**Ph.D. THESIS** **MATHEMATICS**

# ON NON-COMMUTATIVE CONVEXITY IN HILBERT $C^*$-BIMODULES

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by


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# CERTIFICATE

I hereby certify that the thesis entitled "**ON NON-COMMUTATIVE CONVEXITY IN HILBERT $C^*$-BIMODULES**" is a bona fide work carried out by **Syamkrishnan M S.,** under my guidance for the award of Degree of Ph.D., in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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# DECLARATION

I hereby declare that the thesis, entitled "**ON NON-COMMUTATIVE CONVEXITY IN HILBERT $C^*$-BIMODULES**" is based on the original work done by me under the supervision of Dr. A. K. Vijayarajan, Professor, Kerala School of Mathematics and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Kerala School of Mathematics,

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# Contents

# Chapter 1

# Introduction

Non-commutative convexity has been used in the literature to refer to any one of several forms of convexity where convex combinations of operators are studied. $C^*$-convexity and matrix convexity are two well studied forms of non-commutative convexity. Matrix convexity can be viewed as a dimension free version of $C^*$-convexity. While $C^*$-convexity is in the context of a $C^*$-algebras, matrix convexity is in the context of a family of matrix $C^*$-algebras. The early investigators of these concepts, Arveson [1] and Bunce and Salinas [7] were initially concerned about the connections of these concepts with matricial ranges of operators, among other things.

The notion of $C^*$-convexity as we use here was introduced by Loebl and Paulsen in [23] and studied further by several other authors, for example [12–14, 19, 25]. On the other hand, the concept of matrix convex sets was proposed by Wittstock [30] and carried forward by Webster and Winkler [29]. The germ of this concept arose from the properties of matricial ranges of operators. Having an appropriate notion of non-commutative convexity leads naturally to questions related to notions of extreme points and possible analogues of the classical Krein-Milman theorem. In the context of $C^*$-convexity a suitable notion of extreme point called $C^*$-extreme point was put forward by Hopenwasser, Moore and Paulsen [19]. Later, Webster and Winkler [29] defined a notion of extreme points, termed as matrix extreme points, in the case of matrix convex sets. Farenick and Morenz proved a Krien-Milman type theorem [14,27] for $C^*$-convex sets for a particular case. A general Krein-Milman type theorem in the case of matrix convex sets was proved by Webster and Winkler [29], which says that, a matrix convex set $K$ is the matrix convex hull of its matrix extreme points. That is, the matrix extreme points of $K$ span $K$ as matrix convex combinations. Unfortunately, matrix extreme points are not minimal in this sense. Thus, one of the central goals in the study of matrix convexity is to identify minimal sets of matrix extreme points that span matrix convex sets. In [21] Kleski identified a sub collection of matrix extreme points for a matrix convex set, which he termed as *boundary points* (termed as *absolute extreme points* by some other authors). Kleski proved that the set of boundary points of a matrix convex set $K$ is minimal provided it spans $K$. He also proved a connection between boundary points of $K$ and boundary representations of the associated operator system $A(K)$ in finite dimensional setting.We use this result to show that boundary representations for an operator system of the generated $C^*$-algebra are exactly boundary points of a naturally associated matrix convex set in the finite dimensional case. In a recent paper Davidson and Kennedy [8] introduced a new and extensive theory of noncommutative convexity where a Krein-Milman type theorem is proved in full generality and they also established minimality of the set of extreme points concerned.

Let $CP(A, B(H))$ denotes the space of all completely positive maps from a unital $C^*$-algebra $A$ to $B(H)$, the space of all bounded linear operators on a Hilbert space $H$. Exploration of extreme points in this set up led Arveson to the notion of purity of elements and he established a criterion for purity in terms the irreducibility of the Stinespring representation of $A$ on $H$. Replacing $A$ by an operator system $R$ and $B(H)$ by the finite dimensional case ($M_n(\mathbb{C})$ for some positive integer $n$), in a finite dimensional set up of an operator system and matrix algebras, Farenick proved [9] that pure CP maps are precisely matrix extreme points of the associated non-commutative compact convex set of all matrix valued UCP maps on the operator system. Arveson and Farenick in a sense proved that $CP(A, B(H))$ and $CP(R, M_n(C))$ can be viewed as operator (resp. matrix) convex combinations of pure CP maps. But even in the case of operator systems in general, a characterisation of pure elements of $CP(R_1, R_2)$ is still not known where $R_1$ and $R_2$ are arbitrary operator systems.

Note that Arveson's criterion of pure CP maps from a unital $C^*$-algebra to $B(H)$ implies that the identity map between $B(H)$ is pure. A more general question would be about the purity of embeddings (complete order isomorphisms) of operator systems. The above question can in particular be asked about the embedding of an operator system $R$ in its $C^*$-envelope $C^*_e(R)$. Arveson [1] showed that $B(H)$ is an injective operator system by showing that each CP map from $R \to B(H)$ has a completely positive extension to $B(H)$. In [17] Hamana studied inclusions of operator systems into injective operator systems and showed that every operator system $R$ is a subsytem of an injective operator system $I(R)$-the injective envelope of $R$. In the framework $R \subseteq C^*_e(R) \subseteq I(R)$, Farenick and Tessier [11, Theorem 3.2 ] addressed the purity of the inclusion map $I_{ie} : R \to I(R)$ and proved a characterisation of the purity of the embedding in terms of the prime nature of the $C^*$-envelope of $R$. In [18], Hamana established a parallel framework for operator spaces as well, namely $X \subseteq T \subseteq I(X)$ where $X$ is an operator space, $T$ is the *triple envelope* of $X$ and $I(X)$ is the *injective envelope* of $X$. A natural question that arises in this scenario is whether there is a characterisation similar to that of operator systems proved in [11] and we take up this question in this thesis.

The natural counterparts of C*-envelopes and injective envelopes of operator systems for operator spaces are, triple envelopes and injective envelopes. While C*-envelopes and injective envelopes of operator systems are C*-algebras themselves, triple envelopes and injective envelopes of operator spaces display rich bimodule structures over C*-algebras occuring naturally in their construction. Extremity in the form of purity for CP maps associated to operator systems does not have any natural analogue for completely contractive(CC) maps associated to operator spaces. We show that bi-extremity related to *bi-convexity* that we introduce for CC maps in the context of Hilbert $C^*$-bimodules results in a partial parallel result for embeddings of operator spaces into their injective envelopes involving the prime property that we introduce for TROs generated by the operator spaces. In fact, we prove that for an operator space, the canonical embedding of it into its injective envelope is bi-extreme if the triple envelope generated by the operator space is prime.

As an extension of $C^*$-convexity in $C^*$-algebras, Magajna introduced [26] the notion of $A$-$B$ absolute convexity in operator bimodules $B(H)$ over its unital subalgebras and deduced analogues of classical separation theorems for convex subsets of Banach spaces. For general Hilbert $A$-$B$-bimodules($A$ and $B$ are unital C*-algebras), a notion of $A$-$B$ convexity was introduced by Kian and Dehgani [20]. In this thesis we will refer to $A$-$B$ convexity as bi-convexity in general and introduce the notion of bi-extremity associated with bi-convexity.

The thesis broadly divides into two sections distinct though related. The first section deals with boundary representations of operator systems and boundary points of associated matrix convex sets. Here we give a characterisation of boundary representations of an operator system in terms of boundary points of the associated matrix convex set. The second section is on TROs and injective envelopes of operator spaces which exhibit interesting natural Hilbert $C^*$-bimodule structures and this leads us to initiate a study of non-commutative convexity in the setting of Hilbert $C^*$-modules( [22]). We investigate the notion of bi-convex subsets of Hilbert $C^*$-bimodules and observe that the notion is more general than $C^*$-convexity even in the case of trivial $C^*$-bimodules where modules are $C^*$-algebras with self left and right actions. In this case we deduce that unitaries are bi-extreme points in the unit ball of a trivial module, but conversely bi-extreme points are either isometries or co-isometries. We also consider bi-convexity of CC maps between operator spaces and show that (Proposition 4.3.1) the notion is equivalent to the notion of purity and operator extremity of the corresponding Paulsen maps in the special case of embedding of an operator system in its injective envelope. We consider bi-convex combination of CC maps from operator spaces to Hilbert $C^*$-bimodules and define bi-extreme CC maps. Prime TROs are defined in terms of triple ideals and bi-extremity of CC maps on operator spaces is shown to be related to the prime nature of the triple envelope of the operator space under consideration. We deduce that prime TROs occur along with prime $C^*$-algebras. These ideas lead to one of our main results, namely, if the triple envelope of an operator space is a prime TRO, then its embedding in its injective envelope is bi-extreme.

# Chapter 2

# Preliminaries

The main theme of this thesis being non-commutative convexity in $C^*$-algebras, in this chapter we recall all the necessary related concepts that are required to set up the context and to use in later parts of the thesis.

We start with the definition of a $C^*$-algebra. Linear spaces, Banach spaces and Banach algebras we consider in this thesis are over the field of complex numbers $\mathbb{C}$.

An algebra $A$ is a linear space together with vector multiplication which satisfying following conditions. For all $a, b, c \in A$ and a scalar $\alpha$,

1. $a(bc) = (ab)c$;
2. $a(b + c) = ab + ac$;
3. $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Let $A$ be a Banach space over the complex numbers $\mathbb{C}$. The Banach space $A$ is called a Banach algebra if it is also an algebra with identity $1$. That is, there exists a non zero vector denoted by $I \in A$ such that $Ix = xI = x \ \forall x \in A$, satisfying following conditions. For all $x, y \in A$

1. $\|xy\| \leq \|x\|\|y\|$;

2. $\|I\| = 1$.

## 2.1 $C^* - algebras$

A $C^*$-algebra $A$ is a Banach algebra together with a map, $x \to x^*$ called involution with the following properties.
For all $x, y \in A$, $\lambda \in \mathbb{C}$

1. $x^{**} = x$;

2. $(x + y)^* = x^* + y^*$;

3. $(\lambda x^*) = \overline{\lambda} x^*$;

4. $(xy)^* = y^* x^*$;

5. $\|x^* x\| = \|x\|^2$.

Now we will give some important examples.

**Example 2.1.1.** *1. Set of all complex valued continuous functions defined on Compact Hausdorff space $X$, denoted by $C(X)$, where vector addition, vector multiplication ,involution and norm defined as follows. For all $f, g \in C(X)$*

*(a)* $(f + g)x = f(x) + g(x)$;

*(b)* $(fg)x = f(x).g(x)$;

*(c)* $f^*(x) = \overline{f(x)}$.

*(d) for $f \in C(X)$, the supremum norm of $f$ on $C(X)$ is defined as,*

$\|f\|_\infty = Sup\{|f(x)|, x \in X\}$.

*2. The set of all bounded linear operators $B(H)$ defined on Hilbert space $H$, where multiplication defined as composition of operators and involution as adjoint and norm is defined as operator norm,*

*that is, for $T \in B(H)$, $\|T\| = Sup\{\|T(x)\|, x \in H\ with\ \|x\| \leq 1\}$.*

Every commutative $C^*$-algebra with unity is isometrically isomorphic to $C(X)$, for some compact Hausdorff space $X$ in complex plane. In general every $C^*$-algebras can be identified isometrically isomorphic with a subalgebra of $B(H)$, for some Hilbert space $H$.

## 2.2 Operator systems and spaces

Let $A$ be a $C^*$-algebra and $R \subseteq A$, set $R^* = \{a : a^* \in R\}$. If $R = R^*$, then $R$ called self adjoint. A unital self adjoint subspace $R$ of $A$ is called operator system . An element of a $C^*$-algebra is said to be positive if it is self adjoint and its spectrum lies in the nonnegative reals.

An *operator space* $X$ in a $C^*$-algebra is a linear subspace of the $C^*$-algebra. A *ternary ring of operators* (TRO) $T$ is a subspace of the $C^*$-algebra $B(H)$ that is closed under the triple product $(x, y, z) \to xy^*z$ for all $x, y, z \in T$. An important example of a TRO is the space $B(H, K)$, where $H$ and $K$ are Hilbert spaces.

### 2.2.1 Completely positive maps

Let $R$ be an operator system of a $C^*$-algebra $A$ and let $B$ be another $C^*$-algebra. A linear map $\phi : R \to B$ is said to be *positive* if it maps positive elements into positive elements and it is called *completely positive* (CP) if the linear map $I_n \otimes \phi : M_n(\mathbb{C}) \otimes R \to M_n(\mathbb{C}) \otimes B$ is positive for all $n \in \mathbb{N}$, where $I_n$ is the $n \times n$ identity map on $M_n(\mathbb{C})$.

We look at some examples of CP maps. Let $A, B$ be two $C^*$-algebras and $\pi : A \to B$ be a $*$-homomorphism, then $\pi$ is completely positive. An important example of a CP map on a $C^*$-algebra $A$ is given by conjugation with respect to an element, i.e., for $x \in A$, $\phi : A \to A$ defined by $\phi(a) = x^*ax$.

We use $C^*(R)$ to denote the $C^*$-subalgebra of $A$ generated by $R$. Let $S$ be another operator system and $\phi : R \to S$ be a CP map. The map $\phi$ is called *pure* if $\phi' : R \to S$ is any other CP map such that $\phi - \phi'$ is CP, then $\phi' = t\phi$, for some scalar $t \in [0,1]$. A CP map with CP inverse is called a *complete order isomorphism* between two operator systems, which is the natural isomorphism of operator systems. Any abstract operator system can be embedded into $B(H)$, for some Hilbert space $H$ as concrete operator system [28, Theorem 13.1]. Similarly for operator spaces, any operator space can be viewed as an operator space in $B(H)$ for some Hilbert space $H$ [28, Theorem 13.4].

### 2.2.2 Paulsen system

Let $X$ and $Y$ be operator spaces. A linear map $\phi : X \to Y$ is called *completely contractive* (CC) if the linear map $I_n \otimes \phi : M_n(\mathbb{C}) \otimes X \to M_n(\mathbb{C}) \otimes Y$ is contractive for all natural numbers $n$. To each operator space $X \subseteq B(H,K)$ we can associate an operator system $S(X)$ in $B(K \oplus H)$, called the *Paulsen system* of $X$ which is defined by

$$S(X) = \left\{ \begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} : x, y \in X, \lambda, \mu \in \mathbb{C} \right\}$$

where $H$ and $K$ are Hilbert spaces and $I_H$, $I_K$ denote the identity operators on $H$, $K$ respectively. If $X, Y \subseteq B(H,K)$ are operator spaces then a linear map $\phi : X \to Y$ induces a linear map $S(\phi) : S(X) \to S(Y)$ given by

$$S(\phi)\left( \begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} \right) = \begin{bmatrix} \lambda I_K & \phi(x) \\ \phi(y)^* & \mu I_H \end{bmatrix}.$$

Further, if $\phi$ is CC, then $S(\phi)$ is CP [28].

**Lemma 2.2.1.** *[28, Lemma 8.1] Let $A, B$ be two unital $C^*$-algebras, let $X$ be an operator space in $A$, and let $\phi : X \to B$ be a linear map. Define an operator system $S(X) = \left\{ \begin{bmatrix} \lambda I & x \\ y^* & \mu I \end{bmatrix} \lambda, \mu \in \mathbb{C}, x, y \in X \right\}$ and $S(\phi); S(X) \to M_2(B)$ via*

$$S(\phi)\left( \begin{bmatrix} \lambda I & x \\ y^* & \mu I \end{bmatrix} \right) = \begin{bmatrix} \lambda I & \phi(x) \\ \phi(y)^* & \mu I \end{bmatrix}.$$

*If $\phi$ is completely contractive then $S(\phi)$ is completely positive.*

A similar parallel association exists between TROs and $C^*$-algebras: For a TRO $T$ in $B(H, K)$ we can associate a $C^*$-algebra

$$L(T) = \begin{bmatrix} TT^* + \mathbb{C}I_K & T \\ T^* & T^*T + \mathbb{C}I_H \end{bmatrix}$$

in $B(K \oplus H)$, known as the *linking algebra* of $T$ [6]. Note that $TT^* + \mathbb{C}I_K$ and $T^*T + \mathbb{C}I_H$ are $C^*$-algebras in $B(K)$ and $B(H)$ respectively.

## 2.3 Boundary representation of $C^*$-algebras

A representation of a $C^*$-algebra $A$ is a $*$-homomorphism $\pi : A \to B(H)$, where $H$ is a Hilbert space. Let $\pi : A \to B(H)$ be a representation if $\pi$ is injective then $\pi$

is called faithful representation . If closed the linear span of $\pi(A)H = \{\pi(a)h, a \in A, h \in H\}$ is $H$ then, $\pi$ is called non-degenerate representation . Let $\pi : A \to B(H)$ be a representation. If there exists $h \in H$ such that closed linear span of $\pi(A)h = \{\pi(a)h, a \in A\}$ is $H$, then $\pi$ is called cyclic representation and $h$ is said to be cyclic vector. Let $\pi : A \to B(H)$ be a representation. Let $K$ be a subspace of $H$ such that $\pi(a)k \in K\ \forall a \in A, k \in K$ then $K$ is called invariant subspace for $\pi(A)$. If $\pi(A)$ has no nontrivial closed invariant subspace then $\pi$ is called an irreducible representation.

Extensions of CP maps defined on operator systems to the $C^*$-algebra containing it is of at most importance in the theory of operator systems. We recall below the extension theorem due to Arverson on the and unique extension property (UEP).

**Definition 2.3.1.** *A unital completely positive map $\phi : R \to B(H)$ is said to have unique extension property if $\phi$ has a unique completely positive extension $\tilde{\phi} : C^*(R) \to B(H)$, and $\tilde{\phi}$ is a representation of $C^*(R)$ on $H$. An irreducible representation $\pi : C^*(R) \to B(H)$ is said to be a boundary representation of $C^*(R)$ for $R$ if $\pi_{|_R}$ has unique extension property.*

## 2.4 Classical convex sets

Classical convexity in the set up of linear spaces is a geometric concept. An algebraic non commutative analogue this concept to the set up of $C^*$-algebras essentially was initiated by Arveson who also contributed extensively to the associated non commutative extremal theory. We will draw a lot from the work of Arveson and others in the later parts of the thesis.

Let $V$ be a vector space over real or complex numbers and let $K \subset V$ is said to be convex if $\lambda a + (1-\lambda)b \in K$ for all $a, b \in K$ and $\lambda \in [0,1]$.

Let $K \subset V$ be any convex set, a point $x \in K$ is an extreme point of $K$ if there does not exists $\lambda \in (0,1)$ such that $x = \lambda a + (1-\lambda)b$ where $a, b \in K$ and $a \neq b$.

For example, closed unit disc in complex plane is convex. Set of all points with modulus one are its extreme points.

## 2.5 $C^*$-convexity

A non commutative analogue of the classical convexity was formally introduced by Paulsen and Lobel [23] and is named as $C^*$-convexity. A subset $K$ of a $C^*$-algebra $A$ is $C^*$-convex , if $\{x_1, x_2, \cdots, x_n\} \subset K$, and $\{a_1, a_2, \cdots, a_n\} \subset A$ with $\sum_{i=1}^{n} a_i^* a_i = I$, then $\sum_{i=1}^{n} a_i^* x_i a_i \in K$, where $I$ is the identity of $A$.

**Remark 2.5.1.** *It is clear from the definition that classical convexity is a particular instance of $C^*$-convexity where the $C^*$-algebra under consideration is the set of complex numbers. $C^*$-convexity is essentially an 'algebraic' property in the context of algebras as opposed to the 'geometric' property of classical convexity in linear spaces.*

*All singleton sets will form classical convex sets where as in a non commutative $C^*$-algebra any non scalar singleton not a $C^*$-convex set. for this, let $x$ be non scalar ele-*

*ment in a non commutative $C^*$-algebra $A$. For the unitaries, $U$ in $A$ suppose $U^*xU = x$ will imply that $x$ is a scalar multiple of identity.*

**Example 2.5.1.**

- $B = \{T \in B(H) : ||T|| \leq 1\}$, *unit ball in* $B(H)$.

- $K = \{T \in B(H) : 0 \leq T \leq 1\}$, *then $K$ is $C^*$-convex. To show this, let* $T_i \in K$, $a_i \in B(H)$, $i = 1, 2, \cdots, n$ *with* $\sum_{i=1}^{n} a_i^* a_i = I$. $0 \leq T_i \leq 1 \implies 0 \leq \sum_{i=1}^{n} a_i^* T_i a_i \leq \sum_{i=1}^{n} a_i^* a_i = I$.

- *Let $T \in B(H)$, where $H$ is a Hilbert space, and let $C^*(T)$ be the $C^*$-algebra generated by $T$. The $n^{th}$ matrix range of $T$ for any natural number $n$ is defined as* $\mathcal{W}^n(T) = \{\phi(T) | \phi : C^*(T) \to M_n(\mathbb{C}) \text{ is a unital completely positive map}\}$. *It is the study of these sets called generalized numerical ranges of $T$ which preceded the study of $C^*$-convex sets by Arveson. The set $\mathcal{W}^n(T)$ is $C^*$-convex for any $T \in B(H)$. To see this let $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{W}^n(T)$, and $\{a_1, a_2, \cdots, a_n\} \subset M_n(\mathbb{C})$ with $\sum_{i=1}^{n} a_i^* a_i = I_n$. Then we can find $\phi_i : C^*(T) \to M_n(\mathbb{C})$ such that $\phi_i(T) = x_i$, $i = 1, 2, \cdots, n$ then define $\psi(x) = \sum_{i=1}^{n} a_i^* \phi_i(x) a_i$ where $x \in C^*(T)$, then $\psi$ is a completely positive map from $C^*(T)$ to $M_n(\mathbb{C})$ with $\psi(I) = \sum_{i=1}^{n} a_i^* a_i = I_n$, and $\psi(T) = \sum_{i=1}^{n} a_i^* x_i a_i$, then $\psi(T) \in \mathcal{W}^n(T)$ [2, p.301].*

- *Generalising the above idea for an operator system $R$, the set of all $n^{th}$ matrix states of $R$, $CS_n(R) = \{\phi : R \to M_n | \ \phi$ is unital completely positive $\}$ can be seen to be $C^*$-convex, arguing as above.*

### 2.5.1 $C^*$-extreme points of $C^*$-convex sets

As in classical theory, it is interesting to examine the analogous and appropriate notion of extreme points and the resulting possible Krein-Milman type theorems in $C^*$-convexity. The following definition captures the spirit of extremity in the new context which we will see to be the appropriate one.

**Definition 2.5.1.** *Let $K$ be a $C^*$- convex set, $x \in K$ is a proper $C^*$- convex combination of $\{x_1, x_2, \cdots, x_n\}$ if $x = \sum_{i=1}^{n} a_i^* x_i a_i$ where $\sum_{i=1}^{n} a_i^* a_i = I$ and each $a_i$ is invertible. A point $x \in K$ is a $C^*$- extreme point of $K$ if whenever $x$ is a proper $C^*$- convex combination of $\{x_1, x_2, \cdots, x_n\}$, then each $x_i$ is unitarily equivalent to $x$. i.e., there exists unitaries $u_i \in A$ such that $x = u_i^* x_i u_i$.*

Using a result by Loebl [24] that any matrix is linear extreme of its unitary orbit, Paulsen and Loebl proved [23, Proposition 23 ] that $C^*$-extreme points of of $C^*$-convex sets in $M_n$ are actually linear extreme points.

**Proposition 2.5.1.** *[23, Proposition23] If $T$ is a $C^*$-extreme point of a $C^*$-convex set $K \subset M_n$, then $T$ is a linear extreme point.*

We give some examples of $C^*$-extreme point of some $C^*$-convex sets. Consider $B(H)$, the $C^*$-extreme points of closed unit ball are precisely the isometries and co-isometries.

Let $U \in B(H)$ be either an isometry or a co-isometry. Then $U$ is a $C^*$-extreme point of the unit ball in $B(H)$ [19, Theorem 1.1]. The $C^*$-extreme points of the unit ball are precisely the isometries and co-isometries [19, Corollary 1.2]. If $I = \{T \in B(H) : 0 \leq T \leq 1\} \subseteq B(H)$, where H is separable; then the projections are $C^*$-extreme in $I$ [23, Proposition 26]. If $Q$ is a $C^*$-extreme point of $I$, then $Q$ is a projection [23, Proposition 28].

## 2.6 Injective operator systems and spaces

Injectivity is a categorical concept and here we will use the injectivity of operator systems (spaces) in the category of operator systems (spaces) and completely positive maps ( completely contractive maps) to study bi-extremity of embedding maps.

**Definition 2.6.1.** *An operator system $J$ is injective if for every operator system $R$ and every operator system $S$ containing $R$ as an operator subsystem, each completely positive linear map $\phi : R \to J$ has an extension to a completely positive linear map $\Phi : S \to J$.*

*Similarly, an operator space $J$ is injective if, for every operator space $R$ and every operator space $S$ containing $R$ as an operator subspace, each completely contractive linear map $\psi : R \to J$ has an extension to a completely contractive linear map $\Psi : S \to J$.*

Let $S \subset J \subset B(H)$ be any operator systems where $J$ is injective. A linear map $\phi : J \to J$ is a $S$-projection on $J$ if and only if it is unital completely positive, idempotent and $\phi_{|S} = I_S$ - the identity map on $S$. Let $\Phi, \Psi$ be two $S$-projections on $J$. Define a partial ordering $\leq$ by $\Phi \leq \Psi$ if and only if $\Phi \circ \Psi = \Psi \circ \Phi = \Phi$. An $S$-projection which is minimal with respect to this partial ordering $\leq$ will be called a *minimal $S$-projection*. Similarly we can define minimal $X$-projection in the context of an operator space $X$ and CC maps. For operator systems $R$ and $S$, a one-to-one completely positive linear map $\phi : R \to S$ such that $\phi(R)$ is an operator subsystem of $S$ is called a complete order embedding of $R$ into $S$ if $\phi$ is a complete order isomorphism when considered as a map from the operator system $R$ onto the operator system $\phi(R)$. It is known that injective operator systems (spaces) in $B(H)$ corresponds to CP (CC) projections [28, Theorem 15.1].

### 2.6.1 Injective envelope and $C^*$-envelope of operator systems

We recall the notions of injective envelope and $C^*$-envelope of an operator system [11, 17].

**Definition 2.6.2.** *An injective envelope of an operator system $R$ is a pair $(J, \tau)$ consisting of*

1. *an injective operator system $J$, and*
2. *a unital complete order embedding $\tau : R \to J$ such that, for every inclusion $\tau(R) \subseteq Q \subseteq J$ as operator subsystems in which $Q$ is injective, necessarily $Q = J$.*

Let $S \subset B(H)$ and $\phi : B(H) \to B(H)$, be a minimal $S$-projection, then Im$(\phi)$ is the injective envelope of $S$ [17, Theorem 4.1].

**Definition 2.6.3.** *A $C^*$-envelope of an operator system $R$ is a pair $(A, \tau)$ consisting of*

1. *a unital $C^*$ algebra $A$, and*

2. *a unital complete order embedding $\tau : R \to A$ such that $\tau(R)$ generates the $C^*$-algebra $A$*

*such that, for every unital complete order embedding $k : R \to B$ of $R$ into a unital $C^*$-algebra $B$ for which $k(R)$ generates $B$, there exists a unital $*$-homomorphism $\pi : B \to A$ with $\pi \circ k = \tau$.*

$$\begin{tikzcd} R \arrow[r, "\tau"] \arrow[dr, "\kappa"'] & A \\ & B \arrow[u, "\pi"'] \end{tikzcd}$$

The $C^*$-envelope of an operator system $R$ is the smallest $C^*$-algebra that contains $R$ as a sub operator system up to completely isometric isomorphism.

## 2.7 Hilbert $C^*$-modules

A Hilbert $C^*$-module is a generalisation of a Hilbert space with scalar valued inner product replaced by $C^*$-algebra valued inner product.

**Definition 2.7.1.** *Let $B$ be a $C^*$-algebra and $\mathcal{E}$ a complex vector space and a right $B$-module with a sesquilinear map $\langle ., . \rangle_B : \mathcal{E} \times \mathcal{E} \to B$ which is conjugate linear in the first variable and linear in the second variable such that $x, y \in \mathcal{E}, a \in B$*

1. $\langle x, x \rangle_B \geq 0$
2. $\langle x, x \rangle_B = 0 \implies x = 0$
3. $\langle x, y \rangle_B^* = \langle y, x \rangle_B$
4. $\langle x, ya \rangle_B = \langle x, y \rangle_B a$
5. *if $\mathcal{E}$ is complete with respect to the norm $\|x\| = \|\langle x, x \rangle_B\|^{\frac{1}{2}}$, then $\mathcal{E}$ is called a right Hilbert $B$-module.*

*Similarly we can define a left module.*

*We can consider Hilbert $C^*$-modules over possibly different $C^*$-algebras where along with the above axioms, a compatibility condition between the actions of the algebras is necessary.*

**Definition 2.7.2.** *Let $A$ and $B$ be $C^*$- algebras and $\mathcal{E}$ a complex vector space and an $A$-$B$-bimodule. Suppose that we have sesquilinear forms ${}_A\langle ., . \rangle$ and $\langle ., . \rangle_B$ so that $\mathcal{E}$ is both a left Hilbert $A$-module and a right Hilbert $B$-module such that the forms are related by the equation ${}_A\langle x, y \rangle z = x \langle y, z \rangle_B$ for all $x, y, z \in \mathcal{E}$. Then $\mathcal{E}$ is called a Hilbert $C^*$ $A$-$B$-bimodule.*

**Remark 2.7.1.** *The compatibility condition mentioned above ensures that if $\mathcal{E}$ is a Hilbert $C^*$ $A$-$B$-bimodule, then $\|{}_A\langle x, x\rangle\| = \|\langle x, x\rangle_B\|$ for all $x \in \mathcal{E}$ [5, Corollary 1.11].*

**Example 2.7.1.**

1. *Let $A$ be a $C^*$-algebra, then $A$ is a Hilbert $C^*$ $A$-$A$-bimodule where bimodule multiplication is defined by multiplication in $A$ and ${}_A\langle x, y\rangle = xy^*$ and $\langle x, y\rangle_A = x^*y$ are the inner products.*

2. *Consider $B(K, H)$,where $H$ and $K$ are Hilbert spaces. Then $B(K, H)$ is a Hilbert $C^*$ $B(H)$-$B(K)$-bimodule, where inner products are defined by ${}_{B(H)}\langle T, S\rangle = TS^*$, $\langle T, S\rangle_{B(K)} = T^*S$ for all $T, S \in B(K, H)$.*

# Chapter 3

# Matrix convex sets

Let $M_{m\times n}(V)$ denotes the vector space of $m \times n$ matrices over a complex vector space $V$, and set $M_n(V) = M_{n\times n}(V)$. We write $\mathbb{M}_{m\times n} = M_{m\times n}(\mathbb{C})$ and $\mathbb{M}_n = M_{n\times n}(\mathbb{C})$. The following definition of a matrix convex set as a non-commutative analogue of the classical convex set was introduced by Wittstock [30].

**Definition 3.0.1.** *A matrix convex set in a vector space $V$ is a collection $\mathbf{K} = (K_n)$ of subsets $K_n \subset M_n(V)$ such that*

$$\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \in K_n$$

$\forall\, v_i \in K_{n_i}, \gamma_i \in M_{n_i,n}$ *for* $i = 1, 2, \cdots, k$ *satisfying* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$.

From the definition of matrix convex sets, it is clear that if $\mathbf{K} = (K_n)$ is matrix convex, then each set $K_n$ is a $C^*$-convex set in $M_n(V)$, To see this, consider

$\gamma_i \in \mathbb{M}_n, v_i \in K_n$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$, then by definition of matrix convex set $\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \in K_n$. A *compact matrix convex set* is a matrix convex set $\mathbf{K} = (K_n)$ in a locally convex topological vector space $V$ such that each $K_n$ is compact in the product topology in $M_n(V)$.

We will now discuss a few examples of compact matrix convex sets from [29].

**Example 3.0.1.** *Let* $a, b \in \mathbb{R} \cup \{\pm\infty\}$*, define* $[\mathbf{aI}, \mathbf{bI}] = ([aI_n, bI_n])$ *where* $[aI_n, bI_n] = \{\alpha \in M_n | aI_n \leq \alpha \leq bI_n\}$.

To see this, let $\gamma_i \in M_{n_i,n}, \alpha_i \in [aI_{n_i}, bI_{n_i}]$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$, then $\gamma_i^* aI_i \gamma_i \leq \gamma_i^* \alpha_i \gamma_i \leq \gamma_i^* bI_n \gamma_i$ so that $\gamma_i^* \gamma_i aI_n \leq \gamma_i^* \alpha_i \gamma_i \leq \gamma_i^* \gamma_i bI_n$, $\forall i = 1, \cdots, k$ which gives $\sum_{i=1}^{k} \gamma_i^* \gamma_i aI_n \leq \sum_{i=1}^{k} \gamma_i^* \alpha_i \gamma_i \leq \sum_{i=1}^{k} \gamma_i^* \gamma_i bI_n$. Hence $\sum_{i=1}^{k} \gamma_i^* \alpha_i \gamma_i \in [aI_n, bI_n]$.

**Example 3.0.2.** *Let* $T$ *be a bounded operator on a separable Hilbert space* $H$*. Then* $\mathcal{W}(T) = (\mathcal{W}_n(T))$ *is a matrix convex set in* $\mathbb{C}$*, where* $\mathcal{W}_n(T) = \{\phi(T) | \ \phi : B(H) \to \mathbb{M}_n$ *unital completely positive map*$\}$ *is the* $n^{th}$ *matricial range of* $T$.

For this, let $\gamma_i \in M_{n_i,n}, \phi_i(T) \in \mathcal{W}_{n_i}(T)$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$. Since $\phi_i$ is unital completely positive $\gamma_i^* \phi_i \gamma_i$ is completely positive and $\sum_{i=1}^{k} \gamma_i^* \phi_i(I) \gamma_i = \mathbb{I}_n$. Therefore $\sum_{i=1}^{k} \gamma_i^* \phi_i \gamma_i$ is unital completely positive and $\sum_{i=1}^{k} \gamma_i^* \phi_i(T) \gamma_i \in \mathcal{W}_n(T)$.

**Example 3.0.3.** *Consider a concrete operator system* $R$ *acting on a separable infinite dimensional Hilbert space* $H$*. Let* $\mathbf{B} = (B_n)$*, where* $B_n = \{x \in M_n(R) : \|x\| \leq 1\}$*, then* $\mathbf{B}$ *is a matrix convex set in* $R$.

This is a special case of the above example which can be seen as follows. Let $S$ be the unilateral right shift operator on an infinite dimensional separable Hilbert space $\mathscr{H}$. We claim that $B_n = \mathcal{W}_n(S)$ for all $n$. For this we use the proof techniques of [10, Proposition 5.2]. First note that $S^*S \leq 1$ so that $\|\phi(S)\| \leq 1$ for all $\phi(S) \in \mathcal{W}_n(S)$. Thus $\mathcal{W}_n(S) \subseteq B_n$. Now since $\mathbb{T} \subset \sigma(S)$, we have $\sigma(S) \cap \partial\mathcal{W}_1(S) = \mathbb{T}$, where $\partial\mathcal{W}_1(S)$ is the topological boundary of $\mathcal{W}_1(S)$. By [1, Theorem 3.1.2] for each $\lambda \in \sigma(S) \cap \partial\mathcal{W}_1(S)$ there exists a multiplicative state $\phi$ on $C^*(S)$ such that $\lambda = \phi(S)$. Thus $\mathbb{T} \subseteq \mathcal{W}_1(S)$. By passing to matrix convex combinations of the elements of $\mathbb{T}$, every unitary $u \in \mathbb{M}_n$ is an element of $\mathcal{W}_n(S)$. Since the convex hull of the unitary group in $\mathbb{M}_n$ is $B_n$, we have, $B_n \subseteq \mathcal{W}_n(S)$. Hence the claim.

The following example is a well known example of a compact matrix convex set.

**Example 3.0.4.** *For an operator system $R$, denote the set of all $n^{th}$ matrix states of $R$ by $CS_n(R) = \{\phi : R \to \mathbb{M}_n | \ \phi$ is UCP$\}$.*

Then **CS**$(R) = (CS_n(R))$ is a compact matrix convex set in $R^*$-the dual space of $R$, equipped with the weak$^*$-topology [29, Proposition 3.5].

## 3.1 Matrix extreme points

In the section we look at the natural notion of extreme points associated with matrix convex sets.

**Definition 3.1.1.** *Suppose that $\boldsymbol{K} = (K_n)$ is a matrix convex set in $V$. We say that* *a matrix convex combination* $v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \in K_n$ *with* $v_i \in K_{n_i}$, $\gamma_i \in \mathbb{M}_{n_i,n}$ *for* $i = 1, 2, \cdots, k$ *satisfying* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$ *is proper if each* $\gamma_i$ *has a right inverse in* $\mathbb{M}_{n,n_i}$. *Also we say* $v \in K_n$ *is a matrix extreme point if whenever* $v$ *is a proper matrix convex combination of* $v_i \in K_{n_i}$ *for* $i = 1, 2, \cdots, k$, *then each* $n_i = n$ *and* $v = u_i^* v_i u_i$ *for some unitary* $u_i \in \mathbb{M}_n$.

Now we give examples of matrix extreme points of certain matrix convex sets.

**Example 3.1.1.** *[29, Example 2.2] Let* $a, b \in \mathbb{R}$ *then the matrix extreme points of matrix convex set* $K_n = [\mathbf{aI_n}, \mathbf{bI_n}]$ *are* $a, b$.

The matrix extreme points of $K_1 = [a, b]$ are classical extreme points hence are $a$ and $b$. Let $T \in [aI_n, bI_n], n \geq 2$, since $T$ is a diagonalizable matrix and spectrum of $T \in [a, b]$ there exists $U \in \mathbb{M}_n$ such that $T = U^* \begin{bmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{bmatrix} U$, where $x_i \in [a, b]$.

Let $U = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, where $a_i$ is the $i^{th}$ row matrix of $U$. Since $U$ is unitary, $a_i$ s are right invertible, therefore $T = \sum_{i=1}^{n} a_i^* x_i a_i$ is a proper matrix convex combination. But $T$ cannot be unitaraly equivalent to $x_i$, for all $i = 1, \cdots, n$. Hence there is no element in $[aI_n, bI_n], n \geq 2$ which is matrix extreme. Therefore matrix extreme points of $K$ are precisely $a, b$.

**Example 3.1.2.** *[29, Example 2.3] If* $A$ *be a* $C^*$*-algebra, then matrix extreme points* *of the matrix convex set* $\boldsymbol{CS}(A)$ *are exactly the pure matrix states.*

On a matrix convex set $K$, we can define maps called matrix affine maps. With these maps we get operator systems associated with each compact matrix convex set.

**Definition 3.1.2.** *A matrix affine mapping on a matrix convex set* $\mathbf{K} = (K_n)$ *in a vector space* $V$ *is a sequence* $\theta = (\theta_n)$ *of mappings* $\theta_n : K_n \to M_n(W)$ *for some vector space* $W$ *such that*

$$\theta_n(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i) = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i)\gamma_i$$

*for all* $v_i \in K_{n_i}$, $\gamma_i \in \mathbb{M}_{n_i,n}$ *for* $i = 1, 2, \cdots, k$ *satisfying* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = \mathbb{I}_n$.

Let $\mathbf{K} = (K_n)$ and $\mathbf{K}' = (K_n')$ be matrix convex sets in locally convex topological vector spaces $V$ and $W$ respectively. A *matrix affine homeomorphism* $\theta = (\theta_n)$ between $\mathbf{K}$ and $\mathbf{K}'$ is a matrix affine mapping such that each $\theta_n$ is a homeomorphism of the product topologies on either sides. It is easy to see that matrix extreme points and linear extreme points are preserved under matrix affine homeomorphisms.

Given a compact matrix convex set, one can associate an operator system with it in the following way: if $\mathbf{K}$ is a compact matrix convex set in a locally convex space $V$, then $A(\mathbf{K}) = \{f = (f_n) : \mathbf{K} \to \mathbb{C} | f$ matricially affine and $f_1$ continuous$\}$ is an operator system in the sense of Choi-Effros. Following proposition gives the connection between a compact matrix convex set and its associated operator system.

**Proposition 3.1.1.** *[29, Proposition 3.5]*

1. *If* $R$ *is an operator system, then* $\boldsymbol{CS}(R)$ *is a self-adjoint compact convex set in*

*$R^*$, equipped with the weak$^*$ topology, and $A(\boldsymbol{CS}(R))$ and $R$ are isomorphic as operator systems.*

2. *If $K$ is a compact matrix convex set in a locally convex space $V$, then $A(K)$ is an operator system, and $K$ and $\boldsymbol{CS}(A(K))$ are matrix affinely homeomorohic.*

## 3.2 Boundary points

Characterisation of boundary representations for operator systems has been done by several authors. Arveson [3] characterised boundary representations in terms of maximality of UCP maps. Kleski in [21] presents another characterisation that exploits a connection between boundary representations of the operator system $A(\mathbf{K})$ and a certain type of extreme points of the compact convex set $\mathbf{K}$, namely boundary points defined below.

**Definition 3.2.1.** *[21] A boundary point of a matrix convex set $\mathbf{K} = (K_n)$ is an element $b \in K_n$ such that whenever $b$ is a matrix convex combination $b = \sum_{i=1}^{m} \gamma_i^* a_i \gamma_i$, not necessarily proper, of elements $a_i \in K_{n_i}$, then $a_i \sim_u b$ if $n_i \leq n$; otherwise, $a_i \sim_u b \oplus c_i$ for some $c_i \in \mathbf{K}$.*

In the following theorem Kleski [21] established the connection between boundary points of compact matrix convex set $K$ and boundary representations for $A(K)$.

**Theorem 3.2.1.** *[21, Theorem 4.2] Let $\mathbf{K}$ be a compact matrix convex set in a locally convex space $V$. Suppose $A(\mathbf{K})$ acts on a finite dimensional Hilbert space. The* *boundary points of* $\mathbf{K}$ *correspond exactly to the boundary representations for* $A(\mathbf{K})$.

From the definition of a boundary point it is clear that all boundary points are matrix extreme points. But, Kleski [21] gives an example to show that all matrix extreme points need not be boundary points.

**Example 3.2.1.** *[21] Let* $T = T_1 \oplus T_2$*, where* $T_1 = 1, T_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$. *Let* $K = (\mathcal{W}_n(T))$*. We can see that* $T_1 \in (\mathcal{W}_1(T)), T_2 \in (\mathcal{W}_2(T))$ *are matrix extreme points of* $K$*. Since* $T_1$ *can be written as proper compression of* $T_2$*, that is* $T_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} T_2 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. *Therefore* $T_1$ *cannot be a boundary point of* $K$.

Here we give an example to show that there are matrix convex sets for which all matrix extreme points are boundary points.

**Example 3.2.2.** *Consider the matrix convex set* $\boldsymbol{K} = ([a\mathbb{I}_n, b\mathbb{I}_n])$. *Webster and Winkler [29] proved that the set of all matrix extreme points of* $\boldsymbol{K}$*,* $\partial\boldsymbol{K}$ *is* $\{a, b\}$*. Here we show that* $a$ *and* $b$ *are also the boundary points of* $\boldsymbol{K}$*. Without loss of generality assume that* $a = \gamma_i^* v_i \gamma_i$*, where* $\gamma_i \in M_{n_i,1}$ *and* $v_i \in K_{n_i}, \gamma_i^*\gamma_i = 1$*. If* $n_i = 1$*, then there is nothing to prove. If* $n_i = 2$*, then one can reduce the expression* $\gamma_i^* v_i \gamma_i$ *into an expression in the first level in the following way.*

*Since* $v_i \in [a\mathbb{I}_2, b\mathbb{I}_2]$*, we have*

$$v_i = U^* \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} U,$$

*where $U$ is a unitary in $\mathbb{M}_2$ and $a', b' \in [a, b]$.*

*Now consider*

$$\begin{aligned} a =& \gamma_i^* v_i \gamma_i = \gamma_i^* \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \gamma_i = \gamma_i^* U^* \begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix} U \gamma_i \\ =& c_1^* a' c_1 + c_2^* b' c_2, \end{aligned}$$

*where $[c_1, c_2] = \gamma_i^* U^*$. This gives us $a = a'$ and $a = b'$. Similarly one can extend the argument to the case $n_i \geq 2$ and to any matrix convex combination. Thus $a$ is a boundary point of **K**. Similarly $b$ is also a boundary point of **K**.*

This result is similar in nature to Theorem 3.2.1, but instead of a compact convex set, it concerns an operator system. The proof utilizes a result by Farenick that establishes a connection between matrix extreme points of **K** and pure matrix states. Additionally, we employ an extension result on pure matrix states of $A(\mathbf{K})$ and boundary representations established by Kleski.

**Theorem 3.2.2.** *Let $R$ be an operator system acting on a finite dimensional Hilbert space. Then boundary representations for $R$ correspond exactly to boundary points of **CS**$(R)$.*

*Proof.* By Theorem 3.2.1, there is a one to one correspondence between boundary points of $\mathbf{CS}(R)$ and the boundary representations for $A(\mathbf{CS}(R))$. By [29, Proposition

3.5] $R$ and $A(\mathbf{CS}(R))$ are isomorphic as operator systems. That is, there is UCP-map $\phi$ between $R$ and $A(\mathbf{CS}(R))$ such that $\phi$ is bijective and $\phi^{-1}$ is CP. Since $\phi$ and $\phi^{-1}$ are UCP-maps, $\phi$ is a complete isometry. But by [1, Theorem 2.1.2] there is a one to one correspondence between boundary representations for $R$ and boundary representations for $A(\mathbf{CS}(R))$. Thus there is a one to one correspondence between boundary representations for $R$ and the boundary points of $\mathbf{CS}(R)$. $\square$

**Corollary 3.2.1.** *Let $R$ be an operator system acting on a finite dimensional Hilbert space. Then a representation $\pi : C^*(R) \to M_n(\mathbb{C})$ is a boundary representation for $R$ if and only if $\pi_{|_R}$ is a boundary point of $\mathbf{CS}(R)$.*

For a matrix convex set $\mathbf{K}$ for which matrix extreme points and boundary points coincide, the following theorem gives a sufficient condition for a finite dimensional representation to be a boundary representation for $A(\mathbf{K})$.

**Theorem 3.2.3.** *Let $\mathbf{K}$ be a compact matrix convex set and $A(\mathbf{K})$ be the associated operator system which acts on a finite dimensional Hilbert space $H$. Assume that every matrix extreme point in $\mathbf{K}$ is a boundary points of $\mathbf{K}$. If $\pi$ is a representation of $C^*(A(\mathbf{K}))$ on a finite dimensional Hilbert space such that $\pi_{|_{A(\mathbf{K})}}$ is pure, then $\pi$ is a boundary representation.*

*Proof.* We have $\pi_{|_{A(\mathbf{K})}} \in UCP(A(\mathbf{K}), \mathbb{M}_l)$ is pure. By [9] pure CP maps are matrix extreme points. Then by our assumption $\pi_{|_{A(\mathbf{K})}}$ is a boundary point. By Theorem 3.2.1 there is a boundary representation $\rho$ of $C^*(A(\mathbf{K}))$ for $A(\mathbf{K})$ such that $\pi_{|_{A(\mathbf{K})}} = U^*\rho(.)U$

where $U$ is a unitary. Hence $\pi$ is a completely positive extension of $U^*\rho(.)U$. Since $\rho$ is a boundary representation, $U^*\rho(.)U$ is also a boundary representation which gives $\pi = U^*\rho(.)U$. Thus $\pi$ is a boundary representation. $\square$

# Chapter 4

# Bi-convex sets and bi-extreme maps

## 4.1 Bi-convexity in Hilbert $C^*$-bimodules

We consider bi-convex sets in Hilbert $C^*$-bimodules and study their bi-extreme points. We deduce that for a trivial module, unitaries are bi-extreme points of the closed unit ball whereas conversely, bi-extreme points of the closed unit ball of the trivial module $B(H)$, where $H$ is a Hilbert space are shown to be either isometries or co-isometries.

The natural notion of non-commutative convexity in bimodules is bi-convexity and is defined below along the lines of [20].

**Definition 4.1.1.** *Let $\mathcal{E}$ be a Hilbert $C^*$ $A$-$B$-bimodule and $\{x_1, x_2, \cdots, x_n\} \subset \mathcal{E}$. A bi-convex combination of $\{x_1, x_2, \cdots, x_n\}$ is $\sum\limits_{i=1}^{n} a_i^* x_i b_i$ where $a_i \in A, b_i \in B$ such that $\sum\limits_{i=1}^{n} a_i^* a_i = I_A, \sum\limits_{i=1}^{n} b_i^* b_i = I_B$.*
*A subset $K$ of $\mathcal{E}$ is called $A$-$B$-convex if it is closed under all bi-convex combinations.*

*If the algebras under discussion are clear, we will refer to $A$-$B$-convex as bi-convex.*

**Remark 4.1.1.** *(i) Case when $\mathcal{E}$=$A$=$B$=$\mathcal{A}$-a $C^*$-algebra, the $A$-$B$-bimodule $\mathcal{A}$ is called a trivial Hilbert $C^*$-bimodule.*

*(ii) It is to be noted that even in the case of a trivial $C^*$-bimodule, bi-convexity in the $C^*$-bimodule $\mathcal{A}$ is different from the $C^*$- convexity [19, 23] in the $C^*$- algebra $\mathcal{A}$ because of the larger set of coefficients available in bi-convexity. It is easy to see that even in the trivial case, bi-convex sets and $C^*$-convex sets do not coincide though in this case bi-convex sets are $C^*$- convex sets and not conversely.*

**Example 4.1.1.** *Consider $M_2(\mathbb{C})$ as Hilbert $M_2(\mathbb{C}) - M_2(\mathbb{C})$-bimodule. Let $\alpha$ be a scalar, $K = \{\alpha I\}$ then it is easy to see that $K$ is a $C^*$- convex set.*
*Let $a = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, $b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $a^*a = I_2, b^*b = I_2$ but $a^*(\alpha I)b = \alpha a^* \notin K$, hence $K = \{\alpha I\}$ is not $M_2(\mathbb{C}) - M_2(\mathbb{C})$-convex set.*

**Remark 4.1.2.** *Let $X$ be a Hilbert $A$-$B$-bimodule. If $K \subset X$ is $A$-$B$-convex. If $\lambda_i \in [0, 1]$, $i = 1, \cdots, n$, such that $\sum\limits_{i=1}^{n} \lambda_i = 1$. Consider $a_i = \sqrt{\lambda_i} I_A, b_i = \sqrt{\lambda_i} I_B$. we get $\sum\limits_{i=1}^{n} a_i^* a_i = \sum\limits_{i=1}^{n} \lambda_i I_A = I_A, \sum\limits_{i=1}^{n} b_i^* b_i = \sum\limits_{i=1}^{n} \lambda_i I_B = I_B$. Let $x_i \in K$ then $\sum\limits_{i=1}^{n} \lambda_i x_i = \sum\limits_{i=1}^{n} a_i^* x_i b_i \in K$, hence $K$ is convex in the usual sense.*

Now, we will give some examples of bi-convex sets.

**Example 4.1.2.** *1. Let $A$ and $B$ be $C^*$ algebras and $X$ be a Hilbert $A$-$B$-bimodule. If $M$ is a positive scalar, then $S = \{x \in X, \|x\| \leq M\}$ is $A$-$B$-convex. In* *particular, the closed unit ball of $X$ is $A$-$B$-convex [20, Theorem 10].*

2. *Consider $B(K,H)$ as Hilbert $B(H)$-$B(K)$-bimodule. Then for a fixed scalar $r>0$, the set $S=\{T\in B(K,H); 0\leq T^*T\leq rI_K\}$ is $B(H)$-$B(K)$-bi-convex [20, Propositon 4].*

3. *Let $X$ be a Hilbert $A$-$B$-bimodule. Then the subset $S=\{x\in X:\langle x,x\rangle\leq r^2I_A$, for some positive real number $r\neq 1\}$ of $X$ is $A$-$B$- bi-convex [20, Example 8].*

4. *Let $X$ be a Hilbert $A$-$B$- bimodule, $x,y\in X$. Define $A$-$B$-bi-convex segment connecting $x$ and $y$ by $CS_{A,B}(x,y)=\{\sum\limits_{i=1}^{n}a_i^*xb_i+\sum\limits_{i=1}^{m}c_i^*yd_i|\sum\limits_{i=1}^{n}a_i^*a_i+\sum\limits_{i=1}^{m}c_i^*c_i=I_A, \sum\limits_{i=1}^{n}b_i^*b_i+\sum\limits_{i=1}^{m}d_i^*d_i=I_B\}$. If $x,y\in X$, then $CS_{A,B}(x,y)$ is $A$-$B$-bi-convex and contains $x$ and $y$ [20, Proposition 15 ].*

As in the case of any other convexity, a computationally friendly convex combination requires only two elements for testing bi-convexity as proved in Proposition 16 [20] which we quote below:

**Proposition 4.1.1.** *Let $S$ be an $A$-$B$-convex subset of the Hilbert $A$-$B$-bimodule $\mathcal{E}$ and let $\{x_1,x_2,\cdots,x_n\}\subseteq K$. If $z=\sum\limits_{i=1}^{n}a_ix_ib_i$ with $a_i\in A,\ b_i\in B$ and $\sum\limits_{i=1}^{n}a_ia_i^*=I_A$ and $\sum\limits_{i=1}^{n}b_i^*b_i=I_B$, then $z=e_1xf_1+e_2yf_2$, for some $x,y\in S, e_1,e_2\in A$ and $f_1,f_2\in B$ with $e_1e_1^*+e_2e_2^*=I_A$ and $f_1^*f_1+f_2^*f_2=I_B$.*

## 4.2 Bi-extreme points

Now we define bi-extreme points of bi-convex subsets of Hilbert $C^*$-bimodules.

**Definition 4.2.1.** *Let $K$ be a bi-convex subset of a Hilbert $C^*$ $A$-$B$-bimodule $\mathcal{E}$. A bi-convex combination $x = \sum_{i=1}^{n} a_i^* x_i b_i$; $a_i \in A, b_i \in B$ and $x_i \in K$ is said to be a proper representation of $x$ if $a_i$ and $b_i$ are invertible, for all $i = 1, \cdots, n$. An element $x \in K$ is a bi-extreme point of $K$ if for any proper representation $\sum_{i=1}^{n} a_i^* x_i b_i$ of $x$ there exists unitaries $u_i \in A$, $v_i \in B$ such that $x = u_i^* x_i v_i$ for all $i = 1, \cdots, n$.*

**Remark 4.2.1.** *Not that the bi-convex combination in the above definition can be reduced to a combination of two terms in view of 4.1.1. Further using polar decomposition it is enough to consider positive invertible coefficients in the bi-convex combination to check for bi-extreme points in bi-convex sets as explained below. Suppose $X$ is a Hilbert $A$-$B$-bimodule and $K \subseteq X$ a bi-convex set. Consider $z = a_1^* x_1 b_1 + a_2^* x_2 b_2$, where $a_1, a_2 \in A$, $b_1, b_2 \in B$ and $x_1, x_2 \in X$ with $a_1, a_2$ are invertible in $A$ and $b_1, b_2$ are invertible in $B$ such that $\sum_{i=1}^{2} a_i^* a_i = I_A$, $\sum_{i=1}^{2} b_i^* b_i = I_B$. Assume that $a_i = u_i p_i$ and $b_i = v_i q_i$ be the polar decompositions, where $p_i, q_i$ are positive invertible elements and $u_i, v_i$ are unitaries of $A, B$ respectively. Then $z = a_1^* x_1 b_1 + a_2^* x_2 b_2 = p_1 u_1^* x_1 v_1 q_1 + p_2 u_2^* x_2 v_2 q_2 = p_1 y_1 q_1 + p_2 y_2 q_2$, where $y_1, y_2 \in K$. This means that $z$ can be represented as a combination with positive invertible coefficients.*

The following theorem gives examples for bi-extreme points.

**Theorem 4.2.1.** *Unitaries are bi-extreme points of the bi-convex set of closed unit ball of any trivial bimodule.*

*Proof.* Let $\mathcal{A}$ be a trivial bimodule and $K$ denotes the closed unit ball in $\mathcal{A}$. Let $u \in \mathcal{A}$ be any unitary. Let $u = p_1x_1q_1 + p_2x_2q_2$, where $x_i \in K$ and $p_i, q_i \in \mathcal{A}$; $p_i, q_i > 0$ such that $p_1^2 + p_2^2 = I$ and $q_1^2 + q_2^2 = I$. Note that $v_1 = \begin{pmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix}$ are unitaries in $M_2(\mathcal{A})$. For $y = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ we have

$$
\begin{aligned}
v_1yv_2 &= \begin{pmatrix} p_1 & p_2 \\ -p_2 & p_1 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \\
&= \begin{pmatrix} p_1x_1q_1 + p_2x_2q_2 & -p_1x_1q_2 + p_2x_2q_1 \\ -p_2x_1q_1 + p_1x_2q_2 & p_2x_1q_2 + p_1x_2q_1 \end{pmatrix} \\
&= \begin{pmatrix} u & -p_1x_1q_2 + p_2x_2q_1 \\ -p_2x_1q_1 + p_1x_2q_2 & p_2x_1q_2 + p_1x_2q_1 \end{pmatrix}.
\end{aligned}
$$

As $u$ is a unitary and $\|v_1yv_2\| \leq 1$, it follows that $-p_1x_1q_2 + p_2x_2q_1 = 0 = -p_2x_1q_1 + p_1x_2q_2$. Thus,

$$
\begin{aligned}
p_1x_1q_2 &= p_2x_2q_1 \\
x_1 &= p_1^{-1}p_2x_2q_1q_2^{-1} \qquad (4.1)
\end{aligned}
$$

$$\begin{aligned} p_2x_1q_1 &= p_1x_2q_2 \\ x_1 &= p_2^{-1}p_1x_2q_2q_1^{-1} \qquad (4.2) \\ p_1^{-1}p_2x_2q_1q_2^{-1} &= p_2^{-1}p_1x_2q_2q_1^{-1}. \end{aligned}$$

Since $p_1^2 + p_2^2 = I$ and $q_1^2 + q_2^2 = I$, we have $p_1$ commutes with $p_2$ and $q_1$ commutes with $q_2$. Then

$$x_2q_1^2q_2^{-2} = p_1^2p_2^{-2}x_2. \qquad (4.3)$$

Now using $p_1x_1q_2 = p_2x_2q_1$,

$$\begin{aligned} x_2 &= p_1p_2^{-1}x_1q_2q_1^{-1} \qquad (4.4) \\ p_2x_1q_1 &= p_1x_2q_2 \\ x_2 &= p_1^{-1}p_2x_1q_1q_2^{-1}. \qquad (4.5) \end{aligned}$$

From (4.4) and (4.5)

$$\begin{aligned} p_1p_2^{-1}x_1q_2q_1^{-1} &= p_1^{-1}p_2x_1q_1q_2^{-1} \\ x_1q_2^2q_1^{-2} &= p_1^{-2}p_2^2x_1. \qquad (4.6) \end{aligned}$$

Now

$$
\begin{aligned}
u &= p_1x_1q_1 + p_2x_2q_2 \\
&= p_1p_1^{-1}p_2x_2q_1q_2^{-1}q_1 + p_2x_2q_2 \\
&= p_2x_2q_1^2q_2^{-1} + p_2x_2q_2 \\
&= p_2(x_2q_1^2q_2^{-2} + x_2)q_2. && (4.7) \\
&= p_2(p_1^2p_2^{-2}x_2 + x_2)q_2 \\
&= p_2((-I + p_2^{-2})x_2 + x_2)q_2 \\
&= p_2p_2^{-2}x_2q_2 \\
&= p_2^{-1}x_2q_2 && (4.8)
\end{aligned}
$$

where we used $p_1^2p_2^{-2} = p_1^2(I - p_1^2)^{-1} = -I + (I - p_1^2)^{-1} = -I + p_2^{-2}$.

Similarly, using $q_1^2q_2^{-2} = -I + q_2^{-2}$ in equation (4.7) we have

$$
u = p_2x_2q_2^{-1}. \tag{4.9}
$$

Substituting equation (4.4) in $u = p_1x_1q_1 + p_2x_2q_2$ will give

$$
\begin{aligned}
u &= p_1x_1q_1 + p_2p_1p_2^{-1}x_1q_2q_1^{-1}q_2 \\
&= p_1x_1q_1 + p_1x_1q_2^2q_1^{-1} \\
&= p_1(x_1 + x_1q_2^2q_1^{-2})q_1 && (4.10)
\end{aligned}
$$

$$= p_1(x_1 + x_1(-I + q_1^{-2}))q_1$$

$$= p_1 x_1 q_1^{-1}. \tag{4.11}$$

Similarly, $u = p_1^{-1} x_1 q_1$. Then by (4.11) we have $u = p_1 x_1 q_1^{-1} = p_1^{-1} x_1 q_1$. It follows that $p_1^2 x_1 = x_1 q_1^2$. Applying functional calculus with the root function on $\sigma(p_1^2) \cup \sigma(q_1^2)$, we have

$$p_1 x_1 = x_1 q_1 \tag{4.12}$$

Similarly from equations (4.8) and (4.9) we will get

$$p_2 x_2 = x_2 q_2. \tag{4.13}$$

Using equations (4.12) and (4.13), we can see that $u = x_1$ and $u = x_2$ as follows.

$$\begin{aligned} u &= p_1 x_1 q_1 + p_2 x_2 q_2 \\ &= p_1 p_1 x_1 + p_2 p_1^{-1} p_2 x_1 q_1 q_2^{-1} q_2 \\ &= p_1^2 x_1 + p_2^2 p_1^{-1} p_1 x_1 \\ &= p_1^2 x_1 + p_2^2 x_1 \\ &= (p_1^2 + p_2^2) x_1 \\ &= x_1 \end{aligned}$$

and

$$\begin{aligned} u &= p_1 p_1^{-1} p_2 x_2 q_1 q_2^{-1} q_1 + x_2 q_2^2 \\ &= p_2 x_2 q_1^2 q_2^{-1} + x_2 q_2^2 \\ &= x_2 q_2 q_1^2 q_2^{-1} + x_2 q_2^2 \\ &= x_2 (q_1^2 + q_2^2) \\ &= x_2. \end{aligned}$$

Hence $u$ is a bi-extreme point. □

**Remark 4.2.2.** *In the case of the trivial module $B(H)$ where $H$ is a Hilbert space, it can be shown that bi-extreme points of the closed unit ball are either isometries or co-isometries. Let $T$ be a bi-extreme point of the closed unit ball of $B(H)$. Then $T$ can be written as $T = \frac{1}{2}(T_1 + T_2)$, where $T_1, T_2$ are either isometry or co-isometry [16, Problem 107] which is a proper bi-convex combination in the trivial module. Since $T$ is bi-extreme, there exists unitaries $u_i, v_i$ such that $T = u_i^* T_i v_i$, $i = 1, 2$ which implies that $T$ is either an isometry or a co-isometry.*

## 4.3 Bi-extreme CC maps on operator spaces

We describe bi-extremity of CC maps between operator spaces and prove that the notion is equivalent to the notion of purity and operator extremity of the corresponding

Paulsen map for the case of embedding of the Paulsen system in its injective envelope.

In this section, we study extremity of the canonical embedding map from an operator space $X$ into its injective envelope $I(X)$. Let $X \subseteq B(H)$ be an operator space and $S(X)$ be the Paulsen system associated with $X$. Let $\Phi : B(H \oplus H) \to B(H \oplus H)$ be the minimal $S(X)$-projection. Then image of $\Phi$ is the injective envelope of $S(X)$ and is denoted by $I(S(X))$ [17]. We can decompose $\Phi$ as $\Phi = \begin{bmatrix} \sigma & \pi \\ \pi^* & \theta \end{bmatrix}$ where $\sigma, \theta : B(H) \to B(H)$ are CP maps and $\pi : B(H) \to B(H)$ is CC. Let us denote the image of $\sigma$ by $I_{11}(X)$ and image of $\theta$ by $I_{22}(X)$ respectively. As $\Phi$ is a completely positive projection, the maps $\sigma$ and $\theta$ are also completely positive projections on $B(H)$. Thus $I_{11}(X)$ and $I_{22}(X)$ are injective operator systems and hence they are $C^*$-algebras, where the multiplication is defined as follows: for $x, y \in I_{11}(X)$, $x \circ y = \sigma(xy)$ and similarly for $I_{22}(X)$. From the minimality of $\Phi$, we can see that $\pi$ is a minimal $X$-projection and image of $\pi$ is the injective envelope $I(X)$ of the operator space $X$ [18]. We note that $I(X)$ is a Hilbert $C^*$ $I_{11}(X)$ - $I_{22}(X)$-bimodule as explained below.

We have, $I(S(X)) = \left\{ \begin{bmatrix} a & x \\ y^* & b \end{bmatrix} ; a \in I_{11}(X), b \in I_{22}(X), x, y \in I(X) \right\}$. Since $I(S(X))$ is a $C^*$-algebra, for $x, y \in I(X)$, $\begin{bmatrix} 0 & x \\ y^* & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ y^* & 0 \end{bmatrix} = \begin{bmatrix} x \circ y^* & 0 \\ 0 & y^* \circ x \end{bmatrix}$. Therefore the product $x \circ y^* \in I_{11}(X), y^* \circ x \in I_{22}(X)$ gives an $I_{11}(X)$-valued innerproduct $\langle x, y \rangle = x \circ y^*$ which makes $I(X)$ a Hilbert $C^*$- left module over $I_{11}(X)$. Similarly we can define $I_{22}(X)$-valued innerproduct $\langle x, y \rangle = x^* \circ y$ that makes $I(X)$ a Hilbert $C^*$- right module over $I_{22}(X)$. In fact, since there is the required compatibility condition between the above two inner products, $I(X)$ is a Hilbert $C^*$- $I_{11}(X)$ -

$I_{22}(X)$-bimodule.

We require the following lemma for our future discussions. This is in fact a suitable form for our purposes of a similar result in [15].

Let $X$ be an operator space and $\phi : X \to I(X)$ be a CC map. As $S(I(X)) \subseteq I(S(X))$, in the remaining chapters we will consider the corresponding Paulsen map $S(\phi) : S(X) \to S(I(X))$ of $\phi$ as a map $S(\phi) : S(X) \to I(S(X))$.

**Lemma 4.3.1.** *Suppose $\phi : X \to I(X)$ is a completely contractive (CC) map such that $S(\phi) : S(X) \to I(S(X))$ is the corresponding Paulsen map. Let $\Psi : S(X) \to I(S(X))$ be a CP map such that $S(\phi) - \Psi$ is a CP map. If $\Psi(1)$ is an invertible element in the $C^*$-algebra $I(S(X))$, then there exist invertible elements $a \in I_{11}(X)$ and $b \in I_{22}(X)$, and a CC map $\psi : X \to I(X)$ such that*

$$\Psi\left(\begin{bmatrix} \lambda I_H & x \\ y^* & \mu I_H \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \circ \begin{bmatrix} \lambda I_H & \psi(x) \\ \psi(y)^* & \mu I_H \end{bmatrix} \circ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.$$

*Proof.* Let $T \subseteq B(H)$ be the TRO generated by $X$ and $A$ be the $C^*$-algebra generated by $S(X)$. Since $I(S(X))$ is injective, we can extend the CP maps $S(\phi)$ and $\Psi$ to $A$ in such a way that $S(\phi) - \Psi$ is still completely positive. In the following we regard $S(\phi)$ and $\Psi$ as maps from $A$ to $I(S(X))$. Note that $A = \{ \begin{bmatrix} x_{11} + \lambda I_H & x_{12} \\ x_{21} & x_{22} + \mu I_H \end{bmatrix} : x_{11} \in TT^*, x_{12} \in T, x_{21} \in T^*, x_{22} \in T^*T, \lambda, \mu \in \mathbb{C}\}$. Now, let $p$ be a positive element in the $C^*$-algebra $A_1 = TT^* + \mathbb{C}I_H$, such that $p \leq 1$. As

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} \Psi\left(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}\right) &= (S(\phi) - (S(\phi) - \Psi))\left(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\leq (S(\phi) - (S(\phi) - \Psi))\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - (S(\phi) - \Psi)\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\leq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Thus

$$\Psi\left(\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $A_1$ is the span of its positive elements, there exists a CP map

$\phi_1 : A_1 \to B(H)$ such that

$$\Psi\left(\begin{bmatrix} d_1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} \phi_1(d_1) & 0 \\ 0 & 0 \end{bmatrix}.$$

By analogous arguments one obtains a CP map $\phi_2 : A_2 \to B(H)$ such that

$$\Psi\left(\begin{bmatrix} 0 & 0 \\ 0 & d_2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & \phi_2(d_2) \end{bmatrix},$$

where $A_2 = T^*T + \mathbb{C}I_H$. Let $a_{11} = \phi_1(1)$, $b_{11} = \phi_2(1)$ and $\omega = \Psi(1)$. As $\Psi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) \in I(S(X))$, we have $a_{11} \in I_{11}(X)$ and $b_{11} \in I_{22}(X)$. Let $a_{11}^{1/2}$ and $b_{11}^{1/2}$ be the square roots of $a_{11}$ and $b_{11}$ in the $C^*$-algebras $I_{11}(X)$ and $I_{22}(X)$ respectively.

Define a map $\Psi_0 : S(X) \to I(S(X))$ by

$$\Psi_0 = \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix} \circ \Psi \circ \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix}.$$

Recall that $\Phi$ is the minimal $S(X)$-projection. Then

$$\Psi_0\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix} \circ \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix}$$

$$
\begin{aligned}
&= \Phi\left( \begin{bmatrix} a_{11}^{-1/2}a_{11} & 0 \\ 0 & 0 \end{bmatrix} \right) \circ \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix} \\
&= \begin{bmatrix} \sigma(a_{11}^{-1/2}a_{11}) & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} a_{11}^{-1/2} & 0 \\ 0 & b_{11}^{-1/2} \end{bmatrix} \\
&= \Phi\left( \begin{bmatrix} a_{11}^{1/2}a_{11}^{-1/2} & 0 \\ 0 & 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \sigma(a_{11}^{1/2}a_{11}^{-1/2}) & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Similarly, $\Psi_0\left( \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$.

It follows that the two projections $\begin{bmatrix} I_H & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & I_H \end{bmatrix}$ belongs to the multiplicative domain of $\Psi_0$ [4, Proposition 1.3.11]. There exists a completely contractive map $\psi : X \to I(X)$ such that

$$\Psi_0\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}\right) = \begin{bmatrix} \lambda I & \psi(x) \\ \psi(y)^* & \mu I \end{bmatrix}.$$

Then

$$\Psi\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} \lambda I & \psi(x) \\ \psi(y)^* & \mu I \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where $a = a_{11}^{1/2}$ and $b = b_{11}^{1/2}$. □

Now we will study the extremal property of the canonical embedding $\tau : X \to I(X)$.

**Definition 4.3.1.** *Let $\mathcal{E}$ be a Hilbert $C^*$-bimodule over $C^*$-algebras $A$ and $B$ with left $A$ action and right $B$ action ( $\alpha x \beta \in \mathcal{E}\ \forall\ \alpha \in A, x \in \mathcal{E}, \beta \in B$). Suppose that $X$ is an operator space and $\phi : X \to \mathcal{E}$ is a CC map. A bi-convex combination of $\phi$ is an expression $\phi(\cdot) = \sum_{i=1}^{n} \alpha_i^* \phi_i(\cdot)\beta_i$, where $\phi_i : X \to \mathcal{E}$ are CC maps and $\alpha_i \in A, \beta_i \in B$ such that $\sum_{i=1}^{n} \alpha_i^*\alpha_i = I_A, \sum_{i=1}^{n} \beta_i^*\beta_i = I_B$.*

*Such a bi-convex combination is called proper if $\alpha_i, \beta_i$ are right invertible and it is called trivial if $\alpha_i^*\alpha_i = \lambda_i I, \beta_i^*\beta_i = \lambda_i I, \alpha_i^*\phi_i\beta_i = \lambda_i\phi$ for some $\lambda_i \in [0, 1]$.*

*A CC map $\phi : X \to \mathcal{E}$ is bi-extreme if any proper bi-convex combination $\phi = \sum_{i=1}^{n} \alpha_i^*\phi_i\beta_i$ is trivial.*

**Remark 4.3.1.** *An important example and a particular case of interest to us is the* *instance when $\mathcal{E} = I(X)$, $A = I_{11}(X)$ and $B = I_{22}(X)$.*

Let $R$ be an operator system and $I(R)$ be the injective envelope of $R$. We recall operator convexity for maps from $R$ into $I(R)$.

**Definition 4.3.2.** *Let $R$ be an operator system and $\phi : R \to I(R)$ be a unital CP map. An operator convex combination of $\phi$ is an expression $\phi = \sum_{i=1}^{n} \alpha_i^* \phi_i \alpha_i$, where $\phi_i : R \to I(R)$ are UCP maps and $\alpha_i \in I(R)$ such that $\sum_{i=1}^{n} \alpha_i^* \alpha_i = I$.*

*Such an operator convex combination is proper if $\alpha_i$ are right invertible for $i = 1, 2, \cdots, n$ and trivial if $\alpha_i^* \alpha_i = \lambda_i I$ and $\alpha_i^* \phi_i \alpha_i = \lambda_i \phi$ for some $\lambda_i \in [0, 1]$.*

*We say that $\phi$ is operator extreme if any proper convex combination is trivial.*

**Remark 4.3.2.** *The following result along the lines of [15, Proposition. 2.12]gives us a characterization of bi-extremity of a CC map from an operator space into its injective envelope in terms of purity of the corresponding Paulsen map.*

**Proposition 4.3.1.** *Suppose that $\phi : X \to I(X)$ is a CC map and $S(\phi) : S(X) \to I(S(X))$ is the associated CP map. Then the following are equivalent:*

(1) $S(\phi)$ is a pure CP map.

(2) $S(\phi)$ is operator extreme.

(3) $\phi$ is bi-extreme.

*Proof.* (1) $\implies$ (2): Let $S(\phi) = \sum_{i=1}^{n} \alpha_i^* \psi_i \alpha_i$ where $\psi_i : S(X) \to I(S(X))$ are UCP maps and $\alpha_i \in I(S(X))$; $i = 1, 2, \ldots, n$. Then $\alpha_i^* \psi_i \alpha_i \leq S(\phi), \ \forall i$. By purity of $S(\phi)$, there exist $t_i \in [0,1]$ such that $\alpha_i^*\psi_i\alpha_i = t_iS(\phi)$ and hence $\alpha_i^*\alpha_i = t_iI$. Therefore $S(\phi)$ is operator extreme.

(2) $\implies$ (3): Let $\phi = \sum_{i=1}^{n} \alpha_i^*\psi_i\beta_i$ be a proper bi-convex combination. Take $\gamma_i = \alpha_i \oplus \beta_i$ for $i = 1, 2, \ldots, n$. Then $\gamma_i \in I(S(X))$ and $\gamma_i$ is right invertible. Therefore $S(\phi) = \sum_{i=1}^{n} \gamma_i^* S(\psi_i)\gamma_i$ is a proper operator convex combination. By assumption (2), there exist $t_i \in [0,1]$ such that $\gamma_i^*S(\psi_i)\gamma_i = t_iS(\phi)$. It follows that $\alpha_i^*\psi_i\beta_i = t_i\phi$ and hence $\sum_{i=1}^{n} \alpha_i^*\psi_i\alpha_i$ is trivial.

(3) $\implies$ (1): Suppose that $S(\phi) = \Psi_1 + \Psi_2$ for some CP maps $\Psi_1, \Psi_2 : S(X) \to I(S(X))$. Fix $\epsilon > 0$ and define $E_i = (1-\epsilon)\Psi_i + \frac{\epsilon}{2}(S(\phi))$ for $i = 1, 2$. Then $E_1, E_2 : S(X) \to I(S(X))$ are CP maps such that $E_1 + E_2 = S(\phi)$. Note that

$$E_1(1) = (1-\epsilon)\Psi_1(1) + \frac{\epsilon}{2}I \geq \frac{\epsilon}{2}I \text{ as } (1-\epsilon)\Psi_1 \geq 0.$$

Therefore $E_1(1)$ is invertible. Similarly $E_2(1)$ is also invertible. By the lemma 4.3.1

$$E_i\left(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}\right) = \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix} \begin{bmatrix} \lambda I_K & \psi_i(x) \\ \psi_i(y)^* & \mu I_H \end{bmatrix} \begin{bmatrix} a_i & 0 \\ 0 & b_i \end{bmatrix}$$

where $a_1, a_2, b_1, b_2$ are all invertible. Since $E_1 + E_2 = S(\phi)$, $\phi = a_1\psi_1b_1 + a_2\psi_2b_2$. By assumption, the above bi-convex combination is trivial. That is, there exist $t_i \in [0,1]$, $i = 1, 2$ such that $a_i^2 = t_iI, b_i^2 = t_iI$ and $a_i\psi_ib_i = t_i\phi$. It follows that $E_i = t_iS(\phi)$ for $i = 1, 2$. Since this is true for every $\epsilon > 0$, $\Psi_i$ is also a scalar multiple of $S(\phi)$. Hence

$S(\phi)$ is pure. $\square$

## 4.4 Triple ideals and prime TROs

Recall the definition of two sided triple ideal of a TRO. A *two sided triple ideal* of a TRO $T$ is a linear subspace $I$ of $T$ such that $TT^*I + IT^*T \subset I$. Now if $X$ is an operator space and $A$ is the $C^*$-algebra generated by $S(X)$ and $J$ is a two sided ideal of $A$, then $J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$. It is easy to observe that $J_{11}, J_{22}$ are two sided ideals of the $C^*$-algebras $TT^* + \mathbb{C}I$ and $T^*T + \mathbb{C}I$ respectively and $J_{12}, J_{21}$ are triple ideals of the TROs $T$ and $T^*$ respectively.

In the case of embedding of an operator system in its injective envelope, Farenick and Tessier [11] proved an equivalence between purity of the embedding map and prime nature of the $C^*$-envelope of the operator system. In this section we investigate a similar situation for operator spaces where we associate bi-extremity of the corresponding embedding map with an appropriate analogous property of the triple envelope, in terms of triple ideals.

Considering the Hilbert $C^*$-bimodule property of the injective envelope of an operator space, we introduce it's commutant and explore how it is related to the bi-extremity of the embedding of the operator space in its injective envelope. We prove that if the embedding is bi-extreme, then the commutant is trivial.

**Definition 4.4.1.** *Let $X$ be an operator space, its injective envelope $I(X)$ be the Hilbert*

*$C^*$-bimodule over the $C^*$-algebras $I_{11}$ and $I_{22}$. The commutant of $I(X)$ defined by $I(X)' = \{A_1 \oplus A_2 \in I(S(X)), A_1 \in I_{11}(X), A_2 \in I_{22}(X) : A_1 x = xA_2; A_2 x^* = x^* A_1, \forall x \in I(X)\}$.*

The following observation is crucial in our further discussions.

**Remark 4.4.1.** *Elements of $S(I(X))$ commute with the elements of $I(X)'$. For, if $\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \in S(I(X))$ and $A_1 \oplus A_2 \in I(X)'$, then*

$$\begin{aligned}
\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} &= \begin{bmatrix} A_1\lambda & A_1x \\ A_2y^* & A_2\mu \end{bmatrix} \\
&= \begin{bmatrix} \lambda A_1 & xA_2 \\ y^*A_1 & \mu A_2 \end{bmatrix} \\
&= \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.
\end{aligned}$$

**Proposition 4.4.1.** *If the complete order embedding $\tau : X \to I(X)$ is bi-extreme, then the commutant $I(X)'$ is trivial.*

*Proof.* Let $A_1 \oplus A_2$ be a non-trivial positive element in $I(X)'$ such that norm of $A_1 \oplus A_2$

is one. Consider $S(\tau) : S(X) \to I(S(X))$, the associated CP map. Now,

$$\begin{aligned} S(\tau)\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} &= \begin{bmatrix} \lambda & \tau(x) \\ \tau(y^*) & \mu \end{bmatrix} \\ &= \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^{1/2} \begin{bmatrix} \lambda & \tau(x) \\ \tau(y^*) & \mu \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}^{1/2} \\ &+ \left(I - \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\right)^{1/2} \begin{bmatrix} \lambda & \tau(x) \\ \tau(y^*) & \mu \end{bmatrix} \left(I - \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\right)^{1/2}. \end{aligned}$$

Therefore $S(\tau)$ is not pure and hence $\tau : X \to I(X)$ cannot be bi-extreme by Proposition 4.3.1. □

The purity of embedding of an operator system is shown to be equivalent to a certain nature of the generated $C^*$-algebra, namely the *prime* nature. Let us recall the notion of a prime $C^*$-algebra.

**Definition 4.4.2.** *A $C^*$-algebra $A$ is called prime if for any two sided ideals $J$ and $J'$ such that $JJ' = 0$, then $J = 0$ or $J' = 0$.*

**Theorem 4.4.1.** *[11, Theorem 3.2]The following statement are equivalent for the canonical unital complete order embedding $\tau : R \to I(R)$;*

*(1) $\tau$ is pure in the cone $CP(R, I(R))$;*

*(2) the $C^*$-algebra $C_e^*(R)$ is prime.*

To understand a similar scenario in the setup of operator spaces and generated TRO's, we introduce the notion of prime TROs. We show that our notion captures the related structure by showing that prime TROs in a way give rise to prime $C^*$-algebras, which is made precise in a following Proposition.

**Definition 4.4.3.** *A TRO $T$ is called prime if the following conditions hold;*

*(1) If for any two sided ideal $A$ of $TT^*$ and any two sided triple ideal $B$ of $T$ such that $AB = 0$, then $A = 0$ and $B = 0$.*

*(2) If for any two sided ideal $D$ of $T^*T$ and any two sided triple ideal $C$ of $T$ such that $CD = 0$, then $C = 0$ and $D = 0$.*

**Proposition 4.4.2.** *Let $T$ be a TRO generated by an operator space $X$. If $T$ is a prime TRO, then $A = C^*(S(X))$ is a prime $C^*$-algebra.*

*Proof.* Let

$$J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \text{ and } J' = \begin{bmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{bmatrix}$$

are two sided ideals of $A$ such that $JJ' = 0$. Then

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} J'_{11} & J'_{12} \\ J'_{21} & J'_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

will give $J_{11}J'_{12} = 0, J_{12}J'_{22} = 0$ and $J_{21}J'_{11} = 0, J_{22}J'_{21} = 0$. Since $T$ is a prime TRO, it follows that $J_{11} = 0, J_{12} = 0, J_{12} = 0, J_{22} = 0$ or $J'_{21} = 0, J'_{11} = 0, J'_{22} = 0, J'_{21} =$

0. Therefore $J = 0$ or $J' = 0$ and hence $A$ is prime. $\square$

**Remark 4.4.2.** *In a characterisation of pure embedding of an operator system in its injective envelope Farenick and Tessier [11] proves that corresponding $C^*$-envelope is prime. In this chapter we identify a class of TROs for which the embedding of the generating operator space into its injective envelope is bi-extreme.*

Let us briefly recall the construction of triple envelope of an operator space $X$ [17, Theorem 4.1]. Let $\Phi$ be the minimal $S(X)$-projection on $B(H \oplus H)$. Consider $I_\Phi = \{x \in B(H \oplus H); \Phi(x^*x) = \Phi(xx^*) = 0\}$, and $B(H \oplus H)_\Phi = Im\ \Phi + I_\Phi$, where $Im\ \Phi$ denotes the image of $\Phi$. Then $B(H \oplus H)_\Phi$ is a unital $C^*$-subalgebra of $B(H \oplus H)$ and $I_\Phi$ is a closed two sided ideal of $B(H \oplus H)_\Phi$. There is a canonical map

$$k : S(X) \to Im\ \Phi \to B(H \oplus H)_\Phi / I_\Phi = I(S(X)).$$

Then the $C^*$-subalgebra of $I(S(X))$ generated by $k(S(X))$ is the required $C^*$-envelope $C^*_e(S(X))$ of $S(X)$. Hence $C^*_e(S(X))$ is of the form

$$\begin{bmatrix} TT^* + \mathbb{C}I & T \\ T^* & T^*T + \mathbb{C}I \end{bmatrix},$$

where $T$ is a triple subsystem of $I(X)$ generated by $X$. Hence [18, Theorem 3.2(ii)] implies that $T$ is the triple envelope of $X$. Further, if $X$ is an operator space and $\tau : X \to I(X)$ is the canonical embedding of $X$ into its injective envelope $I(X)$, then $S(\tau) : S(X) \to I(S(X))$ is the canonical embedding of the operator system $S(X)$.

**Remark 4.4.3.** *In the following theorem we prove that if the TRO generated by an operator space is prime, then the canonical embedding of the operator space into its injective envelope is bi-extreme.*

**Theorem 4.4.2.** *Let $X$ be an operator space, $\tau : X \to I(X)$ be the canonical embedding map and $T$ be the triple envelope of $X$. If $T$ is a prime TRO, then $\tau$ is bi-extreme.*

*Proof.* Since $T$ is a prime TRO, by the above Proposition $C^*_e(S(X))$ is a prime $C^*$-algebra and hence the Paulsen map $S(\tau) : S(X) \to I(S(X))$ is pure [11]. Then by Proposition 4.3.1, the map $\tau$ is bi-extreme. □

**Remark 4.4.4.** *Given an operator space $X$ and the canonical embedding map $\tau$ from $X$ to $I(X)$, it is not clear whether the bi-extremity of $\tau$ will imply that the triple envelope of $X$ is a prime TRO, unlike in the analogous case of operator systems and $C^*$-envelopes.*

*The theorem 4.2.1 and the theorem 4.4.2 are part of the published work, Bi-extremity of embeddings of operator spaces into their injective envelopes. Syamkrishnan. M.S., Vijayarajan, A.K., Adv. Oper. Theory 7, 52 (2022).*

Chapter 5

# Conclusion

Boundary representations of an operator system in a $C^*$-algebra and boundary points of the associated matrix convex set are studied and possible relations between the two are investigated in this thesis.

Here we established a connection between boundary representations for a finite dimensional operator system and boundary points of the associated matrix convex set. Using this result we characterized boundary representations of the $C^*$-algebra generated by a finite dimensional operator system in terms of boundary points of the associated matrix convex set.

In the subsequent part of the thesis, we study bi-convex subsets of Hilbert $C^*$-bimodules and explore their extremities. Here we introduced the notion of bi-extreme points of bi-convex subsets of Hilbert $C^*$-bimodules and we have proved that unitaries are bi-extreme points of the bi-convex set of closed unit ball of any trivial bimodule. We defined bi-convex CC maps between operator spaces and Hilbert $C^*$- bimodules and defined bi-extreme CC maps. We further introduced the notion of operator convex combination of the map between an operator system and its injective envelope. Here we established equivalent conditions for a CC map to be bi-extreme in terms of purity and operator extremity of the associated Paulsen map. For an operator space $X$, we define commutant of its injective operator space $I(X)$ and prove that if the complete order embedding $\tau : X \to I(X)$ is bi-extreme, then the commutant of $I(X)$ is trivial.

We define prime TROs and proved that if the TRO generated by $X$ is prime, then the $C^*$-algebra generated by its Paulsen system, $S(X)$ is a prime $C^*$-algebra.

Finaly using the above results, for an operator space $X$, we established a sufficient condition for an embedding map $\tau : X \to I(X)$ to be bi-extreme in terms of the prime nature of the triple envelope $T$ of $X$.

It is well known that every Hilbert $C^*$-module can be isometrically embedded into $B(H_1, H_2)$, where $H_1, H_2$ are Hilbert spaces. There are notions of nondegenerate, irreducible and cyclic representations of Hilbert $C^*$-modules. Extending these concepts into Hilbert $C^*$-bimodules will be interesting.

Matrix convexity in the context of Hilbert $C^*$-bimodules is worth exploring. We defined bi-extreme points for bi-convex sets in Hilbert $C^*$-modules. A study of possible relations between $C^*$-extreme points and bi-extreme points in the context of trivial Hilbert $C^*$-bimodule is interesting to study.

# Publications

1. Syamkrishnan. M.S., Vijayarajan, A.K. Bi-extremity of embeddings of operator spaces into their injective envelopes. Adv. Oper. Theory 7, 52 (2022). https://doi.org/10.1007/s43036-022-00214-0.

# Index

# List of Symbols

| | |
|---|---|
| $\mathbb{C}$ | : Set of complex numbers |
| $M_n(\mathbb{C})$ | : Algebra of $n \times n$ complex matrices over $\mathbb{C}$ |
| $H$ | : Hilbert space |
| $B(H)$ | : Algebra of all bounded linear operators on a Hilbert space $H$ |
| $A, B$ | : $C^* -$ algebras |
| $CP(A, B(H))$ | : The space of all completely positive maps from a unital $C^* -$ algebra $A$ to $B(H)$ |
| $R$ | : Operator system |
| $X$ | : Operator space |
| $\sim_u$ | : Unitary equivalence |
| TRO | : Ternary ring of operators |
| $C_e^*(R)$ | : $C^* -$ envelope |

# Bibliography


[1] W. B. Arveson, *Subalgebras of $C^*$-algebras*, Acta Math. **123**(1969) 141-224.

[2] W. B. Arveson,*Subalgebras of C*-algebras II*, Acta Math. **129**(1972), 271-308.

[3] W. B. Arveson, *The noncommutative Choquet boundary*, Journal of the American Mathematical Society **21** (2008), no. 4, 1065-1084.

[4] D.P. Blecher and C. Le Merdy, *Operator algebras and their modules-an operator space approach*, London Mathematical Society Monographs. New Series, **vol.30**, The Clarendon Press Oxford University Press, Oxford, 2004, Oxford Science Publications.

[5] L. G. Brown, J. A. Mingo and N. T. Shen, *Quasi-multipliers and embeddings of Hilbert $C^*$-bimodules*, Canad. J. Math. **46**(1994), no. 6, 1150–1174.

[6] L. G. Brown, P. Green and M.A. Rieffel, *Stable isomorphisms and strong Morita equivalences of $C^*$-algebras*, Pacific J. Math. **71**(1977), 349-363.

[7] J. Bunce and N. Salinas, *Completely positive maps on $C^*$-algebras and the left matricial spectra of an operator*, Duke Math. J., **43** (1976), 747-177.

[8] K. R. Davidson and M. Kennedy, *Noncommutative choquet theory*, arXiv:1905.08436.

[9] D. R. Farenick, *Extremal Matrix states on operator Systems*, Journal of London Mathematical Society **61** (2000), no. 3, 885-892.

[10] D.R. Farenick, *Pure matrix states on operator systems*, Linear Algebra and its Applications **393**(2004), 149-173.

[11] D. R. Farenick and R. Tessier, *Purity of the Embeddings of Operator Systems into their $C^*$ and Injective Envelopes*, preprint, arXiv:2006.08501, 2020.

[12] D.R. Farenick, *Krein-Milman type problem for Compact Matricially Convex Sets*, Linear Algebra and its Applications **162-164** (1992), 325-334.

[13] D.R. Farenick and P.B. Morenz, *$C^*$-Extreme Points of Some Compact $C^*$-Convex Sets*, Proceedings of the American Mathematical Society **118** (1993), 765-775.

[14] D.R. Farenick and P.B. Morenz, *$C^*$-extreme points in the generalised state space of a $C^*$-algebra*, Transactions of the American Mathematical Society **349** (1997), no.5, 1725-1748.

[15] A. H. Fuller, M. Hartz and M. Lupini, *Boundary representations of operator spaces, and compact rectangular matrix convex sets*, Journal of Operator Theory **79**(2018), No.1, 139-172.

[16] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, Princeton, N. J., 1967.

[17] M. Hamana, *Injective envelopes of operator systems*, Publ. R.I.M.S., Kyoto Univ. **15**(1979), no.3, 773–785.

[18] M. Hamana, *Triple envelopes and Shilov boundaries of operator spaces*, Mathematical Journal of Toyama University **22**(1999), 77-93.

[19] A. Hopenwasser, R. L. Moore and V. I. Paulsen, *$C^*$-extreme points*, Transactions of the American Mathematical Society **266**(1981), no.1, 291-307.

[20] M. Kian and M. Dehghani, *A non commutative convexity in $C^*$-bimodules*, Surveys in mathematics and its applications **12**(2017), 7-21.

[21] C. Kleski, *Boundary representations and pure completely positive maps*, Journal of Operator Theory (2014), pages 45-62.

[22] E.C. Lance, *Hilbert $C^*$ modules- a Toolkit for Operator Algebraists*, London Math. Soc. Lecture Note Ser., Vol. **210**, Cambridge Univ. Press, Cambridge, England, 1995.

[23] R.I. Loebl and V. I. Paulsen, *Some remarks on $C^*$-convexity*, Linear Algebra Appl. **78**, 63-78, 1981.

[24] R. I. Loebl, *A remark on unitary orbit*, Acta Math.Sinica,**7**: 401-407 (1979).

[25] B. Magajna, *On $C^*$-extreme points*, Proceedings of the American Mathematical Society **129** (2001), 771-780.

[26] B. Magajna, *$C^*$-convex sets and completely bounded bimodule homomorphisms*, Proc. Roy. Soc. Edinburgh **130A** (2000) 375–387

[27] P. Morenz, *The structure of $C^*$-convex sets*, Canad. J. Math., **46**, 1007–1026, (1994). MR 95k:46095

[28] V.I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, Vol. **78**, Cambridge University Press, Cambridge, 2002.

[29] C. Webster and S. Winkler, *The Krein-Milman theorem in operator convexity*, Transactions of the American Mathematical Society, Vol.351(1999), no 1, 307-322.

[30] G. Wittstock, *On Matrix Order and Convexity*, in functional Analysis, Survey and Recent Results III, Math Studies 90,175-188, 1984.