the optimum value $f(X)$ of the primal, if it exists, is equal to the optimum value $\phi(Y)$ of the dual.

(b) Solve by dual simplex method :

Minimize $2x_1 + 3x_2$

subject to $2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1, x_2 \geq 0.$

(B) (a) What is Caterer problem ? Express it in standard transportation form.

(b) There are forest areas F_1, F_2, F_3, F_4 and timber depots D_1, D_2, D_3. The following table gives the produce of each forest area, the minimum timber required at each depot to attract buyers, and the cost of transportation per unit of timber from each forest area to each depot. Find the distribution of the entire forest produce for minimum cost of transportation.

	D_1	D_2	D_3	
F_1	3	4	6	100
F_2	7	3	8	80
F_3	6	4	5	90
F_4	7	5	2	120
	110	110	60	

11. (A) (a) Describe the branch and bound algorithm to solve an integer linear programming problem.

(b) Solve by the cutting plane method :

Minimize $4x_1 + 5x_2$

subject to $3x_1 + x_2 \geq 2$

$x_1 + 4x_2 \geq 5$

$3x_1 + 2x_2 \geq 7$

x_1, x_2 non-negative integers.

(B) (a) Let X_0 be a solution of the convex programming problem.

Minimize $f(X), X \in E_n$

subject to $g_i(X) \leq 0 \quad i = 1, 2, \ldots n$

where $f(X)$ and $g_i(X)$ are convex functions. Let $G(X) = \left[g_1(X), g_2(X) \ldots g_m(X)\right]^1$ and let the set of points X such that G (X) < 0 be non-empty. Prove that there exists a vector $Y_0 \geq 0$ in E_m such that

$$f(X) + Y_0^1 G(X) \geq f(X_0).$$

(b) Show that in the problem

Maximize $3x_1 + 6x_2 - 4x_1 x_2 - 3x_1^2 - 2x_2^2$

subject to $3x_1 + 2x_2 \leq 4,$

$x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$

the objective function is strictly concave. Hence solve the problem by quadratic programming algorithm.

D 71863 (Pages : 2) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CCSS)

Mathematics

MAT 3E 01—ADVANCED TOPOLOGY

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. 1 Prove that the unit circle in $\mathbb{R}^2$ with usual topology is compact.

2 Give an example of a space that is connected but not locally connected.

3 Prove that a path-connected space is connected.

4 Prove that every complete metric space with no isolated points is countable.

5 Define box in topological spaces. Prove that intersection of a family of boxes is a box.

6 Prove that a subnet of a convergent net is convergent.

7 Prove that the intersection of any family of filters on a set is again a filter on the set.

8 Prove that every compact metric space is complete.

Part B

Answer **either** A **or** B *of each question.*
Each question carries 12 *marks.*

II. A. (a) Prove that every continuous bijection from a compact space onto a Hausdorff space is a homeomorphism.

(b) Prove that every compact Hausdorff space is normal.

B. (a) Prove that every regular, Lindelöff space is normal.

(b) Prove that T4 spaces are Tychnoff spaces.

III. A (a) Prove that a product of topological spaces is Tychnoff if and only if each coordinate space is Tychnoff.

(b) Prove that a topological space X is compact if and only if every net in X has a cluster point in X.

B. Prove that metrisability is a countably productive property.

Turn over

IV. A. (a) Prove that every subspace of the Hilbert cube is a second countable metric space.

(b) Let X and Y be topological spaces and let x_0 be a point in X and $f : X \to Y$ be a function. Prove that f is continuous at x_0 if and only if whenever a net S converges to x_0 in X the net $f \circ S \to f(x_0)$ in Y.

B. (a) Prove that an open subspace of a locally connected space is locally connected.

(b) Let A be a subset of a metric space (X, d) such that A is complete with respect to the metric induced on it. Prove that A is closed in X.

V. A. Prove that a metric space is compact if and only if it is complete and totally bounded.

B. Prove that every contraction of a complete metric space into itself has a unique fixed point.

D 71862 (Pages : 3) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CCSS)

Mathematics

MAT 3C 12—PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Find the corresponding partial differential equation by eliminating the arbitrary function F from the equation $F\left(x+y, x-\sqrt{z}\right)=0$.

2. Show that $z = ax + (y/a) + b$ is a complete integral of $pq = 1$. Show further that this problem has no singular integral.

3. Determine the domain in which the two equations $xp - yq - x = 0$ and $x^2p + q - xz = 0$ are compatible.

4. Find the solution of $xz_y - yz_x = z$ with the initial condition $z(x,0) = f(x), x \geq 0$.

5. Reduce the equation :

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = \frac{y^2}{x}u_x + \frac{x^2}{y}u_y$$

to a Canonical form and solve it.

6. Discuss the motion of a semi-infinite string.

7. State the Dirichlet problem and show that the solution of the Dirichlet problem is stable.

8. State and prove Harnack's theorem.

9. Solve :

$$u_t = u_{xx}, 0 < x < l, t > 0$$
$$u(0,t) = u(l,t) = 0$$
$$u(x,0) = x(l-x), 0 \leq x \leq l.$$

Turn over

10. Transform the problem :

$$\frac{d^2y}{dx^2} + y = x,\ y(0) = 1,\ y'(0) = 0.$$

to a Fredholm integral equation.

11. Show that, if $y(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} + xy = 1$ and the conditions $y(0) = y'(0) = 0$, then $y(x)$ also satisfies the Volterra equation :

$$y(x) = \int_0^x \xi(\xi - x)\, y(\xi)\, d\xi + \frac{1}{2}x^2.$$

12. Write a short note on Neumann series.

(12 × 4 = 48 marks)

Part B

Answer **either** *part A* **or** *part B of each question.*
Each question carries 8 marks.

13. A (i) Let $u(x,y)$ and $v(x,y)$ be two functions of x and y such that $\frac{\partial v}{\partial y} \neq 0$. Show that if, further $\frac{\partial(u,v)}{\partial(x,y)} = 0$, then there exists a relation F $(u, v) = 0$, between u and v not involving x and y explicitly.

(ii) Find the general integral of $(y+1)p + (x+1)q = z$.

B (i) Show that the Pfaffian differential equation :

$$\left(y^2 + yz\right)dx + \left(xz + z^2\right)dy + \left(y^2 - xy\right)dz = 0$$

is integrable and find the corresponding integral.

(ii) Find the complete integral of :

$$zpq = p^2\left(p^2 + xq\right) + q^2\left(q^2 + yp\right).$$

14. A Find the integral surface of the equation $\left(p^2 + q^2\right)x = pz$, passing through the curve :

$$x_0 = 0, y_0 = s^2, z_0 = 2s.$$

B Find by the method of characteristics, the integral surface of $p^2x + qy - z = 0$ containing the initial line $y = 1, x + z = 0$.

15. A (i) State the Neumann problem and show that the solution of the Neumann problem is unique up to the addition of a constant.

(ii) Solve the Neumann problem for a circle.

B Solve the Dirichlet problem for a rectangle and show that it is unique.

16. A (i) Show that the characteristic numbers of a Fredholm equation with a real symmetric kernel are all real.

(ii) Determine the characteristic values of λ and the corresponding functions of the equation : $y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+\xi)\, y(\xi)\, d\xi$.

B (i) Solve by iterative method :

$$y(x) = \lambda \int_0^1 (x+\xi)\, y(\xi)\, d\xi + 1.$$

(ii) Determine the resolvent kernel associated with $K(x,\xi) = x\xi$ in the interval (0, 1) in the form of a power series in λ, obtaining the first three terms.

(4 × 8 = 32 marks)

f

by

by

ver

D 71861 (Pages : 3) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CCSS)

Mathematics

MAT 3C 11—FUNCTIONAL ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Show that the intersection of a countable number of dense open subsets of a complete metric space X is dense in X.

2. Let E be a measurable subset of R. Show that the set of all simple measurable functions on E is dense in $L^{\infty}(E)$.

3. Let Y be a closed subspace of a normed space X and let X/Y be the quotient space with respect to the quotient norm. Show that a sequence $(x_n + Y)$ converges to $x + Y$ in X/Y iff there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X.

4. Show that the three norms $\| \ \|_1, \| \ \|_2$ and $\| \ \|_{\infty}$ defined on K^n are equivalent.

5. Show that a linear map F from a normed space X to a normed space Y is a homeomorphism iff there are $\alpha, \beta > 0$ such that :

   $|\beta \| x \| \leq \|F(x)\| \leq \alpha \| x \|$ for all $x \varepsilon X$.

6. Let X be a normed space and $P \varepsilon BL(X)$ satisfy $P^2 = P$. Show that either $\| P \| = 0$ or $\| P \| > 1$.

7. Let $X = K^2$ with the norm $\| \ \|_{\infty}$ and let $Y = \{(x(1), x(2)) \varepsilon X : x(2) = 0\}$. Define $g \varepsilon Y'$ by $g(x(1), x(2)) = x(1)$. Show that the only Hahn-Banach extension of g to X is given by $f(x(1), x(2)) = x(1)$.

Turn over

8. Define Schander basis for a normed space. Show that if there is Schander basis for a normed space X, then X is separable.

9. Show that the linear space C_{00} cannot be a Banach space in any norm.

10. Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is continuous iff $g \circ F$ is continuous for every $g \varepsilon Y'$.

11. Let X, Y and Z be normed spaces. Show that if $F : X \to Y$ is continuous and $G : Y \to Z$ is closed, then $G \circ F : X \to Z$ is closed.

12. Let X and Y be normed spaces and $F : X \to Y$ be linear. Show that F is an open map iff there exists some $r > 0$ such that for every $y \varepsilon Y$, there is some $x \varepsilon X$ with $F(x) = y$ and $\| x \| \leq r \| y \|$.

(12 × 4 = 48 marks)

Part B

Answer **either** *A* **or** *B of each question.*
Each question carries 8 *marks.*

13. A (a) Show that the $1 \leq p < \infty$, the matrix space l^p is separable, but l^∞ is not separable.

(b) Show that the metric space $L^\infty([a,b])$ is complete.

B (a) Show that every totally bounded set is bounded. Is the converse true ? Justify your answer.

(b) Show that if a metric space X is complete and totally bounded then X is compact.

14. A (a) Show that every finite dimensional subspace of a normed space is complete.

(b) Let X be a normed space such that $\overline{u(0,1)}$ is totally bounded. Show that X is finite dimensional.

B (a) Let X and Y be normed spaces and $F : X \to Y$ be a linear map such that the range R (F) of F is finite dimensional. Show that F is continuous iff the zero space Z (F) of F is closed in X.

(b) Let X and Y be normed spaces and Z be a closed subspace of X. Show that if $\tilde{F} \varepsilon BL(X/Z, Y)$ and if we let $F(x) = \tilde{F}(x + Z)$ for $x \varepsilon X$, then $\tilde{F} \varepsilon BL(X, Y)$ and $\| F \| = \| \tilde{F} \|$.

15. A Let X be a normed space. Show that for every subspace Y of X and every $g \varepsilon Y'$, there is a unique Hahn-Banach extension of g to X iff X' is strictly convex.

B (a) Let X be a normed space and Y be a Banach space. Let X_0 be a dense subspace of X and $F_0 \varepsilon BL(X_0, Y)$. Show that there is a unique $F \varepsilon BL(X, Y)$ such that $F/X_0 = F_0$ and $\| F \| = \| F_0 \|$.

(b) Show that a normed space can be embedded as a dense subspace of a Banach space.

16. A (a) State and prove uniform boundedness principle.

(b) Show that if the domain space X is not Banach, then the uniform boundedness principle may not hold.

B (a) Let X be a normed space and $P : X \to X$ be a projection. Show that P is a closed map iff the subspaces R (P) and Z (P) are closed in X.

(b) Show that if Y is a finite dimensional subspace of a normed space X, then there exists a closed complement of Y in X.

(4 × 8 = 32 marks)

D 71860 (Pages : 3) Name.....

Reg. No.....

THIRD SEMESTER M.Sc. DEGREE EXAMINATION, DECEMBER 2014

(CCSS)

Mathematics

MAT 3C 10—COMPLEX ANALYSIS

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. Find the radius of convergence of the following power series :

   (a) $\sum_{n=0}^{\infty} 2^n z^{n^2}$.

   (b) $\sum_{n=0}^{\infty} \left(\frac{4n^2}{2n+1} - \frac{6n^2}{3n+4} \right)^n z^n$.

2. Discuss the mapping properties of $f(z) = z^2$.

3. Prove that $\log |z|$ is harmonic in $\mathbb{C} - \{0\}$.

4. Let r be the polygon $[1, 1+i, i]$. Express r as a path and calculate :

   $\int_r f$, when $f(z) = |z|^2$.

5. If u is harmonic in $\mathbb{C}$ and $u(z) \geq 0$ for all z in $\mathbb{C}$, prove that u is a constant function.

6. Let $r(t) = 1 + e^{it}, 0 \leq t \leq 2\pi$. Evaluate $\int_r \left(\frac{z}{z-1} \right)^n dz$, for all positive integers n.

7. Let G be an open course set. If r is a closed rectifiable curve in G, show that r is homotopic to zero in G.

8. Let r be a closed rectifiable curve in an open set G. If r is homotopic to zero in G, prove that r is homologous to zero in G.

Turn over

9. Let r be the circle defined by $r(t) = 2\,e^{it}, 0 \le t \le 2\pi$. Show that $\int_r \frac{2z+1}{z^2+z+1}\,dz = 4\pi i$.

10. Let $f(z) = z^2\left(1+z^4\right)^{-1}$. Find the poles of f and obtain the residues at the poles.

11. If f is a meromorphic function on a region G, prove that neither the poles nor the zeros of f have a limit point in G.

12. Using the Residue theorem, evaluate $\int_r \frac{z^2+1}{z(z-1)}\,dz$, where r is the circle given by :

$r(t) = 2\,e^{it}, 0 \le t \le 2\pi$.

(12 × 4 = 48 marks)

Part B

Answer **either** *part A* **or** *part B of each question.*
Each question carries 8 *marks.*

13. A (a) Prove that a power series is analytic in the disk of convergence. (3 marks)

(b) Show that there exists a unique Mobius transformation that carries any three distinct points z_1, z_2, z_3 in $\mathbb{C}_\infty$ respectively to three distinct points w_1, w_2, w_3 in $\mathbb{C}_\infty$.

(2 marks)

(c) If u is a harmonic function on a disk D show that u has a harmonic conjugate in D.

(3 marks)

B (a) If a power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ has a radius of convergence R, then prove that

$R = \lim \left|\frac{a_n}{a_{n+1}}\right|$, if this limit exists.

(2 marks)

(b) If f is a branch of the logarithm in a region G, prove that f is analytic in G and its derivative is $1/z$.

(4 marks)

(c) Show that a Mobius transformation preserves cross ratios. (2 marks)

A (a) Let r be a closed rectifiable curve in $\mathbb{C}$. Show that $n(r,a)$ is a constant for all points a belonging to the same component of $\mathbb{C}-\{r\}$. (2 marks)

(b) State and prove the Cauchy's integral formula (First version). (4 marks)

(c) If $p(z)$ is a non-constant polynomial, show that these exists a point a such that $p(a) = 0$. (2 marks)

B (a) If r is a rectifiable path from $[a, b]$ to $\mathbb{C}$ and if $\phi:[c,d]\to[a,b]$ is a continuous, non-decreasing function with $\phi(c)=a$ and $\phi(d)=b$, then show that $\int_r f = \int_{r\cdot\phi} f$. (2 marks)

(b) Show that $\int_0^{2\pi} \frac{e^{is}}{e^{is}-z}\, ds = 2\pi$, if $|z|<1$. (3 marks)

(c) If f is an analytic function on a region G and if $|f|$ attains a maximum in G, prove that f is constant. (3 marks)

A (a) State and prove Goursat's theorem. (5 marks)

(b) If f has an isolated singularity at $z = a$, prove that it is a removable singularity if and only if $\lim_{z\to a}(z-a)f(z)=0$. (3 marks)

B (a) State and prove the open mapping theorem. (4 marks)

(b) State and prove the Casorati-Weierstrass theorem. (4 marks)

A (a) Show that :

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}.$$ (4 marks)

(b) State and prove Rouche's theorem. (4 marks)

B (a) State and prove the Residue theorem. (3 marks)

(b) State and prove the Maximum modulus theorem (second version). (3 marks)

(c) Is the maximum modulus theorem true for unbounded region ? Justify. (2 marks)

[4 × 8 = 32 marks]