UNIVERSITY OF CALICUT

DEPARTMENT OF MATHEMATICS

Entrance Examination, Ph.D. 2014

Time: Two Hours | Maximum: 100 Marks

PART - A

Answer all questions. Each question carries 2 marks

1. The number of automorphisms of the group $\mathbb{Z}_n$ is

   (a) 1 (b) n (c) $n-1$ (d) $\phi(n)$

2. For any set X, the powerset $\mathbf{P}(X)$ is a ring under the binary operations symmetric difference Δ and intersection $\bigcap$. The characteristic of the ring $(\mathbf{P}(X), \Delta, \bigcap)$ is

   (a) 0 (b) 1 (c) 2 (d) $|X|$

3. For which one of the following field extensions, the Galois group is not cyclic?

   (a) $\mathbb{R} \leq \mathbb{C}$ (b) $\mathbb{Q} \leq \mathbb{Q}(\sqrt[3]{2})$ (c) $\mathbb{Q} \leq \mathbb{Q}(\sqrt{2}, \sqrt{3})$ (d) $\mathbb{Q} \leq \mathbb{Q}(\sqrt[4]{2})$

4. Which one of the following pairs of fields are isomorphic?

   (a) $\mathbb{R}$, $\mathbb{Q}$ (b) $\mathbb{R}$, $\mathbb{C}$ (c) $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ (d) none of these

5. If the matrix of the linear operator T on $\mathbb{R}^2$ relative to the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix},$$

   Then the matrix of T relative to the basis $\{(1,1),(1,-1)\}$ is

   (a) $\begin{pmatrix} 1/2 & 5/2 \\ -3/2 & 5/2 \end{pmatrix}$ (b) $\begin{pmatrix} -1/2 & -5/2 \\ 3/2 & 5/2 \end{pmatrix}$ (c) $\begin{pmatrix} 1/2 & 3/2 \\ 5/2 & 5/2 \end{pmatrix}$ (d) none of these

6. The dimension of the vector space $V = \{a + b\sqrt{2} + c\sqrt{5} : \ a, b, c \in \mathbb{Q})\}$ is

   (a) 0 (b) 1 (c) 2 (d) 3

7. The set of vectors $\{(3,0,4),(0,1,0),(-4,0,3)\}$ is an orthogonal basis for $\mathbb{R}^3$. Find $c_3 \in \mathbb{R}$ such that $\{(0,1,1) = c_1(3,0,4) + c_2(0,1,0) + c_3(-4,0,3)$ for some scalars $c_1, c_2 \in \mathbb{R}$. That is, the third Fourier coefficient of $(0,1,1)$ with respect to the above ordered orthonormal basis of $\mathbb{R}^3$ is

   (a) 3/25 (b) $-3/25$ (c) 25/3 (d) none of these

8. Which one of the following statement is true about the derivative f' of the bounded and differentiable real valued function f defined on [0,1]?

   (a) bounded (b) differentiable (c) continuous (d) none of these

9. Let I be the set of all irrationals in the interval $[0, 1]$. Then

(a) Every subset of I is measurable

(b) There are measurable sets E_n in I, $n = 1, 2, 3, \ldots$ such that $m(\bigcup_{n=1}^{\infty} E_n) = \infty$

(c) All subsets other than the countable subsets have positive measure

(d) The measure of I is ∞

10. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x^2 + 1, y)$ $(x, y) \in \mathbb{R}^2$. Then the derivative of f at (0,0) is

(a) $A(x, y) = (2x, 1)$ (b) $A(x, y) = (0, y)$ (c) $A(x, y) = (1, 0)$ (d) none of these

11. Which one of the following is a completely multiplicative arithmatical function?

(a) $d(n) =$ the number of divisors of n (b) $\sigma(n) =$ the sum of divisors of n
(c) The Liouville's function $\lambda(n)$ (d) None of the above

12. Which one of the following is a quadratic residue mod 13?

(a) 7 (b) 8 (c) 3 (d) none of these

13. If n is a positive integer greater than 1 such that $2^n - 1$ is a prime, then

(a) n is a power of 2 (b) n is prime (c) n is composite (d) None of these

14. Which one of the following is not true?

(a) A metric space is first countable (b) A metric space is second countable
(c) A metrizable space is Hausdorff (d) none of these

15. If f is a continuous function from a topological space X to a toplogical space Y, which one of the following is true?

(a) f is open (b) f is closed (c) f is a quotient map (d) none of these

16. Which one of the following properties hold for the real line with usual topology?

(a) compactness (b) countable compactness (c) Lindelof property (d) none of these

17. The number of non-isomorphic trees on five vertices is

(a) 3 (b) 4 (c) 5 (d) 6

18. Which one of the following sequences is graphic?

(a) $(2, 3, 3, 4, 5)$ (b) $(3, 3, 4, 4)$ (c) $(2, 3, 4, 5)$ (d) $(1, 2, 2, 2, 3)$

19. The point $x_0 = 0$ is an ordinary point of the differential equation $(1 - x^2)y'' - 2y' + 3y = 0$. Begin the process of finding a series solution for this differential. The recurrence relation for the coefficients a_n is

(a) $a_{n+2} = \frac{n^2-n-3}{(n+1)(n+2)}a_n - \frac{2}{n+2}a_{n+1}$ (b) $a_{n+2} = \frac{n^2+n+3}{(n+1)(n+2)}a_n + \frac{2}{n+2}a_{n+1}$

(c) $a_{n+2} = \frac{n^2-n-3}{(n+1)(n+2)}a_n + \frac{2}{n+2}a_{n+1}$ (d) $a_{n+2} = \frac{n}{(n+1)(n+2)}a_n + \frac{1}{n+2}a_{n+1}$

20. If the integral equation

$$y(x) = x - \int_0^x (x-t)y(t)dt$$

is solved by the method of successive approximations, starting with the initial approximation $y_0(x) = x$, then the second approximation is given by

(a) $y_2(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$ (b) $y_2(x) = x + \frac{x^3}{3!}$

(c) $y_2(x) = x - \frac{x^3}{3!}$ (d) $y_2(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

21. A complete integral of the PDE $pqz = p^2(xq+p^2) + q^2(yp+q^2)$ is

(a) $z = ax + by$ (b) $z = ax + by + a^4 + b^4$ (c) $z = ax + by + \frac{a^4+b^4}{ab}$ (d) none of these

22. Let X be a normed space and Y, a subspace of X. Then which one of the following is true?

(a) If X is separable, then its dual space X' is separable

(b) If the dual space X' is separable, then X is separable

(c) If X is separable, then the quotient space X/Y is separable

(d) none of the above

23. Let E_1 and E_2 be subsets of a normed space X and $E_1 + E_2 = \{x + y : x \in E_1,\ x \in E_2\}$. Which one of the following is not true?

(a) If E_1 and E_2 are compact, then $E_1 + E_2$ is compact

(b) If E_1 and E_2 are closed, then $E_1 + E_2$ is closed

(c) If E_1 or E_2 is open, then $E_1 + E_2$ is open

(d) none of the above

24. Let γ be the rectangle with vertices $1-i,\ 1+i,\ -1+i,\ -1-i$. Then $\int_\gamma \frac{1}{z}dz$ is

(a) 0 (b) $2\pi i$ (c) πi (d) 1

25. Let f be analytic in a neighbourhood of $D = \overline{B}(0,1)$. If $|f(z)| < 1$ for $|z| = 1$, then the number of zeros of the function $f(z)$ is

(a) 0 (b) 1 (c) ∞ (d) 2

PART - B

Answer any five questions. Each question carries 10 marks

1. Let f be a function from X to Y, A_1 and A_2 be subsets of X and B_1 and B_2 be subsets of Y. Prove or disprove:

(a) $f(A_1 \bigcap A_2) = f(A_1) \bigcap f(A_2)$.

(b) $f^{-1}(B1 \bigcap B_2) = f^{-1}(B_1) \bigcap f^{-1}(B_2)$.

2. (a) For any positive integer n, the set K, of all permutations in S_n that leaves n fixed, forms a subgroup of S_n. To what known group, K is isomorphic? Justify your claim.

   (b) Prove or disprove: The number of elements in a finite field is prime.

3. (a) Describe explicitly the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$.

   (b) Find all eigen values and a basis for each of the eigen spaces of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

   Is A diagonalizable?

4. (a) Prove that every Borel set is Lebesgue measurable.

   (b) Prove or disprove: Every closed and bounded set in any metric space is compact.

5. (a) Prove that $\phi(n)$ is even for all $n \geq 3$, where ϕ denotes the Euler-totient function.

   (b) Prove that $[2x] - 2[x]$ is either 0 or 1, for every real number x.

6. Reduce the equation $x^2 u_{xx} + y^2 u_{yy} = 0$ into canonical form and solve whenever possible.

7. Prove that the set of all homeomorphisms of a topological space onto itself forms a group under composition of functions.

8. Let $X = \ell^p$ where $1 \leq p \leq \infty$ and $T \in BL(X)$ be given by $T(x) = (0, x(1), \frac{x(2)}{2}, \ldots)$, $x = (x(1), x(2), \ldots) \in X$. Find the eigen values and spectrum of T.

9. An institution is conducting a supplementary examination for 8 students $A, B, \ldots, H$ in 5 courses. The students appearing for each exam is given below. Use graph theory notions to find the minimum number of days required to conduct the examinations, with the assumption that only one examination will be conducted in a day.

Courses :	*Course* 1	*Course* 2	*Course* 3	*Course* 4	*Course* 5
Students :	A, B, C	B, D, E, F, G	A, D, E, H	E, F, G	C, H

10. (a) Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$, for every z in G.Show that either $f \equiv 0$ or $g \equiv 0$.

    (b) Find all possible values of

$$\int_\gamma \frac{dz}{1+z^2},$$

   where γ is any rectifiable curve in $\mathbb{C}$ not passing through $\pm i$.