C 84016 (Pages : 4) Name..................................

Reg. No..................................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 10—OPERATIONS RESEARCH

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part B

Section A

Answer **all** *the questions.*
Each question carries weightage 1.

1. Describe the general problem of mathematical programming.
2. Define convex and concave functions. Give examples for convex functions and concave functions.
3. Define the dual of a linear programming problem. Prove that dual of the dual is the primal problem.
4. Write the general form of a transportation problem.
5. Define cutting planes in integer programming.
6. Explain sensitivity analysis in linear programming.
7. For any feasible flow $\{x_i\}, i = 1, 2, ..., m,$ in the graph, prove that the flow x_0 in the return arc is not greater than the capacity of any cut in the graph.
8. Describe the various characteristics by which games can be classified.

(8 × 1 = 8 weightage)

Section B

Answer any **two** *questions from each unit.*
Each question carries weightage 2.

Unit I

9. Let $K \subseteq E_n$ be a convex set, $X \in K$, and $f(X)$ a convex function. If $f(X)$ has a relative minimum, then prove that it is also a global minimum.

Turn over

10. Let $X \in E_n$ and let $f(X) = X'AX$ be a quadratic form. If $f(X)$ is positive semidefinite, prove that $f(X)$ is a convex function.

11. Solve graphically the linear programming problem :

Minimize $z = x_1 + 3x_2$

subject to $x_1 + x_2 \geq 3$

$-x_1 + x_2 \leq 2$

$x_1 - 2x_2 \leq 2$

$x_1 \geq 0, x_2 \geq 0.$

UNIT II

12. If the primal problem is feasible, prove that it has an unbounded optimum if and only if the dual has no feasible solution, and vice versa.

13. Solve the following problem using dual simplex method :

Minimize $2x_1 + 3x_2$

subject to $2x_1 + 3x_2 \leq 30$

$x_1 + 2x_2 \geq 10$

$x_1 \geq 0, x_2 \geq 0.$

14. Prove that the transportation problem has a triangular basis.

UNIT III

15. By cutting plane method :

Minimize $4x_1 + 5x_2$

subject to $3x_1 + x_2 \geq 2$

$x_1 + 4x_2 \geq 5$

$3x_1 + 2x_2 \geq 7$

x_1, x_2 being non negative integers.

16. Describe the branch and bound methodin integer programming.

17. Describe the notion of dominance in game theory.

(6 × 2 = 12 weightage)

Section C

Answer any **two** *questions.*
Each question carries weightage 5.

18. (a) Let $f(X)$ be a convex differentible function defined in a convex domain $K \subseteq E_n$. Then prove that $f(X_0)$, $X_0 \in K$, is a global minimum if and only if $(X - X_0)' \nabla f(X_0) \geq 0$ for all $X \in K$.

(b) Use simplex method to :

Maximize : $5x_1 - 3x_2 + 4x_3$

subject to constraints $x_1 - x_2 \leq 1$

$-3x_1 + 2x_2 + 2x_3 \leq 1$

$4x_1 - x_2 = 1$

$x_2, x_3 \geq 0$

and x_1, unrestricted in sign.

19. (a) A caterer needs clean tablecovers every day for six days to meet a contract according to the following schedule.

Days	1	2	3	4	5	6
Number of covers	50	60	80	70	90	100

The cost of a new cover is Rs 20 while washing charges are Re 1 for return on the fourth day or later. Rs. 2 for return on the third day and Rs. 3 for the next day. Find the minimum cost schedule for the purchase and washing of table covers, assuming that after the end of the contract the covers are rejected.

(b) Solve the transportation problem for minimum cost starting with the degenerate solution $x_{12} = 30, x_{21} = 40, x_{32} = 20, x_{43} = 60$.

	D_1	D_2	D_3	
O1	4	5	2	30
O_2	4	1	3	40
O_3	3	6	2	20
O_4	2	3	7	60
	40	50	60	

20. (a) Solve the following integer linear programming problem :

$$\text{Maximize } \phi(X) = 3x_1 + 4x_2$$

$$\text{subject to } 2x_1 + 4x_2 \leq 13$$

$$-2x_1 + x_2 \leq 2$$

$$2x_1 + 2x_2 \geq 1$$

$$6x_1 - 4x_2 \leq 15$$

$$x_1, x_2 \geq 0$$

x_1 and x_2 are integers.

(b) Define spanning tree of a graph. Describe an algorithm for finding the spanning tree of minimum length of a graph.

21. (a) Let f (X, Y) be such that both $max_X \, min_Y f(X, Y)$ and $min_Y \, max_X f\{X, Y)$ exist. Then prove that $max_X \, min_Y f(X, Y) \leq min_Y max_X f(X, Y)$.

(b) Solve the game where the pay-off matrix is :

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(2 × 5 = 10 weightage)

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(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 10—OPERATIONS RESEARCH

(2019 Admissions)

Part A

	DD		MM		YEAR				
Date of Examination :	☐	☐	☐	☐	☐	☐	☐	☐	FN/AN

Time : 15 Minutes **Total No. of Questions : 20**

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MTH 2C 10—OPERATIONS RESEARCH

Part A

Multiple Choice Questions :

1. In a Linear Programming Problem, the constraints are :
   (A) Linear.
   (B) Quadratic.
   (C) Cubic.
   (D) Constants.

2. An $\in$-*nbd* of $x_0 \in R^1$ is :
   (A) $\{x_0\}$.
   (B) $(x_0 - \in, x_0 + \in)$.
   (C) $(x_0 + \in, x_0 - \in)$.
   (D) $(-\in, \in)$.

3. The solution of maximize $z = 2x_1 + 3x_2$

   subject to the constraints :

   $$x_1 + 2x_2 \leq 10$$
   $$x_1 + x_2 \leq 6$$
   $$x_1 \leq 4$$
   $$x_1, x_2 \geq 0 \text{ is}$$

   (A) 2, 4.
   (B) 2, 3.
   (C) 2, 5.
   (D) 2, 9.

4. If A is the constraint co-efficient matrix associated with primal and B is the constraint co-efficient matrix.associated with dual, then :
   (A) $B = A^T$.
   (B) $B = A$.
   (C) $B = A^{-1}$.
   (D) $A = B^{-1}$.

5. The dual of dual problem is :
   (A) The unsymmetric dual problem.
   (B) The unsymmetric primal problem.
   (C) The dual problem.
   (D) The primal problem.

6. A transportation problem is :
   (A) Not an L.P.P.
   (B) An L.P.P.
   (C) A dual problem only
   (D) A primal problem only.

7. A system of n linear equtions $Ax = b$ is called a triangular system if the matrix A is :

   (A) Unit matrix.
   (B) Zero matrix.
   (C) Diagonal matrix.
   (D) Triangular matrix.

8. The column co-efficients of the dual constraints are :

   (A) The co-efficients of the primal objective function.
   (B) The column co-efficients of the primal constraints.
   (C) The row co-efficients of the primal constraints.
   (D) None of the above.

9. An initial feasible solution to a T.P. is obtained by :

   (A) Method of penalties.
   (B) North-west corner rule.
   (C) Two-phase simplex method.
   (D) Big M method.

10. In the iteration of simplex method, if there are more than one negative $Z_j - C_j$, then we may choose.

    (A) The most negative of them.
    (B) The largest of them.
    (C) Any one of them.
    (D) None of the above.

11. If for every pair of vertices, there is a chain connecting the two, then the graph is said to be :

    (A) Tree.
    (B) Arborescence.
    (C) Cycle.
    (D) Connected .

12. If v_a is a vertex of a graph, then the set formed by v_a and all other vertices which are connected to v_a by chains, and the set of arcs connecting them, forms a ———— of the graph.

    (A) Circuit.
    (B) Arborescence.
    (C) Component.
    (D) Centre.

13. subset $S \subset E_n$ in said to be convex if :

    (A) For each pair of points $x,\ y \in S$, the line segment joining x, y belongs to S.
    (B) S is closed.
    (C) S is open.
    (D) S is bounded

14. For any *two* points x and y in E_n, the set $\{\lambda x + (1 - \lambda) y, 0 \le \lambda \le 1\}$ is called :

    (A) The circle through x and y.
    (B) A parabola through x and y.
    (C) A halfspace containing x and y.
    (D) Line segment joining the points x and y.

Turn over

15. In R^3, $\{(x_1, x_2, x_3) \in R^3, \; x_1^2 + x_2^2 + x_3^2 \le 1\}$ is :

(A) Convex.
(B) Concave.
(C) A half space.
(D) Unbounded.

16. The intersection of a finite number of convex sets is :

(A) Concave.
(B) Convex.
(C) Sometimes concave and sometimes convex.
(D) Always unbounded.

17. If $f(y_0, z) \le f(y_0, z_0) \le f(y, z_0)$ for all (y, z) in the neighborhood of (y_0, z_0), then the function $f(y, z)$ is said to have a ———— point at (y_0, z_0).

(A) Extreme.
(B) Accumulation.
(C) Boundary.
(D) Saddle.

18. The function $x^2 + y^2$ is :

(A) Negative definite.
(B) Negative semidefinite.
(C) Always constant.
(D) Positive definite.

19. Spanning tree of a graph is :

(A) Unique.
(B) Infinite.
(C) Not a tree.
(D) Not unique.

20. Cutting plane method is applied in :

(A) Transportation problem.
(B) Game theory.
(C) Integer linear programming.
(D) Flow problems.

C 84015 (Pages : 4) Name.....................................

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M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part B

Section A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Verify that the equation $y'' + y' - xy = 0$ has a three-term recursion formula.

2. Find the general solution of the equation $\left(2x^2 + 2x\right) y'' + \left(1 + 5x\right) y' + y = 0$ near its singular point $x = 0$.

3. If $p_n(x)$ denotes the n^{th} degree Legendre polynomial, then show that

$$p_{2n}(0) = (-1)^n \frac{1 \cdot 3 \dots (2n-1)}{2^n \cdot n!}.$$

4. Define gamma function and show that $\overline{|n+1} = n!$ where n is a non-negative integer.

5. Show that $\frac{d}{dx}\left[x J_1(x)\right] = x\, J_0(x)$.

6. Find the critical points of the system :

$$\frac{dx}{dt} = y^2 - 5x + 6, \frac{dy}{dt} = x - y.$$

7. State Picard's theorem.

Turn over

8. Find the normal form of Bessel's equation :

$$x^2 y'' + xy' + \left(x^2 - p^2\right) y = 0.$$

(8 × 1 = 8 weightage)

Section B

Answer any **two** *questions from each of the following 3 units.*
Each question carries 2 weightage.

Unit I

9. Show that $\tan x = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \dots$ by solving the equation $y^1 = 1 + y^2$; $y(0) = 0$ in two ways.

10. Determine the nature of the point $x = \infty$ for Bessel's equation $x^2 y'' + xy' + \left(x^2 - p^2\right) y = 0.$

11. Show that $\int_{-1}^{1} p_m(x)\, p_n(x)\, dx = 0$ if $m \neq n$, where $p_n(n)$ denotes the n^{th} degree Legendre Polynomial.

Unit II

12. If $f(x) = \begin{cases} 1 & , \quad 0 \le x < \frac{1}{2} \\ \frac{1}{2} & , \quad x = \frac{1}{2} \\ 0 & , \quad \frac{1}{2} < x \le 1, \end{cases}$ show that $f(x) = \sum_{n=1}^{\infty} \frac{J_1\left(\frac{\lambda n}{2}\right)}{\lambda_n J_1(\lambda_n)^2} \cdot J_0(\lambda_n x)$, where the $\lambda_n{}'s$ are the positive zeros of $J_0(x)$.

13. Show that $\left(x = e^{4t}, y = e^{4t}\right)$ and $\left(x = e^{-2t}, y = -e^{-2t}\right)$ are two linearly independent solutions of the system :

$$\frac{dx}{dt} = x + 3y, \quad \frac{dy}{dt} = 3x + y.$$

14. Determine the nature and stability properties of the system :

$$\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y.$$

UNIT III

15. Show that if $q(x) < 0$, and $u(x)$ in a non-trivial solution of $u'' + q(x)u = 0$, then $u(x)$ has at most one zero.

16. Using the method of Lagrange multipliers, find the point on the plane $ax + by + cz = d$ that is nearest the origin.

17. Show that the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$ are precisely the continuous solutions of the integral equation $y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]\,dt$.

(6 × 2 = 12 weightage)

SECTION C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. (a) Show that the equation $x^2 y'' + xy' + (x^2 - 1)y = 0$ has only one Frobenius series solution. Then find it.

(b) Show that the solutions of the Legendre's equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$, where n is a non-negative integer, bounded near $x = 1$ are precisely constant multiples of the polynomial $F\left[-n, n+1, 1, \tfrac{1}{2}(1 - x)\right]$.

19. (a) State and prove Liapunov's theorem.

(b) Verify whether the origin (0, 0) is a stable critical point for the system :

$$\frac{dx}{dt} = 2xy + x^3, \frac{dy}{dt} = -x^2 + y^5.$$

20. (a) Describe Picard's method of successive approximations for solving the initial value problem $y' = f(x, y), y(x_0) = y_0$.

(b) Apply Picard's method to solve the initial value problem :

$$\begin{cases} \frac{dy}{dx} = z, & y(0) = 1. \\ \frac{dz}{dx} = -y, & z(0) = 0. \end{cases}$$

21. (a) Obtain Euler's differential equation for an extremal.

(b) Find the curve joining the points (x_1, y_1) and (x_2, y_2) that yields a surface of revolution of minimum area when revolved about the x-axis.

(2 × 5 = 10 weightage)

C 84015 **(Pages : 4 + 4 = 8)** **Name..**

Reg. No..

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admissions)

Part A

	DD		MM		YEAR				
Date of Examination :	☐	☐	☐	☐	☐	☐	☐	☐	FN/AN

Time : 15 Minutes **Total No. of Questions : 20**

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MTH 2C 09—ODE AND CALCULUS OF VARIATIONS

Part A

Multiple Choice Questions :

1. Which of the following are two independent solutions of the differential equation $y^{11} + y = 0$.

   (A) Cos x, sin x.
   (B) Cos x, e^x.
   (C) Sin x, e^x.
   (D) e^x, e^{-x}.

2. Which of the following forms a basis of solutions of the differential equation $y^{11} + 2y^1 + y = 0$.

   (A) x^2, x^3.
   (B) $x^{-1/2}, x^{-3/2}$.
   (C) $x^{-1/2}, x^{3/2}$.
   (D) e^x, sin x.

3. Which of the following pair of functions are linearly independent ?

   (A) 0, tan x $(|x| < \pi/4)$.
   (B) In x, In x^4.
   (C) Sin$^2 x$, sin x^2.
   (D) Cos x, 4 cos x.

4. The general solution of the differential equation $4y^{11} + 4y^1 - 3y = 0$ is :

   (A) $y = \mathrm{A}e^{x/2} + \mathrm{B}e^{-3x/2}$.
   (B) $y = \mathrm{A}e^{x} + \mathrm{B}e^{-x}$.
   (C) $y = \mathrm{A}e^{x/4} + \mathrm{B}e^{-x}$.
   (D) $y = \mathrm{A} \sin x + \mathrm{B} \cos x$.

5. The radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is :

   (A) 0.
   (B) ∞.
   (C) 1.
   (D) $1/n!$.

6. The radius of convergence of the series $\sum_{n=0}^{\infty} x^n$ is :

   (A) ∞.
   (B) 0.
   (C) ½.
   (D) 1.

7. Solution of the differential equation $y^1 = y$ is :

   (A) $y = \sin x$.
   (B) $y = \cos x$.
   (C) $y = \tan^{-1} x$.
   (D) $y = e^x$.

8. $y = (1+x)^p$ is a solution of the differential equation.

(A) $y^1 + 2xy$.

(B) $y^1 + y = 1$.

(C) $y^1 = \frac{py}{(1+x)}$.

(D) $y^1 + 1 + y^2$.

9. $x\,F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, x^2\right)$ equals :

(A) $(1+x)^p$.

(B) $\sin^{-1} x$.

(C) e^x.

(D) $\log(1+x)$.

10. $\lim_{b \to \infty} F\left(a, b, a, \frac{x}{b}\right)$ equals :

(A) $(1+x)^p$.

(B) $\sin^{-1} x$.

(C) $\log(1+x)$.

(D) e^x.

11. $\lim_{a \to \infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right)$ equals.

(A) $(1+x)^p$.

(B) $\sin x$.

(C) $\cos x$.

(D) $\sin^{-1} x$.

12. The n^{th} Legendre polynomial is obtained from :

(A) $F\left(-n, n, 1, \frac{1-x}{2}\right)$.

(B) $F\left(n, n, 1, \frac{1-x}{2}\right)$.

(C) $F\left(-n, n, 0, \frac{1-x}{2}\right)$.

(D) None of the these.

13. The function $E(x, y) = -18x^8 - 12y^4$ is :

(A) Positive definite.

(B) Negative definite.

(C) Positive semi definite.

(D) None of these.

14. Two solutions of the equation $y^{11} + y = 0$ are :

(A) $\sin x$, $2 \sin x$.

(B) $\sin x$, e^x.

(C) e^x, $\log x$.

(D) e^x, $2e^x$.

15. If $y_1(x)$ and $y_2(x)$ are two independent solutions of $y^{11} + P(x) y^1 + Q(x) y = 0$, then the zeros of these functions are :

(A) Distinct and occur alternately. (B) Not distinct.

(C) Orthogonal (D) None of the above.

16. If $q(x) < 0$, and if $U(x)$ is a nontrivial solution of $u^{11} + q(x) u = 0$, then,

(A) $U(x)$ has exactly one zero. (B) $U(x)$ has atleast one zero.

(C) $U(x)$ has atmost one zero. (D) None of the above.

17. The equation in which an unknown function occurs under the integral sign is called :

(A) Differential equation. (B) Integral equation.

(C) Euler's equation. (D) Reciprocal equation.

18. First approximation to the solution of the initial value problem $y^1 = 2x(1 + y)$ with $y(0) = 0$ is :

(A) $y_1 = 1$. (B) $y_1 = 1 + x$.

(C) $y_1 = x^2$. (D) $y_1 = 1 + x^2$.

19. Necessary conditions required for Picard's theorem are :

(A) $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous. (B) Only $f(x, y)$ need to be continuous.

(C) Only $\frac{\partial f}{\partial y}$ need to be continuous. (D) $f(x, y)$ and $\frac{\partial f}{\partial y}$ are piecewise continuous.

20. If 'y' is missing from the function $f\left(x, y, y^1\right)$, then the Euler's equation reduce to :

(A) $\frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right) = 0$. (B) $\frac{d}{dy}\left(\frac{\partial f}{\partial y^1}\right) = 0$.

(C) $\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right) = 0$. (D) $\frac{d}{dx}\left(\frac{\partial f}{\partial x}\right) = 0$.

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M.Sc. (PREVIOUS) DEGREE [CBCSS] EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics–Second Semester

MTH 2C 08—TOPOLOGY

(2019 Admissions)

Part A

	DD		MM		YEAR				
Date of Examination :									FN/AN

Time : 15 Minutes **Total No. of Questions : 20**

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MTH 2C 08—TOPOLOGY

Part A

Multiple Choice Questions :

1. If τ_1 and τ_2 are two topologies on non-empty set X, then ————————— is topological space.

   (A) $\tau_1 \cap \tau_2$.
   (B) $\tau_1 \cup \tau_2$.
   (C) $\tau_1 \setminus \tau_2$.

2. An indiscrete topology has only ————————— elements.

   (A) 1.
   (B) 2.
   (C) 3.

3. Which of the following is true for discrete topology on X ?

   (A) The topology coincides with the power set P (X).
   (B) Is stronger than usual topology on X.
   (C) Neither of (A) and (B).

4. Suppose X is a finite set, then the coinfinite topology on X coincides with :

   (A) Indiscrete topology.
   (B) Discrete topology.
   (C) Neither of (A) and (B).

5. Let X be a set with cofinite topology and x_n be a sequence in X. Then x_n is convergent in X if and only if :

   (A) There is atmost one term in x_n that repeats infinitely.
   (B) x_n only has finitely many distinct terms.
   (C) x_n is eventually constant.

6. A topological space is said to be second countable if :

   (A) It has a countable base.
   (B) It has countable elements.
   (C) It has a finite base.

7. Let X be a set and S is a family of subset of X and / be a topology on X generated by S. Then :

   (A) S is the base for /.
   (B) S is the sub-base for /.
   (C) Neither (A) nor (B) is true.

8. A space is said to be second countable if and only if :

(A) It has countable elements. (B) It has countable sub-base.

(C) Either (A) or (B).

9. Let X and Y be topological spaces. Under which condition a function $f : X \to Y$ is said to be continuous :

(A) It and only if preimages of open sets are open.

(B) If open sets in X are mapped to open sets in Y.

(C) If closed sets in X are mapped to closed sets in Y.

10. Which of the following about boundary of a set is true ?

(A) Boundary of a set is always closed.

(B) Boundary of the set is same as boundary of its complement.

(C) Both (A) and (B).

11. Let A be a subset of a space X. Then the boundary of A is defined as :

(A) $\overline{A} \cap \overline{X \setminus A}$. (B) $A \cap (X \setminus A)$.

(C) $X \setminus \overline{A}$.

12. Which of the following statements about continuous functions are true ?

(A) Inverse of continuous function is always continuous.

(B) Continuous functions is always one-one.

(C) Composition of continuous functions are continuous.

13. Which of the following statements about continuous functions are false ?

(A) Any function from a discrete space is continuous.

(B) Any function into an discrete space is continuous.

(C) Any function into an indiscrete space is continuous.

14. A subset A of a space X is said to be a Lindeloff subset of X if :

(A) Every cover of A by open subsets of X has a countable subcover.

(B) Every cover of A by open subsets of X has a finite subcover.

(C) There exists cover of A by open subsets of X which has countable subcover.

Turn over

15. If $f : X \to Y$ is one-to-one and continuous and Y is Hausdorff, then X is necessarily Hausdorff ?

(A) True. (B) False.

(C) Need not be.

16. Which the following statements are True ?

(A) In a T_1 space, limits of sequences are unique.

(B) Every T_2 space is metrisable.

(C) The real line with the semi-open interval topology is T_2.

17. A space X is said to be regular if :

(A) Every two mutually disjoint closed subsets can be separated from each other by disjoint open sets.

(B) Every point can be separated from every closed subset not containing it by disjoint open sets.

(C) Every distinct points can be separated from each other by disjoint open sets.

18. Let / be the topology on R whose members are φ, R and ail sets of the form (a, ∞) for $a \in R$. Then $(R,)$ / is :

(A) Regular but not normal. (B) Normal but not regular.

(C) Both regular and normal.

19. Which of the following statement is False ?

(A) Normality imply complete regularity.

(B) T_4 space is Tychonoff.

(C) Every completely regular space is regular.

20. If finitely many points (more than one) are moved from the set $S = \{(x, y) : x^2 + y^2 = 1, \}$ then the resulting set is connected :

(A) True. (B) False.

(C) Cant say.

C 84014 (Pages : 3) Name.......................................

Reg. No..................................

M.Sc. (PREVIOUS) DEGREE [CBCSS] EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics–Second Semester

MTH 2C 08—TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part B

Section A

Answer **all** *the questions.*
Each question has weightage 1.

1. Prove that for the usual metric on the real line, an open r-ball is an open interval of length $2r$.
2. Define base for a topology. Can the same topology have more than one base ? Justify your claim.
3. Define closure of a subset of a topological space. Prove that closure of a set of a topological space is a closed subset of the topological space.
4. When do we say that a topological property is divisible ? Prove that the property of being a descrete space is divisible.
5. Define Lebesgue number. State the result which guarantees the existence of such a number for an open cover for a compact metric space.
6. If the topological space X is connected, then prove that X cannot be written as the disjoint union of two non-empty closed sets.
7. Suppose y is an accumulation point of a subset A of a T_1 space X. Then prove that every neighbourhood of y contains infinitely many points of A.
8. Define pointwise convergence and uniform convergence of sequence of functions on a topological space. Show that uniform convergence implies pointwise convergence.

(8 × 1 = 8 weightage)

Turn over

SECTION B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit I

9. Define semi open interval topology on the set of real numbers. Show that semi open interval topology is stronger than the usual topology on the set of real numbers.

10. Prove that if a space is second countable, then eyery open cover of it has a countable subcover.

11. Define derived set of a subset of a topological space. If for any set $A, \bar{A}$ and A' denote respectively the closure and derived sets of A, then prove that $\bar{A} = A \cup A'$.

Unit II

12. Prove that every open surjective map is a quotient map.

13. Prove that every second countable space is separable.

14. Prove that every quotient space of a locally connected space is locally connected.

Unit III

15. Prove that in a Hausdorff space, limits of sequences are unique.

16. Prove that every compact Hausdorff space is T_4.

17. Prove that all T_4 spaces are Tychonoff.

(6 × 2 = 12 weightage)

SECTION C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Prove that metrisability is a hereditary property.

(b) Let (X, τ) be a topological space and $\mathcal{B} \subset \tau$. Then prove that $\mathcal{B}$ is a base for τ if and only if for any $x \in X$ and any open set G containing x, there exists $B \in \mathcal{B}$ such that $x \in B$ and $B \subset G$.

19. (a) Prove that every continuous real valued function on a compact space is bounded and attains its extrema.

(b) Prove that every closed and bounded interval is compact.

20. (a) Let X be a completely regular space . Suppose F is a compact subset of X, C is a closed subset of X and $F \cap C = \phi$. Then prove that there exists a continuous function from X into the unit interval which takes the value 0 at all points of F and the value 1 at all points of C.

(b) Suppose $\mathcal{D}$ is a decomposition of a space X each of whose members is compact and suppose the projection $p : X \to \mathcal{D}$ is closed. Prove that the quotient space $\mathcal{D}$ is Hausdorff or regular according as X is Hausdorff or regular.

21. A be a closed subset of a normal space X and suppose $f : A \to [-1, 1]$ is a continuous function. Then prove that there exists a continuous function $F : X \to [-1, 1]$ such that $F(x) = f(x)$ for all $x \in A$.

(2 × 5 = 10 weightage)

C 84013 **(Pages : 7 + 3 = 10)** **Name.......................................**

Reg. No....................................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Part A

	DD		MM		YEAR				
Date of Examination :	☐	☐	☐	☐	☐	☐	☐	☐	FN/AN

Time : 15 Minutes **Total No. of Questions : 20**

INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. Immediately after the commencement of the examination, the candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Write the Name, Register Number and the Date of Examination in the space provided.
4. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer book.
5. **Candidate should handover this Question paper to the invigilator after 15 minutes and before receiving the question paper for Part B Examination.**

MTH 2C 07—REAL ANALYSIS—II

Part A

Multiple Choice Questions :

1. If X is a set, which one of the following is the smallest σ-algebra of sub-sets of X ?

   (A) $\{\theta, X\}$.
   (B) $\{\theta\}$.
   (C) $\{X\}$.
   (D) None of the above options.

2. If X is a set and $\mathcal{A}$ is an algebra of subsets of X, then which one of the following is true ?

   (A) $\mathcal{P}(X) \subseteq \mathcal{A} \subseteq \{\theta, X\}$.
   (B) $\mathcal{A} \subset \mathcal{P}(X) \subset \{\theta, X\}$.
   (C) $\mathcal{A} \subset \{\theta, X\} \subset \mathcal{P}(X)$.
   (D) $\{\theta, X\} \subseteq \mathcal{A} \subseteq \mathcal{P}(X)$.

3. Which one of the following statements is true ?

   (A) Outer measure of a singleton set is 1.
   (B) Outer measure of a singleton set is 0.
   (C) Outer measure of a countable set is ∞.
   (D) Outer measure of a finite set is the number of elements in the set.

4. A set E is said to be measurable if ———.

   (A) For each set A, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$.
   (B) For each set A, $m^*(A) > m^*(A \cap E) + m^*(A \cap E^C)$.
   (C) For each set A, $m^*(A) < m^*(A \cap E) + m^*(A \cap E^C)$.
   (D) $m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$ for some A.

5. Which one of the following is true ?

   (A) The empty set is not measurable.
   (B) The set of real numbers is not measurable.
   (C) If a set A is measurable then its complement A^C is not measurable.
   (D) If a set A is measurable then its complement A^C is measurable.

6. Let A be any set, and $\{E_1 ....E_n\}$ be a disjoint collection of measurable sets. Then ————.

(A) $m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) \geq \sum_{k=1}^{n} m^*(A \cap E_k)$.

(B) $m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) > \sum_{k=1}^{n} m^*(A \cap E_k)$.

(C) $m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) < \sum_{k=1}^{n} m^*(A \cap E_k)$.

(D) $m^*\left(A \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$.

7. If $\mathcal{A}$ is a σ–algebra of subsets of $\mathbb{R}$, then which one of the following is not true ?

(A) $\mathbb{R} \in \mathcal{A}$.

(B) $\mathcal{A}$ is closed with respect to the formation of complements.

(C) $\mathcal{A}$ is closed with respect to the formation of countable intersections.

(D) $\mathcal{A}$ is not closed with respect to the formation of countable unions.

8. Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of non-empty closed sets of real numbers for which F_1 is bounded. Then ————.

(A) $\bigcap_{n=1}^{n} F_n = F_1$.

(B) $\bigcap_{n=1}^{\infty} F_n = \{0\}$.

(C) $\bigcap_{n=1}^{\infty} F_n = \theta$.

(D) $\bigcap_{n=1}^{\infty} F_n \neq \theta$.

9. An extended real valued function f defined on a set E is said to be Lebesgue measurable if ————.

(A) Provided its domain E is measurable and if for each extended real number c, the set $\{x \in E : f(x) = c\}$ has measure 0.

(B) Provided its domain E is measurable and if for each extended real number c, the set $\{x \in E : f(x) = c\}$ is measurable.

(C) Provided its domain E is measurable and if for each extended real number c, the set $\{x \in E : f(x) = c\}$ has measure 1.

(D) None of the above options.

10. If lower Riemann integral of f over $[a, b]$ is denoted by (R) $\underline{\int_a^b} f$, and upper Riemann integral of f over $[a, b]$ is denoted by (R) $\overline{\int_a^b} f$, then which one of the following is true ?

(A) $(R) \underline{\int_a^b} f = (R) \overline{\int_a^b} f.$

(B) $(R) \underline{\int_a^b} f \geq (R) \overline{\int_a^b} f.$

(C) $(R) \underline{\int_a^b} f \leq (R) \overline{\int_a^b} f.$

(D) $(R) \underline{\int_a^b} f > (R) \overline{\int_a^b} f.$

11. For the step function ψ defined on $[a. b]$, corresponding to the partition $P = \{x_0, x_1 \dots x_n\}$ of $[a, b]$ and numbers $c_1, \dots c_n$ such that for $1 \leq i \leq n$.

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i,$$

the upper Darboux sum for ψ with respect to the partition P is given by ————.

(A) $U(\psi, P) = \sum_{i=1}^{n} c_i (x_i - x_{i-1}).$

(B) $U(\psi, P) = \sum_{i=1}^{n} c_i x_i.$

12. Let E = (0.1] and for a natural number n define $f_n = n \cdot \chi_{\left(0, \frac{1}{n}\right)}$. Then $\{f_n\}$ converges pointwise on E to :

(A) $f \equiv 1$ on E.

(B) $f \equiv 0.5$ on E.

(C) $f \equiv 0.025$ on E.

(D) None of the above options.

13. Let the non-negative function f be integrable over E. Then ————.

(A) f is finite a.e. on E.

(B) f is 0 a.e. on E.

(C) f is constant a.e. on E.

(D) None of the above options.

14. For an extended real-valued function f on E, positive part f^+ of f is given by ————.

(A) $f^+(x) = \max\{-f(x), 0\}$ for all $x \in$ E.

(B) $f^+(x) = \max\{f(x), 0\}$ for all $x \in$ E.

(C) $f^+(x) = -\max\{f(x), 0\}$ for all $x \in$ E.

(D) $f^+(x) = \min\{f(x), 0\}$ for all $x \in$ E.

15. For an extended real-valued function f on E, negative part f^- of f is given by ————.

(A) $f^-(x) = \min\{f(x), 0\}$ for all $x \in$ E.

(B) $f^-(x) = \max\{f(x), 0\}$ for all $x \in$ E.

(C) $f^-(x) = \min\{-f(x), 0\}$ for all $x \in$ E.

(D) $f^-(x) = \max\{-f(x), 0\}$ for all $x \in$ E.

16. Let f be integrable over E. If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then ————.

(A) $\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n \to \infty} \int_{E_n} f.$

(B) $\int_{\bigcup_{n=1}^{\infty} E_n} f < \lim_{n \to \infty} \int_{E_n} f.$

(C) $\int_{\bigcup_{n=1}^{\infty} E_n} f > \lim_{n \to \infty} \int_{E_n} f.$

(D) None of the above options.

17. The divided difference function $\mathrm{Diff}_h f$ on $[a, b]$ is defined by ————.

(A) $\mathrm{Diff}_h f(x) = \frac{f(x+h) - f(x)}{h}.$

(B) $\mathrm{Diff}_h f(x) = \frac{f(x+h) + f(x)}{h}.$

(C) $\mathrm{Diff}_h f(x) = \frac{f(x-h) - f(x)}{h}.$

(D) $\mathrm{Diff}_h f(x) = \frac{f(x-h) + f(x)}{h}.$

18. For a real-valued function f and an interior point x of its domain, the lower derivative of f at x is defined by ————.

(A) $\underline{D} f(x) = \lim_{h \to 0} \left[\inf_{0 < |t| \le h} \frac{f(x+t) + f(x)}{t} \right].$

(B) $\underline{D} f(x) = \lim_{h \to 0} \left[\inf_{0 < |t| \le h} \frac{f(x+t) + f(x)}{2t} \right].$

(C) $\underline{D} f(x) = \lim_{h \to 0} \left[\inf_{0 < |t| \le h} \frac{f(x+t) - f(x)}{2t} \right].$

(D) $\underline{D} f(x) = \lim_{h \to 0} \left[\inf_{0 < |t| \le h} \frac{f(x+t) - f(x)}{t} \right].$

19. Let f be an increasing function on the closed, bounded interval $[a, b]$. The for each $a > 0$, ———.

(A) $m^*\{x \in (a, b) \mid \bar{D} f(x) \geq \alpha\} > \frac{1}{\alpha} \cdot [f(b) - f(a)]$.

(B) $m^*\{x \in (a, b) \mid \bar{D} f(x) \geq \alpha\} \geq \frac{1}{\alpha} \cdot [f(b) - f(a)]$.

(C) $m^*\{x \in (a, b) \mid \bar{D} f(x) \geq \alpha\} = \frac{1}{\alpha} \cdot [f(b) - f(a)]$.

(D) $m^*\{x \in (a, b) \mid \bar{D} f(x) \geq \alpha\} \leq \frac{1}{\alpha} \cdot [f(b) - f(a)]$.

20. If the function f is monotone on the open interval (a, b), then it is ———.

(A) Differentiable at rational points in (a, b).

(B) Differentiable on (a, b).

(C) Differentiable almost everywhere on (a, b).

(D) Differentiable at irrational points in (a, b).

C 84013 (Pages : 3) Name.......................................

Reg. No....................................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part B

Section A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Show that if $m^*(\mathrm{A}) = 0$, then $m^*\left(\mathrm{A} \cup \mathrm{B}\right) = m^*(\mathrm{B})$ for any set B.

2. If $\{\mathrm{A}_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then prove that :

$$m\left(\bigcup_{k=1}^{\infty} \mathrm{A}_k\right) = \lim_{k \to 0} m\left(\mathrm{A}_k\right).$$

3. Prove that a real-valued function that is continuous on its measurable domain is measurable.

4. Let $\{\mathrm{E}_i\}_{i=1}^{n}$ be a finite collection of measurable subsets of a set of finite measure E. If $\phi = \sum_{i=1}^{n} a_i \cdot \chi_{\mathrm{E}i}$ on E, where a_i is a real number for $1 \le i \le n$, then prove that :

$$\int_{\mathrm{E}} \phi = \sum_{i=1}^{n} a_i \cdot m\left(\mathrm{E}_i\right).$$

5. Let f be a measurable function on E. Prove that f^+ and f^- are integrable over E if and only if $|f|$ is integrable over E.

6. Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converge pointwise a.e. on E to f and f is finite a.e. on E. Prove that $\{f_n\} \to f$ in measure on E.

Turn over

7. Let f and g be real-valued functions on (a, b). Show that, on (a, b),

$$\bar{D}(f+g) \le \bar{D}f + \bar{D}g.$$

8. Show that the sum of two absolutely continuous functions is again an absolutely continuous function.

(8 × 1 = 8 weightage)

Section B

Answer any **six** *questions by choosing* **two** *questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that the union of a countable collection of measurable sets is measurable.

10. Prove that the Lebesgue measure is countably additive.

11. Suppose that f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable and g is continuous. Is the composition $f \circ g$ necessarily measurable ? Justify your answer.

Unit II

12. Let f be a non-negative measurable functions on E. Prove that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

13. Let f and g be integrable over E. Prove that

$$\int_E (f+g) = \int_E f + \int_E g.$$

14. Let f be a bounded function on a set of finite measure E. If f is Lebesgue integrable over E, then prove that f is measurable.

Unit III

15. Let f be a monotone function on the open interval (a, b). Prove that f is continuous except possibly at a countable number of points in (a, b).

16. Prove that a function f on a closed bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

17. State and prove Riesz-Fischer theorem.

(6 × 2 = 12 weightage)

Section C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Prove that the collection of measurable sets is a σ-algebra that contains the σ-algebra of Borel sets.

19. (a) State and prove the Bounded Convergence theorem.

    (b) Let f be a measurable function on E. Suppose there is a non-negative function g that is integrable over E and dominates f in the sense that :

$$|f| \leq g$$

on E. Prove that f is integrable over E and

$$\left|\int_E f\right| \leq \int_E |f|.$$

20. (a) Let f be integrable over E and g be a bounded measurable function on E. Show that $f \cdot g$ is integrable over E.

    (b) State and prove the Vitali Covering Lemma.

21. Let the function f be continuous on the closed bounded interval $[a, b]$. Prove that f is absolutely continuous on $[a, b]$ if and only if the family of divided difference functions $\{\text{D}if\, f_h f\}_{0 < h \leq 1}$ is uniformly integrable over $[a, b]$.

(2 × 5 = 10 weightage)

C 84012 (Pages : 4) Name.....................................

Reg. No..................................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 06—ALGEBRA—II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

Part B

Section A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. Show that $Z \times \{0\}$ is a prime ideal of $Z \times Z$.

2. Let E be an extension field of a finite field F, where F has q elements. Let $\alpha \in E$ be algebraic over F of degree n. Prove that $F(\alpha)$ has q^n elements.

3. Show that a finite extension field E of a field F is an algebraic extension of F.

4. Show that the map $\sigma : Q(\sqrt{2}) \to Q(\sqrt{2})$ defined by $\sigma(a + b\sqrt{2}) = a - b\sqrt{2}$ for $a, b \in Q$ is an automorphism and determine the field fixed by σ.

5. Prove that if E is an algebraic extension of a perfect field F, then E is perfect.

6. Find the degree over Q of the splitting field over Q of the polynomial $x^3 - 1$ in $Q[x]$.

7. Find $\phi_8(x)$ over Z_3.

8. Show that $x^5 - 2$ is solvable by radicals over Q.

(8 × 1 = 8 weightage)

Turn over

Section B

Answer any **two** *questions from each of the following* 3 *units.*
Each question carries 2 *weightage.*

Unit I

9. Show that if F is a field, then every ideal in $F[x]$ is principal.

10. Let E be an algebraic extension of a field F. Show that there exist a finite number of elements $\alpha_1, ... \alpha_n$ in E such that $E = F(\alpha_1, ..., \alpha_n)$ iff E is a finite extension of F.

11. Find the degree and a basis for the field extension $Q(\sqrt{2}, \sqrt{3}, \sqrt{18})$ over Q.

Unit II

12. Show that a finite field of p^n elements exists for every prime power p^n.

13. Let E be a field, and let F be a sub-field of E show that the set G (E/F) of all automorphisms of E leaving F fixed forms a subgroup of all automorphisms of E and that $F \le E_{G(E/F)}$.

14. Show that if E is a finite extension of F, then E is separable over F iff each α in E is separable over F.

Unit III

15. Let $s_1, ..., s_n$ be the elementary symmetric functions in the indeterminates $y_1, y_2, ..., y_n$. Show that every symmetric function of $y_1, ..., y_n$ over F is a rational function of the elementary symmetric functions.

16. Describe the group of the polynomial $(x^4 - 1) \in Q[x]$ over Q.

17. Show that the Galois group of the n^{th} cyclotomic extension of Q has $\phi(n)$ elements.

(6 × 2 = 12 weightage)

Section C

Answer any **two** *questions.*
Each question carries 5 *weightage.*

18. (A) Let R be a commutative ring with unity. Show that an ideal M is a maximal ideal of R iff R/M is a field.

(b) Is $\frac{Q[x]}{\langle x^2 - 6x + 6 \rangle}$ a field ? Justify your answer.

19. (a) Let E be an extension field of F, and let $\alpha \in E$, where α is algebraic over F. Then :

(i) Show that there is an irreducible polynomial $p(x) \in F[x]$ such that $p(\alpha) = 0$.

(ii) Show that this irreducible polynomial $p(x)$ is uniquely determined upto a constant factor in F and is a polynomial of minimal degree ≥ 1 in $F[x]$ having α as a zero.

(iii) Show that if $f(\alpha) = 0$ for $f(x) \in F[x]$, with $f(x) \neq 0$, then $p(x)$ divides $f(x)$.

(b) Show that the field C of complex numbers is algebraically closed.

20. (a) Let F be a field, and let α and β be algebraic over F with degree $(\alpha, F) = n$. Show that the map $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ defined by :

$$\psi_{\alpha,\beta}\left(c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1}\right) = c_0 + c_1\beta + \ldots + c_{n-1}\beta^{n-1}$$

for $c_i \in F$ is an isomorphism of $F(\alpha)$ onto $F(\beta)$ iff α and β are cojugate over F.

(b) Let $f(x) \in R[n]$. Show that if $f(a+bi) = 0$ for $(a+bi) \in C$, where $a, b \in R$, then $f(a-bi) = 0$.

Turn over

21. (a) Let F be a field of characteristic 0, and let $a \in$ F. Show that if K is the splitting field of $x^n - a$ over F, then G (K/F) is a solvable group.

(b) Verify whether the splitting field of $x^{17} - 5$ over Q has a solvable Galois group.

(2 × 5 = 10 weightage)

C 84012 (Pages : 4 + 4 = 8) **Name.......................................**

Reg. No...................................

M.Sc. (PREVIOUS) DEGREE (CBCSS) EXAMINATION, APRIL/MAY 2020

(PVT/SDE)

M.Sc. Mathematics—Second Semester

MTH 2C 06—ALGEBRA—II

(2019 Admissions)

Part A

	DD		MM		YEAR				
Date of Examination :	☐	☐	☐	☐	☐	☐	☐	☐	FN/AN

Time : 15 Minutes **Total No. of Questions : 20**

INSTRUCTIONS TO THE CANDIDATE

1. This Question Paper carries Multiple Choice Questions from 1 to 20.
2. Immediately after the commencement of the examination, the candidate should check that the question paper supplied to him/her contains all the 20 questions in serial order.
3. Write the Name, Register Number and the Date of Examination in the space provided.
4. Each question is provided with choices (A), (B), (C) and (D) having one correct answer. Choose the correct answer and enter it in the main answer book.
5. **Candidate should handover this Question paper to the invigilator after 15 minutes and before receiving the question paper for Part B Examination.**

MTH 2C 06—ALGEBRA—II

Part A

Multiple Choice Questions :

1. Which of the following is a not a group ?

(A) $(R, +)$.

(B) $(Z, -)$.

(C) $(R^*, .)$

(D) $(Q^*, .)$

2. $\{1, i, -i, -1\}$ is ———.

(A) Semigroup.

(B) Subgroup.

(C) Cyclic Group.

(D) Abelian group.

3. Every group of order 4 is :

(A) Cyclic.

(B) Abelian.

(C) Non-abelian.

(D) None of the above.

4. Which of the following is not a Field ?

(A) Z_4.

(B) Z_7.

(C) Z_5.

(D) Z_2.

5. Which of the following ring is of Characteristic zero ?

(A) Z_4.

(B) Z_7.

(C) R.

(D) Z_2.

6. Which of the following is not a Prime Ideal of Z ?

(A) 5Z.

(B) Z_2.

(C) 3Z.

(D) 2Z.

7. Which of the following is a prime field ?

(A) Z.

(B) R.

(C) Q.

(D) C.

8. Which of the following is an example of Transcendental number ?

(A) $\sqrt{2}$.
(B) π.
(C) i.
(D) 2.

9. Number of ideals of a Field is :

(A) 0.
(B) 2.
(C) 1.
(D) 3.

10. Which of these are not constructible numbers ?

(A) $\sqrt{2}$.
(B) π.
(C) $\sqrt{3}$.
(D) 4.

11. Find generator of Z^*_{11}.

(A) 2.
(B) 4.
(C) 5.
(D) 3.

12. Select the number which is not an element of $Q\left(\sqrt{2}\right)$:

(A) 1.
(B) 4.
(C) $\sqrt[3]{2}$.
(D) 2.

13. Let E is $Q\left(\sqrt{3}, \sqrt{7}\right)$ and F is Q. Then index of E over F is :

(A) 2.
(B) 3.
(C) 4.
(D) 1.

14. Find splitting of $x^3 - 1$ over Q :

(A) $Q\left(i\sqrt{3}, \sqrt[3]{2}\right)$.
(B) $Q(\zeta)$.
(C) $Q\left(\sqrt{2}\right)$.
(D) R.

15. Which of the following is a Fermat Prime ?

(A) 2.
(B) 6.
(C) 5.
(D) 8.

16. Which of the following is a not a cyclic group ?

(A) $Z_3 \times Z_3$.

(B) $Z_3 \times Z_2$.

(C) $Z_5 \times Z_4$.

(D) $Z_3 \times Z_5$.

17. Find the number of subgroups of S_3 of order 2 :

(A) 2.

(B) 4.

(C) 3.

(D) 1.

18. Find number of proper normal subgroups of Z_6 :

(A) 2.

(B) 4.

(C) 3.

(D) 1.

19. Number of elements in S_n is :

(A) $n!$.

(B) $\frac{n!}{2}$.

(C) n.

(D) $n+1$.

20. Number of zero divisors of Q is :

(A) 1.

(B) 3.

(C) 2.

(D) 0.

C 83523 (Pages : 4) Name.......................................

Reg. No...................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let A be a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Prove that $\| A \| < \infty$.

2. Let f and g be differentiable real functions in $\mathbb{R}^3$. Prove that

$$\nabla (fg) = f \nabla g + g \nabla f,$$

where $(\nabla (fg))(x)e_j = \sum_{i=1}^{3} (D_i (fg))(x)e_j, \; j = 1, 2, 3....$

3. Show that the curve

$$\gamma (t) = \left(\cos^3 t \cos 3t, \cos^3 t \sin 3t\right), t \in \mathbb{R}$$

is a closed curve which has exactly one self intersection.

4. Is $\gamma (t) = (t, \cosh t)$ regular for all $t \in \mathbb{R}$? Justify your answer.

5. Find the equation of the tangent plane of the surface patch $\sigma (u, v) = \left(u, v, u^2 - v^2\right)$ at $(1, 1, 0)$.

6. If S is a surface and $p \in S$, then prove that the derivative at p of the identity map $S \to S$ is the identity map $T_pS \to T_pS$, where T_pS is the tangent space of S at p.

Turn over

7. Calculate the Gauss map $\mathcal{G}$ of the paraboloid S with equation $z = x^2 + y^2$.

8. Prove that mean curvature of a surface S is a smooth function on S.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*

Each question carries 4 marks.

9. Prove that the set of all invertible linear operators on a vector space $\mathbb{R}^n$ is an open subset of the set of all linear operators on a vector space $\mathbb{R}^n$.

10. If the tangent vector of a parametrized curve is constant, then prove that the image of the curve is a part of the straight line.

11. Verify the Frenet-Serret equations for the curve $r(t) = \left(\frac{4}{5}\cos t, 1-\sin t, -\frac{3}{5}\cos t\right)$.

12. Let S_1 and S_2 be surfaces and let $f: S_1 \to S_2$ be a diffeomorphism. If σ_1 is an allowable surface patch on S_1, then prove that $f \circ \sigma_1$ is an allowable surface patch on S_2.

13. Show that the Weingarten map changes sign when the orientation of the surface changes.

14. Find the principal curvature of the helicoid $\sigma(u, v) = (v\cos u, v\sin u, \lambda u)$, where λ is a constant.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 marks.

Unit I

15. A (a) Let X be vector space and let $\dim X = n$. Prove that a set E of n vectors spans X if and only if E is independent.

(b) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

B (a) Let X be vector space with dimension n. Prove that every basis of X consists of n vectors.

(b) Let A be a linear transformation from $\mathbb{R}^{m+n}$ to $\mathbb{R}^n$ and let $A_x : \mathbb{R}^n \to \mathbb{R}^n$ defined by $A_x(h) = A(h, 0), h \in \mathbb{R}^n, 0 \in \mathbb{R}^m$ be a linear transformation. If A_x is invertible, then prove that $A(h, k) = 0$ can be solved for h if k is given.

Unit II

16. A (a) If $\gamma(t)$ is a regular curve, then prove that its arc-length s, starting at any point of γ, is a smooth function of t.

(b) Show that the total signed curvature of a closed plane curve is an integer multiple of 2π.

B (a) Let $k:(\alpha,\beta)\to\mathbb{R}$ be any smooth function with $\alpha,\beta\in\mathbb{R}$ and $\alpha<\beta$. Prove that there is a unit-speed curve $k:(\alpha,\beta)\to\mathbb{R}^2$ whose signed curvature is k.

(b) Let γ be a unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Prove that γ is a parametrization of a circle.

Unit III

17. A (a) Prove that the unit cylinder

$$S=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1\right\}$$

is a smooth surface.

(b) Prove that unit circle is a smooth surface.

B (a) If S_1, S_2 and S_3 are surfaces and $f_1: S_1\to S_2$ and $f_2: S_2\to S_3$ are smooth maps, then prove that for all $p\in S_1$,

$$D_p(f_2\circ f_1)=D_{f_1(p)}f_2\circ D_p f_1.$$

(b) Prove that Möbius band is not an orientable surface.

Unit IV

18. A (a) Let p be a point of a surface S, let $\sigma(u,v)$ be a surface patch of S with p in its image and let $L\,du^2+2M\,du\,dv+N\,dv^2$ be the second fundamental form of σ. Prove that for any $v,w\in T_pS$,

$$\langle v,w\rangle = L\,du(v)du(w)+M\left(du(v)\,dv(w)+du(w)\,dv(v)\right)+N\,dv(v)\,dv(w).$$

(b) Prove that the Gaussian curvature of a ruled surface is negative or zero.

Turn over

B (a) With usual notations prove that the Gaussian curvature is

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)}.$$

(b) If p is a point of a surface S, then prove that there is an orthonormal basis of the tangent plane T_pS consisting of principal vectors.

(4 × 12 = 48 marks)

C 83522 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 09—TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 marks.

1. Give an example of a Hausdorff topology on the set $X = \{a, b, c\}$.

2. Define d on $\mathbb{R}^2$ by the rule $d\left((x_1, x_2), (y_1, y_2)\right) = \max\left\{|x_1 - y_1|, |x_2 - y_2|\right\}$. Describe the open balls in this space.

3. Prove that in a topological space, the closure of the closure of a set is same as the closure of the set.

4. Discuss the lifting problem in continuous functions.

5. Prove that the property of being a discrete space is divisible.

6. If the topological space X is connected, then prove that X cannot be written as the disjoint union of two non-empty closed sets.

7. Suppose y is an accumulation point of a subset A of a T_1 space X. Then prove that every neighbourhood of y contains infinitely many points of A.

8. Prove that the intersection of any family of boxes is a box.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Prove that in a topological space, the family of all closed sets has the property that it is closed under finite unions.

10. Prove that the usual topology on the Euclidean plane R^2 is strictly weaker than the topology induced by lexicographic ordering.

Turn over

11. Prove that every open surjective map is a quotient map.

12. Prove that every separable space satisfies the countable chain condition.

13. Prove that a topological space X is T_1 space if and only if for any $x \in X$, the singleton set $\{x\}$ is closed.

14. Prove that every T_3 space is a T_1 space.

(4 × 4 = 16 marks)

Part C

Answer **either** (A) **or** (B) *part of the following questions.*
Each question carries 12 *marks.*

15. (A) (a) Prove that in the co-countable topology, the only convergent sequences are those which are eventually constant.

(b) Determine the topology induced by the discrete metric on a set.

(B) (a) If a space is second countable, prove that every open cover of it has a countable subcover.

(b) Let (X, τ) be a topological space and $\mathbb{B} \subset \tau$. Then prove that $\mathbb{B}$ is a base for τ if and only if for any $x \in X$ and any open G containing x, there exists $B \in \mathbb{B}$ such that $x \in B$ and $B \subset G$.

16. (A) (a) Let (X, τ) be a topological space and $\mathcal{C}$ be the family of all closed subsets of X. Prove or disprove that $\mathcal{C}$ is the complement of τ in P (X), the set of all subsets of X.

(b) Define dense subset of a topological space. State and prove a necessary and sufficient condition for a set to be dense.

(B) Let X and Y be two topological spaces and $f : X \to Y$ be a function and $x_0 \in X$. Prove that the following are equivalent :

(i) f is continuous at x_0.

(ii) The inverse image (under f) of every neighbourhood of $f(x_0)$ in Y is a neighbourhood of x_0 in X.

(iii) For every subset $A \subset X$, $x_0 \in \bar{A}$, implies $f(x_0) \in \overline{f(A)}$.

17. (A) (a) Prove that the product topology is the weak topology determined by the projection functions.

(b) Prove that every closed surjective map is a quotient map.

(B) (a) Let $f : X \to Y$ be a continuous surjective map. Then prove that if X is connected, so is Y.

(b) Prove that every closed and bounded interval is compact.

18. (A) (a) Prove that in a Hausdorff space, limits of sequences are unique.

(b) Distinguish between regular topological spaces and completely regular topological spaces. Prove that every completely regular space is regular.

(B) (a) For any sets Y, I and J, with usual notations, prove that $(Y^I)^J = Y^{I \times J}$, upto a bijection.

(b) Prove that if the topological product of an indexed family of topological spaces is non-empty, then each co-ordinate space is embeddable in it.

(4 × 12 = 48 marks)

C 83521 (Pages : 4) Name..

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition on the rectangle $-1 \leq x \leq 2$ and $1 \leq y \leq 2$.

2. Show that $x = 1$ is a regular singular point of the equation $x^3 (x-1) y'' - 2(x-1) y' + 3xy = 0$.

3. Show that $F'(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x)$.

4. Prove that $\int x^p J_{p-1}(x)\, dx = x^p J_p(x) + c$.

5. Describe the phase portrait of the system :

$$\frac{dx}{dt} = 1, \frac{dy}{dt} = 2.$$

6. Show that a function of the form $ax^3 + bx^2 y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Consider the initial value problem $y' = 2x(1+y)$, $y(0) = 0$. Starting with $y_0(x) = 0$, apply Picard's method to calculate $y_1(x)$, $y_2(x)$, $y_3(x)$ and $y_4(x)$.

8. Express $\sin^{-1}(x)$ in the form of a power series $\sum a_n x^n$ by solving $y' = \left(1 - x^2\right)^{-1/2}$, $y(0) = 0$ in two ways.

9. Find the general solution of the equation $\left(x^2 - 1\right)y'' + (5x+4)y' + 4y = 0$ near its singular point $x = -1$.

10. Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ occur alternately, in the sense that between each pair of consecutive positive zeros of either there is exactly one zero of the other.

11. Find the curve joining the points (x_1, y_1) and (x_2, y_2) that yields a surface of revolution of minimum area when revolved about the x-axis.

12. Using the method of Lagrange multipliers, find the point on the plane $ax + by + cz = d$ that is nearest the origin.

13. Consider the non-linear system :

$$\frac{dx}{dt} = y\left(x^2 + 1\right), \frac{dy}{dt} = 2xy^2.$$

Find the critical points and the differential equation of the paths.

14. Determine the nature and stability properties of the critical point $(0, 0)$ for the system :

$$\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y.$$

(5 × 4 = 20 marks)

Part C

Answer **either** *A* **or** *B of each of the following three questions.*
Each question carries 16 *marks.*

15. A Let $f(x, y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is an interior point of R, then show that there exists a number $h > 0$ with the property that the initial value problem $y' = f(x, y), y(x_0) = y_0$ has a unique solution $y = y(x)$ on the interval $|x - x_0| \le h$.

B (a) Solve Chebyshev's equation

$$\left(1 - x^2\right) y'' - xy' + p^2 y = 0,$$ where p is a constant.

(b) Show that the equation $x^2 y'' + xy' + \left(x^2 - 1\right) y = 0$ has only one Fröbenius series solution and find it.

16. A (a) Derive Rodrigne's formula for Legendre polynomials :

$$p_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} \left(x^2 - 1\right)^n.$$

(b) Verify that $p_n(x)$ defined by part (a) satisfies the Legendre's equation $\left(1 - x^2\right) y'' - 2xy' + n(n+1) y = 0,$ where n is a non-negative integer.

B (a) State and prove the orthogonality properties of the Bessel functions.

(b) If $f(x) = x^p$ for the interval $0 \le x < 1$, show that its Bessel series in the functions $J_p(\lambda_n x)$, where the λ_n's are the positive zeros of $J_p(x)$, is

$$x^p = \sum_{n=1}^{\infty} \frac{2}{\lambda_n J_{p+1}(\lambda_n)} J_p(\lambda_n x).$$

Turn over

17. A (a) Find the general solution of the system :

$$\frac{dx}{dt} = 7x + 6y, \quad \frac{dy}{dt} = 2x + 6y.$$

(b) Find the critical points of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - \left(x^3 + x^2 - 2x\right) = 0.$$

B (a) State and prove Liapanov's theorem.

(b) Show that $(0,0)$ is an unstable critical point for the system :

$$\frac{dx}{dt} = 2xy + x^3, \quad \frac{dy}{dt} = -x^2 + y^5.$$

(3 × 16 = 48 marks)

C 83520 (Pages : 4) Name..................................

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. If $m^*A = 0$, then prove that $m^*(A \cup B) = m^*B$.

2. If f is a measurable function, then prove that for each extended real number α, the set $\{x : f(x) = \alpha\}$ is measurable.

3. Let $\phi = \sum_{i=1}^{n} a_i \lambda_{E_i}$ with $E_i \cap E_j = \phi$ for $i \neq j$ and suppose that each set E_i is a measurable set of finite measure. Prove that :

$$\int \phi = \sum_{i=1}^{n} a_i \, mE_i$$

4. If f and g are integrable over E, then show that $f + g$ is integrable over E and

$$\int_E f + g = \int_E f + \int_E g.$$

5. Let f be the function defined by $f(x) = |x|$. Find $D^+ f(0)$, $D_- f(0)$, $D^- f(0)$ and $D_- f(0)$.

6. If f is absolutely continuous, then prove that f has a derivative almost everywhere.

Turn over

7. Let $(X, \mathscr{B}, \mu)$ be a measure space. Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu E_1 = \mu E_2$ provided E_1 and E_2 belongs to $\mathscr{B}$.

8. Let μ be a measure on an algebra Q and let μ^* be the outer measure induced by μ. If $A \in Q$, show that $\mu^* A = \mu A$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let $\langle A_n \rangle$ be a countable collection of sets of real numbers. Prove that :

$$m^*(\cup A_n) \leq \sum m^* A_n.$$

10. Let $\langle f_n \rangle$ be a sequence of measurable functions with the same domain of definition. Show that the functions $\sup\{f_1, f_2, ..., f_n\}$, $\inf\{f_1, f_2, ..., f_n\}$, $\sup_n f_n$, $\overline{\lim} f_n$ are measurable functions.

11. If f is integrable over E, show that $|f|$ is integrable over E and

$$\left|\int_E f\right| \leq \int_E |f|.$$

12. Show that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$.

13. Let f be an integrable function on $[a, b]$ and suppose that :

$$F(x) = F(a) + \int_a^x f(t)\, dt.$$

Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

14. If υ_1 and υ_2 are two finite signed measures, then prove that :

$$|\upsilon_1 + \upsilon_2| \leq |\upsilon_1| + |\upsilon_2|.$$

where $\upsilon \leq \mu$ means $\upsilon E \leq \mu E$ for all measurable sets E.

(4 × 4 = 16 marks)

Part C

Answer **either** (A) **or** (B) *of each question.*
Each question carries 12 *marks.*

15. (A) (a) Prove that the outer measure of an interval is its length.

(b) Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2.$$

(B) (a) Show that the cantor ternary set has measure zero.

(b) Let f be a measurable function defined on an interval $[a, b]$, and assume that f takes the values $\pm\infty$ only on a set of measure zero. Prove that for any $\epsilon > 0$, there is a step function g and a continuous function h such that :

$$|f - g| < \epsilon \text{ and } |f - h| < \epsilon$$

except on a set of measure less than ϵ.

16. (A) (a) Define the Lebesgue integral of a bounded measurable function on a measurable set E.

(b) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then show that f is measurable and

$$R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx.$$

(c) State and prove Bounded convergence theorem.

(B) (a) Let $\langle f_n \rangle$ be a sequence of non-negative measurable functions that converge to f and suppose $f_n \leq f$ for each n. Prove that :

$$\int f = \lim \int f_n.$$

(b) If $\langle f_n \rangle$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then prove that :

$$\int_E f \leq \underline{\lim} \int_E f_n.$$

17. (A) (a) State and prove Vitali's lemma.

(b) If f is integrable on $[a, b]$ and

$$\int_a^x f(t)\, dt = 0$$

for all $x \in [a, b]$, then prove that $f(t) = 0$ *a.e.* in $[a, b]$.

(B) (a) Let f be on increasing real - valued function on the interval $[a, b]$. Show that f is differentiable *a.e.* and

$$\int_a^b f'(x)\, dx \leq f(b) - f(a).$$

(b) Prove that every absolutely continuous function is the indefinite integral of its derivative.

18. (A) (a) Show that, if f is integrable, then the set $\{x : f(x) \neq 0\}$ is of σ-finite measure.

(b) State and prove Radon - Nikodym theorem.

(B) (a) Prove that the class $\mathscr{B}$ of μ^*-measurable sets is a σ-algebra.

(b) Let E be a measurable subset of X × Y such that $\mu \times \upsilon(E)$ is finite. Then for almost all x, prove that the set E_x is a measurable subset of Y. Prove also that the function g defined by :

$$g(x) = \upsilon(E_x)$$

is a measurable function defined for almost all x and $\int g\, d\mu = \mu \times \upsilon(E)$.

(4 × 12 = 48 marks)

C 83519 (Pages : 3) Name..

Reg. No.....................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 06—ALGEBRA–II

(2019 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question is of 2 *marks.*

1. Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_3$ be a homomorphism with $\phi(1) = 2$. Find *ker* ϕ.
2. Consider the normal series $(0) < \langle 5 \rangle < \mathbb{Z}_{15}$. Give another normal series of $\mathbb{Z}_{15}$ which is isomorphic to this series.
3. Verify whether $\sqrt{2}$ and $\sqrt{3}$ are conjugates over the rationals $\mathbb{Q}$.
4. Find the order of the Galois group $G\left(\mathbb{Q}\sqrt{2}, \sqrt{3}/\mathbb{Q}\right)$.
5. Let K be an extension of degree 3 over $\mathbb{Z}_2$. Give a generator of the Galois group $G(K/\mathbb{Z}_2)$.
6. Describe the fourth cyclotomic polynomial over $\mathbb{Q}$.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question is of 4 *marks.*

7. Let $\phi : G \to G'$ be a homomorphism of groups and H be a normal subgroup of G'. Show that $\phi^{-1}(H)$ is a normal subgroup of G.
8. Show that the group $\mathbb{Z}$ of integers has no composition series.
9. Show that every group of order 35 has a normal subgroup of order 5.
10. Let $\overline{F}$ be an algebraic closure of a field F and $\alpha \in F$. Let ψ be an automorphism of $\overline{F}$ leaving elements of F fixed. Show that $\phi(\alpha)$ is a conjugate of α.

Turn over

11. Let E be a splitting field over F. Let $p(x) \in F[x]$ be irreducible with one zero in E. Show that every zero of $p(x)$ is in E.

12. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a separable extension of $\mathbb{Q}$.

13. Describe the splitting field K of $x^4 - 2$ over $\mathbb{Q}$ and find the degree of K over $\mathbb{Q}$.

14. Describe the Galois group of the 8th cyclotomic polynomial over $\mathbb{Q}$.

(5 × 4 = 20 marks)

Part C

Answer Part A **or** *Part B of each question.*
Each question is of 16 *marks.*

15. A Let G be a group and H, N be normal subgroups of G. Show that

(a) $H \vee N = HN$.

(b) HN is a normal subgroup of G.

(c) HN/N is isomorphic to $H/(H \cap N)$.

B Let G be a finite group of order n and p be a prime dividing n. Let e be the identity of the group G and $X = \{(g_1, g_2, \ldots, g_p) : g_i \in G \text{ and } g_1 g_2 \ldots g_p = e\}$. Show that

(a) $|X| = n^{p-1}$.

(b) For a permutation $\sigma = (1\,2 \ldots p), (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(p)}) \in X$ whenever $(g_1, g_2, \ldots, g_p) \in X$.

(c) There exists $a \in G$ such that $a \neq e$ and $a^p = e$.

16. A (a) Let α, β be algebraic over a field F with $irr(\alpha, F) = irr(\beta, F)$. Prove that $F(\alpha)$ and $F(\beta)$ are isomorphic.

(b) Let α be algebraic over a field F and $\psi : F(\alpha) \to \overline{F}$ be an isomorphism leaving elements of F fixed. Show that $\psi(\alpha)$ is a conjugate of α.

B (a) Define index {E : F} of a finite extension E of a field F.

(b) Let F < E < K where K is a finite extension of F. Show that {K : F} = {K : E}{E : F}.

(c) Find the index $\{\mathbb{Q}(\sqrt{2},\sqrt{3}):\mathbb{Q}\}$.

17. A Let K be a finite normal extension of F and F < E < K. Show that

(a) K is a normal extension of E.

(b) G(K/E) is a subgroup of G(K/F).

(c) if $\alpha,\beta \in G(K/F)$ are such that both belong to the same left coset of G(K/E) in G(K/F), then α and β are equal on E.

B (a) Define symmetric function in n variables.

(b) Let E = F $(s_1, s_2, ..., s_n)$ where $s_1, s_2, ..., s_n$ are the elementary symmetric functions in $y_1, y_2, ..., y_n$. Let K be the field of all symmetric functions in $y_1, y_2, ..., y_n$. Show that

i) K = E.

ii) F ($y_1, y_2, ..., y_n$) is finite normal extension of K.

(3 × 16 = 48 marks)

C 83518 (Pages : 4) Name.......................................

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let A be a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$. Prove that A is uniformly continuous.

2. Find the Cartesian equation of the parametrized curve $\gamma(t) = (\cos^2 t, \sin^2 t)$.

3. Find the arc length of the curve $\gamma(t) = (e^t \cos t, e^t \sin t)$ starting at $(1, 0)$.

4. Is $\sigma(u, v) = (u + u^2, v, v^2)$ a regular surface patch ? Justify your answer.

5. Show that the curve

   $$\gamma(t) = (\cos^3 t \cos 3t, \cos^3 t \sin 3t), t \in \mathrm{R}$$

   is a closed curve which has exactly one self intersection.

6. Calculate the first fundamental form of the surface $\sigma(u, v) = (u - v, u + v, u^2 + v^2)$.

7. Prove that the second fundamental form of a plane is zero.

8. Show that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Prove that a linear operator A on a finite dimensional vector space X is one to one if and only if the range of A is all of X.

10. Let f be a differentiable real function in an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. If $f'(x) = 0$ for every $x \in E$, then prove that f is constant on E.

11. Compute the curvature of the curve

$$\gamma(t) = \left(\cos^3 t, \sin^3 t\right).$$

12. Verify the Frenet-Serret equations for the curve

$$\gamma(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{t}{\sqrt{2}}\right).$$

13. Prove that the ellipsoid

$$\frac{x^2}{p^2} + \frac{y^2}{q^2} + \frac{z^2}{r^2} = 1$$

where p, q, r are non-zero constants, is a smooth surface.

14. Compute the second fundamental form of $\sigma(u, v) = \left(u, v, u^2 + v^2\right)$.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 marks.

Unit I

15. A (a) Let $E \subset \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into $\mathbb{R}^m$ and g is differentiable at $f(x_0)$, then prove that

the map $F : E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(b) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

B State and prove implicit function theorem.

UNIT II

16. A (a) Prove that a parametrized curve has a unit-speed reparametrization if and only if it is regular.

(b) Prove that any regular plane curve γ whose curvature is a positive constant is part of a circle.

B (a) Let $\gamma(s)$ be a unit-speed plane curve and let $\varphi(s)$ be a turning angle for γ. Prove that the signed curvature of γ is given by

$$\kappa_s = \frac{d\varphi}{ds}.$$

(b) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere-vanishing curvature. Prove that its torsion is given by

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\| \dot{\gamma} \times \ddot{\gamma} \|^2},$$

where $\times$ denote the vector product and dot denote $\frac{d}{dt}$.

UNIT III

17. A (a) Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$ and let (u, v) be co-ordinates in U. Prove that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

(b) Prove that the quadric

$$x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = 5$$

is a hyperboloid of one sheet.

B (a) Let $f : S_1 \to S_2$ be a diffeomorphism. Prove that the linear map $D_p f : T_p S_1 \to T_{f(p)} S_2$ is invertible for all $p \in S_1$.

(b) Show that every compact surface is orientable.

UNIT IV

18. A (a) Prove that the Weingarten map is self adjoint.

(b) Show that the Gaussian curvature of a surface S is a smooth function on S.

B (a) κ_1 and κ_2 are the principal curvatures of a surface, then prove that the mean and Gaussian curvatures are given by

$$H = \frac{1}{2}(\kappa_1 + \kappa_2) \text{ and } K = \kappa_1 \kappa_2.$$

(b) Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

(4 × 12 = 48 marks)

C 83517 (Pages : 3) Name..

Reg. No.....................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 09—TOPOLOGY

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 *marks.*

1. Prove that for the usual metric on the real line, an open r-ball is an open interval of length $2r$.

2. Define cofinite topology and co-countable topology on a set. Among the two which is stronger topology ? Justify your answer.

3. Define closure of a subset of a topological space. Prove that closure of a set of a topological space is a closed subset of the topological space.

4. Prove that in a topological space, composition of two continuous functions is a continuous function.

5. Define absolute property of a subset of a topological space. Write an example for an absolute property.

6. When do we say that a topological space is disconnected ? Give an example for a disconnected topological space.

7. Prove that in a Hausdorff space, limits of sequences are unique.

8. Justify the term 'box' geometrically for products of copies of the real line.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let $\{x_n\}$ be a sequence in a metric space $(X ; d)$. Then prove that $\{x_n\}$ converges to y in X if and only if for every open set U containing y, there exists a positive integer N such that for every integer $n \geq N, x_n \in U$.

Turn over

10. Find the boundary of the set N of natural numbers in the real line with usual topology.

11. Prove that a function $e : X \to Y$ is an embedding if and only if it is continuous, one-to-one and for every open set V in X, there exist an open set W in Y such that $e(V) = W \cap Y$.

12. Prove that the topological product of any finite number of connected spaces is connected.

13. Prove that regularity is a hereditary property.

14. Prove that projection functions are open.

(4 × 4 = 16 marks)

Part C

Answer either A **or** *B part of the following questions.*
Each question carries 12 *marks.*

15. A (a) Prove that the usual topology on the Euclidean plane R^2 is strictly weaker than the topology induced by lexicographic ordering.

(b) Determine the topology induced by a discrete metric on a set.

B (a) Define second countable space. Prove that if a space is second countable, then every open cover of it has a countable subcover.

(b) Define hereditary property with reference to topological spaces. Prove that second countability is a hereditary property.

16. A (a) Prove that a subset A of a space X is dense in X if and only if for every non-empty open subset B of X, $A \cap B \neq \phi$.

(b) Define interior of a subset of a topological space. Let X be a space and $A \subset X$. Then prove that interior of A is the union of all open sets contained in A. Also prove that it is the largest open subset of X contained in A.

B (a) For any subset A of a space X, with usual notations prove that $\bar{A} = A \cup A'$.

(b) Suppose $f : X \to Y$ is continuous at a point $x_0 \in X$. Prove that whenever a sequence $\{x_n\}$ converges to x_0 in X, the sequence $\{f\{x_n\}\}$ converges to $f\{x_0\}$ in Y.

17. A (a) Prove that every open surjective map is a quotient map.

(b) Prove that the composite of two quotient maps is a quotient map.

B (a) Prove that every second countable space is first countable. Is the converse true ? Justify your claim.

(b) Prove that a subset of the set of real numbers is connected if and only if it is an interval.

18. A (a) Prove that all metric spaces are T_3 spaces.

(b) Define Tychnoff space. Prove that every Tychnoff space is T_3.

B (a) For any sets Y, I and J, with usual notations, prove that $\left(Y^I\right)^J = Y^{I \times J}$.

(b) Let $X = \prod_{i \in I} X_i$, each X_i being a topological space. Suppose $\{x_n\}$ is a sequence in X and that $x \in X$. Then prove that $\{x_n\}$ converges to x in X if and only if for each $i \in I$, the sequence $\{\pi_i(x_n)\}$ converges to $\pi_i(x)$ in X_i.

(4 × 12 = 48 marks)

C 83516 (Pages : 4) Name..

Reg. No.....................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Show that e^x and e^{-x} are linearly independent solutions of the equation $y'' - y = 0$ on any interval.

2. Determine the nature of the point $x = 0$ for the equation $x^3 y'' + (\sin x)\, y = 0$.

3. Show that $F'(a, b, c, x) = \frac{ab}{c} F(a+1, b+1, c+1, x)$.

4. Define Gamma functions and show that $\sqrt{(n+1)} = n!$ for any integer $n \geq 0$.

5. Find the critical points of the system :

   $$\frac{dx}{dt} = e^y, \quad \frac{dy}{dt} = e^y \cos x.$$

6. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive definite or negative definite.

7. Find the external for the integral $I = \int_{x_1}^{x_2} \left[y^2 - (y^1)^2 \right] dx.$

8. Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition on the rectangle $1 \leq x \leq 2$ and $-1 \leq y \leq 0$.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*

Each question carries 4 marks.

9. Find the general solution of the equation $\left(1-x^2\right)y''-2xy'+2y=0$.

10. Show that $\tan = x+\frac{1}{3}x^3+\ldots$ by solving the equation $y'=1+y^2$; $y(0)=0$ in two ways.

11. Determine the nature of the point $x=\infty$ for Legendre's equation $\left(1-x^2\right)y''-2xy'+p(p+1)y=0$.

12. If $f(x)=x^p$ for the interval $0\le x<1$, show that its Bessel series in the functions $J_p(\lambda_n x)$, where the λ_n's are the positive zeros of $J_p(x)$, is

$$x^p=\sum_{n=1}^{\infty}\frac{2}{\lambda_n J_{p+1}(\lambda_n)}J_p(\lambda_n x).$$

13. Describe the relation between the phase portraits of the systems :

$$\begin{cases}\dfrac{dx}{dt}=F(x,y)\\ \dfrac{dy}{dt}=G(x,y)\end{cases}\quad \text{and} \quad \begin{cases}\dfrac{dx}{dt}=-F(x,y)\\ \dfrac{dy}{dt}=-G(x,y)\end{cases}.$$

14. Find the curve of fixed length L that joins the points $(0,0)$ and $(1,0)$, lies above the x-axis, and encloses the maximum area between itself and the x-axis.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each of the following four questions.*
Each question carries 12 marks.

15. A (a) Find a particular solution of the equation $y''+2y'+y=e^{-x}\log x$.

(b) Solve Legendre's equation $\left(1-x^2\right)y''-2xy'+p(p+1)y=0$, where p is a constant.

B Find two independent Fröbenius series solutions of the equation $x^2y'' - x^2y' + \left(x^2 - 2\right)y = 0$.

16. A (a) Determine the general solution of the hypergeometric equation $x(1-x)y'' + \left[c - (a+b+1)x\right]y' - aby = 0$ near the singular point $x = 0$.

(b) Find the general solution of the equation $\left(x^2 - 1\right)y'' + (5x+4)y' + 4y = 0$ near the singular point $x = -1$.

B (a) Show that $\int_{-1}^{1} p_m(x)\, p_n(x)\, dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \dfrac{2}{2n+1} & \text{if} \quad m = n \end{cases}$,

where $p_n(x)$ is the n^{th} degree Legendre polynomial.

(b) Let $f(x)$ be a function defined on the interval $-1 \leq x \leq 1$. Determine the polynomial $p(n)$ of degree $\leq n$ which minimize the value of the integral $I = \int_{-1}^{1} \left[f(x) - p(x)\right]^2 dx$.

17. A (a) Find the general solution of the system :

$$\frac{dx}{dt} = 7x + 6y, \quad \frac{dy}{dt} = 2x + 6y.$$

(b) Determine the nature and stability properties of the critical point $(0, 0)$ for the system :

$$\frac{dx}{dt} = -3x + 4y, \quad \frac{dy}{dt} = -2x + 3y.$$

B (a) Show that $(0, 0)$ is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -2x + xy^3, \quad \frac{dy}{dt} = -x^2y^2 - y^3.$$

(b) Determine the system which is equivalent to the van der Pol equation

$\frac{d^2x}{dt^2} + \mu\left(x^2 - 1\right)\frac{dx}{dt} + x = 0$. Investigate the stability properties of the critical point $(0, 0)$

for the system when $\mu < 0$ and $\mu > 0$.

18. A (a) Explain Picard's method of successive approximations to solve the initial value problem.

(b) Apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$ for the equation

$y' = x + y,\ y(0) = 1$ where $y_0(x) = 1$.

B Solve the initial value problem by Picard's method, and compare the result with the exact solution :

$$\frac{dy}{dx} = z, \qquad y(0) = 1$$
$$\frac{dz}{dx} = -y, \qquad z(0) = 0.$$

(4 × 12 = 48 marks)

C 83515 (Pages : 4) Name.................................

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. (Mathematics)

MAT 2C 07—REAL ANALYSIS—II

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks

1. Let $\mathcal{C}$ be an algebra on a set X. Prove that θ and X are in $\mathcal{C}$.

2. Is the set $\left\{3+\left(\sqrt{2}\right)^n : n = 1, 2, 3, ...\right\}$ measurable ? Justify your answer.

3. Let $f : [1, 5] \to \mathbb{R}$ be defined by $f(x) = 5$. Prove that f is measurable.

4. Let f and g be bounded measurable functions defined on a set E of finite measure. If $f \leq g$ *a.e.*, then prove that :

$$\int_E f \leq \int_E g.$$

5. Let f be an integrable function over a measurable set E and let $c \in \mathbb{R}$. Prove that cf is integrable $\int_E cf = c\int_E f$.

6. Let f and g be bounded functions. Prove that $D^+(f+g) \leq D^+ f + D^+ g$.

7. Prove that every measurable subset of a negative set is negative.

8. Prove that the total variation of a signed measure on a measure space $(X, \mathcal{B})$ is a measure on $(X, \mathcal{B})$.

(8 × 2 = 16 marks)

Turn over

Part B

*Answer any **four** questions.*
Each question carries 4 marks.

9. Let E be a subset of $\mathbb{R}$. Prove that the function $\chi_E : \mathbb{R} \to \mathbb{R}$ defined by :

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is measurable if and only if E is measurable.

10. Let f and g be two extended real valued functions defined on a measurable set E. If f and g are measurable, then prove that their product $f \cdot g$ defined by :

$$(f \cdot g)(x) = f(x)\, g(x)$$

is measurable.

11. Let f and g be integrable over a measurable set E. Then prove that $f + g$ is integrable over E and

$$\int_E f + g = \int_E f + \int_E g.$$

12. If f is of bounded variation on $[a, b]$, then prove that $T_a^b(f) = P_a^b(f) + N_a^b(f)$ where $P_a^b(f), N_a^b(f), T_a^b(f)$ are the positive, negative, total variation of f on $[a, b]$.

13. If f is absolutely continuous, then prove that f has a derivative almost everywhere.

14. Give an example to show that the Hahn decomposition need not be unique.

(4 × 4 = 16 marks)

Part C

Answer (A) or (B) of the following questions.
Each question carries 12 marks.

Unit I

15. (A) (a) Let $\mathcal{C}$ be an algebra of subsets and let $\{A_i\}$ be a sequence of sets in $\mathcal{C}$. Prove that there is a sequence $\{B_i\}$ of sets in $\mathcal{C}$ such that $B_n \cap B_m = \phi$ for $n \neq m$ and

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

(b) If E_1 and E_2 are measurable, then prove that $E_1 \cup E_2$ is measurable.

(B) (a) Prove that every Borel set is measurable.

(b) Let f be an extended real valued function whose domain is measurable. Prove that the following are equivalent :

(i) For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

(ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

(iv) For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable.

UNIT II

16. (A) (a) Let f be a bounded real valued function defined on a measurable set E of finite measure. Prove that f is measurable if and only if :

$$\inf_{f \to \psi} \int_E \psi(x)\, dx = \inf_{f \geq \varphi} \int_E \varphi(x)\, dx$$

for all simple functions φ and ψ.

(b) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable on $[a, b]$, then prove that f is a measurable function.

(B) (a) Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and let $f = \lim f_n$ *a.e.*. Prove that

$$\int f = \lim \int f_n.$$

(b) Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Prove that :

$$\int_E f = \lim \int_E f_n.$$

UNIT III

17. (A) (a) Let f be an increasing real valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere, the derivative f' is measurable and

$$\int_a^b f'(x)\, dx \leq f(b) - f(a).$$

(b) Is $x^2 + 2$ bounded variation on [1, 2] ? Justify your answer.

(B) (a) If f is integrable on $[a, b]$, then prove that the function F defined by :

$$F(x) = \int_a^b f(t)\, dt$$

is of bounded variation on $[a, b]$.

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continuous.

UNIT IV

18. (A) (a) Let υ be a signed measure on a measure space $(X, \mathcal{B})$. If E is a measurable set such that $0 < \upsilon(E) < \infty$, then prove that there is a positive set A contained in E with $\upsilon(A) > 0$.

(b) Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure and ν a σ-finite measure defined on $\mathcal{B}$. Prove that there exists a measure ν_0, singular with respect to μ, and a measure ν_1, absolutely continuous with respect to μ, such that $\nu = \nu_0 + \nu_1$.

(B) (a) Let μ be a σ-finite measure on an algebra $\mathcal{A}$ and let μ^* be the outer measure generated by μ. Prove that a set E is μ^* measurable if and only if E is the proper difference $A \sim B$ of a set A in $\mathcal{A}_{\sigma\delta}$ and a set B with $\mu^*(B) = 0$.

(b) Let F be a monotone increasing function which is continuous on the right. Prove that there is a unique Baire measure μ such that for all a and b we have

$$\mu(a, b] = F(b) - F(a).$$

(4 × 12 = 48 marks)

C 83514 (Pages : 3) Name..................................

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2020

(CCSS)

M.Sc. Mathematics

MAT 2C 06—ALGEBRA–II

(2017 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question is of 2 marks.

1. Let the symmetric group S_3 act on $X = \{1, 2, 3\}$ by $\sigma \cdot x = \sigma(x)$ of all $\sigma \in S_3$ and $x \in X$. Find the orbit of 1 in this action.
2. Verify whether $\mathbb{Z}_6$ and S_3 are isomorphic groups.
3. Let G be a group of order 12 and H be a Sylow 2-subgroup of G. Find the order of H.
4. Find a Sylow 2-subgroup of the cyclic group $\mathbb{Z}_{36}$.
5. Verify whether $\sqrt{2}$ and $\sqrt{3}$ are conjugates over the field $\mathbb{Q}$ of rationals.
6. Verify whether $\mathbb{Q}\left(\sqrt[3]{2}\right)$ is a splitting field over $\mathbb{Q}$.
7. Find the order of the Galois group $G(K/\mathbb{Q})$ where K is the splitting field of $x^3 - 2$ over $\mathbb{Q}$.
8. Verify whether $\mathbb{Z}_2$ is a Galois group.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question is of 4 marks.

9. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{0,1\}$. Describe an action of G on the set {1, 2} such that (1, 1).1 = 2.
10. Verify whether the series $(0) \leq 12\mathbb{Z} \leq 6\mathbb{Z} \leq \mathbb{Z}$ and $(0) \leq 12\mathbb{Z} \leq 3\mathbb{Z} \leq \mathbb{Z}$ are isomorphic.
11. Let G be a group of order 100. Show that G has a normal subgroup of order 25.
12. Verify whether there exists an automorphism ϕ of $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ such that $\phi\left(\sqrt{2}\right) = \sqrt{3}$.

Turn over

13. Describe the automorphism group $G\left(\mathbb{Q}\left(\sqrt{2}+\sqrt{3}\right)/\mathbb{Q}\right)$.

14. Verify whether a regular 7-gon is constructible with straight edge and compass.

(4 × 4 = 16 marks)

Part C

Answer Part A **or** *Part B of each question.*
Each question is of 12 marks.

15. A (a) Let G be a group and X be a G-set. For each $g \in G$, let $\sigma_g : X \to X$ be defined by $\sigma_g(x) = gx$ for all $x \in X$. Show that σ_g is a permutation of X.

(b) Let X be a set and S_x be the group of all permutations on X. For a group G let $\phi : G \to S_x$ be a homomorphism. Show that $gx = \phi(g)(x)$ gives an action of G on X.

B (a) Let N be a normal subgroup of a group G and $\gamma : G \to G/N$ be the canonical homomorphism. Show that

i. If H is a normal subgroup of G containing N then $\gamma(H)$ is a normal subgroup of G/N.

ii. If K is a normal subgroup of G/N then $\gamma^{-1}(K)$ is a normal subgroup of G.

(b) Find all normal subgroups of $\mathbb{Z}_{24}/N$ where N is the subgroup generated by 4.

16. A (a) Let X be a finite G-set and let

$X_G = \{x \in X : gx = x \text{ for all } g \in G\}$.

Prove that $|X| = |G| + \sum_{i=1}^{k} |C_i|$ where C_i are the orbits of X such that $|C_i| \geq 2$.

(b) Show that if G is a group of order p^n where p is a prime, then $|X| = |X_G| (\text{mod } p)$.

B Let G be a group generated by $A = \{a_1, a_2, ..., a_n\}$. Let G' be any group and let $b_1, b_2, ..., b_n$ be elements of G'. Show that

i) If G is a free group on A then there exists a homomorphism $\phi : G \to G'$ such that $\phi(a_i) = b_i$ for all i.

ii) If ϕ_1, ϕ_2 are homomorphisms from G to G' such that $\phi_1(a_i) = \phi_2(a_i)$ for all i then $\phi_1 = \phi_2$.

iii) Give one example each of G, G' A and $b_1, b_2, ..., b_n$ such that there is no homomorphism $\phi : G \to G'$ satisfying $\phi(a_i) = b_i$ for all i.

17. A (a) Let α, β be algebraic over F with $irr(\alpha, F) = irr(\beta, F)$ and let $\deg(\alpha, F) = n$. Show that $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ defined by

$$c_0 + c_1\alpha + \ldots + c_{n-1}\alpha^{n-1} \mapsto c_0 + c_1\beta + \ldots + c_{n-1}\beta^{n-1}$$

is an isomorphism of fields.

(b) Let α be algebraic over F and $\psi : F(\alpha) \to \bar{F}$ be an isomorphism leaving F fixed. Show that $\psi(\alpha)$ is a conjugate of α over F.

B (a) Let E be a finite extension of a field F. Show that the number of isomorphisms of E onto a subfield of $\bar{F}$ leaving F fixed is less than or equal to [E : F}.

(b) Let $F < E < \bar{F}$. Show that if E is a splitting field over F then every automorphism of $\bar{F}$ leaving F fixed maps E onto E.

18. A (a) Define normal extension of a field.

(b) Let E, K be extensions of a field F such that $F \leq E \leq K \leq \bar{F}$. Show that

i) if K is a finite normal extension of F then K is a finite normal extension of E.

ii) G (K/E) is a subgroup of G (K/F).

B (a) Define symmetric rational functions in n indeterminates $y_1, y_2, \ldots, y_n$.

(b) Let $s_1, s_2, \ldots, s_n$ be elementary symmetric functions in $y_1, y_2, \ldots, y_n$ and K be the field of all elementary symmetric functions in $y_1, y_2, \ldots, y_n$. Show that $K = F(s_1, s_2, \ldots, s_n)$.

(4 × 12 = 48 marks)

C 63430 (Pages : 3) Name..................................

Reg. No..................................

SECOND SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper IX—P.D.E. AND INTEGRAL EQUATIONS

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. Eliminate the parameters a and b and find the corresponding partial differential equation from $2z = (ax + y)^2 + b$.
2. Determine the Monge cone with vertex at (0, 0, 0) for the equation $p^2 + q^2 = 1$.
3. Show that the solution to the Dirichlet problem is stable.
4. Show that the kernel $K(x,\xi) = 1 + \xi + 3x\xi$ has a double characteristic number associated with (– 1, 1), with only one characteristic function.

(4 × 4 = 16 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

I. (a) Show that the general solution of the quasi-linear equation $P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$, where P, Q and R are given continuously differentiable functions of x, y and z is $F(u, v) = 0$, where F is an arbitrary function of u and v and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the solutions of the system :

$$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}.$$

(b) Show that the Pfaffian differential equation :

$$yzdx + \left(x^2y - zx\right)dy + \left(x^2z - xy\right)dz = 0$$

is integrable and find the corresponding integral.

II. (a) Determine a necessary and sufficient condition for the compatibility of the two equations $f(x,y,z,p,q) = 0$ and $g(x,y,z,p,q) = 0$.

(b) Find the complete integral of $\left(p^2 + q^2\right)y = qz$.

Turn over

III. (a) Find the general integral of the differential equation $(x-y)y^2p+(y-x)x^2q=\left(x^2+y^2\right)z$ and the particular solution through $xz=a^2, y=0$.

(b) Solve the Cauchy problem for $zz_x+z_y=1$ with the initial conditions $x=s, y=s, z=\frac{1}{2}s, 0\le s\le 1$.

Unit II

IV. (a) Reduce the equation $y^2u_{xx}-2xyu_{xy}+x^2u_{yy}=\frac{y^2}{x}u_x+\frac{x^2}{y}u_y$, to a canonical form and solve it.

(b) Show that $v(x,y;\alpha,\beta)=\frac{(x+y)[2xy+(\alpha-\beta)(x-y)+2\alpha\beta]}{(\alpha+\beta)^3}$ is the Riemann function for the second order partial differential equation $u_{xy}+\frac{2}{(x+y)}\left(u_x+u_y\right)=0$.

V. (a) State and prove the maximum and minimum principles for harmonic functions.

(b) Show that the solution for the Dirichlet problem for a circle of radius a is given by the Poisson integral formula.

VI. (a) Solve the Neumann problem for the upper half plane.

(b) Solve : $u_t=ku_{xx}, 0<x<l, \quad t>0$

$u(o,t)=u(l,t)=0, \quad t>0$

$u(x,0)=f(x), \quad 0\le x\le l.$

Unit III

VII. (a) Transform the problem $\frac{d^2y}{dx^2}+xy=1, y(0)=y(1)=0$ to the integral equation $y(x)=\int_0^1 G(x,\xi)\,\xi y(\xi)d\xi-\frac{1}{2}x(1-x)$, where $G(x,\xi)=\begin{cases}x(1-\xi) & \text{when } x<\xi\\ \xi(1-x) & \text{when } x>\xi.\end{cases}$

(b) Determine $p(x)$ and $q(x)$ in such a way that the equation $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ is equivalent to the equation $\frac{d}{dx}\left(p\frac{dy}{dx}\right)+qy=0$, then showing that the equation can be written in the self-adjoint form $\frac{d}{dx}\left(\frac{1}{x^2}\frac{dy}{dx}\right)+\frac{2}{x^4}y=0$.

VIII. Consider the equation $y(x) = F(x) + \lambda \int_0^{2\pi} \cos(x+\xi) y(\xi) d\xi$

(i) Determine the characteristic values of λ and the corresponding characteristic functions.

(ii) Express the solution in the form

$$y(x) = F(x) + \lambda \int_0^{2\pi} \Gamma(x,\xi;\lambda)\, F(\xi)\, d\xi$$

where λ is not characteristic.

IX. (a) Solve by iterative method :

$$y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)\, y(\xi)\, d\xi.$$

(b) Determine the resolvent kernel associated with $K(x,\xi) = \cos(x+\xi)$ in $(0, 2\pi)$, in the form of a power series in λ.

(4 × 16 = 64 marks)

C 63429 (Pages : 2) Name..................................

Reg. No..................................

SECOND SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper VIII—TOPOLOGY—I

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. (a) Prove that open balls in a metric space are open sets.

(b) Define nearness relation on a set. Write an example of a nearness relation.

(c) Let X be a compact space and $f: X \rightarrow Y$ is continuous and onto. Then prove that Y is compact.

(d) Write an example of a T_2 space that is not T_3.

($4 \times 4 = 16$ marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Define the co-countable topology. Prove that in the co-countable topology, the only convergent sequences are those which are eventually constant.

(b) Prove that if a space is second countable, then every open cover of it has a countable subcover.

III. (a) Prove that metrisability is a hereditary property.

(b) Prove that a discrete space is second countable if and only if the underlying set is countable.

IV. (a) Define continuous function on a topological space. Prove that compositions of continuous functions are continuous.

(b) Define derived set of a set in a topological space. Obtain the derived set of the open interval (0, 1) in the set of real numbers with usual topology.

Unit II

V. (a) Define product topology. Prove that product topology is the weak topology determined by the projection functions.

(b) Define quotient map with respect to a topological space. Prove that every open surjective map is a quotient map.

Turn over

VI. (a) Prove that every continuous real-valued function on a compact space is bounded and attains its extrema.

(b) If X is Lindeloff space and $A \subset X$ is closed in X then prove that A in its relative topology is Lindeloff.

VII. (a) Prove that an interval in R with usual topology is connected.

(b) Define path connected space. Prove that every path-connected space is connected.

Unit III

VIII. (a) Suppose y is an accumulation point of a subset A of a T_1 space. Then prove that every neighbourhood of y contains infinitely many points.

(b) Define regular and completely regular spaces. Prove that every completely regular space is regular.

IX. (a) Prove that a compact subset in a Hausdorff space is closed.

(b) Prove that every compact Hausdorff space is T_4.

X. (a) State and prove Urysohn's lemma.

(b) Prove that all T_4 spaces are completely regular.

($4 \times 16 = 64$ marks)

C 63428 (Pages : 2) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper VII—REAL ANALYSIS—II

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. (a) Define contraction and give an example of a contraction of $\mathbb{R}$ into $\mathbb{R}$.

(b) Prove that the outer measure m^* is translation invariant.

(c) Let A be a subset of $\mathbb{R}$. Prove that the characteristic function $\mathcal{X}_A$ of the set A is measurable if and only if A is measurable.

(d) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure and let $f_n \to f$. Prove that $\{f_n\}$ converges to f in measure.

($4 \times 4 = 16$ marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 marks.

Unit I

II. (a) Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then prove that $\dim X \leq r$.

(b) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

III. (a) Let $E \subset \mathbb{R}^n$ be an open set and let $f : E \to \mathbb{R}^m$ be a mapping differentiable at a point $x \in E$. Prove that the partial derivatives $(D_j f_i)(x)$ exist and $f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i$, where $1 \leq j \leq n$.

(b) If [A] and [B] are n by n matrices, then prove that $\det([B][A]) = \det[B]\det[A]$.

IV. State and prove implicit function theorem.

Turn over

Unit II

V. (a) Prove that outer measure of an interval is its length.

(b) Let $\{E_n\}$ be a sequence of measurable sets such that $E_{n+1} \subset E_n$ for each n. If $m(E_1)$ is finite, then prove that $m\left(\bigcap_{n=1}^{\infty} E_i\right) = \lim_{n\to\infty} m(E_n)$.

VI. (a) Prove that there exists a non-measurable subset of $\mathbb{R}$.

(b) Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Prove that $\sup_n f_n$ is measurable.

VII. (a) State and prove Fatou's lemma.

(b) Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge to f. If $f_n \leq f$ for each n, then prove that $\int f = \lim \int f_n$.

Unit III

VIII. (a) Let f and g be integrable over a measurable set E. Prove that the function $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.

(b) Let g be integrable over a measurable set E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Prove that $\int_E f = \lim \int_E f_n$.

IX. (a) Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real valued functions on $[a, b]$.

(b) Let f be a real valued function defined on the interval $[a, b]$ and let $a \leq c \leq b$. Prove that $T_a^b(f) = T_a^c(f) + T_c^b(f)$, where $T_a^b(f)$ denote the total variation of f on $[a, b]$.

X. (a) If f is absolutely continuous on $[a, b]$ and $f'(x) = 0$ a.e., then prove that f is a constant.

(b) Prove that every absolutely continuous function is the indefinite integral of its derivative.

[4 × 16 = 64 marks]

C 63427 (Pages : 2) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. (SSE) DEGREE EXAMINATION, MARCH 2019

Mathematics

Paper VI—ALGEBRA—II

(2003 Admissions)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. (a) Find all prime ideals of the ring $\mathbb{Z}_6$.

(b) Find a zero of $x^2 - 2$ in the field $\mathbb{Q}[x]/N$ where N is the ideal generated by $x^2 - 2$ in $\mathbb{Q}[x]$.

(c) Show that if α and β are constructible reals with $\alpha\beta$ is constructible.

(d) Find a primitive fifth root of unity in $\mathbb{Z}_{11}$.

($4 \times 4 = 16$ marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 16 *marks.*

Unit I

II. (a) Let R be a ring with unity and N be an ideal of R. Show that if N contains a unit then N = R.

(b) Let R be a commutative ring with unity and M be an ideal of R such that R/M is a field. Show that M is a maximal ideal of R.

III. (a) Let R be a ring with unity 1. Show that $\phi : \mathbb{Z} \to R$ defined by $\phi(n) = n.1$ is a homomorphism of rings.

(b) Show that if R is a ring with unity and if characteristic of R is n then R contains a subring isomorphic to $\mathbb{Z}_n$.

IV. (a) Show that every finite extention E of a field F is an algebraic extension of F.

(b) Let $E = F(\alpha)$ be an algebraic extension of F. Show that E is a finite extension of F.

Unit II

V. (a) Let E be a finite extension of degree n of a finite field F and let $|F| = q$. Show that $|E| = q^n$.

(b) Show that every finite field of characteristic p contains exactly p^n elements for some positive integer n.

Turn over

VI. (a) Let α, β be algebraic over F and let $irr(\alpha, F) = irr(\beta, F)$. Show that there exists an isomorphism $\psi_{\alpha,\beta} : F(\alpha) \to F(\beta)$ such that $\psi(\alpha) = a$ for all $a \in F$.

(b) Let $\overline{F}$ be an algebraic closure of F and $\alpha \in \overline{F}$. Let $\psi : F(\alpha) \to \overline{F}$ be an isomorphism such that $\psi(a) = a$ for all $a \in F$. Show that $\psi(\alpha)$ is a conjugate of α.

VII. (a) Let $\overline{F}$ be an algebraic closure of F and $E \leq \overline{F}$ be a splitting field over F. Show that every automorphism of $\overline{F}$ leaving F fixed maps E onto E.

(b) Let E be a splitting field over F. Show that $\{E : F\} = |G(E/F)|$ where $\{E : F\}$ is the index and G(E/F) is the group of all automorphisms of E leaving F fixed.

Unit III

VIII. (a) Define normal extension and verify whether $\mathbb{Q}(\alpha)$ is a normal extension of $\mathbb{Q}$ where α is the real cube root of 2.

(b) Let K be a finite normal extension of a field F and let $F \leq E \leq K$. Show that

(i) K is a finite normal extension of E.

(ii) G(K/E) is a subgroup of G(K/F).

IX. (a) Define elementary symmetric functions.

(b) Prove that for any field F, $F(y_1, y_2, \ldots y_n)$ is a finite normal extension of $F(s_1, s_2, \ldots s_n)$, where $s_1, s_2, \ldots, s_n$ are elementary symmetric functions in $y_1, y_2, \ldots, y_n$.

X. (a) Define extension by radicals.

(b) Let K be a finite normal extension of a field F and also an extension by radicals over F. Show that G(K/F) is a solvable group.

(c) Show that $2x^5 - 5x^4 + 5$ is not solvable by radicals over $\mathbb{Q}$.

(4 × 16 = 64 marks)

C 4752 **(Pages : 3)** Name.......................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 10—OPERATIONS RESEARCH

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *the questions.*
Each question has weightage 1.

1. Verify whether $f(x) = x^2$ is a convex function or not.
2. Write the general form of a linear programming problem in two variables. Describe the method of solving such a problem using graphical method.
3. Prove that a vertex of the set of all feasible solutions S_F of a linear programming problem is a basic feasible solution.
4. Describe the concept of loop in a transportation array.
5. Define chain and path in graphs. Prove that path is a chain, but every chain is not a path.
6. Define cutting planes in integer programming.
7. State the minimax theorem in game theory.
8. What do we do in sensitivity analysis in linear programming problems ?

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit I

9. Define the dual of a linear programming problem. Prove that dual of the dual is the primal problem.

10. Define multiplier vector and simplex multipliers. Explain their relevance in simplex method of solving linear programming problems.

11. Solve graphically the linear programming problem :

    Maximize $4x_1 + 2x_2$ subject to $x_1 + x_2 \le 8, x_1 = 4, \; x_1 \ge 0, \; x_2 \ge 0$. Does the optimal solution change if the constraint $x_1 = 4$ is changed to $x_1 \ge 4$?

Unit II

12. If the primal problem is feasible, prove that it has an unbounded optimum if and only if the dual has no feasible solution, and vice versa.

13. Using dual simplex method

    maximise $2x_1 + 3x_2$ subject to $2x_1 + 3x_2 \le 30, x_1 + 2x_2 \ge 10, x_1 \ge 0, x_2 \ge 0$.

14. Prove that the transportation problem has a triangular basis.

Unit II

15. Describe the terms: chain, path, cycle, circuit, component and strongly connected with reference to graphs.

16. Describe the branch and bound method in integer programming.

17. Examine the payoff matrix $\begin{pmatrix} 1 & 3 \\ -2 & 10 \end{pmatrix}$ for saddle point.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Let $f(X)$ be a convex differentible function defined in a convex domain $K \subseteq E_n$. Then prove that $f(X_0), X_0 \in K$, is a global minimum if and only if $(X - X_0)' \nabla f(X_0) \geq 0$ for all $X \in K$.

(b) Use simplex method to verify that the problem : Maximize $f(X) = 2x_1 + x_2$ subject to the constraints $x_1 - x_2 - x_3 \leq 1$, $x_1 - 2x_2 + x_3 \leq 2$, $x_1 \geq 0, x_2 \geq 0$, $x_3 \geq 0$, has no finite optimal solution.

19. (a) Discuss the Caterer problem in operations research.

(b) Solve the transportation problem for minimum cost with the cost co-efficients, demands and supplies as given in the following table. Obtain three optimal solutions :

	D_1	D_2	D_3	D_4	
O_1	1	2	– 2	3	70
O_2	2	4	0	1	38
O_3	1	2	– 2	5	32
	40	28	30	42	

20. (a) Solve the following integer linear programming problem :

Maximize $\phi(X) = 3x_1 + 4x_2$; subject to $2x_1 + 4x_2 \leq 13$, $-2x_1 + x_2 \leq 2$, $2x_1 + 2x_2 \geq 1$,

$6x_1 - 4x_2 \leq 15$, $x_1, x_2 \geq 0$, x_1 and x_2 are integers.

(b) By cutting plane method : Minimize $4x_1 + 5x_2$ subject to $3x_1 + x_2 \geq 2$, $x_1 + 4x_2 \geq 5$, $3x_1 + 2x_2 \geq 7$; x_1, x_2 being non negative integers.

21. For an $m \times n$ matrix game, prove that both $\max_X \min_Y E(X, Y)$ and $\min_Y \max_X E(X, Y)$ exist and are equal.

(2 × 5 = 10 weightage)

C 4751 (Pages : 3) Name..

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 09—ODE AND CALCULUS OF VARIATIONS

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that $x = 1$ is a regular singular point of the equation $x^2\left(x^2-1\right)^2 y'' - x(1-x)y' + 2y = 0$.

2. Show that $\cos x = \lim_{a \to \infty} F\left(a, a, \frac{1}{2}, \frac{-x^2}{4a^2}\right)$.

3. Use Rodrigue's formula to obtain $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$.

4. Show that $J_{-m}(x) = (-1)^m J_m(n)$ for any non-negative integer m.

5. Describe the phase portrait of the system :

$$\frac{dx}{dt} = x, \frac{dy}{dt} = 0.$$

6. Show that (0, 0) is an asymptotically stable critical point for the system :

$$\frac{dx}{dt} = -3x^3 - y, \frac{dy}{dt} = x^5 - 2y^3.$$

Turn over

7. Verify whether $f(x,y) = y^{1/2}$ satisfies a Lipschitz condition on the rectangle $|x| \le 1$ and $0 \le y \le 1$.

8. Find the external for the integral $I = \int_{x_1}^{x_2} \left[y^2 - (y')^2 \right] dx$.

(8 × 1 = 8 weightage)

Part B

Answer any **two** *questions from each of the following* 3 *units.*
Each question carries 2 *weightage.*

Unit I

9. Express $\sin^{-1}(x)$ in the form of a power series by solving the equation $y' = (1 - x^2)^{-1/2}$; $y(0) = 0$ in two ways.

10. Determine the nature of the point $x = \infty$ for Legendre's equation $(1 - x^2)y'' - 2xy' + (p+1)y = 0$, where p is a constant.

11. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$. Determine the polynomial p (x) of degree x that minimizes the integral $I = \int_{-1}^{1} [f(x) - p(x)]^2 dx$.

Unit II

12. Show that between any two positive zeros $J_0(x)$ there is a zero of $J_1(x)$ and that between any two positive zeros of $J_1(x)$ there is a zero of $J_0(x)$.

13. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y.$$

14. Show that (0, 0) is a simple critical point for the system $\frac{dx}{dt} = x + y - 2xy, \frac{dy}{dt} = -2x + y + 3y^2$ and determine its nature and stability properties.

Unit III

15. Consider the initial value problem $y' = 2x(1+y), y(0) = 0$, starting with $y_0(x) = 0$, apply Picard's method to calculate $y_1(x), y_2(x), y_3(x)$.

16. Let $u(x)$ be any non-trivial solution of $u'' + q(x)u = 0$, where $q(x) > 0$ for all $x > 0$. Show that if $\int_1^\infty q(x)\,dx = \infty$, then $u(x)$ has infinitely many zeros on the positive x-axis.

17. A curve in the first quadrant joins (0, 0) and (1, 0) and has a given area beneath it. Show that the shortest such curve is an arc of a circle.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. (a) Solve Legendre's equation $\left(1-x^2\right)y'' - 2xy' + p(p+1)y = 0$, where p is a constant.

(b) Show that equation $4x^2y'' - 8x^2y' + \left(4x^2+1\right)y = 0$ has only one Frobenius series solution and find it.

19. (a) Derive Rodrigue's formula for Legendre polynomials ; $P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$.

(b) Obtain $J_p(x)$, the Bessel function of first kind of order p.

20. (a) Find the general solution of the system :

$$\frac{dx}{dt} = 4x - 2y, \frac{dy}{dt} = 5x + 2y.$$

(b) Find the critical points and the differential equation of the paths of the system :

$$\frac{dx}{dt} = y\left(x^2+1\right); \frac{dy}{dt} = 2xy^2.$$

21. (a) Let $f(x,y)$ and $\frac{\partial f}{\partial y}$ be continuous functions of x and y on a closed rectangle R with sides parallel to the axes. If (x_0, y_0) is any interior point of R, then show that there exists a number $h > 0$ with the property that the initial value problem $y' = f(x,y), y(x_0) = y_0$ has one solution $y = y(x)$ on the interval $|x - x_0| \leq h$.

(b) Show that if $y(x)$ is a non-trivial solution of $y'' + q(x)y = 0$, then $y(x)$ has an infinite number of positive zeros if $q(x) > \frac{k}{x^2}$ for some $k > 1/4$.

(2 × 5 = 10 weightage)

C 4750 **(Pages : 3)** Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 08—TOPOLOGY

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *the questions.*
Each question has weightage 1.

1. Define co-finite topology and co-countable topology on a set. Among the two which is stronger topology ? Justify your answer.
2. Define base for a topology. Give an example for a base for the usual topology on the set of real numbers.
3. Prove that in a topological space, the closure of the closure of a set is same as the closure of the set.
4. When do we say that a topological property is divisible ? Prove that the property of being a finite space is divisible.
5. Prove that every separable space satisfies the countable chain condition.
6. Prove that the topological product of any finite number of connected spaces is connected.
7. Write an example of a Hausdorff topology on the set $X = \{1, 2, 3\}$.
8. State Urysohn's lemma.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each unit.*
Each question has weightage 2.

Unit I

9. Prove that intersection of two open sets in a metric space is open.
10. Determine the topology induced by a discrete metric on a set.
11. If a space is second countable, then prove that every open cover of it has a countable subcover.

Unit II

12. Prove that every closed surjective map is a quotient map.
13. Let X be a compact space and suppose $f : X \to Y$ is continuous and onto. Then prove that Y is compact.
14. Prove that every second countable space is first countable.

Unit III

15. Suppose y is an accumulation point of a subset A of a T_1 space X. Then prove that every neighbourhood of y contains infinitely many points of A.
16. Prove that a compact subset of a Hausdorff space is closed.
17. Let A, B be subsets of a space X and suppose there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Then prove that there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. (a) Prove that the usual topology on the Euclidean plane R^2 is strictly weaker than the topology induced by lexicographic ordering.

    (b) Determine the topology induced by a discrete metric on a set.

19. (a) Prove that if the space (X, T) has a base $\mathbb{B}$ of cardinality α, then the cardinality of T cannot exceed 2^{α}.

(b) Let (X, T) be a topological space and $\mathbb{B} \subset T$. Then prove that $\mathbb{B}$ is a base for T if and only if for any $x \in X$ and any open set G containing x, there exists $B \in \mathbb{B}$ such that $x \in B$ and $B \subset G$.

20. (a) Let (X, d) be a compact metric space and U be an open cover of X. Then prove that there exists a positive real number r such that for any $x \in X$, there exits $V \in U$ such that $B(x, r) \subset V$.

(b) Let $f : X \to Y$ be a continuous surjection. Then if X is connected, prove that Y is also connected.

21. A be a closed subset of a normal space X and suppose $f : A \to [-1, 1]$ is a continuous function. Then prove that there exists a continuous function $F : X \to [-1, 1]$ such that $F(x) = f(x)$ for all $x \in A$.

(2 × 5 = 10 weightage)

C 4749 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 07—REAL ANALYSIS—II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Let A be the set of irrational numbers in the interval [0, 1], Prove that $m^*(A) = 1$.
2. Prove that a monotone function that is defined on an interval is a measurable function.
3. If $\{f_k\}_{k=1}^{n}$ is a finite family of measurable functions with common domain E, then prove that the functions max $\{f_1 ..., f_n\}$ and min $\{f_1, ..., f_n\}$ are also measurable.
4. Let f and g be bounded measurable functions on a set of finite measure E. If $f \le g$ on E, then prove that $\int_E f \le \int_E g$.
5. Let E be a set of finite measure and let $\delta > 0$ be given. Prove that E is the disjoint union of a finite collection of sets, each of which has measure less than δ.
6. If $\{f_n\} \to f$ in measure on E, then prove that there is a subsequence $\{f_{n_k}\}$ that converges pointwise a.e. on E to f.

Turn over

7. Find the upper and lower derivatives of f at $x = 0$ for the function $f(x) = |x|$, for all real numbers x.

8. Give an example of a Cauchy sequence of real numbers that is not rapidly Cauchy.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions by choosing* **two** *questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that every interval is a Borel set.

10. Show that the Cantor set is an uncountable set of measure zero.

11. Let $\{f_n\}$ be a sequence of measurable functions on E that converges point wise a.e. on E to the function f. Prove that f is measurable.

Unit II

12. Show that the function f defined on [0, 1] by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational is not Riemann integrable over [0, 1], but it is Lebesgue integrable over [0, 1].

13. State and prove the Monotone Convergence theorem.

14. Let E have finite measure, $\{f_n\} \to f$ in measure on E and g is a measurable function on E that is finite a.e. on E. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure.

Unit III

15. Let f and g be real-valued functions on (a, b). Show that, on (a, b),

$$\underline{D}f + \underline{D}g \leq \underline{D}(f+g) \leq \overline{D}(f+g) \leq \overline{D}f + \overline{D}g.$$

16. State and prove Jensen's Inequality.

17. Let E be a measurable set and $1 \leq p \leq \infty$. If the functions f and g belong to $L^P(E)$, then prove that their sum $f + g$ also belong to $L^P(E)$. Also prove that

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. Define the outer measure $m^*(A)$ of a set $A \subset \mathbb{R}$ and give an example. Prove that the outer measure of an interval is its length.

19. (a) State and prove Egoroff's theorem.

    (b) Let f be a bounded function on a set of finite measure E. Prove that f is Lebesgue integrable over E if and only if it is measurable.

20. (a) Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

    (b) Let E be of finite measure. Suppose the sequence of functions $\{f_n\}$ is uniformly integrable over E. If $\{f_n\} \to f$ pointwise a.e. on E, then prove that f is integrable over E and $\lim_{n \to \infty} \int_E f_n = \int_E f$.

21. (a) If the function f is monotone on the open interval (a, b), then prove that it is differentiable almost everywhere on (a, b).

    (b) Prove that a function f defined on a closed, bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

(2 × 5 = 10 weightage)

C 4748 (Pages : 3) Name..................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2021

(CBCSS)

Mathematics

MT 2C 06—ALGEBRA—II

(2019 Admissions)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
3. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries 1 weightage.

1. Show that a commutative ring with unity is a field iff it has no proper non-trivial ideals.
2. Show that $\sqrt{1+\sqrt{3}}$ is algebraic over Q.
3. Show that doubling the cube is impossible.
4. What is the order of $G\left(Q\left(\sqrt[3]{2}\right)/Q\right)$?
5. Prove that if E is an algebraic extension of a perfect field F, then E is perfect.
6. Show that the Galois group of the p^{th} cyclotomic extension of Q for a prime p is cyclic of order $p-1$.
7. Show that the regular 18-gon is not constructible.
8. Show that the polynomial x^5-1 is solvable by radicals over Q.

($8 \times 1 = 8$ weightage)

Turn over

Part B

Answer any **two** *questions from each of the following* 3 *units.*
Each question carries 2 *weightage.*

Unit I

9. Let E be a simple extension $F(\alpha)$ of a field F, and let α be algebraic over F. Let the degree of irr (α, F) be $n \geq 1$. Show that every element β of $E = F(\alpha)$ can be uniquely expressed in the form $\beta = b_0 + b_1 \alpha + \dots + b_{n-1} \alpha^{n-1}$, where the b_i are in F.

10. Show that $Q\left(2^{1/2}, 2^{1/3}\right) = Q\left(2^{1/6}\right)$.

11. Show that a field F is algebraically closed iff every non-constant polynomial in $F[x]$ factors in $F[x]$ into linear factors.

Unit II

12. Find all the primitive 18^{th} roots of unity in $GF(19)$.

13. Let F be a finite field of characteristic p. Show that the map $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism.

14. Show that if K is a finite extension of E and E is a finite extension of F, then K is separable over F iff K is separable over E and E is separable over F.

Unit III

15. State the Main Theorem of Galois Theory.

16. Find $\phi_{12}(x)$ *in* $Q[x]$.

17. Let F be a field of characteristic zero and F contains all the n^{th} roots of unity. Show that if K is the splitting field of $x^n - a$ over F for some $a \in F$, then $G(K|F)$ is a soluble group.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. (a) Let F be a field. Show that an ideal $< p(x) > \neq \{0\}$ of $F[x]$ is maximal iff $p(x)$ is irreducible over F.

    (b) Show that $\frac{z_5[x]}{< x^3 + 3x + 2 >}$ is a field.

19. (a) Show that if E is finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and $[K:F] = [K:E][E:F]$.

    (b) Show that if E is a finite extension of F, then $\{E:F\}$ divides $[E:F]$.

20. State and prove the theorem of the conjugation isomorphisms.

21. Let K be the splitting field of $x^4 + 1$ over Q :

    (i) Describe the group $G(K|Q)$; and

    (ii) Give the group and field diagrams for K over Q.

(2 × 5 = 10 weightage)

C 4577 (Pages : 4) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE (SUPPLEMENTARY) EXAMINATION APRIL 2021

(CUCSS)

Mathematics

MT 2C 10—OPERATIONS RESEARCH

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Each question carries a weightage of 1.

1. Show that every affine function $f(x) = ax + b,\ x \in \mathbb{R}$ is convex.
2. Define an optimal solution of LPP.
3. Which quadrant the graph of $x \leq 2$ and $y \geq 2$ will be situated ?
4. What have been constructed for Operations Research problems and methods for solving the models that are available in many cases ?
5. Define degenerate basic feasible solution.
6. The solution of any transportation problem is obtained in how many stages ?
7. Which problem is a subclass of a linear programming problem ?
8. When a feasible solution is called a basic feasible solution in a transportation problem ?
9. Define degenerate transportation problem.
10. Where the right hand side constant of a constraint in a primal problem does appear in the corresponding dual ?
11. What is integer programming model ?
12. What is the full form of PERT and CPM ?
13. What are the three time estimates in PERT ? Define them.
14. What is float in network diagram ?

(14 × 1 = 14 weightage)

Turn over

Part B

Answer any **seven** *questions.*
Each question carries a weightage of 2.

15. Prove that the set of feasible solutions is a closed convex set bounded from below and also prove that it has at least one vertex.

16. Prove that if $f(X)$ is minimum at more than one of the vertices of the set of feasible solutions, then it is minimum at all those points which are the convex linear combinations of these vertices.

17. Solve the following LPP by graphically :

Maximize $Z = 120x_1 + 80x_2$

subject to the conditions $2x_1 + x_2 \leq 6$

$7x_1 + 8x_2 \leq 28$

and $x_1, x_2 \geq 0$.

18. Prove that the value of the objective function $f(X)$ for any feasible solution of the primal is not less than the value of the objective function Φ (Y) for any feasible solution of the dual.

19. Prove that the transportation problem has a triangular basis.

20. Explain with example 'North West Corner Rule' for transportation problem.

21. Find the initial basic feasible solution of the following transportation problem using Vogel's approximation method :

	1	2	3	4	*Available*
A	5	8	3	6	30
B	4	5	7	4	50
C	6	2	4	6	20
Required	30	40	20	10	

22. Draw the network defined by the nodes N = {1, 2, 3, 4, 5, 6} and arcs :

A = {(1, 2), (1 ,5), (2, 3), (2, 4), (3, 5), (3, 4), (4, 2), (4, 6), (5, 2), (5, 6)}. From the network determine a path and a tree.

23. Find the minimum path from v_0 to v_8 in the graph with arcs and arc lengths are given below :

Arc	(0,1)	(0, 2)	(0, 3)	(1, 2)	(1, 4)	(1, 5)	(2, 3)	(2, 5)	(3, 5)	(3, 6)	(4, 7)
Length	2	6	8	3	10	8	1	1	2	4	3

Arc	(5, 4)	(5, 7)	(6, 5)	(6, 7)	(6, 8)	(7, 4)	(7, 6)	(7, 8)
Length	1	5	4	6	7	2	1	10

24. Solve the following game whose pay-off matrix is :

		B					
		I	II	III	IV	V	VI
	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
A	III	4	3	7	– 5	1	2
	IV	4	3	4	– 1	2	2
	V	4	3	3	– 2	2	2

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*

Each question carries a weightage of 4.

25. Solve the following LPP by simplex method :

Maximize $Z = 4x_1 + 3x_2 + 6x_3$

subject to the conditions $2x_1 + 3x_2 + 2x_3 \leq 440$

$4x_1 + 3x_3 \leq 470$

$2x_1 + 5x_2 \leq 430$

and $x_1, x_2, x_3 \geq 0.$

26. Solve the following LPP :

Maximize $Z = x_1 + 1.5x_2 + 5x_3 + 2x_4$

subject to the conditions $3x_1 + 2x_2 + x_3 + 4x_4 \leq 6$

$2x_1 + x_2 + 5x_3 + x_4 \leq 4$

$2x_1 + 6x_2 - 4x_3 + 8x_4 = 0$

$x_1 + 3x_2 - 2x_3 + 4x_4 = 0$

and $x_1, x_2, x_3, x_4 \geq 0$.

27. Find the optimum solution to the following transportation problem :

	1	2	3	4	5	*Available*
A	7	6	4	5	9	40
B	8	5	6	7	8	30
C	6	8	9	6	5	20
D	5	7	7	8	6	10
Required	30	30	15	20	5	

28. For the following activity data draw the network, find the critical path and the three floats for each activity :

Activity	1 - 2	1 - 3	2 - 3	2 - 5	3 - 4	3 - 6	4 - 5	4 - 6	5 - 6	6 - 7
Duration (weeks)	15	15	3	5	8	12	1	14	3	14

(2 × 4 = 8 weightage)

C 4576 (Pages : 3) Name.................................

Reg. No.............................

SECOND SEMESTER M.Sc. DEGREE (SUPPLEMENTARY) EXAMINATION APRIL 2021

(CUCSS)

Mathematics

MT 2C 09—ODE AND CALCULUS OF VARIATIONS

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *the questions.*
Each question carries 1 *weightage.*

1. Find a power series solution of the form $\sum a_n x^n$ of the equation $y' = 2xy$.

2. Determine the nature of the point $x = 0$ for the equation $xy'' + (\sin x)\, y = 0$.

3. Write down the hypergeometric series.

4. Determine the nature of the point $x = -\infty$ for the Legendre's equation

$$\left(1 - x^2\right) y'' - 2xy' + p\,(p+1)\,y = 0.$$

5. Show that $\frac{d}{dx}\left[J_a(x)\right] = \frac{1}{2}\left(J_{a-1}(x) - J_{a+1}(x)\right)$.

6. Prove that $P_n(1) = 1$.

7. Express $J_2(x)$ in terms of $J_0(x)$ and $J_1(x)$.

8. Explain the phase portrait of the system $\frac{dy}{dt} = x$ and $\frac{dy}{dt} = 0$.

9. Determine whether the function $-x^2 - 4xy - 5y^2$ is positive definite, negative definite or neither.

Turn over

10. Find the critical points of the system : $\frac{dy}{dx} = e^y$ and $\frac{dy}{dt} = e^y \cos x$.

11. Solve the following differential equation $\frac{dx}{dt} = tx$ with $x(0) = 1$. using Picard iteration.

12. Write down an integral equation for unknown function.

13. State Strum comparison theorem.

14. Find an externals point $y(x)$ of the functional $I = \int_0^1 \left(y' - y^2\right) dx, y(0) = 0, y(1) = 2$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries 2 weightage.

15. Find the general solution of the differential equation $(x^2 - 1)\, y'' + (5x + 4)\, y' + 4y = 0$ at the singular point $x = -1$.

16. Find the judicial equation and its roots of the equation $x^3 y'' + (\cos 2x - 1)\, y' + 2xy = 0$.

17. Some differential equations are of the hypergeometric type even though they may not appear to be so. Find the general solution of $\left(1 - e^x\right) y'' + \frac{1}{2} y' + e^x y = 0$ near the singular point $x = 0$ by changing the independent variable to $t = e^x$.

18. Prove that $\frac{d}{dx}\left[x^p J_p(x)\right] = x^p J_{p-1}(x)$.

19. Find the critical points of $\frac{d^2x}{dt^2} + \frac{dx}{dt} - \left(x^3 + x^2 - 2x\right) = 0$.

20. Determine the nature of stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = -3x + 4y, \frac{dy}{dt} = -2x + 3y.$$

21. Show that the system is almost linear and (0, 0) is a stable critical point of the system :

$$\frac{dx}{dt} = -x - xy^2, \frac{dy}{dt} = -y - x^2y.$$

22. If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of $y'' + P(x)y' + Q(x) = 0$, then prove that the functions and distinct and occur alternately in the sense that $y_1(x)$ vanishes exactly once between any two successive zeros of $y_2(x)$ and conversely.

23. Apply the Picard's method to the initial value problem $y' = x + y, y(0) = 1$ with $y_0(x) = \cos x$.

24. Show that $f(x, y) = xy^2$ satisfies a Lipschitz condition any rectangle $a \le x \le b$ and $c \le y \le d$.

(7 × 2 = 14 Weightage)

Part C

Answer any **two** *questions.*
Each question carries 4 *weightage.*

25. Calculate the independent Fröbenius series solutions of the equation

$$4xy'' + 2y' + y = 0.$$

26. Find the general solution of each of the following differential equations near the indicated singular point.

(a) $x(1-x)y'' + \left(\frac{3}{2} - 2x\right)y' + 2y = 0 ; x = 0.$

(b) $(2x^2 + 2x)y'' + (1 + 5x)y' + y = 0 ; x = 0.$

27. Find the general solution of the system $\frac{dx}{dt} = x - 2y, \frac{dy}{dt} = 4x + 5y.$

28. State and prove Picard's theorem

(2 × 4 = 8 weightage)

C 4575 (Pages : 3) Name..

Reg. No.....................................

SECOND SEMESTER M.Sc. DEGREE (SUPPLEMENTARY) EXAMINATION APRIL 2021

(CUCSS)

Mathematics

MT 2C 08—TOPOLOGY

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each question carries a weightage of 1.

1. Define co-finite Topology.
2. Define neighbourhood of a point in a topological space.
3. Give an example of τ which is not a topology.
4. What do you mean by a metric topology ?
5. Let $X = \{a,b,c,d\}; = \{\varnothing,\{a\},\{b,c\},\{a,b,c\},\{b,c,d\},\{X\}\}; A = \{a,c,d\}$, find $d(A)$.
6. Which of the following statement is true ?
   i) Limiting points are also the adherent points.
   ii) Adherent points are also limiting points.
7. Define closure of a set A.
8. Define quotient map.
9. What is meant by sub-base for τ.
10. Define regular at a point $x \in X$.
11. Difference between normal and regular on the space X.
12. Define semi-open interval topology.
13. What is meant by weaker topology ?
14. What is meant by separable on the space ?

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each question carries a weightage of 2.

15. Let A, B be subsets of a topological space (X,τ). Prove the following :

    (i) A is dosed in X if and only if $\bar{A} = A$.

    (ii) $\bar{\bar{A}} = \bar{A}$.

    (iii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

16. Let X be a space and $A \subset X$, then prove that int(A) is the union of all open sets contained in A.

17. Prove that for a subset A of a space X, $\bar{A} = \{y \in X :$ every neighbourhood of y meets A non-vacuously $\}$.

18. Prove that if f is continuous at x_0, then the inverse image (under f) of every neighbourhood of $f(x_0)$ in Y is a neighbourhood of x_0 in X.

19. Let $\{(Y_i, \tau_i) : i \in I\}$ be an index family of topological spaces X and $\{f_i : i \in I\}$ an indexed family of functions from Y_i into X. Prove that there exists a unique largest topology on U on X which makes each f_i continuous.

20. Let X have the weak topology determined by a family $\{f_i : X \to Y_i | i \in I\}$ of functions, where each Y_i is a topological space, I being an index set. Then prove that for any space Z, a function $g : Z \to X$ is continuous if and only if for each $i \in I$, the composite $f_i \circ g : Z \to Y_i$ is continuous.

21. Prove that if X is a compact space and $A \subset X$ is closed in X, then A in its relative topology, is also compact.

22. Prove that the space X is a T_1-space, then for any $x \in X$, the singleton set $\{x\}$ is closed.

23. Prove that the topological space X is regular, then for any $x \in X$ and open set G containing x there exist an open set H containing x such that $\bar{H} \subset G$.

24. Prove that regularity is a hereditary property.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 4.

25. a) Let (X,d) be a metric space, then prove that the intersection of any finite number of open set is open.

b) Let (X,d) be a metric space. Prove that for any given disjoint points $x,y \in X$ there exist an open sets U, V such that $x \in U, y \in V$ and $U \cap V = \varnothing$.

26. a) Prove that if a space is second countable then every open cover of it has a countable sub-cover.

b) Let X be a set and B a family of its subsets which covers X, there exists a topology on X with B as a base. Then prove that for any $B_1, B_2 \in B, B_1 \cap B_2$ can be expressed as the union of some members of B.

27. a) Prove that every quotient space of a discrete space is discrete.

b) Prove that every continuous image of a compact space is compact.

28. a) If y is an accumulation point of a subset A of a T_1-space X, then prove that every neighbourhood of y contains infinitely many points of A.

b) Prove that in a Hausdorff space, limits of sequences are unique.

(2 × 4 = 8 weightage)

C 4573 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE (SUPPLEMENTARY) EXAMINATION APRIL 2021

(CUCSS)

Mathematics

MT2C06—ALGEBRA—II

(2016 Admissions)

Time : Three Hours Maximum : 36 Weightage

Part A

Answer **all** *questions.*
Each questions carries 1 *weightage.*

1. Let $R = \left\{ \begin{bmatrix} a & b \\ -b & -a \end{bmatrix} : \det(A) \neq 0 \text{ and } a, b \in \mathbb{R} \right\}$ be a commutative ring with identity prime ideals in R.

2. Is $\sqrt{\pi}$ an algebraic over $\mathbb{Q}\left(\pi^2\right)$? Justify your answer.

3. Let $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$ be a extension field of $\mathbb{Q}$. Is $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}, \sqrt{5}\right)$ a simple ? Why ?

4. Find a basis for $\mathbb{Q}\left(w, i, \sqrt[3]{2}\right)$ over $\mathbb{Q}$, where w is cube root of unity.

5. Find the number of isomorphism of $\mathbb{Q}\left(\sqrt[4]{2}, i\right)$ onto to a subfield of $\bar{\mathbb{Q}}$ leaving $\mathbb{Q}$ fixed.

6. Find the number of subfields of a field $\mathbb{Z}_3[x]/\langle x^2+1\rangle$.

7. Find all conjugate of $\sqrt{2+\sqrt{i}}$ over $\mathbb{Q}$.

8. If α, β are constructible and β is nonzero, then α/β is constructible.

Turn over

9. Is $x^4 + x^2 - 1$ a separable over $\mathbb{Q}$? Why ?

10. Give an example of two finite normal extensions K_1 and K_2 of the same field F such that K_1 and K_2 are not isomorphic fields but $G(K_1/F) \cong G(K_2/F)$.

11. Express $x_1^2 + x_2^2 + x_3^2 + x_4^2$ as polynomials in the elementary symmetric functions in x_1, x_2, x_3, x_4.

12. Prove or disprove : the regular 7-gon is constructible.

13. Prove or disprove : $x^3 - 2$ is solvable by radical over $\mathbb{Q}$.

14. Find $\Phi_{14}(x)$.

(14 × 1 = 14 weightage)

Part B

Answer any **seven** *questions.*
Each questions carries 2 weightage.

15. Find a prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maximal.

16. Let E be an extension of $\mathbb{Z}_2$ and let $\alpha \in E$ be algebraic of degree 3 over $\mathbb{Z}_2$. Classify the groups $(\mathbb{Z}_2(\alpha), +)$ and $((\mathbb{Z}_2(\alpha))^*, \cdot)$.

17. Prove that an algebraically closed field F has no proper algebraic extension.

18. Determine whether or not the polynomial $x^2 - 6$ is irreducible in $\mathbb{Z}_{11}[x]$.

19. Find all automorphism of $\mathbb{Q}$.

20. Find the splitting field of $x^3 + x^2 + 1$ over $\mathbb{Z}_2$.

21. Describe the group of the polynomial $x^3 - 1 \in \mathbb{Q}[x]$ over Q.

22. Show that in $\mathbb{Q}[x]$, $\Phi_{2n}(x) = \Phi_n(-x)$ for odd integer $n > 1$.

23. Find $\Phi_8(x)$ over $\mathbb{Z}_3$.

24. If α and β are constructible real numbers, prove that $\alpha + \beta$ is constructible.

(7 × 2 = 14 weightage)

Part C

Answer any **two** *questions.*
Each questions carries 4 *weightage.*

25. Prove that every field contains either a subfield is isomorphic to $\mathbb{Z}_p$ for some prime p or a subfield is isomorphic to Q.

26. (a) Show that every finite field is perfect.

(b) If F is a field, prove that every ideal in F [x] is principal ideal.

27. Find the Galois group of $x^4 - 2$ over $\mathbb{Q}$ and exhibit the correspondence between the subgroups of the Galois group and the intermediate fields.

28. Let F be a field of characteristic zero, and let $a \in F$, If K is a splitting field of $x^n - a$ over F, prove that G (K/F) is a solvable.

(2 × 4 = 8 weightage)

C 3981 (Pages : 4) Name..................................

Reg. No...............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*

Each question carries 2 marks.

1. Let A and B be a linear transformations from the vector space $\mathbb{R}^n$ to itself. Prove that

   $\| BA \| \le \| B \| \| A \|$.

2. Let φ_1 and φ_2 be contractions of a metric space X to itself. Is $\varphi \circ \varphi$ a contraction ? Justify your answer.

3. Find the arc length of the curve $\gamma(t) = \left(e^t \cos t, e^t \sin t\right)$ starting at (1, 0).

4. Show that the total signed curvature of any regular plane curve $\gamma(t)$ is a smooth function of t.

5. Is $\sigma(u, v) = (u, u, uv), u, v \in \mathbb{R}$ a regular surface patch ? Justify your answer.

6. Show that $x^2 + y^2 + z^2 = 1$ is a smooth surface.

7. Calculate the first first fundamental form of the surface :

   $\sigma(u, v) = \left(u - v, u + v, u^2 + v^2\right)$.

8. Show that the normal curvature of any curve on a sphere of radius r is $\pm \frac{1}{r}$.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let Ω denote the set of all invertible linear operators on a vector space $\mathbb{R}^n$. Prove that Ω is an open subset of $\mathbb{R}^n$.

10. Let f be a differentiable real function on an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. If $f'(x) = 0$ for every $x \in E$, then prove that f is constant on E.

11. Show that any regular plane curve γ whose curvature is a positive constant is part of a circle.

12. Show that any open disc in the xy-plane is a surface.

13. Show that every compact surface is orientable.

14. If the second fundamental form of a surface patch σ is zero everywhere, then prove that σ is an open subset of a plane.

(4 × 4 = 16 marks)

Part C

Answer (A) **or** *(B) of the following questions.*
Each question carries 12 marks.

Unit I

15. (A) (a) Let X be a vector space and let dim X = n. Prove that a set E of n vectors spans X if and only if E is independent.

(b) Let $E \subset \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into $\mathbb{R}^m$ and g is differentiable at $f(x_0)$, then prove that the map $F : E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(B) (a) If X is a complete metric space and if φ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\varphi(x) = x$.

(b) Let f map an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$. Prove that f is continuously differentiable in E if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m, 1 \le j \le n$.

UNIT II

16. (A) (a) If $\gamma(t)$ is a regular curve, then prove that its arc-length s, starting at any point of γ, is a smooth function of t.

(b) Prove that the total signed curvature of a closed plane curve is an integer multiple of 2π.

(B) (a) Prove that any parametrization of a regular curve is regular.

(b) Let $\gamma(t)$ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Prove that its torsion is given by :

$$\tau = \frac{(\dot{\gamma} \times \ddot{\gamma}) \cdot \dddot{\gamma}}{\|(\dot{\gamma} \times \ddot{\gamma}\|^2}$$

where $\dot{\gamma}$ is $\frac{d\gamma}{dt}$.

UNIT III

17. (A) (a) Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$ and let (u, v) be co-ordinates in U. Prove that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

(b) Show that the quadric

$$x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = 5$$

is a hyperboloid of one sheet.

(B) (a) Let $f : S_1 \to S_2$ be a diffeomorphism. Prove that the linear map $D_p f : T_p S_1 \to T_{f(p)} S_2$ is invertible for all $p \in S_1$.

(b) Prove that every compact surface is orientable.

UNIT IV

18. (A) (a) Prove that the second fundamental form of a surface is a bilinear form.

(b) Show that the Gaussian curvature of a surface S is a smooth function on S.

(B) (a) If k_1 and k_2 are the principal curvatures of a surface, then prove that the mean and Gaussian curvatures are given by :

$$H = \frac{1}{2}(k_1 + k_2) \text{ and } K = k_1 k_2.$$

(b) Let S be a connected surface of which every point is an umbilic. Prove that S is an open subset of a plane or a sphere.

(4 × 12 = 48 marks)

C 3980 (Pages : 3) Name.......................................

Reg. No...................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 *marks.*

1. Prove that in a metric space, the union of any family of open sets is open.
2. Prove that a sequence in a cofinite topology is convergent only if there is at most one term of the sequence which repeats infinitely often.
3. Prove that a set in a topological space is open if and only if its complement is closed.
4. Prove that in a topological space, composition of two continuous functions is a continuous function.
5. Prove that the product topology is the weak topology determined by the projection functions.
6. When do we say that a topological space is disconnected? Give an example for a disconnected topological space.
7. Suppose y is an accumulation point of a subset A of a T_1 space X. Prove that every neighbourhood of y contains infinitely many points of A.
8. Show that a large box is a box which has only finitely many 'short' sides.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

9. Let $\{x_n\}$ be a sequence in a metric space (X : d). Then prove that $\{x_n\}$ converges to y in X if and only if for every open set U containing y, there exists a positive integer N such that for every integer $n \geq N, x_n \in U$.
10. Prove that second countability is a hereditary property.

Turn over

11. Let X be a space and $A \subset X$. Then prove that int (A) is the union of all open sets contained in A.

12. Prove that there is a one-to-one correspondence between the set of topologies on a set and the set of all nearness relations on that set.

13. Define divisible property. Prove that the property of being a discrete space is divisible.

14. Prove that the intersection of a finite number of large boxes is a large box.

(8 × 2 = 16 marks)

Part C

Answer **either** *A* **or** *B part of the following questions.*
Each question carries 12 *marks.*

15. A (a) Define semi-open interval topology on the set of real numbers. Prove that this topology is stronger than the usual topology on this set.

(b) Determine the topology induced by a discrete metric on a set.

B (a) Let X be a set, $\mathcal{J}$ a topology on X and S a family of subsets of X. Prove that S is a sub-base for $\mathcal{J}$ if and only if S generates $\mathcal{J}$.

(b) Prove that a discrete space is second countable if and only if the underlying set is countable.

16. A (a) Prove that a subset A of a space X is dense in X if and only if for every non-empty open subset B of X, $A \cap B \neq \phi$.

(b) For a subset A of a space X, prove that $\bar{A} = A \cup A', \bar{A}$ is the closure of A and A' is the derived set of A.

B (a) Define embedding in topological spaces. Prove that a function $e : X \to Y$ is an embedding if and only if it is continuous and one-to-one and for every open set V in X, there exists an open subset W in Y such that $e(V) = W \cap Y$.

(b) For any three spaces X_1, X_2, X_3 prove that $X_1 \times (X_2 \times X_3)$ is homeomorphic to $(X_1 \times X_2) \times X_3$.

17. A (a) Prove that every closed, surjective map is a quotient map.

(b) Prove that every continuous real-valued function on a compact space is bounded and attains its extrema.

B (a) Define countable chain condition. Prove that every separable space satisfies the countable chain condition.

(b) Prove that every closed and bounded interval is compact.

18. A. (a) Define Tychnoff space. Prove that every Tychnoff space is a T_3 space.

(b) For a topological space X, prove that the following statements are equivalent :

(i) X is regular.

(ii) For any $x \in X$ and any open set G containing x there exists an open set H containing x such that $\bar{H} \subset G$.

(iii) The family of all closed neighbourhoods of any point of X forms a local base at that point.

B (a) If a product is non-empty, prove that each projection function is onto.

(b) Prove that projection functions are open.

(4 × 12 = 48 marks)

C 3979 **(Pages : 3)** Name.......................................

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. State Picard's theorem.

2. Show that $x = 1$ is a regular singular point for the equation $x^2\left(x^2-1\right)^2 y''-x(1-x)y'+2y=0$.

3. Define Gamma function and show that $\Gamma(n+1)=n!$ for any integer $n \geq 0$.

4. Find the stationary function of $\int_0^4 \left[xy'-(y')^2\right]dx$ which is determined by the boundary conditions $y(0)=0,\ y(4)=3$.

5. Describe the phase portrait of the system :

   $\frac{dx}{dt}=x, \frac{dy}{dt}=0.$

6. Show that (0, 0) is an asymptotically stable critical point for the system :

   $\frac{dx}{dt}=-2x+xy^3, \frac{dy}{dt}=-x^2y^2-y^3.$

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question carries 4 marks.

7. Show that the solutions of the initial value problem $y'=f(x,y), y(x_0)=y_0$ are precisely the continuous solutions of the integral equation $y(x)=y_0+\int_{x_0}^{x} f\left[t,y(t)\right]dt.$

Turn over

8. Find a series solution $y_1(x)$ for the equation $y'' + y' - xy = 0$, $y_1(0) = 1$, $y'_1(0) = 0$.

9. Find the indicial equation and its roots for the equation $x^3 y'' + (\cos 2x - 1) y' + 2xy = 0$.

10. Find the general solution of the equation $(1 - e^x) y'' + \frac{1}{2} y' + e^x y = 0$ near the singular point $x = 0$ by changing the independent variable to $t = e^x$.

11. Let $f(x)$ be a function defined on the interval $-1 \le x \le 1$. Determine the polynomial $p(x)$ which minimizes the value of the integral $I = \int_{-1}^{1} [f(x) - p(x)]^2 \, dx$.

12. A curve in the first quadrant joins (0, 0) and (1, 0) and has a given area beneath it. Show that the shortest such curve is an arc of a circle.

13. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt} = 5x + 2y, \quad \frac{dy}{dt} = -17x - 5y.$$

14. Investigate the stability properties of the critical point (0, 0) for the van der Pol equation :

$$\frac{d^2x}{dt^2} + \mu(x^2 - 1)\frac{dx}{dt} + x = 0, \ \mu > 0.$$

(5 × 4 = 20 marks)

Part C

Answer **either** A **or** B *of each of the following three questions.*
Each question carries 16 *marks.*

15. A (a) Let $f(x, y)$ be a continuous function that satisfies a Lipschitz condition :

$|f(x, y_1) - f(x, y_2)| \le k |y_1 - y_2|$ on a strip defined by $a \le x \le b$ and $-\infty < y < \infty$. If (x_0, y_0) is any point on the strip, then show that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has one and only one solution $y = y(x)$ on $a \le x \le b$.

(b) Verify whether the function $f(x, y) = xy^2$ satisfies a Lipschitz condition on any strip $a \le x \le b$, $-\infty < y < \infty$.

B (a) Show that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$ by solving the equation $y' = 1 + y^2$; $y(0) = 0$ in two ways.

(b) Show that the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ has two independent Frobenius series solutions and find them.

16. A (a) Show that the solutions of the Legendre's equation $\left(1 - x^2\right)y'' - 2xy' + n(n+1)y = 0$, where n is a non-negative integer, bounded near $x = 1$ are precisely constant multiples of the polynomials $F\left[-n, n+1, \frac{1}{2}(1-x)\right]$.

(b) Obtain Bessel function $J_p(x)$ of first kind of order p.

B (a) Derive Euler's differential equation for an extremal.

(b) Find the curve of fixed length L that joins the points (0, 0) and (1, 0), lies above the x-axis and encloses the maximum area between itself and the x-axis.

17. A (a) Find the general solutions of the system :

$$\frac{dx}{dt} = 5x + 4y, \frac{dy}{dt} = -x + y.$$

(b) If $a_1b_2 - a_2b_1 \neq 0$, show that the system :

$$\frac{dx}{dt} = a_1x + b_1y + c_1, \frac{dy}{dt} = a_2x + b_2y + c_2$$; has a single isolated critical point (x_0, y_0).

B (a) Consider the non-linear system :

$$\frac{dx}{dt} = -x, \frac{dy}{dt} = 2x^2y^2.$$

(i) Find the critical points.

(ii) Find the differential equation of paths.

(iii) Solve this differential equation to find the paths.

(iv) Sketch a few of the paths.

(3 × 16 = 48 marks)

C 3978 (Pages : 3) Name.......................................

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS – II

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Prove that the outer measure m^* is translation invariant.
2. If f and g are measurable real-valued functions defined on the same domain, then prove that $f + g$ is also measurable.
3. If f and g are bounded measurable functions defined on a set E of finite measure and if $f = g$ a.e., then prove that :

$$\int_E f = \int_E g.$$

4. Let f be a non-negative measurable function. Show that :

$$\int f = \sup \int \phi \text{ over all simple functions } \phi \le f .$$

5. Let f be the function defined by $f(0) = 0$ and $f(x) = x \sin\left(\frac{1}{n}\right)$ for $x \neq 0$. Find $D^+ f(0)$ and $D_- f(0)$.
6. Define an absolutely continuous function. If f is absolutely continuous on $[a, b]$, then prove that f is of bounded variation on $[a, b]$.
7. Prove that every measurable subset of a positive set is positive.
8. Let E be a set for which $\mu \times \vartheta(E) = 0$. Prove that for almost all $x, \vartheta(E_x) = 0$.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Let $\mathscr{C}$ be any collection of subsets of a set X. Prove that there exists a smallest $\sigma-$algebra containing $\mathscr{C}$.

10. Prove that the interval (a,∞) is measurable.

11. If f and g are non-negative measurable functions defined on a measurable set E, then prove that :

$$\int_E f + g = \int_E f + \int_E g.$$

12. Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real-valued functions on $[a, b]$.

13. If f is integrable on $[a, b]$ and $\int_a^z f(t)\, dt = 0$ for all $x \in [a,b]$, then $f(t) = a$ a.e. in $[a, b]$.

14. If $E_i \in \mathscr{B}, \mu E_1 < \infty$ and $E_i \supset E_i + 1$, then prove that :

$$\mu\left(\bigcap_{i-1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n), \text{where} (X, \mathscr{B}, \mu) \text{ is a measure space.}$$

(4 × 4 = 16 marks)

Part C

Answer **either** A **or** B *of each question.*
Each question carries 12 *marks.*

15. A (a) Prove that the outer measure of an interval is its length.

(b) Let $\langle E_i \rangle$ be a sequence of disjoint measurable sets and A any set. Prove that :

$$m^*\left(A \cap \bigcup_{i-1}^{\infty} E_i\right) = \sum_{i-1}^{\infty} m^*(A \cap E_i).$$

B (a) Prove that there exists a non-measurable set.

(b) Let E be a measurable set of finite measure, and let $\langle f_n \rangle$ a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E, $f_n(x) \to f(x)$. Prove that for given $\in > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$

$$| f_n(x) - f(x) | < \in.$$

16. A (a) Let f be defined and bounded on a measurable set E with mE finite. Prove that a necessary and sufficient condition for f to be measurable is :

$$\inf_{f \leq \psi} \int_E \psi(x)\, dx = \sup_{f \geq \phi} \int_E \phi(x)\, dx$$

for all simple functions ϕ and ψ.

(b) Prove that a bounded function f on $[a, b]$ is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.

B (a) State and prove Fatou's Lemma.

(b) Let $\langle g_n \rangle$ be a sequence of integrable functions which converges a.e. to an integrable function g. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\langle f_n \rangle$ converges to f a.e. If $\int g = \lim \int g_n$, then prove that $\int f = \lim \int f_n$.

17. A (a) Let f be an increasing real-valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere.

(b) If f is absolutely continuous on $[a, b]$, and $f'(x) = 0$, a.e., then prove that f is constant.

B (a) Let f be an integrable function on $[a, b]$ and suppose that :

$$F(x) = F(a) + \int_a^x f(t)\, dt.$$

Prove that $F'(x) = f(x)$ for almost all x in $[a, b]$.

(b) Prove that every absolutely continuous function is the indefinite integral of its derivative.

18. A (a) Let E be a measurable set such that $0 < \vartheta E < \infty$. Prove that there is a positive set A continued in E with $\vartheta A > 0$.

(b) State and prove Radon-Nikodym theorem.

B (a) Let F be a monotone increasing function which is continuous on the right. Prove that there is a unique Baire measure μ such that for all a and b,

$$\mu\,(a, b] = F(b) - F(a).$$

(b) Let x be a point of X and E a set in $R_{\sigma\delta}$, where $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \vartheta)$ are two complete measure spaces. Prove that the x cross-section E_x is a measurable subset of Y.

(4 × 12 = 48 marks)

C 3977 **(Pages : 3)** **Name..**

Reg. No......................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA—II

(2019 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question is of 2 *marks.*

1. Let $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_4$ be a homomorphism with $\phi(1) = 1$. Let ker $\phi = K$. List all elements of $\mathbb{Z}_{12}/K$.

2. Give a composition series for the group $\mathbb{Z}_{15}$.

3. Verify whether $1+\sqrt{2}$ and $1-\sqrt{2}$ are conjugates over the rationals $\mathbb{Q}$.

4. Find the number of automorphisms of $\mathbb{Q}(\alpha)$ where α is the real cube root of 2.

5. Let K be an extension of degree 3 over $\mathbb{Z}_2$. Give a generator of the Galois group $G(K/\mathbb{Z}_2)$.

6. Describe the 8^{th} cyclotomic polynomial over the rationals $\mathbb{Q}$.

(6 × 2 = 12 marks)

Part B

Answer any **five** *questions.*
Each question is of 4 *marks.*

7. Let N be a normal subgroup of a group G and $\gamma : G \to G/N$ be the canonical homomorphism. Show that if K is normal in G/N then $\gamma^{-1}(K)$ is a normal subgroup of G.

8. Show that the symmetric group S_3 is solvable.

9. Find the number of Sylow 2 subgroups of S_3.

Turn over

10. Show that if α and β are conjugates over $\mathbb{Q}$ then $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$ are isomorphic fields.

11. Show that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$ is a splitting field over $\mathbb{Q}$.

12. Let $F < E < K$ be extensions of fields such that K is a finite extension of F. Show that if K is separable over F then K is separable over E.

13. Let K be a splitting field of $x^4 + 1$ over $\mathbb{Q}$. Describe the Galois group $G(K/\mathbb{Q})$.

14. Show that the regular 7-gon is not constructible.

(5 × 4 = 20 marks)

Part C

Answer part A **or** *part B of each question.*
Each question is of 16 *marks.*

15. A Let G be a group and H, K be normal subgroups of G with $K \leq H$. Show that :

(a) $HK = H$;

(b) H/K is a normal subgroup of G/K ; and

(c) G/H is isomorphic to $(G/K)/(H/K)$.

B Let G be a finite group of order n and p be a prime dividing n. Let e be the identity of the group G and $X = \{(g_1, g_2, \ldots, g_p) : g_i \in G \text{ and } g_1 g_2 \ldots g_p = e\}$. Show that :

(a) $|X| = n^{p-1}$;

(b) For a permutation $\sigma = (1\,2 \ldots p)$ and $H = \langle \sigma \rangle$ the map

$(g_1, g_2, \ldots, g_p) \mapsto (g_{\sigma(1)}, g_{\sigma(2)}, \ldots, g_{\sigma(p)})$ induces an action of H on X ; and

(c) There exists $a \in G$ such that $a \neq e$ and $a^p = e$.

16. A (a) Let E be a field and σ be an automorphism of E. Show that $E_\sigma = \{a \in E : \sigma(a) = a\}$ is a subfield of E.

(b) Let F be a finite field of characteristic p. Show that :

i $\sigma_p : F \to F$ defined by $\sigma_p(a) = a^p$ for $a \in F$ is an automorphism of F.

ii The fixed field of σ_p is isomorphic to $\mathbb{Z}_p$.

B (a) Define splitting field.

(b) Let E be a splitting field over F. Show that every irreducible polynomial in F [x] with one zero in E splits over E.

(c) Let $\bar{F}$ be an algebraic closure of F and $E < \bar{F}$ be a splitting field over F. Show that every isomorphism of E into $\bar{F}$ leaving elements of F fixed is an automorphism of E.

17. A Let K be a finite normal extension of F with Galois group $G(K/F)$. For each intermediate field E with $F < E < K$ let $\lambda(E) = G(K/E)$. Show that :

(a) Fixed field of $G(K/E)$ in K is E ;

(b) λ is one to one on the set all intermediate fields ; and

(c) If E is a normal extension of F then $G(K/E)$ is a normal subgroup of $G(K/F)$.

B (a) Define symmetric function in n variables.

(b) Let $E = F(s_1, s_2, \ldots\ldots, s_n)$ where $s_1, s_2, \ldots\ldots, s_n$ are the elementary symmetric functions in $y_1, y_2, \ldots\ldots, y_n$. Show that the Galois group of $F(y_1, y_2, \ldots\ldots, y_n)$ over E is isomorphic to the symmetric group S_n.

(3 × 16 = 48 marks)

C 3976 (Pages : 4) Name..................................

Reg. No..............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 10—MULTIVARIABLE CALCULUS AND GEOMETRY

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

1. Let A be a linear transformation from the space $\mathbb{R}^n$ to the space $\mathbb{R}^m$ and B be a linear transformation from the space $\mathbb{R}^m$ to the space $\mathbb{R}^k$. Prove that

$$\| BA \| \le \| B \| \| A \|.$$

2. Let A be a linear transformation from the vector space $\mathbb{R}^n$ to the vector space $\mathbb{R}^m$ and let $x \in \mathbb{R}^n$. Prove that $A'(x) = A$.

3. Find the arc length of the curve $\gamma(t) = (t, \cosh t)$ starting at the point (0, 1).

4. Show that the total signed curvature of any regular plane curve $\gamma(t)$ is a smooth function of t.

5. Is $\sigma(u, v) = (u, u, uv), u, v \in \mathbb{R}$ a regular surface patch ? Justify your answer.

6. Show that $x^2 + y^2 + z^4 = 1$ is a smooth surface.

7. Prove that the second fundamental form of a plane is zero.

8. Define Gaussian curvature of a surface S at a point $p \in S$.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*

Each question carries 4 marks.

9. Let Ω denote the set of all invertible linear operators on a vector space $\mathbb{R}^n$. Prove that the mapping $A \to A^{-1}$ is continuous on Ω.

10. Give an example to show that a given level curve can have both regular and non-regular parametrization.

11. Prove that any regular plane curve γ whose curvature is a positive constant is part of a circle.

12. Show that any open disc in the xy-plane is a surface.

13. Prove that every compact surface is orientable.

14. If the second fundamental form of a surface patch σ is zero everywhere, then prove that σ is an open subset of a plane.

(4 × 4 = 16 marks)

Part C

Answer (A) **or** *(B) of the following questions.*
Each question carries 12 marks.

Unit I

15. (A) (a) Let $E \subset \mathbb{R}^n$ be an open set and the map $f : E \to \mathbb{R}^k$ be differentiable at $x_0 \in E$. If g maps an open set containing $f(E)$ into R^m and g is differentiable at $f(x_0)$, then prove that the map $F : E \to \mathbb{R}^m$ defined by $F(x) = g(f(x))$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0)) f'(x_0)$.

(b) Show that composition of two contractions is again a contraction.

(B) State and prove implicit function theorem.

UNIT II

16. (A) (a) If $\gamma(t)$ is a regular curve, then prove that its arc-length s, starting at any point of γ, is a smooth function of t.

(b) Prove that the total signed curvature of a closed plane curve is an integer multiple of 2π.

(B) (a) Prove that any parametrization of a regular curve is regular.

(b) Let γ be a regular curve in $\mathbb{R}^3$ with nowhere vanishing curvature. Prove that the image of γ is contained in a plane if and only if the torsion τ is zero at every point of the curve.

UNIT III

17. (A) (a) Let $\sigma : U \to \mathbb{R}^3$ be a patch of a surface S containing a point $p \in S$ and let (u, v) be co-ordinates in U. Prove that the tangent space to S at p is the vector subspace of $\mathbb{R}^3$ spanned by the vectors σ_u and σ_v.

(b) Show that the quadric

$$x^2 + y^2 - 2z^2 - \frac{2}{3}xy + 4z = 5$$

is a hyperboloid of one sheet.

(B) (a) Let $f : S_1 \to S_2$ be a diffeomorphism. Prove that the linear map $D_p f : T_p S_1 \to T_{f(p)} S_2$ is invertible for all $p \in S_1$.

(b) Prove that the transition maps of a smooth surface are smooth.

UNIT IV

18. (A) (a) Prove that the second fundamental form of a surface is a bilinear form.

(b) Show that the Weingarten map W of a surface satisfies the quadratic equation

$$W^2 - 2HW + K = 0$$

in the usual notation.

(B) (a) If k_1 and k_2 are the principal curvatures of a surface, then prove that the mean and Gaussian curvatures are given by :

$$H = \frac{1}{2}(k_1 + k_2) \text{ and } K = k_1 k_2.$$

(b) Prove that the principal curvatures at a point of a surface are the maximum and minimum values of the normal curvature of all curves on the surface that pass through the point.

(4 × 12 = 48 marks)

C 3973 (Pages : 4) Name..............................

Reg. No..............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 07—REAL ANALYSIS—II

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 marks.

(1) Find the Lebesgue measure of the set $[0,1]-\left\{\frac{1}{2^n}: n=1,2,3,\ldots\right\}$.

(2) Let $f(x)=\begin{cases}0 & \text{if } x\notin[0,1]\\ 1 & \text{if } x\in[0,1]\end{cases}$.

Is f measurable ? Justify your answer.

(3) Let f, g be bounded measurable functions defined on a measurable set E. If $f=g$ a.e., then prove that $\int_E f=\int_E g$.

(4) Let f be an integrable function over a measurable set E and let $c\in\mathbb{R}$. Prove that cf is integrable and $\int_E cf=c\int_E f$.

(5) Let f and g be functions. Prove that $D^+(f+g)\le D^+f+D^+g$.

(6) Let $f(x)=\begin{cases}0 & \text{if } x=0\\ x^2\sin\left(\frac{1}{x^2}\right) & \text{if } x\neq 0.\end{cases}$

Is f bounded variation on $[-1,1]$? Justify your answer.

Turn over

(7) Prove that every measurable subset of a positive set is positive.

(8) Let μ be a complete measure on a measure space $(X, \mathcal{B})$. If $E_1 \in \mathcal{B}$ and $\mu(E_1 \Delta E_2) = 0$, then prove that $E_2 \in \mathcal{B}$ where $E_1 \Delta E_2$ denote the symmetric difference of E_1 and E_2.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question carries 4 *marks.*

(9) If E_1, E_2 are Lebesgue measurable, then prove that $E_1 \cup E_2$ is Lebesgue measurable.

(10) Let $a \in \mathbb{R}$. Prove that the interval (a, ∞) is measurable.

(11) Let φ and ψ be measurable simple functions which vanish outside a set of finite measure. Prove that $\int \varphi + \psi = \int \varphi + \int \psi$.

(12) Let f be a non-negative measurable function. Show that $\int f = 0$ if and only if $f = 0$ a.e..

(13) If f is absolutely continuous, then prove that f has a derivative almost everywhere.

(14) Give an example to show that the Hahn decomposition need not be unique.

(4 × 4 = 16 marks)

Part C

Answer A **or** *B of the following questions.*
Each question carries 12 *marks.*

UNIT I

(15) A (a) Let $\{A_n\}$ be a countable collection of sets of real numbers. Prove that

$$m^*\left(\bigcup_n A_n\right) \le \sum_n m^*(A_n).$$

(b) Let f be an extended real valued function whose domain is measurable. Prove that the following are equivalent :

(i) For each real number α the set $\{x : f(x) > \alpha\}$ is measurable.

(ii) For each real number α the set $\{x : f(x) \geq \alpha\}$ is measurable.

(iii) For each real number α the set $\{x : f(x) < \alpha\}$ is measurable.

(iv) For each real number α the set $\{x : f(x) \leq \alpha\}$ is measurable.

B (a) Let $\{E_n\}$ be an infinite sequence of measurable sets such that $E_{n+1} \subset E_n$ for each n. If $m(E_1)$ is finite, then prove that

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n \to \infty} m(E_n).$$

(b) Let E be a measurable set of finite measure and let $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x) \to f(x)$. Prove that for given $\varepsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \notin A$ and all $n \geq N$.

UNIT II

(16) A (a) Let f be a bounded function defined on $[a, b]$. If f is Riemann integrable, then prove that f is measurable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$,

where $R\int_a^b f(x)\,dx$ denotes the Riemann integral of f.

(b) If $\{f_n\}$ is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then prove that $\int_E f \leq \underline{\lim} \int_E f_n$.

Turn over

B (a) Let $\{f_n\}$ be a sequence of measurable functions defined on a set E of finite measure and suppose that there is a real number M such that $|f_n(x)| \le M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each $x \in E$, then prove $\int_E f = \lim \int_E f_n$.

(b) Let f be non-negative which is integrable on a measurable set E. Prove that for given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $m(A) < \delta$ we have $\int_A f < \varepsilon$.

UNIT III

(17) A (a) Let f be an increasing real valued function on the interval $[a, b]$. Prove that f is differentiable almost everywhere, the derivative f' is measurable and $\int_a^b f'(x)\,dx \le f(b) - f(a)$.

(b) If f is integrable on $[a, b]$, then prove that the function F defined by $F(C) = \int_a^x f(t)\,dt$ is bounded variation on $[a, b]$.

B (a) Prove that a function f is of bounded variation on $[a, b]$ if and only if f is the difference of two monotone real valued functions on $[a, b]$.

(b) Prove that a function F is an indefinite integral if and only if it is absolutely continuous.

UNIT IV

(18) A (a) Let $(X, \mathcal{B})$ be a measurable space, $\{\mu_n\}$ a sequence of measures that converge set wise to a measure μ and let $\{f_n\}$ be a sequence of non-negative measurable functions that converge pointwise to the function f. Prove that $\int f\,d\mu \le \underline{\lim} \int f_n\,d\mu_n$.

(b) Let μ be a signed measure on the measurable space $(X, \mathcal{B})$. Prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \theta$.

B (a) Let μ be a measure on an algebra D and let μ^* be the outer measure induced by μ. Prove that the restriction $\bar{\mu}$ of μ^* to the μ^*-measurable sets is an extension of μ to a σ-algebra containing D.

(b) If μ is a finite Borel measure on the real line, then prove that its cumulative distribution function F is a monotone increasing bounded function which is continuous on the right.

(4 × 12 = 48 marks)

C 3974 **(Pages : 3)** Name.......................................

Reg. No..................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 08—ORDINARY DIFFERENTIAL EQUATIONS

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question carries 2 *marks.*

1. Verify that $y = x^2$ is one solution of $x^2y'' + xy' - 4y = 0$, and find the general solution.

2. Define ordinary point of a linear differential equation $y'' + p(x)y' + Q(x)y = 0$. Give an example of a linear differential equation with $x = 1$ as its ordinary point.

3. Prove that $J_{-7}(x) = -J_7(x)$.

4. Show that $P_{2n}(0) = (-1)^n . \frac{1.3\ldots(2n-1)}{2^n \cdot n!}$, where $P_n(x)$ denotes the n^{th} degree Legendre polynomial.

5. Describe the phase portrait of the system $\frac{dx}{dt} = x, \frac{dy}{dt} = 0$.

6. Show that a function of the form $ax^3 + bx^2y + cxy^2 + dy^3$ cannot be either positive or negative definite.

7. Find the extremal for the integral $I = \int_{x_1}^{x_2} \sqrt{2 + (y')^2} . dx$.

8. State Picard's theorem.

(8 × 2 = 16 marks)

Turn over

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Show that if $y_1(x)$ and $y_2(x)$ are any two solutions of the equation $y''+P(x)y'+Q(x)y=0$ on $[a,b]$, then they are linearly dependent on this interval $[a,b]$ iff their Wronskian $W(y_1,y_2)$ is identically zero.

10. Express $\sin^{-1}(x)$ in the form of a power series in x by solving $y'=\left(1-x^2\right)^{-1/2}$; $y(0)=0$ in two ways.

11. Prove that the positive zeros of $J_p(x)$ and $J_{p+1}(x)$ occur alternately, in the sense that between each pair of consecutive positive zeros of either there is exactly one zero of the other.

12. Determine the nature and stability properties of the critical point (0, 0) for the system :

$$\frac{dx}{dt}=-3x+4y\,;\ \frac{dy}{dt}=-2x+3y.$$

13. Show that (0, 0) is an asymptotically stable critical point for the system :

$$\frac{dx}{dt}=-2x+xy^3\,;\ \frac{dy}{dt}=-x^2y^2-y^3.$$

14. Explain Picard's method of successive approximation of solving initial value problem $y'=f(x,y), y(x_0)=y_0$, where $f(x, y)$ is an arbitrary function defined and continuous in some neighborhood of the point (x_0,y_0).

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B of each of the following questions.*

Each question carries 12 marks.

15. A (a) Find a particular solution of the equation :

$$y''+2y'+y=e^{-x}\log x.$$

(b) Find the general solution of Chebyshev's equation $\left(1-x^2\right)y''-xy'+p^2y=0$, where p is a constant.

B Show that the equation $x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=0$ has two independent Fröbenius series solutions and find them.

16. A (a) Find the general solution of the equation $(x^2 - x - 6)y'' + (5 + 3x)y' + y = 0$, near its singular point $x = 3$.

(b) Derive Rodrigue's formula for n^{th} degree Legendre polynomial.

B (a) Determine the nature of the point $x = \alpha$ for Bessel's equation $x^2y'' + xy' + (x^2 - p^2)y = 0$.

(b) State and prove the orthogonal property of Bessel functions.

17. A Find the general solution of the system :

$$\frac{dx}{dt} = -4x - y, \frac{dy}{dt} = x - 2y.$$

B For the non-linear system $\frac{dx}{dt} = y(x^2 + 1), \frac{dy}{dt} = -x(x^2 + 1)$:

(i) find the critical points.

(ii) find the differential equation of the paths.

(iii) solve the equation in (ii) to find the paths.

(iv) sketch a few of the paths.

18. A (a) Obtain Euler's differential equation.

(b) A curve in the first quadrant joins (0, 0) and (1, 0) and has a given area beneath it. Show that the shortest such curve is an arc of s circle.

B Solve the following initial value problem by Picard's method, and compare the result with the exact solution :

$$\begin{cases} \dfrac{dy}{dx} = z, & y(0) = 1 \\ \dfrac{dz}{dx} = -y, & z(0) = 0. \end{cases}$$

(4 × 12 = 48 marks)

C 3975 (Pages : 3) Name..............................

Reg. No..............................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 09—TOPOLOGY

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *the questions.*
Each question carries 2 marks.

1. Define cofinite topology and co-countable topology. Among the two topologies on a set X which is stronger topology. Justify your answer.
2. Define base for a topology. Give an example for a base for the usual topology on the set of real numbers.
3. Prove that in a topological space, the closure of the closure of a set is same as the closure of the set.
4. Prove that every open surjective map is a quotient map.
5. Let X_1 and X_2 be connected topological spaces and $X = X_1 \times X_2$ with product topology. Then prove that X is connected.
6. Prove that a topological space X is T_1 space if and only if for any $x \in X$, the singleton set $\{x\}$ is closed.
7. Prove that every T_3 space is a T_1 space.
8. Prove that the intersection of a finite number of large boxes is a large box.

($8 \times 2 = 16$ marks)

Part B

Answer any **four** *questions.*
Each question carries 4 marks.

9. Define open sets and closed sets in metric spaces. Can the same set be both open and closed ? Justify the claim.
10. Prove that in a topological space, the family of all closed sets has the property that it is closed under finite unions.

11. Prove that a function $e : X \to Y$ is an embedding if and only if it is continuous and one-to-one and for every open set V in X, there exists an open subset W of Y such that $e(V) = W \cap Y$.

12. Prove that the property of being a finite space is divisible.

13. Justify the terms 'box' and 'wall' geometrically for product of copies of real line.

14. Prove that all metric spaces are T_4 spaces.

(4 × 4 = 16 marks)

Part C

Answer **either** *A* **or** *B part of the following questions.*
Each question carries 12 *marks.*

15. A (a) Let $\{x_n\}$ be a sequence in a metric space $(X ; d)$. Then prove that $\{x_n\}$ converges to y in X, if and only if for every open set U containing y there exists a positive integer N such that for every integer $n \geq N, x_n \in U$.

(b) Prove that the semi-open interval topology is stronger than the usual topology on the set of real numbers.

B (a) If a space is second countable, prove that every open cover of it has a countable subcover.

(b) Let (X, T) be a topological space and $B \subset T$. Then prove that B is a base for T if and only if for any $x \in X$ and any open G containing x, there exists $B \in B$ such that $x \in B$ and $B \subset G$.

16. A (a) Let (X, τ) be a topological space and $\mathcal{C}$ be the family of all closed subsets of X. Prove or disprove that $\mathcal{C}$ is the complement of τ in P (X), the set of all subsets of X.

(b) Define dense subset of a topological space. State and prove necessary and sufficient condition for a set to be dense.

B (a) For a subset A of a space X, prove that :

$$\bar{A} = \{y \in X : \text{every neighborhood of } y \text{ meets A non - vacuously}\}.$$

(b) For any three spaces X_1, X_2, X_3 prove that $X_1 \times (X_2 \times X_3)$ is homeomorphic to $(X_1 \times X_2) \times X_3$.

17. A (a) Prove that every closed surjective map is a quotient map.

(b) When do we say that a space satisfies the countable chain condition ? Prove that every second countable space satisfies the countable chain condition.

B (a) Prove that every continuous image of a compact space is compact.

(b) Prove that every closed and bounded interval is compact.

18. A (a) Prove that all metric spaces are T_4 spaces.

(b) Prove that the product of a finite number of regular spaces is regular.

B For a topological space X, prove that the following statements are equivalent :

(i) X is regular.

(ii) For any $x \in X$ and any open set G containing x, there exists an open set H containing x such that $\overline{H} \subset G$

(iii) The family of all closed neighbourhoods of any point of X forms a local base at that point.

(4 × 12 = 48 marks)

C 3972 (Pages : 4) Name.....................................

Reg. No.................................

SECOND SEMESTER P.G. DEGREE EXAMINATION, APRIL 2021

(CCSS)

Mathematics

MAT 2C 06—ALGEBRA—II

(2017 Admission onwards)

Time : Three Hours Maximum : 80 Marks

Part A

Answer **all** *questions.*
Each question is of 2 marks.

1. Let $X = \{1, 2, 3\}$ be a set and $G = \mathbb{Z}_5$ be the cyclic group with generator a. Define $a \cdot 1 = 1, a \cdot 2 = 3$ and $a \cdot 3 = 2$. Verify whether this extends to an action of G on X.

2. Verify whether $\mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_3 \times \mathbb{Z}_4$ are isomorphic groups.

3. Find the number of 3—Sylow subgroups of the symmetric group S_3.

4. Verify whether $a_1 a_2^{-1} a_2^5 a_3^2 a_3^{-1}$ is a reduced word.

5. Let $E = \mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right)$. Verify whether every element of E is of the form $a\sqrt{2} + b\sqrt{3}$ with $a, b \in \mathbb{Q}$.

6. Give an example of an algebraic extension of $\mathbb{Q}$ which is not a splitting field over $\mathbb{Q}$.

7. Find the order of the Galois group $G(K/\mathbb{Q})$ where K is the splitting field of $x^4 - 1 \in \mathbb{Q}[x]$.

8. Let K be a normal extension of a field F and $F \leq E \leq K$. Suppose that the Galois groups $G(K/F)$ and $G(E/F)$ are isomorphic. Is it necessary that $K = E$. Justify your answer.

(8 × 2 = 16 marks)

Part B

Answer any **four** *questions.*
Each question is of 4 marks.

9. Let G be a group, H a subgroup of G and L_H be the set of all left cosets of H. Show that L_H is a G–set.

10. Verify whether the series $(0) \leq \langle 3 \rangle \leq \mathbb{Z}_{15}$ and $(0) \leq \langle 5 \rangle \leq \mathbb{Z}_{15}$ are isomorphic.

11. Let G be a group of order 45. Show that G has a normal subgroup of order 9.

12. Show that group $\mathbb{Z}$ of integers under addition is a free group.

13. Let F be a finite field of characteristic p. Let $\sigma_p : F \to F$ be the automorphism given by $\sigma_p(a) = a^p$. Show that the fixed field of σ_p is $\mathbb{Z}_p$.

14. Find α such that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\alpha)$.

(4 × 4 = 16 marks)

Part C

Answer part A **or** *part B of each question.*
Each question is of 12 marks.

15. (A) (a) Let G be a finite group and X be a G–set. For $x \in X$ show that the number of elements in the orbit Gx of x is equal to the number of left cosets of the isotropy group G_x.

(b) Let S_3 act on the set {1, 2, 3} with usual action. Find the number of elements in the orbit of $\sigma = (1\ 2)$.

(B) (a) Define sub-normal series and isomorphic subnormal series.

(b) Find a non-trivial subgroup H of S_4 such that $H \neq A_4$ and $(1) < H < A_4 < S_4$ is a subnormal series of S_4.

(c) Give an example of a normal series and a refinement of it.

16. (A) (a) Define p–group and give an example.

(b) Show that every finite p–group is solvable.

(c) Show that the centre of a finite p–group is non-trivial.

(B) (a) Show that every group is a homomorphic image of a free group.

(b) Describe $\mathbb{Z}_5$ as a homomorphic image of a free group.

17. (A) (a) Let σ be an automorphism of a field E. Show that $E_\sigma = \{a \in E : \sigma(a) = a\}$ is a subfield of E.

(b) Let E be an extension of a field F. Show that :

(i) The set G (E/F) of all automorphisms of E leaving F fixed is a group under function composition.

(ii) The fixed field of G (E/F) contains F.

(B) (a) Define separable extension.

(b) Let E be a finite extension of F. Show that E is separable over F if and only if each $\alpha \in E$ is separable over F.

(c) Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a separable extension of $\mathbb{Q}$.

18. (A) (a) Let K be a finite normal extension of field F. Show that the Galois map λ with λ (E) = G (K/E) for F < E < K is a one to one map.

(b) Find all intermediate fields E such that $\mathbb{Q} < E < K$ where K is the splitting field of $x^4 + 1 \in \mathbb{Q}[x]$.

(B) Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ and K be the splitting field of $f(x)$ over $\mathbb{Q}$.

(a) Describe all the zeros of $f(x)$.

(b) Show that $K = \mathbb{Q}(\alpha, i)$ where $\alpha = \sqrt[4]{2}$ is the real positive fourth root of 2.

(c) Show that $[K : \mathbb{Q}] = 8$.

(d) Give a sub-field E of K such that $[E : \mathbb{Q}] = 2$.

(4 × 12 = 48 marks)