Shikhi M. “A study on common neighbor polynomial of graphs.”
Thesis. Department of Mathematics, University of Calicut, 2019.

DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON COMMON NEIGHBOR POLYNOMIAL OF GRAPHS**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

University of Calicut,

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ACKNOWLEDGEMENT

First and foremost I like to express my sincere gratitude to my research supervisor, Dr. V. Anil Kumar, Professor, Department of Mathematics, University of Calicut for giving me the privilege of doing research under his guidance. Though he gave me complete freedom of thought, he was available for discussions at any time. Heartfelt gratitude to him for his friendly approach and skillful guidance, only through which I could complete my research work within the minimum period.

My sincere gratitude to Dr. Preethi Kuttipulackal, Associate Professor and Head, Department of Mathematics, University of Calicut for the affectionate support that she has given me. I am also grateful to the faculty members of the department Dr. Raji Pilakkat (Professor), Dr. Sini P (Assistant Professor), Dr. Sreekanth K and Ms. Priya K for the mental support and encouragement. I would like to thank Dr. Ramachandran P. T. (Former Head), Department of Mathematics for extending necessary help at different stages of my work. Further, I acknowledge my gratitude to the non teaching staff and Librarian of the department for their timely support during the entire period of my study in the department.

I am thankful to the Department of Higher Education, Kerala for sanctioning FDP deputation and to University Grants Commission(UGC) for providing financial assistance for FDP. Further, I extend my gratitude to the Principal, all the teachers, students and non-teaching staff of Thunchan Memorial Government College, Tirur for the kindful support.

Heartfelt thanks to my research friends Dr. Latheesh, Dr. Shyama, Mr. Vineesh, Mr. Libeeshkumar, Mrs. Jalsiya, Ms. Sruthy, Mrs. Neethu, Mrs. Naheeda, Mrs. Noushida, Mr. Ajeesh, Mr. Midhun, Mr. Saleel, Mr. Fazil and Ms. Saira for their cheerful companionship and discussions they had with me. Further, sincere thanks to all the research fellows, M.Phil. and M.Sc. students of the department of Mathematics, University of Calicut for their vibrant presence which make my time in the department a memorable one.

This thesis is dedicated to my beloved father Mr. Rajagopalam M, whose blessings, as I believe, turned all the hard situations of my life favorable to me. His divine presence acted as a driving force during the journey of my research work. I bow my head before his loving memories.

Last but not the least, I thank my mother Prabhavathi K.M., my spouse Dr. Anilkumar A.K. and my loving kids Mihir A.K. and Malha A.K. for the whole hearted support and encouragement given to me for the entire period of my study.

Above all, sincere gratitude to the Almighty God whose kindful blessings helped me to successfully complete my research work.

Contents

List of symbols

G	A simple finite graph
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
$N(G,i)$	i-common neighbor set of G
$\|S\|$	Cardinality of the set S
$N[G;x]$	Common neighbor polynomial of G
$N^m[G;x]$	m^{th} derivative of $N[G;x]$
K_n	Complete graph on n vertices
P_n	Path on n vertices
C_n	Cycle on n vertices
$K_{m,n}$	Complete bipartite graph with $m+n$ vertices
$B_{n,n}$	Bistar graph on $2n+2$ vertices
$K_{n_1,n_2,\ldots,n_m}$	Complete m-partite graph
$L_{m,n}$	Lollipop graph
W_n	Wheel graph
H_n	Helm

WB_n	Web graph
S_n	Shell graph
B_N	Bow graph
BF	Butterfly graph
F_n	Friendship graph
$d_u(G)$	Degree of the vertex u in G
$B_{n,1}$	n-Barbell graph
B_n	Bipartite cocktail party graph
$W_n^{(m)}$	Windmill graph
$D_n^{(m)}$	Dutch windmill graph
$C_n \odot P_m$	Armed crown of C_n and P_m
$f_{n\times m}$	Flower graph
$C_p \odot C_q^t$	Chaplet graph
$S_{n,m}$	Snake graph
$RK_{n\times n}$	$n \times n$ square rook's graph
$P_n(m_1, m_2, \ldots, m_n)$	Caterpillar tree
$H \vee K$	Join of the graphs H and K
$H \circ K$	Corona of the graphs H and K
$H \square K$	Cartesian product of the graphs H and K
L_n	Ladder graph
CL_n	Circular ladder graph
B_m	m-book graph
$H \times K$	Tensor product of the graphs H and K
$S(G)$	Splitting graph of G
$Sh(G)$	Shadow graph of G

$\mu(G)$	Mycielski graph of G
$\overset{\mathcal{N}}{\sim}$	CNP-equivalent
$[G]_{\mathcal{N}}$	CNP-equivalent class of G
$p(G)$	Disjoint union of p copies of G
$\overline{G}$	Complement of G
$G+H$	Disjoint union of the graphs G and H
$\mathcal{N}$	Number of real common neighbor roots of G
$N_r(G,i)$	Generalized i-common neighbor set of G
$N_r[G;x]$	Generalized common neighbor polynomial of G
$[G]_{\mathcal{N}_r}$	CNP_r-equivalent class of G
$clr(v)$	Cluster of the vertex v
$W(G)$	Wiener index of G
$WW(G)$	Hyper wiener index of G
$H(G,x)$	Hosoya polynomial of G
$D_3[n]$	Nanostar dendrimer of third generation
D_k	PAMAM dendrimer of $k^(th)$ generation
$\mathcal{K}(G)$	Kronecker double cover of G

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Introduction

Graph theory is one of the well flourished branches of Mathematics. Originating from the modeling and negative resolution of famous Konigsberg bridge problem by Leonard Euler[23], graph theory has entrenched as one of the best tool to model network systems involved in complex real life problems. Beauty of graph theory lies in its wide scope of applications in the fields ranging from network theory, chemistry and operational research to architecture and linguistics. Performing as a translator of real life problems to mathematical models, graph theory has an astounding position amidst various branches of applied mathematics.

Among various branches of graph theory, graph polynomials is one of the well studied concepts as they are used to unveil the structural properties of graphs. Roughly speaking, a graph polynomial is a polynomial assigned to a graph whose coefficients are the indicators of some graph theoretic parameters. It can be defined as a function from the set of all finite graphs to the polynomial ring over the set of real numbers such that isomorphic graphs are assigned to the same polynomial.

In the present work, emphasizing on the structural similarity of pairs of nodes in a network system, a new graph polynomial is introduced named as 'Common neighbor polynomial of graphs'. While modeling the structure of a social network system, usually pairs of individuals with shared interests are represented by pairs of vertices with common neighbors. The number of such common neighbors serves as a measure of consensus and proclivities between the corresponding pair of individuals. Moreover, it is conjectured that two persons having one or more common acquaintances are more likely to be acquainted in future[7]. Hence the study of common neighbors of pairs of nodes in a network system is significant in predicting the possibility of future links as well as in clustering analysis.

An overview of the thesis

The thesis comprises an introductory chapter together with nine chapters in which a new graph polynomial called "Common Neighbor Polynomial of graphs" is introduced and studied in a detailed manner. In the introductory chapter, a concise description is given detailing the motivational facts behind the introduction of the new graph polynomial. Moreover, a blueprint of the upcoming chapters is also provided.

In **chapter 1**, the terminology and notations that will appear in the subsequent chapters are detailed. Basic graph theoretic definitions are explained in the first section. Second section of the chapter describes some important graph operations. Section 1.3 includes an introduction to the theory of graph polynomials along with some theorems on polynomials which are beneficial in the study of roots of polynomials.

In **chapter 2**, a new graph polynomial called 'Common neighbor polynomial' is introduced whose coefficients are the cardinalities of i-common neighbor sets which are defined as subsets of $V(G) \times V(G)$. The definition of i-common neighbor set and common neighbor polynomial of graphs is introduced in section 2.2. Let $G(V, E)$ be a graph of order n. Then for $0 \leq i \leq n-2$, the i-common-neighbor set of G is defined as $N(G, i) := \{(u, v) : u, v \in V, u \neq v \text{ and } |N(u) \cap N(v)| = i\}$. The common-neighbor polynomial of G denoted by $N[G; x]$ is defined as $N[G; x] = \sum_{i=0}^{(n-2)} |N(G, i)|x^i$. In section 2.3, the common neighbor polynomial of many well known graph classes are identified. The common neighbor polynomial of strongly regular graphs and trees are studied in section 2.4 and 2.5 respectively. The common neighbor polynomial of some special graph constructions are discussed in section 2.6.

The common neighbor polynomial of graphs obtained by the unary graph operations such as splitting graph, shadow graph or mycielsky graph of a given graph are discussed in **chapter 3**.

Binary graph operations are used to model the action between two network systems. Binary graph operations are usually known as graph products in which two initial graphs are acted together according to some specific rules to produce a new graph. **Chapter 4** provides explicit formulae to find common neighbor polynomial of some well known graph products such as join, corona, cartesian product, rooted product and tensor product of graphs in terms of the common neighbor polynomial of the parent graphs.

Structural equivalence of network systems is one of the prime concerns of network analysis. Usually in graph theory, isomorphic graphs are referred to as equal graphs. But, the existence of isomorphism may not be a criteria for identifying two graphs as equivalent as far as structural equivalence is concerned. From this point of view, CNP-equivalent classes of graphs are defined and studied in **chapter 5**. Two graphs G and H are said to be CNP-equivalent ($G \overset{\mathcal{N}}{\sim} H$) if and only if $N[G;x] = N[H;x]$. Obviously, the relation $\overset{\mathcal{N}}{\sim}$ is an equivalence relation on the class $\mathcal{G}$ of all simple finite graphs. The set of all graphs CNP-equivalent to a graph G is denoted as $[G]_{\mathcal{N}}$ and is defined as $[G]_{\mathcal{N}} = \{H \in \mathcal{G} : N[H;x] = N[G;x]\}$. A graph H is said to be $CNP-$unique if $[H]_{\mathcal{N}} = \{H\}$. Some CNP-equivalent classes of graphs are identified in section 5.2. In section 5.3, it is showed that graph classes like complete graphs and complete bipartite graphs are CNP-unique graphs.

While introducing a new graph polynomial, it is customary to verify whether it can be the graphical model of a stable physical system. A polynomial all of whose non zero roots lie in the open left half plane is said to be stable with respect to the closed right half plane and such a polynomial is called a Hurwitz polynomial. Identification of Hurwitz polynomials are beneficial in control systems theory as they represent the characteristic equations of stable linear systems. In **chapter 6**, we identify the conditions under which the common neighbor polynomial of some graph classes becomes a Hurwitz polynomial.

Chapter 7 focuses on the real roots of common neighbor polynomial of graphs. The roots of common neighbor polynomial of a graph G are called the common neighbor roots of G. The number of real common neighbor roots of a graph G where the multiplicities counted, is denoted by $\mathcal{N}(G)$. In chapter 7, we study the number of real common neighbor roots of some well known graph classes.

In **chapter 8** we generalize the concepts of i-common neighbor sets and common neighbor polynomial of graphs and define generalized i-common neighbor sets and generalized common neighbor polynomial of graphs. In section 8.2. The generalized common neighbor polynomial of some well known graph classes are identified. Moreover, some characterizations on graphs in terms of generalized common neighbor polynomial of graphs are also discussed. In section 8.3, we define the simplicial complexes of graphs and introduce the concept of cluster of a vertex in a graphs. In the light of these concepts, generalized i-common neighbor sets of graphs is studied.

Chapter 9 spot lighted on the significance of common neighbor polynomial of graphs in some applied areas. In section 9.1, we study common neighbor polynomial of graphs incorporated with chemical graph theory. Structural analysis of chemical molecules is a prime concern of mathematical chemistry. The common neighbor polynomial of nanostar dendrimers and PAMAM dendrimers are studied in subsections 9.1.1 and 9.1.2 respectively. In 9.1.3, the Hosoya polynomial of graphs with diameter not more than three is derived using the common neighbor polynomial of corresponding graphs. Section 9.2 deals with the significance of common neighbor polynomial of graphs in network data clustering. In 9.2.1, we discuss the Shared Nearest Neighbor(SNN) clustering and explains the way in which the common neighbor polynomial of graphs is useful in the formation of meaningful clusters. In section 9.3, we establish a relation which connects common neighbor polynomial of a graph with the adjacency matrix of the graphs. Making use of this relation, a C^{++} program is developed for generating coefficients of common neighbor polynomial of a graph and is provided as an Appendix.

In the concluding chapter of the thesis, some directions for further research are included. Also this chapter includes a list of publications and bibliography.