# *ESTIMATION OF STRESS-STRENGTH RELIABILITY FOR SOME DISTRIBUTIONS BASED ON RECORD VALUES*

*A Thesis submitted to the*

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*in*

**STATISTICS**

*by*

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# DECLARATION

I do hereby declare that the thesis titled "**ESTIMATION OF STRESS-STRENGTH RELIABILITY FOR SOME DISTRIBUTIONS BASED ON RECORD VALUES**" is the result of investigations carried out by me under the supervision of **Dr. P. Anilkumar**, in P G & Research Department of Statistics, Farook College, Kozhikode. The results reported in this thesis have not been submitted in whole or part for any degree in any university to the best of my knowledge.

Place:

Date:

(Juvairiyya R M)

# CERTIFICATE

Certified that the work reported in this thesis entitled **ESTIMATION OF STRESS-STRENGTH RELIABILITY FOR SOME DISTRIBUTIONS BASED ON RECORD VALUES** that is being submitted by Smt. Juvairiyya R M for the award of Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by her under my supervision and guidance in the P G & Research Department of Statistics, Farook College, Kozhikode. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma of any other university or institution. Also certified that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check.


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# Table of contents

# List of figures

# List of tables

# List of Abbreviations

| | |
|---|---|
| pdf | Probability density function |
| cdf | Cumulative distribution function |
| MLE | Maximum likelihood estimator |
| MSE | Mean square error |
| ICD | Inverse Chen distribution |
| MCMC | Markov chain Monte Carlo |
| EL | Expected length |
| CP | Coverage probability |
| i.i.d | Independently and identically distributed |

# Chapter 1

# Introduction

## 1.1 Stress-strength reliability

Stress strength model has become an important topic in applied statistics which resulted in a flood of research articles of many authors across the continents during the last two decades. The main force behind these developments is the term stress-strength. If a random variable X represents the stress experienced by a system and another random variable Y represents the strength of the system and if $X > Y$ the system fails. Then $P(X < Y)$, the probability of not failing is defined as stress-strength reliability. This can be considered as an assessment of reliability of the system. The particular problem of applied statistics termed by stress-strength reliability is one of the developments in the field of research of the model $P(X < Y)$. For a more detailed study of stress strength reliability the reader is referred to Kotz [1].

The seed of this idea was introduced by Birnbaum [2]. The first paper with $P(X < Y)$ in its title came from Birnbaum and McCarty [3]. The specific term stress-strength first appeared in the title of

Church and Harris [4]. Govindarajulu [5] among others considered estimation of $P(X < Y)$ in the non-parametric set up. It was Owen et al. [6] who first considered $P(X < Y)$ under certain parametric assumptions. Later, majority of common distributions were considered for estimation of $P(X < Y)$.

If X and Y are two random variables, the notion $P(X < Y)$ has a number of implications. $P(X < Y)$ has some relationship to the broader field of (partial) ordering of distributions. A variable Y is said to be stochastically larger than a variable X, if the cumulative distribution function (cdf) of Y is less than or equal to that of X. That is $F_Y(t) \leq F_X(t)$. Consequently, $P(X < Y) \geq 1/2$. Thus the model $P(X < Y)$ can be considered as originated from a problem of classical non-parametric tests of equality of two distributions.

It is noted that to compare between two populations we use the expression of difference of means $(\mu_1 - \mu_2)/\sigma$ which is useful only under the assumption of normality. But to cope up with the variety of complex real life situations it is required to use appropriate non-normal models; classic as well as recently developed. Hence it was realized that the expressions of the type $P(X < Y)$ or $P(X > Y)$ can be useful for examining the probability of inequality type relations between two random variables under various conditions and situations. Natural application of these probabilities involve comparison of two random variables representing the state of affairs in two different situations or different time intervals. The specific practical problem of applied statistics highlighted by the term "stress-strength" associated with a working system made this probability the topic of interest for a bulk of research papers appeared in the literature.

**Applications of the stress-strength parameter ($R$)**

- **In life testing experiments:** When X and Y represent the lifetimes of two devices $P(X < Y)$ gives the probability that the device with lifetime X fails before the other.

- **Two-treatment comparisons:** Treatment I is assigned to one group of individuals of size $n$ and treatment II to the other group of size $m$. If $X$ and $Y$ represent the remission times with these two treatments say $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ respectively, the researcher is interested to estimate $P(X < Y)$ which gives the probability that treatment II is better than treatment I (Kotz [1]).

- **Rocket engines:** Here $X$ represents the maximal chamber pressure generated by ignition of a solid propellant, while $Y$ is assumed to be the strength of the rocket chamber so that $P(X < Y)$ is simply the probability of successful firing of an engine (Kotz [1]).

- **Response Models:** A certain unit like a receptor in a human eye or any other organ operates only if it is stimulated by a source of random magnitude $Y$ and the stimulus exceeds a lower threshold specific for that unit, $X$. In this case P(unit functions) is equivalent to the $P(X < Y)$,a stress-strength relationship (Kotz [1]).

- **As a general measure of difference between populations:** Expressions of the type $P(X < Y)$ or $P(X > Y)$ can be useful for examining the probability of inequality type relations between two random variables under various conditions and situations.

**Note:** Different types of expressions for stress-strength reliability are prevalent in literature. In the stress-strength context if $X$ represents the stress experienced by a system and $Y$ represents the strength of the system $P(X < Y)$ is termed as stress-strength reliability. If it is vice versa $P(X > Y)$ is the stress-strength reliability. In situations other than stress-strength it makes no difference whether we consider $P(X < Y)$ or $P(X > Y)$. Stress-strength reliability (stress-strength parameter) is usually denoted by $R$.

In industry many products may fail under stress. For example, a wooden bar breaks when sufficient force is applied to it, a light bulb dies under stress of time, an electronic equipment stops functioning in an environment of very high temperature. In such experiments for getting the precise failure point, measurements may be made sequentially and only values larger (or smaller) than all previous ones are recorded. This type of data are called record data. In the manufacturing industry, a researcher might be interested to find the minimum failure stress of the product sequentially. So he would like to search for the smallest of all the previous observations. Hence inference procedures for $R$ based on record data are also relevant in the stress-strength context. Motivated by this fact a new field of enquiry was opened for researchers. A detailed review of literature on inference procedures of $R$ based on record values is given in Section 1.4.

## 1.2 Record values

Records were unexplored until Chandler[7] introduced the theory of record values and studied some of its properties. At that time he formulated the ground work for a mathematical study of records. Since then abundant literature was devoted to the study of records. Record values and associated statistics have an important role in many real life applications involving data relating to meteorology, hydrology, sports and life tests. A meteorologist may be interested in noting down the amount of rainfall greater or smaller than previous ones. In some experiments an observation is stored only if it is an upper (lower) record, because the measurement saving can be important especially when the sample size is large, costly or all (some portion) of the data is damaged. In situations such as hydrology, sports, financial markets, industry and traffic analysis, only records are observed and noted. When all or some part of the data are destroyed record data is very useful for measurement saving.

Many authors have considered records and associated statistics. For example see Ahsanullah[8], Balakrishnan et al. [9] and so on. Recently Asgharzadeh [10] has considered interval estimation for the two- parameter Pareto distribution based on records. Wang and Ye [11] studied inference on Weibull distribution based on record values. Wang et al. [12] have discussed inference on Gompertz distribution under records. Interval estimation for Gumbel distribution using climate records was considered by Asgharzadeh et al. [13]. Estimation based on lower record values from exponentiated Pareto distribution was studied by Sanggyeong Yoon et al. [14].

The standard record value process corresponds to an infinite sequence of independent and identically distributed (i.i.d) continuous random variables. In that case we shall define record values.

**Definition**

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with common distribution function $F$ which is continuous. An observation $X_j$ is called an upper record , if it exceeds all previous observations. That is if $X_j > X_i$ for all $i < j$. Similarly an observation $X_j$ is called a lower record , if it is less than all previous observations. That is if $X_j < X_i$ for all $i < j$.

### 1.2.1 Distribution of $n^{th}$ upper record

Let $\{X_j^*, j \geq 1\}$ denote a sequence of i.i.d exponential random variables. Due to the lack of memory property of the exponential distribution, we find that the differences between successive records follow i.i.d standard exponential distribution. Thus $\{J_n^*\} = \{R_n^* - R_{n-1}^*, n \geq 1\}$ are i.i.d exponential random variables. Thus $R_n^*$, $n = 0, 1, 2, \ldots$ the $n^{th}$ record corresponding to an i.i.d $Exp(1)$ sequence has $Gamma(n+1, 1)$ distribution. This result can be used to derive the distribution of $R_n$, the $n^{th}$record corresponding to an i.i.d sequence of random variables $\{X_j\}$ with common continuous cdf $F$ (see Arnold et al. [15]).

Note that if $X$ has cumulative distribution function $F$, then

$$\begin{aligned} -\log(1-F(x)) &\overset{d}{=} X^* \\ \text{where } X^* &\sim exp(1), \\ \Rightarrow \quad 1-F(x) &\overset{d}{=} e^{-X^*}, \\ \text{or } X &\overset{d}{=} F^{-1}(1-e^{-X^*}). \end{aligned}$$

Thus $X$ is a monotone increasing function of $X^*$ and consequently, the $n^{th}$ record of $\{X_n\}$, $R_n$ is related to the $n^{th}$ record of $R_n^*$ of the exponential sequence by

$$R_n \overset{d}{=} F^{-1}(1-e^{-R_n^*}).$$

Now the density function of $R_n$ is given as follows.

$$R_n^* \sim Gamma(n+1,1),$$

so that

$$f_{R_n^*}(r_n^*) = \frac{1}{\Gamma(n+1)} e^{-r_n^*}(r_n^*)^n.$$

Since

$$\begin{aligned} r_n &= F^{-1}(1-e^{-r_n^*}), \\ F(r_n) &= 1-e^{-r_n^*}, \\ r_n^* &= -\log(1-F(r_n)). \end{aligned}$$

Now the jacobian,

$$\left|\frac{dr_n^*}{dr_n}\right| = \frac{f(r_n)}{1-F(r_n)},$$

so that

$$f_{R_n}(r_n) = f(r_n^*)\left|\frac{dr_n^*}{dr_n}\right| = \frac{1}{\Gamma(n+1)}\left[-\log(1-F(r_n))\right]^n f(r_n). \tag{1.1}$$

We can transform upper records to lower records by replacing the original sequence $\{X_j\}$ by $\{-X_j, j \geq 1\}$. Thus if $L_n$ is the $n^{th}$ lower record of the above sequence, the density function of $L_n$ is given by

$$f_{L_n}(l_n) = \frac{1}{\Gamma(n+1)}\left[-logF(l_n)\right]^n f(l_n). \tag{1.2}$$

### 1.2.2 Joint density of records

The joint pdf of a set of records $(R_0^*, R_1^*, \ldots, R_n^*)$ is given by (see Arnold et al. [15])

$$f_{R_0^*, R_1^*, \ldots, R_n^*}(r_0^*, r_1^*, \ldots, r_n^*) = e^{-r_n^*}, \;\; 0 < r_0^* < r_1^* < \ldots < r_n^*.$$

Applying the transformation $R_n \stackrel{d}{=} F^{-1}(1-e^{-R_n^*})$ coordinate wise, we obtain the joint pdf of the set of records $R_0, R_1, \ldots, R_n$ corresponding to an i.i.d $F$ sequence as

$$\begin{aligned} f_{R_0, R_1, \ldots, R_n}(r_0, r_1, \ldots, r_n) &= \frac{\prod_{i=0}^{n} f(r_i)}{\prod_{i=0}^{n-1}(1-F(r_i))} \\ &= f(r_n)\prod_{i=0}^{n-1}\frac{f(r_i)}{1-F(r_i)}, \;\; -\infty < r_0 < r_1 < \ldots < r_n < \infty. \end{aligned} \tag{1.3}$$

Applying the transformation $L_n = -R_n$, the joint density of the set of lower records $L_0, L_1, \ldots, L_n$ corresponding to an i.i.d sequence is given by

$$f_{L_0,L_1,\ldots,L_n}(l_0,l_1,\ldots,l_n) \quad = \quad f(l_n)\prod_{i=0}^{n-1}\frac{f(l_i)}{F(l_i)}, \;\; \infty > l_0 > l_1 > \ldots > l_n > -\infty. \qquad (1.4)$$

# 1.3 Methods used for estimation of $R$

## 1.3.1 Maximum likelihood estimation

The maximum likelihood estimation is definitely the most popular procedure for estimation of stress-strength reliability, $R = P(X < Y)$. Assume that a random vector $(X,Y)$ has the probability density function (pdf) $f(x,y|\theta)$ with an unknown scalar or vector valued parameter $\theta$. We want to calculate $R = R(\theta) = P(X < Y)$ as a function of $\theta$.

$$R(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y|\theta)I(x<y)dxdy. \qquad (1.5)$$

If $X$ and $Y$ are independent with pdf's $f_X(x)$ & $f_Y(y)$ and cdf's $F_X(x)$ & $F_Y(y)$ respectively, (1.5) can be rewritten in the following ways as

$$\begin{aligned} \text{either } R &= \int_{-\infty}^{\infty} F_X(z|\theta)f_Y(z|\theta)dz, \\ \text{or } R &= \int_{-\infty}^{\infty} (1 - F_Y(z|\theta)f_X(z|\theta)dz. \end{aligned} \qquad (1.6)$$

Having derived an expression for $R(\theta)$ we construct the maximum likelihood estimator (MLE) of the unknown parameter $\theta$ by maximizing the logarithm of the likelihood function $\log L(\theta|\underset{\sim}{x},\underset{\sim}{y})$.

Then due to the invariance property of maximum likelihood estimators, the MLE of $R$ has the form $\hat{R} = R(\hat{\theta})$, where $\hat{\theta}$ is the MLE of $\theta$.

### 1.3.2 Bayes estimation

Bayesian approach considers parameter $\theta$ (scalar or vector) as a random variable (vector) with the pdf $\pi(\theta)$ called the prior pdf. This pdf is based on some knowledge available to the person carrying out the inference before data has been obtained.

Let $\pi(\theta)$ be a prior pdf of $\theta$. Then the posterior pdf of $\theta$ is given by

$$\pi(\theta|\underset{\sim}{X},\underset{\sim}{Y}) = \frac{f(\underset{\sim}{X},\underset{\sim}{Y}|\theta)\pi(\theta)}{\int_\theta f(\underset{\sim}{X},\underset{\sim}{Y}|\theta)\pi(\theta)d\theta},$$

where $f(\underset{\sim}{X},\underset{\sim}{Y}|\theta)$ denote the joint pdf of the data $(\underset{\sim}{X},\underset{\sim}{Y})$; $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$, $\underset{\sim}{Y} = (Y_1, Y_2, \ldots, Y_n)$. Under squared error loss function the Bayes estimator $R$ can be obtained as the expectation of $R = R(\theta)$ with respect to the posterior pdf $\pi(\theta|\underset{\sim}{X},\underset{\sim}{Y})$ as

$$\bar{R} = \int_\theta R(\theta)\pi(\theta|\underset{\sim}{X},\underset{\sim}{Y})d\theta.$$

Another method of determining Bayes estimator is to derive the posterior pdf of $R$ first and then find the estimator $\bar{R}$ as an expectation over this posterior pdf. The pdf $\pi_R(R|\underset{\sim}{X},\underset{\sim}{Y})$ of $R$ can be obtained using a one-to-one transformation.

**The choice of a prior**:

Prior information about the parameter $\theta$ is summarized by the prior distribution $g(\theta)$. First step of any Bayesian analysis is the choice of the prior. There are several ways of choosing the prior.

1. **Conjugate prior**:

   A class $P$ of prior distributions is said to be conjugate family for $F$, a class of pdf's, $f(x,y|\theta)$, if the posterior distribution is in class $P$. A pdf $g(\theta)$ is called natural conjugate to the likelihood function $L(\theta)$ if $g(\theta)$ and $L(\theta)$ are proportional to a function of $\theta$. A simple and elegant method of obtaining a conjugate prior for a family of distributions $f(x|\theta)$ which admits a sufficient statistic $T(x_1,x_2,\ldots,x_n) = T$ is as follows. From factorization theorem, we have likelihood function

   $$L(x|\theta) = u(x_1,x_2,\ldots,x_n)\hat{v}(T(x_1,x_2,\ldots,x_n),\theta) \propto \hat{v}(T,\theta)$$

   which implies that there exists a pdf $g(\theta|T) = g(\theta)$ such that $g(\theta) \propto \hat{v}(T,\theta), \theta \in \Omega$. Comparing $L(x|\theta)$ and $g(\theta)$ it follows that there exists an interesting relationship between the family of priors and likelihood. That is $g(\theta) \propto L(x|\theta)$ (see A.K.Bansal[16]).

2. **Non-informative prior**:

   When there is no information about the parameter the choice is the non-informative prior. Jeffreys [17] proposed the following prior distribution $g(\theta)$:

   - Rule A: If $\Omega = (-\infty,\ \infty)$, take $g(\theta)$ to be a constant.i.e. $\theta$ is assumed to be uniformly distributed.

   - Rule B: If $\Omega = (0,\ \infty)$, take $g(\theta) \propto \frac{1}{\theta}$ i.e. $\log\theta$ is assumed to be uniformly distributed.

   Rule A is invariant under any linear transformation $u = a\theta + b$. Rule B is invariant under any exponential transformation $u = \theta^k,\ k \neq 0$. Rules A and B are members of a large family of priors $g(\theta)$ where $\theta$ may be a scalar or vector valued parameter and proportional to the positive square root of the determinant of the Fisher information matrix.

$$g(\theta) \propto |I(\theta)|^{1/2}$$

If $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, then under some assumptions (see Lehmann & Casella [18]), $I(\theta)$ is the matrix with $(i, j)^{th}$ element,

$$I_{ij}(\theta) = -E\left[\frac{\delta^2}{\delta\theta_i\delta\theta_j}\log f(X,Y|\theta)\right]$$

Jeffreys's prior is invariant under any one-to-one reparameterization.

#### 1.3.2.1 Lindley's approximate method to find Bayes estimator

In Bayesian estimation often we have a ratio of two integrals which cannot be expressed in a closed form and requires numerical approximation. Lindley [19] developed an asymptotic approximation for this as follows.

Let $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$, $L(\theta)$ be the logarithm of the likelihood function, $v(\theta)$ be the prior distribution of $\theta$ and $u(\theta)$ be an arbitrary function of $\theta$. Then the Bayes estimator of $u(\theta)$ is the posterior expectation $u(\theta)$ given the data $x = (x_1, x_2, \ldots, x_n)$, which is given by

$$I = E(u(\theta|x)) = \frac{\int_\Omega u(\theta)v(\theta)e^{L(\theta)}d\theta}{\int_\Omega v(\theta)e^{L(\theta)}d\theta}. \tag{1.7}$$

Let $w(\theta) = u(\theta)v(\theta)$ and

$$L_k(\hat{\theta}) = \left[\frac{\delta^k L(\theta)}{\delta\theta^k}\right]_{\theta=\hat{\theta}}, \;\; w_k(\hat{\theta}) = \left[\frac{\delta^k w(\theta)}{\delta\theta^k}\right]_{\theta=\hat{\theta}}, \;\; W_i(\hat{\theta}) = \frac{w_i(\hat{\theta})}{w(\hat{\theta})}, \;\; w(\hat{\theta}) \neq 0$$

Clearly $L_1(\hat{\theta}) = 0$ at the MLE $\hat{\theta}$.

Expanding $L(\theta)$, $w(\theta)$ and $v(\theta)$ by Taylor's series about $\hat{\theta}$ and substituting in (1.7) we obtain

$$E(u(\theta|x)) \approx \left[u + \frac{1}{2}\sum_i\sum_j(u_{ij} + u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i\sum_j\sum_k\sum_l L_{ijk}u_i\sigma_{ij}\sigma_{kl}\ldots\right],$$

evaluated at $\theta = \hat{\theta}$ where

$$\rho = \log(v(\theta)),\ \ \rho_i = \frac{\delta\rho}{\delta\theta_i},\ \ u_i = \frac{\delta u}{\delta\theta_i},\ \ u_{ij} = \frac{\delta^2 u}{\delta\theta_i\delta\theta_j}.$$

$$L_{ij} = \frac{\delta^2 L}{\delta\theta_i\delta\theta_j},\ \ \sigma = \left\{-L_{ij}\right\}^{-1},\ \ L_{ijk} = \frac{\delta^3 L}{\delta\theta_i\delta\theta_j\delta\theta_k},\ \ \ i,j,k,l = 1,2,\ldots,m.$$

In particular for $\theta = (\theta_1, \theta_2)$, we have

$$\begin{aligned} E(u(\theta_1,\theta_2|x) \;&\approx\; u + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22} + u_{12}\sigma_{12} + u_{21}\sigma_{21}) + \rho_1(u_1\sigma_{11} + u_2\sigma_{21}) + \rho_2(u_1\sigma_{12} + u_2\sigma_{22}) \\ &+\; \frac{1}{2}[L_{30}(u_1\sigma_{11}^2 + u_2\sigma_{11}\sigma_{12}) + L_{21}\left\{3u_1\sigma_{11}\sigma_{12} + u_2(\sigma_{11}\sigma_{22} + 2\sigma_{12}^2)\right\} \\ &+\; L_{12}\left\{3u_2\sigma_{22}\sigma_{21} + u_1(\sigma_{11}\sigma_{22} + 2\sigma_{21}^2)\right\} + L_{03}(u_1\sigma_{21}\sigma_{22} + u_2\sigma_{22}^2)] \end{aligned} \tag{1.8}$$

evaluated at the MLE $(\hat{\theta}_1, \hat{\theta}_2)$.

If $\theta_1$ and $\theta_2$ are apriori independent, $\sigma_{ij} = 0,\ \ i \neq j$ and we have

$$\begin{aligned} E[u(\theta|x)] \;&=\; u + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + \rho_1 u_1\sigma_{11} + \rho_2 u_2\sigma_{22} \\ &+\; \frac{1}{2}\left(L_{30}u_1\sigma_{11}^2 + L_{03}u_2\sigma_{22}^2 + L_{21}u_2\sigma_{11}\sigma_{22} + L_{12}u_1\sigma_{22}\sigma_{11}\right) + \ldots. \end{aligned} \tag{1.9}$$

evaluated at $(\hat{\theta}_1, \hat{\theta}_2)$.

#### 1.3.2.2 Markov chain Monte Carlo method

It is known that MCMC originated in Statistical Physics literature and the first MCMC algorithm is the Metropolis algorithm published by Metropolis et al. [20] in Journal of Chemical Physics. It comes from the same group of scientists who developed Monte Carlo methods, mostly physicists working on Mathematical Physics and atomic bomb. The primary focus of this algorithm is the computation of the integral of high dimensions for which numerical integration is impossible. The first algorithm is associated with the second computer called 'MANIAC' built in Los Almos, New Mexico under the direction of Metropolis, a physicist as well as a mathematician. The Metropolis algorithm was later generalized by Hastings [21] as a statistical simulation tool in high-dimensional case. More about this can be had from Robert and Casella [22].

Markov chain Monte Carlo Method (MCMC) is essentially Monte Carlo integration using Markov chains. MCMC is a computer-driven sampling method. In Bayesian method as well as frequentist approach one needs to integrate over high-dimensional probability distributions to make inference about the model parameters. Using MCMC a distribution can be characterized without knowing all of the distribution's mathematical properties. MCMC is very useful in Bayesian inference in taking random samples from posterior distributions, finding the mean of a posterior distribution etc. which can not be directly calculated. The name MCMC combines two properties: Monte Carlo and Markov chain.

Let us discuss the Monte Carlo method.

If $g(\theta|y)$ is the posterior distribution of $\theta$

$$E(h(\theta)|y) = \int h(\theta)g(\theta|y)d\theta.$$

Suppose we are able to simulate independent observations $\theta^1, \theta^2, \ldots, \theta^m$ from the posterior density, then the Monte Carlo estimate of the posterior mean is given by the sample mean

$$\hat{h} = \frac{\sum_{j=1}^{m} h(\theta^j)}{m}.$$

The associated simulated variance of the estimate is given by

$$\hat{v_h} = \frac{\sum_{j=1}^{m} (h(\theta^j) - \hat{h})^2}{(m-1)m}.$$

The Markov chain property of MCMC is that the random samples are drawn using a markov chain in which each new sample depends upon the one just before it and not any other previous samples. The MCMC sampling method sets up an irreducible, aperiodic Markov chain for which the stationary distribution is the posterior distribution. A general way of constructing a Markov chain is by Metropolis-Hastings algorithm.

A Metropolis-Hastings algorithm (refer Metropolis et al. [20] and Hastings [21]) begins with an initial value $\theta^t$ given the $(t-1)^{th}$ value in the sequence, $\theta^{t-1}$. This rule consists of a proposal density which simulates a candidate value $\theta^*$ and the computation of an acceptance probability $P$ which indicates the probability that the candidate value will be accepted to be the next value in the sequence.

**Algorithm**

Step 1: Simulate a candidate value $\theta^*$ from a proposal density $p(\theta^*|\theta^{t-1})$

Step 2: Compute the ratio $R = \frac{g(\theta^*)p(\theta^{t-1})}{g(\theta^{t-1})p(\theta^*)}$

Step 3: Compute the acceptance probability $P = min(R, 1)$

Step 4: Sample a value $\theta^t$ such that $\theta^t = \theta^*$ with probability $P$, otherwise $\theta^t = \theta^{t-1}$

Step 5: Repeat the steps 1 to 4 until we get the required number of samples.

If the proposal density is symmetric about the origin, the ratio has the simple form $R = \frac{g(\theta^*)}{g(\theta^{t-1})}$. We often choose normal density as proposal density. Then the ratio of posterior densities $R = \frac{g(\theta^*)}{g(\theta^{t-1})}$ is found. To decide whether to accept or reject the sampled observation $\theta^*$ generate an observation $U$ from Uniform(0,1). If $U < R, \theta^t = \theta^*$, otherwise $\theta^t = \theta^{t-1}$.

#### 1.3.2.3 Gibbs sampling

Let the joint posterior distribution $[\theta|data]$ is of high dimension; $\theta = (\theta_1, \theta_2, \ldots, \theta_p)$. In this case it cannot be obtained analytically.

We define the set of conditional distributions

$$[\theta_1|\theta_2 \ldots, \theta_p, data],$$

$$[\theta_2|\theta_1, \ldots, \theta_p, data], \ldots,$$

$$[\theta_p|\theta_1, \ldots, \theta_{p-1}, data].$$

We set up a Markov chain simulation algorithm from the joint posterior distribution by simulating individual parameters from the set of $p$ conditional distributions. Simulating in turn one value of each individual parameters from these distributions is called one cycle of Gibbs sampling. Under general conditions, draws from this simulated algorithm will converge to the target distribution (the joint posterior of $\theta$) of interest. In situations where it is not possible to draw samples directly from the conditional distribution one can use Metropolis algorithm.

### 1.3.3 Interval estimation

In applications where the mere knowledge about a point estimator is not sufficient, that is in situations where variability of the point estimator is to be known, one needs an interval which covers the unknown value of $R$ with a high probability of at least $(1-\alpha)$, where $\alpha > 0$ and small.

**Definition**

Let the statistics $L(X,Y)$ and $U(X,Y)$ be such that

$$P\left(L(X,Y) < R < U(X,Y)\right) \geq 1-\alpha, \;\; 0 < \alpha < 1.$$

The interval $(L(X,Y), \;\; U(X,Y))$ is called the confidence interval for $R$ with the lower and upper bounds $L(X,Y)$ and $U(X,Y)$ respectively with confidence coefficient $1-\alpha$. There are four methods for construction of confidence intervals; exact method, asymptotic method, bootstrap method and Bayesian method.

#### 1.3.3.1 Exact method of interval estimation

To obtain the confidence interval by exact method it is required to find a pivotal quantity. A random variable $Q(\underset{\sim}{X},\underset{\sim}{Y})$ is said to be a pivotal quantity, if the distribution of $Q(\underset{\sim}{X},\underset{\sim}{Y})$ is independent of all the parameters. If one can find $a$ and $b$ such that

$$P(a < Q(\underset{\sim}{X},\underset{\sim}{Y}) < b) \geq 1-\alpha,$$

from which we can deduce that

$$P\left(L(\underset{\sim}{X},\underset{\sim}{Y}) < \theta < U(\underset{\sim}{X},\underset{\sim}{Y})\right) \;\geq\; 1-\alpha$$

that gives a $100(1-\alpha)\%$ confidence interval for $\theta$.

#### 1.3.3.2 Asymptotic confidence interval

This method is suggested (refer Arnold [15]) for the cases when the construction of exact confidence interval of $R$ is impossible. The asymptotic confidence interval is based on the central limit theorem which is stated as under "mild regularity conditions" a point estimator $\hat{R}$ of $R$ whether it is MLE or UMVUE is asymptotically normal with mean $R$ and the variance $\sigma_R^2$ as the sample sizes $n_1$ and $n_2$ of $X$ and $Y$ respectively turn to infinity. So if we can construct an estimator $\hat{\sigma_R^2}$ of $\sigma_R^2$, we have

$$P\left(-z_{\frac{\alpha}{2}} < \frac{(\hat{R}-R)}{\hat{\sigma_R}} < z_{\frac{\alpha}{2}}\right) \approx 1-\alpha,$$

where $z_\alpha$ is the upper $\alpha$ cut off point of the standard normal distribution of the equation

$$\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = \alpha.$$

Then a $100(1-\alpha)\%$ asymptotic confidence interval for $R$ is given by

$$\left(\hat{R} - z_{\frac{\alpha}{2}}\hat{\sigma_R},\;\; \hat{R} + z_{\frac{\alpha}{2}}\hat{\sigma_R}\right).$$

#### 1.3.3.3 Bootstrap confidence interval

The bootstrap method is a type of re-sampling method introduced by Bradley Efron[23] in which sample data is considered as population. From this sample data, we re-sample it with replacement large number of times. The resultant sampling distribution often approximate the true sampling distribution of the statistic. From this bootstrapped sampling distribution, we can estimate parameter value, standard errors (standard deviation of sample statistic) and then, calculate the confidence interval.

Let $\theta$ be a population parameter of interest which belongs to unknown population distribution $F$. $\hat{\theta}$ is a statistic that estimate the parameter $\theta$ and we have to find the sampling distribution of $\hat{\theta}$ from the fitted distribution function $\hat{F}$. We draw bootstrap samples and calculate estimates of $\hat{\theta}$ from these bootstrap samples to obtain the sampling distribution $\hat{F}^*$ of $\hat{\theta}^*$. This sampling distribution is a good approximation of $\hat{F}$.

There are two types of bootstrap sampliing; parametric and non-parametric.

**Nonparametric bootstrap**

When we are unable to determine from which probability distribution function the sample data may have generated, we use empirical distribution function (which is based on observed sample data) to simulate bootstrap samples. In practice we draw a random sample of the same size as that of original sample with replacement to get a bootstrap sample. This process is repeated until we get the required number of bootstrap samples.

**Parametric Bootstrap**

We assume that the observed data may have come from some known probability distribution function ( i.e normal, gamma or poisson or any). So, instead of using observed data ( as a non-parametric bootstrap) we can use the assumed distribution function with probable parameter estimates ( which most likely the maximum likelihood estimates) for bootstrap re-sampling. We find the estimate of $\theta$ from the original sample data and then this estimate is used to generate a bootstrap sample. Again the parameter is estimated from the bootstrap sample using which the second bootstrap sample is obtained. This process is repeated $B$ times to get $B$ bootstrap samples.

Three types of confidence intervals are used here.

1. **Standard normal interval:**

   The simplest $100(1-\alpha)\%$ confidence interval is the standard normal interval given by

$$(\hat{\theta} - z_{1-\frac{\alpha}{2}} \hat{se}_{boot}, \hat{\theta} + z_{1-\frac{\alpha}{2}} \hat{se}_{boot})$$

   where $\hat{se}_{boot}$ is the bootstrap estimate of standard error based on $\hat{\theta}_1{}^*, \ldots, \hat{\theta}_B^*$.

2. **Basic pivotal interval:** The $100(1-\alpha)\%$ basic pivotal confidence interval is

$$\left(2\hat{\theta} - \hat{\theta}^*_{(1-\frac{\alpha}{2})B}\ ,\ \ 2\hat{\theta} - \hat{\theta}^*_{(\frac{\alpha}{2})B}\right)$$

   where $\hat{\theta}^*_\beta$ is the $\beta$ quantile of $\hat{\theta}_1^*, \ldots, \hat{\theta}_B^*$.

3. **Percentile bootstrap (Boot-p) interval:**

   Let $G(x) = P(\hat{\theta^*} \leq x)$ be the cdf of $\hat{\theta^*}$. Define $\hat{\theta}_{boot}(x) = G^{-1}(x)$ for a given x. Then the $100(1-\alpha)\%$ bootstrap percentile interval for $\theta$ is defined by

$$\left(\hat{\theta}_{boot(\frac{\alpha}{2})}, \hat{\theta}_{boot(1-\frac{\alpha}{2})}\right)$$

   where the confidence limits are $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of the bootstrap sample $\hat{\theta}_1^*, \ldots, \hat{\theta}_B^*$.

#### 1.3.3.4 Bayesian method

The concept of confidence interval in Bayesian set up is different from frequentist confidence interval. In frequetist approach it is known that the interval covers $R$ with probability of at least $(1-\alpha)$ where as in Bayesian method one knows that $R$ is inside the interval with a given probability. This is because the fact that under Bayesian model $R$ is cosidered to be a random variable. To distinguish between Bayesian and classical interval estimators the former is referred to as credible intervals rather than confidence intervals. So we derive $100(1-\alpha)\%$ posterior credible intervals for the parameters of interest based on the marginal posterior distribution of the parameter. Such credible intervals can be obtained analytically or using a Markov chain Monte Carlo (MCMC) method. When a marginal distribution is not symmetric, a $100(1-\alpha)\%$ highest posterior density (HPD) is more desirable. An HPD interval has two main properties.

1. The density for every point inside the interval is greater than that for every point outside the interval.
2. For a given probability content (say$(1-\alpha)$), the interval is of shortest length.

Consider a Bayesian posterior density having the form

$$\pi(\theta,\phi|D) = \frac{1}{C(D)} L(\theta,\phi,D)\,\pi(\theta,\phi)$$

where $D$ denotes data, the parameter $\theta$ is one-dimensional and $\phi$ may be multi-dimensional vector of parameters other than $\theta$ in the model. Let $\pi(\theta|D)$ and $\Pi(\theta|D)$ be the marginal posterior density function and marginal posterior cumulative distribution function (cdf) of $\theta$ respectively. We assume that $\theta$ is unimodal and also $\theta$ can be generated from $\pi(\theta|D)$ using direct random generation scheme or a MCMC algorithm. To obtain exact credible and HPD intervals, closed forms of $\pi(\theta|D)$ and

$\Pi(\theta|D)$ are assumed to be known. For obtaining a $100(1-\alpha)\%$ credible interval, we calculate $\theta^{(\alpha/2)}$ and $\theta^{(1-\alpha/2)}$ such that

$$\Pi(\theta^{\alpha/2}|D) = \frac{\alpha}{2}$$

and

$$\Pi(\theta^{1-\alpha/2}|D) = 1 - \frac{\alpha}{2}.$$

Then a $100(1-\alpha)\%$ for $\theta$ is

$$(\theta^{\alpha/2}, \theta^{1-\alpha/2}).$$

Because $\pi(\theta|D) = \int \pi(\theta, \psi|D) d\psi$ while $\psi$ is high-dimensional, $\pi(\theta|D)$ is often analytically unobtainable. Chen and Shao [24] proposed a simple Monte Carlo method to estimate HPD intervals for which it is not required to evaluate the marginal posterior densities analytically or numerically. This method requires only an MCMC sample generated from the marginal posterior distribution of the parameter of interest.

Let $\{\theta_i, i = 1, 2, \ldots, n\}$ be an ergodic MCMC sample from $\pi(\theta|D)$. Then a $100(1-\alpha)\%$ Bayesian credible interval is $\left(\theta_{([(\frac{\alpha}{2})n])}, \theta_{([(1-\frac{\alpha}{2})n])}\right)$, where $\theta_{([(\frac{\alpha}{2})n])}$ and $\theta_{([(1-\frac{\alpha}{2})n])}$ are the $[(\frac{\alpha}{2})n]^{th}$ smallest and $[(1-\frac{\alpha}{2})n]^{th}$ smallest of $\{\theta_i\}$ respectively.

To obtain a $100(1-\alpha)\%$ HPD interval, we let $\theta_j$ be the smallest of $\{\theta_i\}$ and denote

$$R_j(n) = \left(\theta_{(j)}, \theta_{(j+[(1-\alpha)n])}\right)$$

for $j = 1, 2, \ldots, n - [(1-\alpha)n]$. Then, $R_{j^*}(n)$ which has smallest interval width among all $R_j(n)'$s is a $100(1-\alpha)\%$ HPD interval for $\theta$.

## 1.4 Review of literature

Birnbaum et al. [2] was one of the first researchers who dealt with the model $P(X < Y)$ in stress-strength context. Since then various research papers have come out in this area. Inference procedures on $P(X < Y)$ has been considered by researchers for different distributions based on random samples. Jeevanand and Nair [25] considered estimating $P(X < Y)$ from exponential samples containing spurious observations. Jeevanand [26] also studied Bayesian estimation of reliability under stress-strength model for the Marshall-Olkin bivariate exponential distribution. Kundu and Gupta[27] considered estimation of $P(X < Y)$ for generalized exponential distribution and for Weibull distribution[28]. Estimation of $P(X < Y)$ for the three parameter generalized exponential distribution was considered by Raqab et al. [29]. Jiang and Wong [30] studied inference of $P(X < Y)$ for right truncated exponential distribution. Stress-strength reliability of two parameter life time distribution based on progressively censored samples was considered by Shoaee and Khorram [31]. Rezaei et al. [32] studied the estimation of $R$ when X and Y are two independent generalized Pareto random variables with common scale parameter and different shape parameters. Some other works on random samples include estimation of stress strength reliability of the standard two-sided power distribution by Cetinkaya and Genc [33], reliability estimation in stress strength models : an MCMC approach by Soliman et al. [34] and estimation of $P(Y < X)$ for Weibull distribution under progressive Type II censoring by Valiollahi et al. [35].

Abundant research studies have appeared in the literature in the area of estimation of the stress-strength model using record values. Asgharzadeh et al. [36] considered interval estimation for the two parameter bathtub shaped lifetime distribution based on records. Likelihood and Bayesian estimation of $P(X<Y)$ was considered by Baklizi [37]. He also studied inference on $P(X<Y)$ in the two parameter Weibull model based on records [38] and estimation of $Pr(X<Y)$ using Record Values in the One and Two Parameter Exponential Distributions[39]. Basirat et al. [40] made a study on estimation of stress-strength parameter using record values from proportional hazard rate models. Interval estimation of stress-strength reliability based on lower record values from Inverse Rayleigh distribution was considered by Tarvirdizade & Hussain [41]. Tarvirdizade & Ahmedpur [42] studied estimation of the stress-strength reliability for the two-parameter bathtub shaped lifetime distribution based on upper record values. Statistical inference of $P(X<Y)$ for the Burr Type XII distribution based on records was presented by Nadar et al. [43]. Here we would like to present a work on "Estimation of stress-strength reliability for some distributions based on record values".

## 1.5 Outline of the thesis

In this thesis we present the estimation procedures of stress strength parameter based on record values for some distributions namely; Pareto Type I, Pareto Type II, Pareto Type IV, inverse Chen, exponentiated inverse Chen and inverse Weibull . The thesis is structured as follows.

Chapter 2 describes the estimation of $R=P(X>Y)$ based on upper record values when $X$ and $Y$ are independent Pareto Type I random variables with same scale parameter and different shape parameters. Maximum likelihood and Bayesian methods are used for the estimation of $R$ when the scale parameter is known. The corresponding confidence intervals are also found. A simulation study is conducted to investigate the estimation procedures. A real data analysis is done to illustrate the methods proposed. We also considered likelihood inference of $R$ for Pareto Type II and Type IV distributions. Numerical examples are presented for illustrating the proposed methods followed by some conclusions.

Chapter 3 deals with the estimation of $R = P(X > Y)$ when $X$ and $Y$ are from inverse Chen distribution with different first shape parameters and same second shape parameter based on lower record values. MLE, Bayes estimator and the confidence intervals are obtained when the second shape parameter $\beta$ is known. When $\beta$ is unknown MLE, Bayes estimator using MCMC method and an HPD interval are found. An approximate confidence interval and bootstrap confidence intervals are also obtained. A Monte Carlo simulation is conducted to investigate the estimation procedures. A real data analysis is presented for illustrative purpose. Further we discuss estimation of $R$ for exponentiated inverse Chen distribution. Maximum likelihood estimator and Bayes estimator are obtained when the second shape parameter $\beta$ is known as well as unknown. A numerical example is also presented. Finally some conclusions are given.

Chapter 4 considers inverse Weibull distribution with same shape parameter and different scale parameters for estimation of $R = P(X < Y)$ based on lower record values. When the shape parameter is known MLE, Bayes estimator and the corresponding confidence intervals are derived. For unknown shape parameter MLE, Bayes estimator using MCMC method and HPD interval are obtained. To illustrate the methods presented a numerical example is given followed by some conclusions.

In Chapter 5 summary of the thesis and final conclusions with some directions for future work are given.

# Chapter 2

# Estimation of stress-strength reliability for Pareto distribution based on upper record values

## 2.1 Introduction

The problem of estimating stress-strength reliability has many applications in a variety of fields. If X and Y are two random variables representing the life lengths of a product with same guarantee period produced by two companies $P(X > Y)$ represents the probability that one is better than the other. This probability can be well described when $X$ and $Y$ follow Pareto Type I distribution. Pareto distributions provide a family of fat-tailed distributions which can be used to describe income distributions. It is also quite popular in modelling a wide variety of other social and economic distributions.

Jeevanand [44] studied the problem of estimating $R = P(X_2 < X_1)$ when $X_1$ and $X_2$ has the bivariate Pareto distribution which arises in the context of a two-component system that is subject to shocks arriving from independent sources according to Poisson process with different intensities. Rezaei et al. [32] studied the estimation of $R = P(X < Y)$ when X and Y are two independent generalized Pareto random variables with common scale parameter and different shape parameters.

The organization of this chapter is as follows. In Section 2 we derive the maximum likelihood estimator and the corresponding confidence interval for the stress-strength parameter, $R$ for Pareto Type I distribution. Section 3 describes the Bayesian inference in which Lindley's method is used and the confidence interval is obtained. A simulation study is conducted to investigate and compare the performance of point estimators and interval estimators in Section 4. A real data analysis for illustrating the estimation methods is presented in Section 5. In Section 6 likelihood inference of $R$ for Pareto Type II distribution followed by a numerical example is described. Section 7 deals with estimation of $R$ for Pareto Type IV distribution by maximum likelihood method along with a numerical example. Finally the conclusions drawn are given in Section 8.

## 2.2 Likelihood inference

The probability density function (pdf) and the cumulative distribution function (cdf) of the Pareto Type I distribution with scale parameter $\alpha$ and shape parameter $\beta$ (both positive) are given by

$$\begin{aligned} f(x) &= \frac{\beta\alpha^{\beta}}{x^{\beta+1}}, \qquad x \geq \alpha, \alpha > 0, \beta > 0, \\ F(x) &= 1 - (\frac{\alpha}{x})^{\beta}, \qquad x \geq \alpha, \alpha > 0, \beta > 0 \end{aligned} \tag{2.1}$$

Let X and Y be two independent random variables from Pareto Type I distribution with parameters

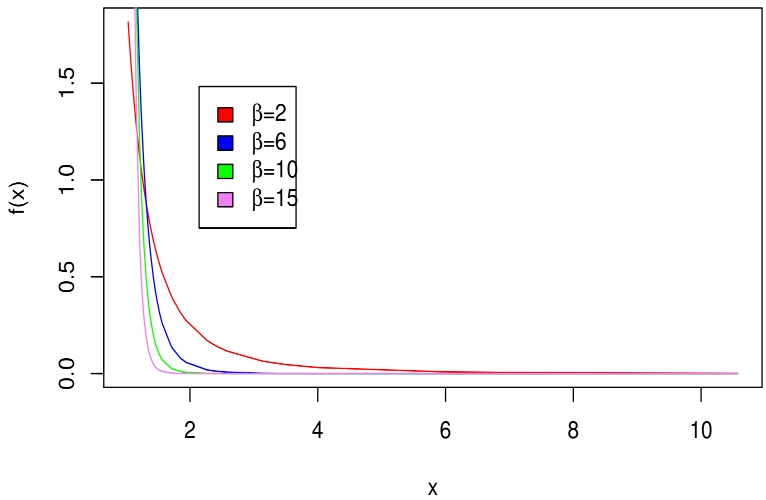


Fig. 2.1 pdf of Pareto Type I with $\alpha = 1$

$\alpha, \beta_1$ and $\alpha, \beta_2$ respectively. Then using (2.1)

$$R = P(X > Y) = \int_{\alpha}^{\infty} \int_{y}^{\infty} \frac{\beta_1 \alpha^{\beta_1}}{x^{\beta_1+1}} \frac{\beta_2 \alpha^{\beta_2}}{y^{\beta_2+1}} dxdy = \frac{\beta_2}{\beta_1 + \beta_2}.$$

We are interested in estimating R based on upper record values on both variables. Let $\underset{\sim}{r} = (r_1, r_2, ......, r_n)$ be a set of upper records from distribution of X with pdf 'f' and cdf 'F' and let $\underset{\sim}{s} = (s_1, s_2, ......, s_m)$ be an independent set of upper records from distribution of Y with pdf 'g' and cdf 'G'. The likelihood functions are given by

$$L(\alpha, \beta_1 | \underset{\sim}{r}) = f(r_n) \prod_{i=1}^{n-1} \left( \frac{f(r_i)}{1 - F(r_i)} \right), \quad 0 < r_1 < r_2 < ..... < r_n < \infty \tag{2.2}$$

$$L(\alpha, \beta_2 | \underset{\sim}{s}) = g(s_m) \prod_{i=1}^{m-1} \left( \frac{g(s_i)}{1 - G(s_i)} \right), \quad 0 < s_1 < s_2 < ..... < s_m < \infty \tag{2.3}$$

Substituting f,F,g and G the joint likelihood and the joint log likelihood are respectively given by

$$L(\alpha,\beta_1,\beta_2|\underset{\sim}{r},\underset{\sim}{s}) = \frac{\beta_1^n\alpha^{\beta_1}}{r_n^{\beta_1+1}}\prod_{i=1}^{n-1} r_i^{-1}\frac{\beta_2^m\alpha^{\beta_2}}{s_m^{\beta_2+1}}\prod_{i=1}^{m-1} s_i^{-1} \tag{2.4}$$

$$\begin{aligned} l(\alpha,\beta_1,\beta_2|\underset{\sim}{r},\underset{\sim}{s}) &= n\log\beta_1+\beta_1\log\alpha-\sum_{i=1}^{n-1}\log r_i-(\beta_1+1)\log r_n \\ &+ m\log\beta_2+\beta_2\log\alpha-\sum_{i=1}^{m-1}\log s_i-(\beta_2+1)\log s_m \end{aligned} \tag{2.5}$$

$$\frac{\delta l}{\delta\beta_1} = 0 \quad\Rightarrow\quad \frac{n}{\beta_1}=\log\left(\frac{r_n}{\alpha}\right) \tag{2.6}$$

$$\frac{\delta l}{\delta\beta_2} = 0 \quad\Rightarrow\quad \frac{m}{\beta_2}=\log\left(\frac{s_m}{\alpha}\right) \tag{2.7}$$

when $\alpha$ (the guarantee period) is known,

$$\hat{\beta}_1=\frac{n}{\log(r_n/\alpha)} \quad , \quad \hat{\beta}_2=\frac{m}{\log(s_m/\alpha)}. \tag{2.8}$$

Then the MLE of R is given by $\hat{R}=\frac{\hat{\beta}_2}{\hat{\beta}_1+\hat{\beta}_2}$.

**Remark**: While estimating shape parameters using record values it is assumed that the scale parameters are common and known. The procedure is same if we assume that the scale parameters $\alpha_1$ and $\alpha_2$ are assumed to be different but known. On the other hand if $\alpha_1$ and $\alpha_2$ are assumed to be unknown we have to estimate them from the upper record data. But due to the nature of the scale parameter $(x \geq \alpha)$ in the model $\alpha$ cannot be estimated consistently from upper record values alone. Hence the assumption that the scale parameters are known cannot be relaxed. In the data set analysed we are using a data with same scale parameter. Intuitively this is an appropriate situation when we are comparing the performance of two products with same warranty period.

We shall study the distribution of $\hat{\beta}_1$ and $\hat{\beta}_2$. Consider first $\hat{\beta}_1 = z_1 = \frac{n}{\log(r_n/\alpha)}$.

We know that the p.d.f of $R_n$ is given by

$$
\begin{aligned}
f_{R_n}(r_n) &= \frac{1}{(n-1)!} f(r_n)(-\log(1-F(r_n)))^{n-1} \\
&= \frac{1}{(n-1)!} \frac{\beta_1^n \alpha^{\beta_1}}{r_n^{\beta_1+1}} (\log(r_n/\alpha))^{n-1}, \qquad r_n > \alpha.
\end{aligned} \tag{2.9}
$$

Therefore the p.d.f of $z_1 = \hat{\beta}_1$ is given by

$$
f_{z_1}(z_1) = \frac{(n\beta_1)^n}{(n-1)!} \frac{\exp^{\frac{-n\beta_1}{z_1}}}{z_1^{n+1}} \qquad , \quad z_1 > 0. \tag{2.10}
$$

Here $z_1 \sim InvGamma(n, n\beta_1)$.

Similarly $z_2 \sim InvGamma(m, m\beta_2)$.

Therefore we can find the p.d.f of

$$
\begin{aligned}
\hat{R} &= \frac{\hat{\beta}_2}{\hat{\beta}_1 + \hat{\beta}_2} \\
&= \frac{z_2}{z_1 + z_2} = \frac{1}{1 + z_1/z_2}.
\end{aligned} \tag{2.11}
$$

Consider $\frac{z_1}{z_2}$. By the properties of the inverted Gamma distribution and its relation with the gamma distribution, we have $\frac{n\beta_1}{z_1} \sim Gamma(n,1)$ and $\frac{m\beta_2}{z_2} \sim Gamma(m,1)$.

Hence $\frac{2n\beta_1}{z_1} \sim \chi^2_{2n}$ and $\frac{2m\beta_2}{z_2} \sim \chi^2_{2m}$. Since the two random variables are independent we have

$$
\frac{\left(\frac{2m\beta_2}{2mz_2}\right)}{\left(\frac{2n\beta_1}{2nz_1}\right)} = \frac{\beta_2 z_1}{\beta_1 z_2} = \frac{R}{1-R} \frac{\hat{\beta}_1}{\hat{\beta}_2} \sim F(2m, 2n).
$$

This can be used to construct the following $(1-\alpha)\%$ confidence interval for R.

$$\left(\left(1+\frac{\hat{\beta}_1}{\hat{\beta}_2 F_{\frac{\alpha}{2},2m,2n}}\right)^{-1}\ ,\ \left(1+\frac{\hat{\beta}_1}{\hat{\beta}_2 F_{1-\frac{\alpha}{2},2m,2n}}\right)^{-1}\right). \tag{2.12}$$

## 2.3 Bayesian inference

It is also interesting how supplementary information other than upper record values available can be incorporated. A convenient vehicle is the method of Bayesian inference. We have considered both informative and non-informative priors . The sampling distribution of $\beta_1$ and $\beta_2$ and hence $R$ will instigate the use of an appropriate conjugate prior. We assume the conjugate family of prior distribution to be Gamma family of distributions.

So,

$$\begin{aligned}\pi(\beta_1) &= \frac{1}{\Gamma(\gamma_1)}\theta_1^{\gamma_1}\beta_1^{\gamma_1-1}\exp^{-\theta_1\beta_1}, \quad \beta_1>0;\ \ \theta_1,\gamma_1>0,\\ \pi(\beta_2) &= \frac{1}{\Gamma(\gamma_2)}\theta_2^{\gamma_2}\beta_2^{\gamma_2-1}\exp^{-\theta_2\beta_2}, \quad \beta_2>0;\ \ \theta_2,\gamma_2>0.\end{aligned} \tag{2.13}$$

Using the priors and the likelihood function(2.4), the posterior distributions of $\beta_1$ and $\beta_2$ are obtained as

$$\begin{aligned}\beta_1|\underset{\sim}{r} &\sim Gamma\left(n+\gamma_1\ ,\ \theta_1+\log\left(\frac{r_n}{\alpha}\right)\right)\\ \beta_2|\underset{\sim}{s} &\sim Gamma\left(m+\gamma_2\ ,\ \theta_2+\log\left(\frac{s_m}{\alpha}\right)\right)\end{aligned} \tag{2.14}$$

Since $\beta_1$ and $\beta_2$ are independent, using standard transformation techniques and after some manipulations the posterior p.d.f of R is given by

$$f_R(r)=C\frac{(1-r)^{n+\gamma_1-1}r^{m+\gamma_2-1}}{\left[(1-r)\left(\theta_1+\log\frac{r_n}{\alpha}\right)+r\left(\theta_2+\log\frac{s_m}{\alpha}\right)\right]^{n+m+\gamma_1+\gamma_2}}\ ,\ \ 0<r<1$$

where $C = \frac{\Gamma(n+m+\gamma_1+\gamma_2)}{\Gamma(n+\gamma_1)\Gamma(m+\gamma_2)} \left(\theta_1 + \log\left(\frac{r_n}{\alpha}\right)\right)^{n+\gamma_1} \left(\theta_2 + \log\left(\frac{s_m}{\alpha}\right)\right)^{m+\gamma_2}$.

Under squared error loss function, the Bayes estimator of $R$ is the expected value of $R$. This expected value contains an integral which is not obtainable in a simple closed form. Therefore using the approximate method of Lindley [19], we can find the approximate Bayes estimator $\bar{R_B}$ relative to squared error loss function. By approximate method of Lindley the Bayes estimator for $u(\theta)$ for a prior $v(\theta)$ is given by

$$\begin{aligned}
\frac{\int_\theta u(\theta)v(\theta)\exp^{L(\theta)}\,d\theta}{\int_\theta v(\theta)\exp^{L(\theta)}\,d\theta} &= [u^* + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + \rho_1 u_1 \sigma_{11} + \rho_2 u_2 \sigma_{22} \\
&+ \frac{1}{2}[\sigma_{11}\sigma_{22}(u_1 L_{12} + u_2 L_{21}) + u_1 \sigma_{11}^2 L_{30} \\
&+ u_2 \sigma_{22}^2 L_{03}] + O(\frac{1}{n^2})]_{at\,\hat{\theta}} \qquad (2.15)
\end{aligned}$$

where

$$u(\theta) = \frac{\beta_2}{\beta_1+\beta_2};\ \ \rho = \log(\pi(\beta_1)\pi(\beta_2)) = logC + (\gamma_1 - 1)log\beta_1 - \theta_1\beta_1 + (\gamma_2 - 1)log\beta_2 - \theta_2\beta_2,$$

$u^*$ is the MLE of $u(\theta)$ and $L(\theta)$ is the logarithm of likelihood function. C is independent of $\beta_1$ and $\beta_2$.

Further

$$u_1 = \frac{\delta u}{\delta\beta_1} = \frac{-\beta_2}{(\beta_1+\beta_2)^2};\ u_2 = \frac{\delta u}{\delta\beta_2} = \frac{\beta_1}{(\beta_1+\beta_2)^2};\ u_{11} = \frac{\delta^2 u}{\delta\beta_1^2} = \frac{2\beta_2}{(\beta_1+\beta_2)^3};\ u_{22} = \frac{\delta^2 u}{\delta\beta_2^2} = \frac{-2\beta_1}{(\beta_.1+\beta_2)^3}.$$

$$\rho_1 = \frac{\gamma_1 - 1}{\beta_1} - \theta_1;\ \ \rho_2 = \frac{\gamma_2 - 1}{\beta_2} - \theta_2.$$

$$\sigma = [-L_{ij}]^{-1}\ \ where\ \ L_{ij} = \left[\frac{\delta^2 L}{\delta\beta_i\delta\beta_j}\right].$$

$\sigma = \begin{bmatrix} \frac{\beta_1^2}{n} & 0 \\ 0 & \frac{\beta_2^2}{m} \end{bmatrix}$; $L_{30} = \frac{\delta^3 L}{\delta \beta_1^3} = \frac{2n}{\beta_1^3}$; $L_{03} = \frac{\delta^3 L}{\delta \beta_2^3} = \frac{2m}{\beta_2^3}$.

Substituting in eqn (2.15) we get the Bayes estimator as

$$\bar{R_B} = \hat{R} + \hat{R}(1-\hat{R}) \left[ \frac{\left(1-\hat{R}-\gamma_1+\hat{\beta}_1\theta_1\right)}{n} + \frac{\left(\gamma_2-\hat{R}-\hat{\beta}_2\theta_2\right)}{m} \right]. \tag{2.16}$$

Further more, it follows from (2.14) that $2(\theta_1 + \log\frac{r_n}{\alpha})\left(\beta_1|\underset{\sim}{r}\right) \sim \chi^2_{2(n+\gamma_1)}$ and $2(\theta_2 + \log\frac{s_m}{\alpha})\left(\beta_2|\underset{\sim}{s}\right) \sim \chi^2_{2(m+\gamma_2)}$. It follows that $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, the posterior distribution of $R$ is equal to that of $(1+AW)^{-1}$ where $W \sim F_{2(n+\gamma_1),2(m+\gamma_2)}$ and $A = \frac{(n+\gamma_1)(\theta_2+\log\frac{s_m}{\alpha})}{(m+\gamma_2)(\theta_1+\log\frac{r_n}{\alpha})}$. Therefore a Bayesian $(1-\alpha)\%$ confidence interval for $R$ is given by

$$\left( \left(AF_{1-\frac{\alpha}{2},2(n+\gamma_1),2(m+\gamma_2)}+1\right)^{-1}, \left(AF_{\frac{\alpha}{2},2(n+\gamma_1),2(m+\gamma_2)}+1\right)^{-1} \right). \tag{2.17}$$

When we have no information about the parameter we use a non informative prior. Jeffreys Invariant prior is a non-informative prior. We assume a Jeffreys Invariant prior (see Subsection 1.3.2). So,

$$\begin{aligned} \pi(\beta_1) &\propto \frac{1}{\beta_1}; \quad \beta_1 > 0, \\ \pi(\beta_2) &\propto \frac{1}{\beta_2}; \quad \beta_2 > 0. \end{aligned} \tag{2.18}$$

Using the priors and the likelihood functions, the posterior distributions of $\beta_1$ and $\beta_2$ are obtained as

$$\begin{aligned} \beta_1|\underset{\sim}{r} &\sim Gamma\left(n\ ,\ \log\left(\frac{r_n}{\alpha}\right)\right) \\ \beta_2|\underset{\sim}{s} &\sim Gamma\left(m\ ,\ \log\left(\frac{s_m}{\alpha}\right)\right). \end{aligned} \tag{2.19}$$

Since $\beta_1$ and $\beta_2$ are independent, then using standard transformation techniques and after some manipulations the posterior p.d.f of R is given by

$$f_R(r) = C\frac{(1-r)^{n-1}r^{m-1}}{\left[(1-r)\log\left(\frac{r_n}{\alpha}\right)+r\log\left(\frac{s_m}{\alpha}\right)\right]^{n+m}}, \quad 0<r<1$$

where $C = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\left(\log\left(\frac{r_n}{\alpha}\right)\right)^n\left(\log\left(\frac{s_m}{\alpha}\right)\right)^m$.

Under squared error loss function, the Bayes estimator of $R$ is the expected value of $R$. This expected value contains an integral which is not obtainable in a simple closed form. Therefore using the approximate method of Lindley [19], we can find the approximate Bayes estimator $\bar{R_B}$ relative to squared error loss function. By the approximate method of Lindley the Bayes estimator for $u(\theta)$ for a prior $v(\theta)$ and a likelihood function is given by

$$\begin{aligned}\frac{\int_\theta u(\theta)v(\theta)\exp^{L(\theta)}\,d\theta}{\int_\theta v(\theta)\exp^{L(\theta)}\,d\theta} &= [u^* + \frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22})+\rho_1u_1\sigma_{11}+\rho_2u_2\sigma_{22}\\ &+ \frac{1}{2}[\sigma_{11}\sigma_{22}(u_1L_{12}+u_2L_{21})+u_1\sigma_{11}^2L_{30}\\ &+ u_2\sigma_{22}^2L_{03}]+O(\frac{1}{n^2})]_{at\,\hat{\theta}}\end{aligned} \tag{2.20}$$

where

$$u(\theta)=\frac{\beta_2}{\beta_1+\beta_2};\ v(\theta)=\frac{1}{\beta_1\beta_2};\ \rho = logv(\theta) = log\left(\frac{1}{\beta_1\beta_2}\right);$$

$u^*$ is the MLE of $u(\theta)$and $L(\theta)$ is the logarithm of likelihood function.

Further

$$u_1=\frac{\delta u}{\delta\beta_1}=\frac{-\beta_2}{(\beta_1+\beta_2)^2};\ u_2=\frac{\delta u}{\delta\beta_2}=\frac{\beta_1}{(\beta_1+\beta_2)^2};\ u_{11}=\frac{\delta^2u}{\delta\beta_1^2}=\frac{2\beta_2}{(\beta_1+\beta_2)^3};\ u_{22}=\frac{\delta^2u}{\delta\beta_2^2}=\frac{-2\beta_1}{(\beta_1+\beta_2)^3}.$$

$$\rho_1 = \frac{\delta\rho}{\delta\beta_1} = \frac{-1}{\beta_1};\ \ \rho_2 = \frac{\delta\rho}{\delta\beta_2} = \frac{-1}{\beta_2}.$$

$$\sigma = [-L_{ij}]^{-1} \ \ where \ \ L_{ij} = \left[\frac{\delta^2 L}{\delta\beta_i\delta\beta_j}\right].$$

$\sigma = \begin{bmatrix} \frac{\beta_1^2}{n} & 0 \\ 0 & \frac{\beta_2^2}{m} \end{bmatrix};\ L_{30} = \frac{\delta^3 L}{\delta\beta_1^3} = \frac{2n}{\beta_1^3};\ \ L_{03} = \frac{\delta^3 L}{\delta\beta_2^3} = \frac{2m}{\beta_2^3}.$
Substituting in eqn(2.20) we get the Bayes estimator as

$$\bar{R_B} = \hat{R} + \hat{R}(1-\hat{R})\left[\frac{(1-\hat{R})}{n} - \frac{\hat{R}}{m}\right].$$

Further more, it follows from (2.19) that

$$2\log(\frac{r_n}{\alpha})\left(\beta_1|\underset{\sim}{r}\right) \sim \chi^2_{2n}$$

and

$$2\log(\frac{s_m}{\alpha})\left(\beta_2|\underset{\sim}{s}\right) \sim \chi^2_{2m}.$$

It follows that $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, the posterior distribution of $R$ is equal to that of $(1+AW)^{-1}$ where

$$W \sim F_{2n,2m}$$

and $A = \frac{n\log(\frac{s_m}{\alpha})}{m\log(\frac{r_n}{\alpha})}$. Therefore a Bayesian $(1-\alpha)\%$ credible interval for $R$ is given by

$$\left(\left(AF_{1-\frac{\alpha}{2},2n,2m}+1\right)^{-1},\left(AF_{\frac{\alpha}{2},2n,2m}+1\right)^{-1}\right). \tag{2.21}$$

which happens to be the same as the confidence interval based on MLE.

## 2.4 A simulation study

In this section a Monte Carlo simulation study is conducted to investigate and compare the performance of point estimators and confidence intervals presented in this chapter. The performance of MLE's and Bayes estimators is compared in terms of their biases and mean square errors (MSE's). We consider only one case, when the scale parameter $\alpha$ is known. We use the parameter values $(\beta_1, \beta_2) = (4,1), (2,2), (1,3), (1,9)$. Therefore $R_{exact} = 0.2, 0.5, 0.75, 0.9$. To compute the Bayes estimators two methods, one with conjugate prior and the other with Jeffreys invariant prior are considered. The results based on 2000 replications are given here.

When the scale parameter is known we obtain the average estimates, biases and MSE's of the MLE and the approximate Bayes estimator of R. We also compute the expected length for the confidence intervals obtained by using the ML method and Bayes method with conjugate prior as well as Jeffreys prior. The results are tabulated in Table 3.1 and Table 3.2. From the simulation results, it is observed that as the sample size (n,m) increases the biases and the MSE's decrease. Thus the consistency properties of all the methods are verified. It is observed that the bias of the estimators become negative for values of R larger than 0.5.

It is also observed that the intervals based on all methods are maximized when R = 0.5 and they become shorter and shorter as we move away to smaller and larger values. Increasing the sample size on either variable also results in shorter intervals. Except for $R = 0.5$ expected length for non-informative prior is shorter than that for conjugate prior.

Table 2.1 Average estimates(AVR),biases and MSE's of the estimators of R

| Methods | R | MLE | | | Bayes non-inform | | | Bayes Conjugate | | |
|---|---|---|---|---|---|---|---|---|---|---|
| (n,m) | | AVR | Bias | MSE | AVR | Bias | MSE | AVR | Bias | MSE |
| (4,4) | 0.2 | 0.2254 | 0.0254 | 0.0164 | 0.2440 | 0.0440 | 0.0166 | 0.3028 | 0.1028 | 0.0211 |
| (4,7) | | 0.2124 | 0.0124 | 0.0111 | 0.2363 | 0.0363 | 0.0125 | 0.2999 | 0.0999 | 0.0189 |
| (4,10) | | 0.2075 | 0.0075 | 0.0096 | 0.2333 | 0.0333 | 0.0112 | 0.2972 | 0.0972 | 0.0172 |
| (7,4) | | 0.2298 | 0.0298 | 0.0138 | 0.2363 | 0.0363 | 0.0129 | 0.2636 | 0.0636 | 0.0125 |
| (7,7) | | 0.2161 | 0.0161 | 0.0089 | 0.2279 | 0.0279 | 0.0092 | 0.2616 | 0.0616 | 0.0110 |
| (7,10) | | 0.2101 | 0.0101 | 0.0068 | 0.2239 | 0.0239 | 0.0073 | 0.2590 | 0.0590 | 0.0094 |
| (10,4) | | 0.2323 | 0.0323 | 0.0130 | 0.2337 | 0.0337 | 0.0118 | 0.2492 | 0.0492 | 0.0097 |
| (10,7) | | 0.2179 | 0.0179 | 0.0077 | 0.2247 | 0.0247 | 0.0077 | 0.2472 | 0.0472 | 0.0083 |
| (10,10) | | 0.2127 | 0.0127 | 0.0059 | 0.2214 | 0.0214 | 0.0061 | 0.2453 | 0.0453 | 0.0072 |
| (4,4) | 0.5 | 0.5019 | 0.0019 | 0.0275 | 0.5019 | 0.0019 | 0.0228 | 0.5019 | 0.0019 | 0.0112 |
| (4,7) | | 0.4908 | -0.0090 | 0.0227 | 0.5036 | 0.0036 | 0.0193 | 0.5258 | 0.0258 | 0.0116 |
| (4,10) | | 0.4863 | -0.0140 | 0.0207 | 0.5043 | 0.0043 | 0.0179 | 0.5339 | 0.0339 | 0.0114 |
| (7,4) | | 0.5145 | 0.0145 | 0.0231 | 0.5014 | 0.0014 | 0.0196 | 0.4777 | -0.0220 | 0.0115 |
| (7,7) | | 0.5027 | 0.0027 | 0.0172 | 0.5026 | 0.0026 | 0.0152 | 0.5021 | 0.0021 | 0.0106 |
| (7,10) | | 0.4978 | -0.0020 | 0.0142 | 0.5029 | 0.0029 | 0.0128 | 0.5103 | 0.0103 | 0.0096 |
| (10,4) | | 0.5209 | 0.0209 | 0.0206 | 0.5024 | 0.0024 | 0.0176 | 0.4716 | -0.0280 | 0.0109 |
| (10,7) | | 0.5089 | 0.0089 | 0.0142 | 0.5034 | 0.0034 | 0.0128 | 0.4954 | -0.0050 | 0.0095 |
| (10,10) | | 0.5046 | 0.0046 | 0.0119 | 0.5044 | 0.0044 | 0.0109 | 0.5040 | 0.0040 | 0.0086 |
| (4,4) | 0.75 | 0.7296 | -0.0200 | 0.0186 | 0.7123 | -0.0380 | 0.0179 | 0.6675 | -0.0830 | 0.0183 |
| (4,7) | | 0.7245 | -0.0260 | 0.0162 | 0.7203 | -0.0300 | 0.0146 | 0.7008 | -0.0490 | 0.0121 |
| (4,10) | | 0.7225 | -0.0280 | 0.0151 | 0.7237 | -0.0260 | 0.0132 | 0.7138 | -0.0360 | 0.0101 |
| (7,4) | | 0.7434 | -0.0070 | 0.0139 | 0.7193 | -0.0300 | 0.0146 | 0.6685 | -0.0820 | 0.0167 |
| (7,7) | | 0.7387 | -0.0110 | 0.0109 | 0.7275 | -0.0230 | 0.0107 | 0.7014 | -0.0490 | 0.0108 |
| (7,10) | | 0.7371 | -0.0130 | 0.0092 | 0.7314 | -0.0190 | 0.0086 | 0.7143 | -0.0360 | 0.0085 |
| (10,4) | | 0.7506 | 0.0006 | 0.0117 | 0.7238 | -0.0260 | 0.0125 | 0.6720 | -0.0780 | 0.0148 |
| (10,7) | | 0.7459 | -0.0040 | 0.0084 | 0.7321 | -0.0180 | 0.0085 | 0.7044 | -0.0460 | 0.0091 |
| (10,10) | | 0.7443 | -0.0060 | 0.0072 | 0.7360 | -0.0140 | 0.0071 | 0.7174 | -0.0330 | 0.0072 |
| (4,4) | 0.9 | 0.8817 | -0.0180 | 0.0063 | 0.8648 | -0.0350 | 0.0078 | 0.7662 | -0.1340 | 0.0269 |
| (4,7) | | 0.8804 | -0.0200 | 0.0054 | 0.8720 | -0.0280 | 0.0058 | 0.8198 | -0.0800 | 0.0117 |
| (4,10) | | 0.8799 | -0.0200 | 0.0051 | 0.8751 | -0.0240 | 0.0051 | 0.8413 | -0.0590 | 0.0075 |
| (7,4) | | 0.8909 | -0.0090 | 0.0041 | 0.8727 | -0.0270 | 0.0055 | 0.7752 | -0.1250 | 0.0217 |
| (7,7) | | 0.8899 | -0.0100 | 0.0031 | 0.8799 | -0.0200 | 0.0037 | 0.8258 | -0.0740 | 0.0094 |
| (7,10) | | 0.8898 | -0.0100 | 0.0027 | 0.8833 | -0.0170 | 0.0029 | 0.8460 | -0.0540 | 0.0059 |
| (10,4) | | 0.8954 | -0.0050 | 0.0032 | 0.8769 | -0.0230 | 0.0045 | 0.7809 | -0.1190 | 0.0188 |
| (10,7) | | 0.8946 | -0.0050 | 0.0023 | 0.8842 | -0.0160 | 0.0028 | 0.8300 | -0.0700 | 0.0079 |
| (10,10) | | 0.8943 | -0.0060 | 0.0019 | 0.8873 | -0.0130 | 0.0022 | 0.8498 | -0.0500 | 0.0049 |

Table 2.2 Expected length

| (n,m) | R | **MLE** | | **Bayes non-inform** | | **Bayes Conjugate** | |
|---|---|---|---|---|---|---|---|
| | | 95%EL | 90%EL | 95%EL | 90%EL | 95%EL | 90%EL |
| (4,4) | | 0.4571 | 0.3837 | 0.4571 | 0.3837 | 0.4894 | 0.4143 |
| (4,7) | | 0.4181 | 0.3478 | 0.4181 | 0.3478 | 0.4513 | 0.3792 |
| (4,10) | | 0.4001 | 0.3316 | 0.4001 | 0.3316 | 0.4329 | 0.3625 |
| (7,4) | | 0.3976 | 0.3360 | 0.3976 | 0.3360 | 0.4119 | 0.3489 |
| (7,7) | 0.2 | 0.3464 | 0.2901 | 0.3464 | 0.2901 | 0.3652 | 0.3070 |
| (7,10) | | 0.3229 | 0.2693 | 0.3229 | 0.2693 | 0.3424 | 0.2868 |
| (10,4) | | 0.3732 | 0.3164 | 0.3732 | 0.3164 | 0.3789 | 0.3214 |
| (10,7) | | 0.3165 | 0.2659 | 0.3165 | 0.2659 | 0.3280 | 0.2761 |
| (10,10) | | 0.2900 | 0.2428 | 0.2900 | 0.2428 | 0.3026 | 0.2539 |
| | | | | | | | |
| (4,4) | | 0.5846 | 0.5031 | 0.5846 | 0.5031 | 0.5739 | 0.4941 |
| (4,7) | | 0.5348 | 0.4584 | 0.5348 | 0.4584 | 0.5232 | 0.4484 |
| (4,10) | | 0.5122 | 0.4383 | 0.5122 | 0.4383 | 0.4993 | 0.4274 |
| (7,4) | | 0.5346 | 0.4581 | 0.5345 | 0.4581 | 0.5233 | 0.4485 |
| (7,7) | 0.5 | 0.4704 | 0.4011 | 0.4704 | 0.4011 | 0.4624 | 0.3942 |
| (7,10) | | 0.4409 | 0.3753 | 0.4409 | 0.3753 | 0.4332 | 0.3687 |
| (10,4) | | 0.5131 | 0.4391 | 0.5131 | 0.4391 | 0.5003 | 0.4282 |
| (10,7) | | 0.4411 | 0.3756 | 0.4411 | 0.3756 | 0.4334 | 0.3688 |
| (10,10) | | 0.4058 | 0.3447 | 0.4058 | 0.3447 | 0.3996 | 0.3394 |
| | | | | | | | |
| (4,4) | | 0.4993 | 0.4228 | 0.4993 | 0.4228 | 0.5129 | 0.4364 |
| (4,7) | | 0.4388 | 0.3729 | 0.4388 | 0.3729 | 0.4424 | 0.3763 |
| (4,10) | | 0.4130 | 0.3517 | 0.4130 | 0.3517 | 0.4110 | 0.3499 |
| (7,4) | | 0.4569 | 0.3834 | 0.4568 | 0.3834 | 0.4731 | 0.3996 |
| (7,7) | 0.75 | 0.3849 | 0.3242 | 0.3849 | 0.3242 | 0.3933 | 0.3320 |
| (7,10) | | 0.3534 | 0.2983 | 0.3534 | 0.2983 | 0.3575 | 0.3020 |
| (10,4) | | 0.4377 | 0.3659 | 0.4377 | 0.3659 | 0.4534 | 0.3817 |
| (10,7) | | 0.3593 | 0.3014 | 0.3593 | 0.3014 | 0.3686 | 0.3101 |
| (10,10) | | 0.3235 | 0.2719 | 0.3235 | 0.2719 | 0.3292 | 0.2771 |
| | | | | | | | |
| (4,4) | | 0.3142 | 0.2560 | 0.3142 | 0.2560 | 0.4158 | 0.3466 |
| (4,7) | | 0.2593 | 0.2146 | 0.2593 | 0.2146 | 0.3170 | 0.2648 |
| (4,10) | | 0.2378 | 0.1982 | 0.2378 | 0.1982 | 0.2746 | 0.2299 |
| (7,4) | | 0.2796 | 0.2256 | 0.2796 | 0.2256 | 0.3811 | 0.3151 |
| (7,7) | 0.9 | 0.2193 | 0.1799 | 0.2192 | 0.1799 | 0.2778 | 0.2303 |
| (7,10) | | 0.1949 | 0.1613 | 0.1949 | 0.1613 | 0.2342 | 0.1948 |
| (10,4) | | 0.2634 | 0.2115 | 0.2634 | 0.2115 | 0.3632 | 0.2991 |
| (10,7) | | 0.2005 | 0.1639 | 0.2005 | 0.1639 | 0.2577 | 0.2129 |
| (10,10) | | 0.1749 | 0.1442 | 0.1749 | 0.1442 | 0.2133 | 0.1769 |

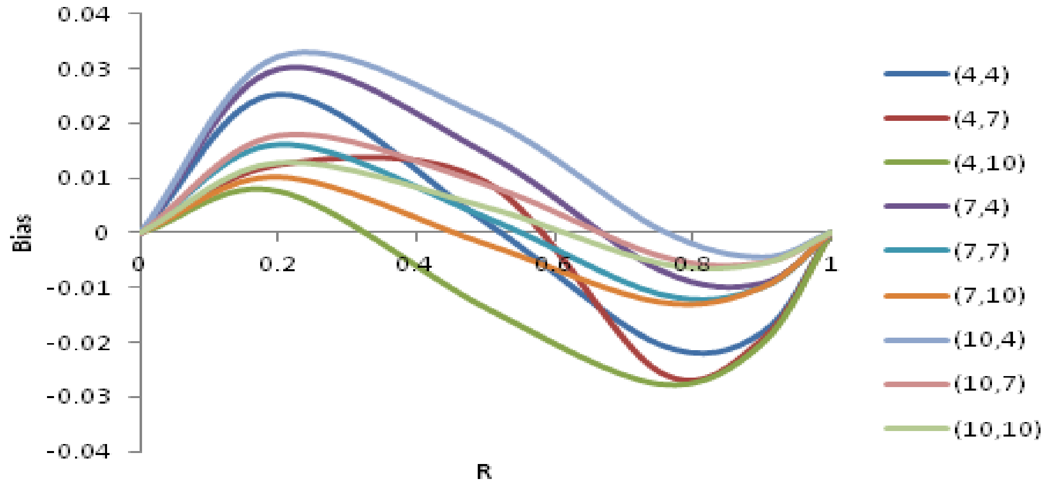


Fig. 2.2 Curve of bias of $\hat{R}$ against $R$

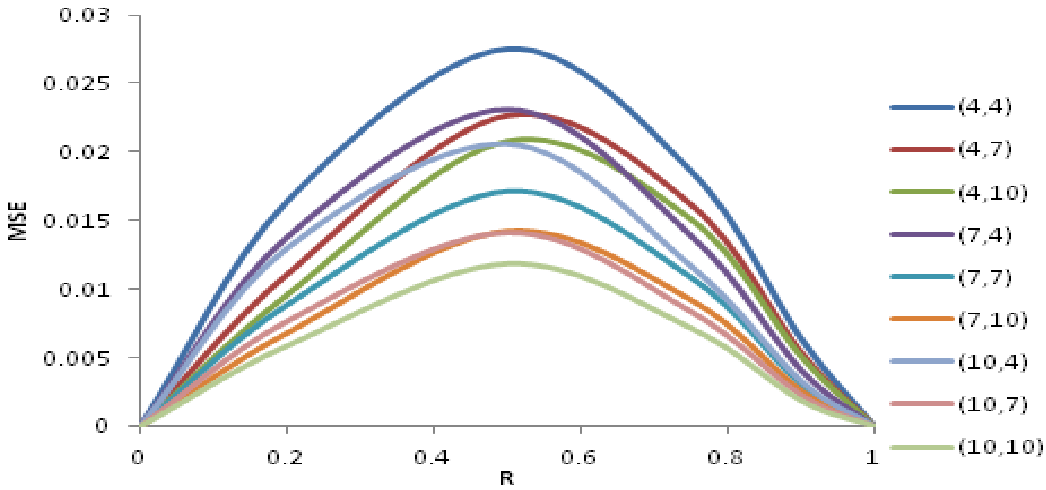


Fig. 2.3 Curve of MSE of $\hat{R}$ against $R$

## 2.5 Data analysis

In this section we analyze the real data to illustrate the use of our proposed estimation methods. The data from Crowder [45] give the lifetimes of the steel specimens tested at two different stress levels.

DataSet 1: (38.5 stress level): $60, 51, 83, 140, 109, 106, 119, 76, 68, 67$

DataSet 2: (38 stress level): $100, 90, 59, 80, 128, 117, 177, 98, 158, 107$

We fit the Pareto Type I distribution to the two data sets separately. The estimated parameters (based on

ML methods), Kolmogorov-Smirinov(K-S) distances between the fitted and the empirical distribution functions and corresponding p-values are presented in Table 2.3.

From the Table it is clear that Pareto distribution with common scale parameter fits quite well to both data. For the above data we observe the upper record values as follows

$$r : 60, 83, 140$$

$$s : 100, 128, 177$$

Based on these record values and assuming $\alpha = 51$, we obtain the MLE's of $\beta_1$ and $\beta_2$ from (2.8) as 3.9611 and 3.2146 respectively. Therefore the MLE of $R$ becomes $\hat{R} = 0.4479$. The corresponding 95% confidence interval based on (2.12) is equal to $(0.1224, 0.8253)$. In the Bayesian inference, for the first estimator we take the values of the hyper parameters as $\gamma_1 = \gamma_2 = \theta_1 = \theta_2 = 0.5$. Then we obtain the Bayes estimator $\bar{R}_1 = 0.4775$. Also the 95% credible interval from (2.17) is (0.1295, 0.8121). Using a non-informative prior, the Bayes estimator $\bar{R}_2 = 0.4544$. The corresponding 95%

Table 2.3 K S distances and p values

| Data set | Scale parameter | shape parameter | K-S distance | p-value |
|---|---|---|---|---|
| 1 | 51 | 2.0173 | 0.2233 | 0.6247 |
| 2 | 51 | 1.3592 | 0.3577 | 0.119 |

credible interval from (2.21) is calculated as (0.1224,0.8253).

## 2.6 Pareto Type II distribution

If the standard Pareto distribution is allowed to have an additional location parameter it is called Pareto Type II distribution. The p.d.f of a Pareto Type II distribution with location parameter $\mu$, scale parameter $\alpha$ and shape parameter $\beta$ is given by

$$f(x) = \frac{\beta}{\alpha}\left(1+\frac{x-\mu}{\alpha}\right)^{-(\beta+1)}, \quad \mu \in R, x \geqslant \mu, \alpha, \beta > 0 \tag{2.22}$$

and the c.d.f is given by

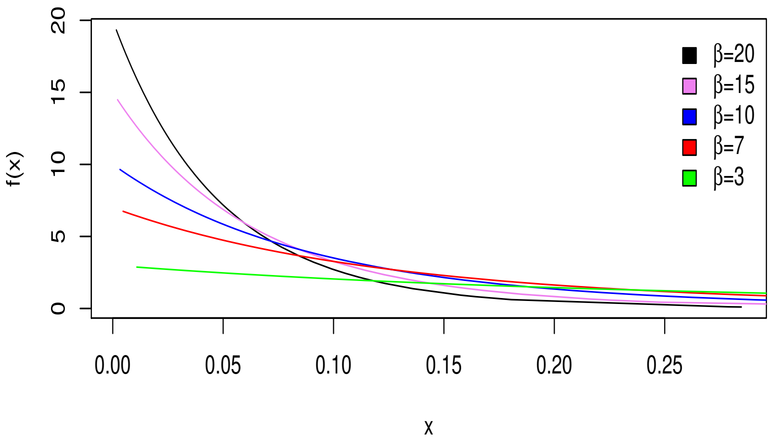


Fig. 2.4 pdf of Pareto Type II with $\mu = 0$ and $\alpha = 1$

$$F(x) = 1-\left(1+\frac{x-\mu}{\alpha}\right)^{-\beta}. \tag{2.23}$$

Let

$$X \sim ParetoTypeII(\mu, \alpha, \beta_1) \qquad \text{and}$$

$$Y \sim ParetoTypeII(\mu, \alpha, \beta_2) \tag{2.24}$$

$$R = P(X > Y) = \int_{\mu}^{\infty}\int_{\mu}^{\infty} \frac{\beta_1}{\alpha}\left(1+\frac{x-\mu}{\alpha}\right)^{-(\beta_1+1)} \frac{\beta_2}{\alpha}\left(1+\frac{y-\mu}{\alpha}\right)^{-(\beta_2+1)} dxdy$$
$$= \frac{\beta_2}{\beta_1+\beta_2} \tag{2.25}$$

We would like to estimate $R = P(X > Y)$ based on upper records from both the distributions of $X$ and $Y$.

### 2.6.1 Likelihood inference

Let $\underset{\sim}{r} = (r_1, r_2, ..., r_n)$ be a set of upper records from the distribution of $X$ and $\underset{\sim}{s} = (s_1, s_2, ..., s_n)$ be an independent set of upper records from the distribution of $Y$. The likelihood function based on upper records is given by

$$L(\mu,\alpha,\beta_1,\beta_2) = \frac{\beta_1^n}{\alpha^n}\left(1+\frac{r_n-\mu}{\alpha}\right)^{-\beta}\prod_{i=1}^{n}\left(1+\frac{r_i-\mu}{\alpha}\right)^{-1}$$
$$\times \frac{\beta_2^m}{\alpha^m}\left(1+\frac{s_m-\mu}{\alpha}\right)^{-\beta}\prod_{i=1}^{m}\left(1+\frac{s_i-\mu}{\alpha}\right)^{-1}. \tag{2.26}$$

The log likelihood is given by

$$l = n\log\beta_1 - n\log\alpha - \beta_1\log\left(1+\frac{r_n-\mu}{\alpha}\right) - \sum_{i=1}^{n}\log\left(1+\frac{r_i-\mu}{\alpha}\right)$$
$$+ m\log\beta_2 - m\log\alpha - \beta_2\log\left(1+\frac{s_m-\mu}{\alpha}\right) - \sum_{i=1}^{m}\log\left(1+\frac{s_i-\mu}{\alpha}\right). \tag{2.27}$$

$$\frac{\partial l}{\partial\beta_1} = 0 \quad \Rightarrow \quad \frac{n}{\beta_1} - \log\left(1+\frac{r_n-\mu}{\alpha}\right) = 0$$
$$\frac{\partial l}{\partial\beta_2} = 0 \quad \Rightarrow \quad \frac{m}{\beta_2} - \log\left(1+\frac{s_m-\mu}{\alpha}\right) = 0. \tag{2.28}$$

When $\alpha$ & $\mu$ are known, the MLE's of $\beta_1$ & $\beta_2$ are given by

$$\bar{\beta_1} = \frac{n}{\log\left(1+\frac{r_n-\mu}{\alpha}\right)} \qquad ; \qquad \bar{\beta_2} = \frac{m}{\log\left(1+\frac{s_m-\mu}{\alpha}\right)}. \tag{2.29}$$

Then the MLE of $R$ is obtained as

$$\hat{R} = \frac{\hat{\beta_2}}{\hat{\beta_1}+\hat{\beta_2}}. \tag{2.30}$$

When $\alpha$ is unknown, from $\frac{\partial l}{\partial \alpha} = 0$, we obtain the MLE of $\alpha$ as the solution of a non-linear equation of the form $\alpha = h(\alpha)$ where

$$\begin{aligned} h(\alpha) = \frac{1}{m+n}[ & \frac{n(r_n-\mu)}{\left(1+\frac{r_n-\mu}{\alpha}\right)\log\left(1+\frac{r_n-\mu}{\alpha}\right)} + \sum_{i=1}^{n} \frac{r_i-\mu}{\left(1+\frac{r_i-\mu}{\alpha}\right)} \\ & + \frac{m(s_m-\mu)}{\left(1+\frac{s_m-\mu}{\alpha}\right)\log\left(1+\frac{s_m-\mu}{\alpha}\right)} + \sum_{i=1}^{m} \frac{s_i-\mu}{\left(1+\frac{s_i-\mu}{\alpha}\right)}]. \end{aligned} \tag{2.31}$$

The Lomax distribution is a particular case of Pareto Type II distribution that has been shifted so that its support begins at zero [46]. The problem of estimating $R = P(X > Y)$ for this distribution is discussed by Mahmoud et al. [47].

### 2.6.2 Numerical example

In this section we present the analysis of a data generated from Pareto Type II distribution to illustrate the proposed methods. First we generate 20 observations from Pareto Type II $(\mu = 10, \alpha = 2, \beta = 3)$ and 20 observations from Pareto Type II $(\mu = 10, \alpha = 2, \beta = 4)$ and is reported in Table 2.4.

Table 2.4 Pareto Type II data

| x | y | x | y | x | y | x | y |
|---|---|---|---|---|---|---|---|
| 10.16 | 10.15 | 14.54 | 10.22 | 11.11 | 14.5 | 10.67 | 10.73 |
| 12.27 | 10.42 | 12.61 | 10.41 | 10.53 | 10.04 | 13.04 | 10.06 |
| 10.37 | 10.09 | 12.34 | 10.03 | 10.49 | 10.23 | 11.54 | 10.47 |
| 11.76 | 10.34 | 10.46 | 11.01 | 10.64 | 10.44 | 15.25 | 10.63 |
| 10.44 | 10.13 | 11 | 10.08 | 10.09 | 10.1 | 10.06 | 10.26 |

The upper records from these two sets of data are

$$r: 10.16,\ 12.27,\ 14.54,\ 15.25.$$

$$s: 10.15,\ 10.42,\ 11.01,\ 14.5.$$

It is assumed that $\mu$ and $\alpha$ are known and is equal to 10 & 2 respectively. Then the MLE's of $\beta$ from the two sets are 3.1059 and 3.3936 respectively. Hence the MLE of $R$ is 0.5221.

When $\alpha$ is assumed to be unknown the mle of $\alpha$ is 2.7797. Then the MLE's of $\beta$ from the two sets are 3.7707 and 4.1548 respectively. Hence the MLE of $R$ is 0.5242.

## 2.7 Pareto Type IV distribution

In Pareto Type IV we have an inequality parameter in addition. The pdf and c.d.f of a Paerto Type IV distribution with scale parameter $\alpha$, location parameter $\mu$, shape parameter $\beta$ and inequality parameter $\gamma$ is given respectively by

$$f(x) = \frac{\beta}{\gamma\alpha}\left(1+\left(\frac{x-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-(\beta+1)}\left(\frac{x-\mu}{\alpha}\right)^{\frac{1}{\gamma}-1}$$

$$F(x) = 1-\left(1+\left(\frac{x-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-\beta} \qquad x \geqslant \mu \qquad \alpha,\mu,\beta,\gamma > 0$$

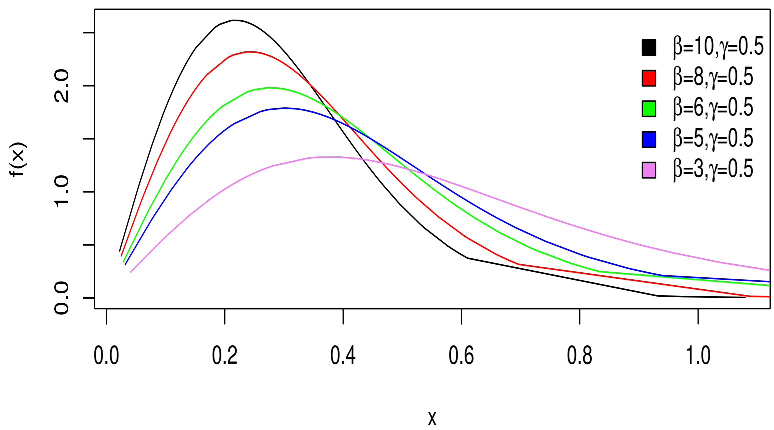


Fig. 2.5 pdf of Pareto Type IV with $\mu = 0 and\ \alpha = 1$

Let

$$X \sim ParetoTypeIV(\mu, \alpha, \gamma, \beta_1) \qquad \text{and}$$

$$Y \sim ParetoTypeIV(\mu, \alpha, \gamma, \beta_2) \tag{2.32}$$

Then

$$\begin{aligned} R &= P(X > Y) \\ &= \int_{\mu}^{\alpha} \left(1 + \left(\frac{y-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-\beta_1} \frac{\beta_2}{\gamma\alpha} \left(1 + \left(\frac{y-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-(\beta_2+1)} \left(\frac{y-\mu}{\alpha}\right)^{\frac{1}{\gamma}-1} dy \\ &= \frac{\beta_2}{\beta_1 + \beta_2} \end{aligned} \tag{2.33}$$

We would like to consider estimation of $R = P(X > Y)$ based on upper records.

### 2.7.1 Likelihood inference

Let $\underset{\sim}{r} = (r_1, r_2, ..., r_n)$ be a set of upper records from the distribution of $X$ and $\underset{\sim}{s} = (s_1, s_2, ..., s_n)$ be an independent set of upper records from the distribution of $Y$. The likelihood function based on lower records is given by

$$L(\mu,\alpha,\gamma,\beta_1,\beta_2) = \frac{\beta_1^n}{\gamma^n\alpha^n}\left(1+\left(\frac{r_n-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-\beta_1}\prod_{i=1}^{n}\left(1+\left(\frac{r_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-1}\left(\frac{r_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}-1}$$
$$\times\frac{\beta_2^m}{\gamma^m\alpha^m}\left(1+\left(\frac{s_m-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-\beta_2}\prod_{i=1}^{m}\left(1+\left(\frac{s_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)^{-1}\left(\frac{s_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}-1} \quad (2.34)$$

and the log likelihood is given by

$$l = n\log\beta_1 - n\log\gamma - n\log\alpha - \beta_1\log\left(1+\left(\frac{r_n-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right) - \sum_{i=1}^{n}\left[\log\left(1+\left(\frac{r_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right) - (\frac{1}{\gamma}-1)\log\left(\frac{r_i-\mu}{\alpha}\right)\right]$$
$$+m\log\beta_2 - m\log\gamma - m\log\alpha - \beta_2\log\left(1+\left(\frac{s_m-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right) - \sum_{i=1}^{m}\left[\log\left(1+\left(\frac{s_i-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right) - (\frac{1}{\gamma}-1)\log\left(\frac{s_i-\mu}{\alpha}\right)\right]$$
$$(2.35)$$

$$\frac{\partial l}{\partial\beta_1} = 0 \Rightarrow \text{the MLE of}\beta_1,\ \hat{\beta}_1 = \frac{n}{\log\left(1+\left(\frac{r_n-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)} \quad (2.36)$$

$$\frac{\partial l}{\partial\beta_2} = 0 \Rightarrow \text{the MLE of}\beta_2,\ \hat{\beta}_2 = \frac{m}{\log\left(1+\left(\frac{s_m-\mu}{\alpha}\right)^{\frac{1}{\gamma}}\right)} \quad (2.37)$$

Then the MLE of $R$ is obtained as

$$\hat{R} = \frac{\hat{\beta}_2}{\hat{\beta}_1+\hat{\beta}_2} \quad (2.38)$$

### 2.7.2 Numerical example

In this section we present the analysis of a data generated from Pareto Type IV distribution to illustrate the proposed methods. For this purpose 20 observations each from Pareto Type IV$(\mu = 10, \alpha = 4, \beta = 3, \gamma = 2)$ and Pareto Type IV$(\mu = 10, \alpha = 4, \beta = 2, \gamma = 2)$ are generated and tabulated (Table 2.5).

Table 2.5 Pareto Type IV data

| x | y | x | y | x | y | x | y |
|---|---|---|---|---|---|---|---|
| 10.16 | 10.3 | 10.2 | 10.07 | 11.01 | 13.59 | 10.01 | 10.01 |
| 10.03 | 10.01 | 30.58 | 18.24 | 11.23 | 13.5 | 10.45 | 10.12 |
| 15.15 | 10.85 | 16.83 | 10.18 | 10.29 | 10.34 | 19.24 | 10.06 |
| 10.13 | 10.07 | 15.47 | 12.91 | 10.24 | 10.02 | 12.38 | 10.04 |
| 13.1 | 10.01 | 10.21 | 11.47 | 10.42 | 15.29 | 37.53 | 10.44 |

The upper records from these two sets of data are

$$r : 10.16, 15.15, 30.58, 37.53.$$

$$s : 10.3, 10.85, 18.24.$$

It is assumed that $\mu,\ \alpha$ and $\gamma$ are known and are equal to 10,4 and 2 respectively. Then the MLE's of $\beta$ from the two sets are 3.1069 and 3.3706 respectively. Hence the MLE of $R$ is 0.5203.

## 2.8 Conclusion

This chapter considered the estimation of stress-strength reliability $R = P(X > Y)$ based on upper record values where $X$ and $Y$ are independent random variables from Pareto Type I distribution with same scale parameter but different inequality parameters. The results for estimation of R by maximum likelihood estimation and Bayesian approach are reported when the common scale parameter is known. Further the likelihood inference of $R$ for Pareto Type II & Type IV distributions are discussed. We didn't discuss the estimation of $R$ for Pareto Type III distribution due to the intractability of an explicit expression for $R$. Moreover, in the case of Pareto Type II and Type IV we think that the estimation of $R$ in Bayesian method can be studied using E M algorithm which we haven't attempted here.

From the simulation results, it is observed that as the sample size (n, m) increases the biases and the MSE's decrease. Thus the consistency properties of all the methods are verified. It is observed that the bias of the estimators become negative for values of R larger than 0.5. It is also observed that the interval based on MLE is maximized when R = 0.5 and it becomes shorter and shorter as we move away to smaller and larger values. Increasing the sample size on either variable also results in shorter intervals. The simulation results of Bayesian interval estimation are influenced by the values of the hyperparameters. All the simulations and real data analysis are done in *R* software.

# Chapter 3

# Estimation of stress-strength reliability for inverse Chen distribution based on lower record values

## 3.1 Introduction

An increasing demand for life time distributions to describe complex practical situations gave rise to a number of life time models. The new life time distribution with bathtub-shape or increasing failure rate function proposed by Chen [48] was one among them. The bathtub hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans.

The cumulative distribution function (cdf) of Chen distribution is given by

$$F(y) = 1 - e^{\lambda(1-e^{y^{\beta}})}, \;\; y > 0, \lambda, \beta > 0$$

and hence probability density function (pdf) is given by

$$f(y) = \lambda \beta y^{\beta-1} e^{\left[y^{\beta}+\lambda(1-e^{y^{\beta}})\right]}, \;\; y > 0, \lambda, \beta > 0.$$

If a random variable $Y$ has a Chen distribution, then the distribution of $X = \frac{1}{Y}$ is termed as an Inverse Chen distribution (ICD). Its cumulative distribution function (cdf) is given by

$$F(x) = e^{\lambda(1-e^{x^{-\beta}})}, \;\; x > 0, \lambda, \beta > 0 \tag{3.1}$$

and the probability density function (pdf) of Inverse Chen distribution (ICD) is

$$f(x) = \lambda \beta x^{-(\beta+1)} e^{\left[x^{-\beta}+\lambda(1-e^{x^{-\beta}})\right]}, \;\; x > 0, \lambda, \beta > 0. \tag{3.2}$$

In literature a few papers have been published for estimation of stress-strength reliability for Chen distribution based on random samples. Asgharzadeh et al. [36] considered interval estimation for the two parameter bathtub shaped lifetime distribution based on records. Estimation of stress-strength reliability for the two-parameter bathtub shaped life time distribution based on upper record values was presented by Bahman Tarvirdizade [42]. In this chapter we consider the estimation of $R = P(X > Y)$ based on lower records when X and Y are independent but not identically distributed inverse-Chen random variables.

The organization of this chapter is as follows. In Section 2 the likelihood inference of the stress-

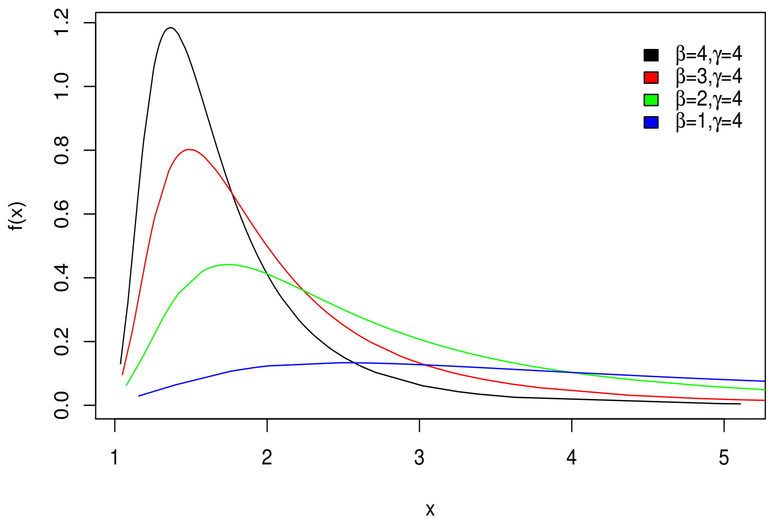


Fig. 3.1 pdf of inverse Chen distribution

strength parameter followed by asymptotic confidence interval and bootsrap confidence intervals are discussed. Section 3 describes the Bayesian inference. In Section 4 a simulation study is conducted to investigate and compare the performance of point estimators presented in this chapter. Section 5 presents a real data analysis for the illustration of the proposed estimation methods . Section 6 deals with the estimation of $R$ for exponentiated inverse Chen distribution followed by a numerical example. Finally the conclusions drawn are given in Section 7.

## 3.2 Likelihood inference

Let X and Y be two independent random variables from inverse Chen distribution with parameters $(\gamma,\beta)$ and $(\delta,\beta)$ respectively. Then using (3.1) and (3.2)

$$R = P(X > Y) = \int_0^\infty P(Y < X|X = x)f(x)dx = \frac{\gamma}{\delta+\gamma}.$$

We would like to estimate R based on lower record values on both variables. Let $\underset{\sim}{r} = (r_1, r_2, ......, r_n)$ be a set of lower records from distribution of X with pdf f and cdf F and let $\underset{\sim}{s} = (s_1, s_2, ......, s_m)$ be an independent set of lower records from distribution of Y with pdf g and cdf G. Then the likelihood functions are given by

$$L(\gamma,\beta|\underset{\sim}{r}) = f(r_n)\prod_{i=1}^{n-1}\left(\frac{f(r_i)}{F(r_i)}\right), r_1 > r_2 > ..... > r_n > 0. \quad (3.3)$$

$$L(\delta,\beta|\underset{\sim}{s}) = g(s_m)\prod_{i=1}^{m-1}\left(\frac{g(s_i)}{G(s_i)}\right), s_1 > s_2 > ..... > s_m > 0. \quad (3.4)$$

Substituting f, F, g and G the joint likelihood and the joint log likelihood are respectively given by

$$\begin{aligned} L(\beta,\gamma,\delta|\underset{\sim}{r},\underset{\sim}{s}) &= \gamma^n\beta^n e^{\gamma\left(1-e^{r_n^{-\beta}}\right)}\prod_{i=1}^{n} r_i^{-(\beta+1)}e^{r_i^{-\beta}} \\ &\quad \delta^m\beta^m e^{\delta\left(1-e^{s_m^{-\beta}}\right)}\prod_{i=1}^{m} s_i^{-(\beta+1)}e^{s_i^{-\beta}} \quad (3.5) \\ l(\beta,\gamma,\delta|\underset{\sim}{r},\underset{\sim}{s}) &= nlog\gamma + nlog\beta + \gamma\left(1-e^{r_n^{-\beta}}\right) + \sum_{i=1}^{n}\left[-(\beta+1)logr_i + r_i^{-\beta}\right] \\ &+ mlog\delta + mlog\beta + \delta\left(1-e^{s_m^{-\beta}}\right) + \sum_{i=1}^{m}\left[-(\beta+1)logs_i + s_i^{-\beta}\right] \quad (3.6) \end{aligned}$$

$$\frac{\delta l}{\delta \gamma} = 0 \Rightarrow \quad \frac{n}{\gamma} + \left(1 - e^{r_n^{-\beta}}\right) = 0 \tag{3.7}$$

$$\frac{\delta l}{\delta \delta} = 0 \Rightarrow \quad \frac{m}{\delta} + \left(1 - e^{s_m^{-\beta}}\right) = 0 \tag{3.8}$$

$$\begin{aligned} \frac{\delta l}{\delta \beta} &= \frac{(m+n)}{\beta} - \gamma r_n^{-\beta} e^{r_n^{-\beta}} log r_n - \sum_{i=1}^{n} \left( log r_i (1 + r_i^{-\beta}) \right) \\ &- \delta s_m^{-\beta} e^{s_m^{-\beta}} log s_m - \sum_{i=1}^{m} \left( log s_i (1 + s_i^{-\beta}) \right) = 0 \end{aligned} \tag{3.9}$$

### 3.2.1 When shape parameter $\beta$ is known

Under the assumption that the shape parameter $\beta$ is known, the MLE's of $\gamma$ and $\delta$, say respectively, can be obtained from (3.7) and (3.8) as

$$\hat{\gamma} = \frac{-n}{\left(1 - e^{r_n^{-\beta}}\right)} \ ; \quad \hat{\delta} = \frac{-m}{\left(1 - e^{s_m^{-\beta}}\right)}. \tag{3.10}$$

Then the MLE of R is given by

$$\hat{R} = \frac{\hat{\gamma}}{\hat{\gamma} + \hat{\delta}}.$$

We shall study the distribution of $\hat{R}$. For this we first obtain the distribution of

$$\hat{\gamma} = \frac{-n}{1 - e^{r_n^{-\beta}}}.$$

Since the pdf of $R_n$ is given by

$$\begin{aligned} f_{R_n}(r_n) &= \frac{1}{(n-1)!} f(r_n) \left[-\log F(r_n)\right]^{n-1} \\ &= \frac{1}{(n-1)!} \gamma \beta r_n^{-(\beta+1)} e^{\left[r_n^{-\beta} + \gamma(1 - e^{r_n^{-\beta}})\right]} \left[-\gamma(1 - e^{r_n^{-\beta}})\right]^{n-1}, \ r_n > 0. \end{aligned} \tag{3.11}$$

the pdf of $Z_1 = \hat{\gamma}$ is given by

$$f_{Z_1}(z_1) = \frac{(n\gamma)^n e^{-\frac{n\gamma}{z_1}}}{\Gamma(n) z_1^{n+1}}, \ \ z_1 > 0. \tag{3.12}$$

This is identified as the inverted gamma distribution.

$$Z_1 \sim IGamma(n, n\gamma).$$

Similarly, for $Z_2 = \hat{\delta}$ we can obtain

$$Z_2 \sim IGamma(m, m\delta).$$

Therefore we can find the pdf of

$$\hat{R} = \frac{\hat{\gamma}}{\hat{\delta} + \hat{\gamma}} = \frac{Z_1}{Z_2 + Z_1} = \frac{1}{1 + \frac{Z_2}{Z_1}}.$$

Consider $\frac{Z_2}{Z_1}$. By the properties of inverse gamma distribution and the relation with the gamma distribution, we have

$$\frac{n\gamma}{z_1} \sim Gamma(n, 1); \quad \frac{m\delta}{z_2} \sim Gamma(m, 1).$$

Hence

$$\frac{2n\gamma}{z_1} \ \sim \ \chi^2_{2n}; \ \ \frac{2m\delta}{z_2} \ \sim \ \chi^2_{2m}.$$

Clearly, by the independence of two $\chi^2$ random variables, we have

$$\frac{2n\gamma/2nz_1}{2m\delta/2mz_2} = \frac{\gamma z_2}{\delta z_1} = \frac{R}{1-R} \sim F(2n,2m)$$

This fact leads us to the construction of the following $(1-\alpha)\%$ confidence interval for $R$.

$$\left( \left(1+\frac{\hat{\delta}}{\hat{\gamma}F_{\frac{\alpha}{2},2n,2m}}\right)^{-1}, \left(1+\frac{\hat{\delta}}{\hat{\gamma}F_{1-\frac{\alpha}{2},2n,2m}}\right)^{-1} \right) \tag{3.13}$$

### 3.2.2 When shape parameter $\beta$ is unknown

This subsection discusses likelihood inference of $R$ when all of the parameters $\gamma, \delta$ and $\beta$ are unknown. In this case by using (3.7) and (3.8) we obtain

$$\hat{\gamma} = \frac{-n}{\left(1-e^{r_n^{-\hat{\beta}}}\right)} \; ; \quad \hat{\delta} = \frac{-m}{\left(1-e^{s_m^{-\hat{\beta}}}\right)} \tag{3.14}$$

where $\hat{\beta}$ is the MLE of the parameter $\beta$ which is obtained by solving the following non-linear equation

$$\begin{aligned} \frac{(m+n)}{\beta} &- \gamma r_n^{-\beta} e^{r_n^{-\beta}} logr_n - \sum_{i=1}^{n} \left( logr_i (1+r_i^{-\beta}) \right) \\ &- \delta s_m^{-\beta} e^{s_m^{-\beta}} logs_m - \sum_{i=1}^{m} \left( logs_i (1+s_i^{-\beta} \right) = 0 \end{aligned} \tag{3.15}$$

Therefore $\hat{\beta}$ can be obtained as a solution of the non-linear equation of the form $h(\beta) = \beta$ where

$$h(\beta) = (m+n)\left[\frac{-nr_n^{-\beta}e^{r_n^{-\beta}}logr_n}{\left(1-e^{r_n^{-\beta}}\right)} - \frac{ms_m^{-\beta}e^{s_m^{-\beta}}logs_m}{\left(1-e^{s_m^{-\beta}}\right)} + \sum_{i=1}^{n} logr_i(1+r_i^{-\beta}) + \sum_{i=1}^{m} logs_i(1+s_i^{-\beta})\right]^{-1} \quad (3.16)$$

Since $\hat{\beta}$ is a fixed point solution of this non-linear equation, it can be obtained by using an iterative procedure as $h(\beta_j) = \beta_{j+1}$ where $\beta_j$ is the $j^{th}$ iteration of $\hat{\beta}$.

The iteration procedure should be stopped when $|\beta_j - \beta_{j+1}|$ is sufficiently small. Once we obtain $\hat{\beta}, \hat{\gamma}$ and $\hat{\delta}$ can be deduced from (3.7) and (3.8).

Therefore the MLE of $R$ is computed to be

$$\hat{R} = \frac{\hat{\gamma}}{\hat{\delta}+\hat{\gamma}}.$$

Since the study of the distribution of $R$ is very complicated and difficult, it is not possible to obtain exact confidence interval of $R$. In this case we propose some confidence interval based on the asymptotic distribution of $\hat{R}$ and the bootstrap method.

### 3.2.3 Asymptotic confidence interval

In this subsection we obtain approximate confidence interval of $R$ based on the asymptotic distribution of $\hat{R}$ in which it is required to calculate Fisher information matrix. We obtain the observed information matrix, since the expected information matrix is very complicated and will require numerical integration. The 3x3 observation matrix is given by

$$I(\gamma,\delta,\beta) = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

where

$$I_{11} = -\frac{\delta^2 l}{\delta\gamma^2} = \frac{n}{\gamma^2};\ \ I_{12} = I_{21} = -\frac{\delta^2 l}{\delta\gamma\delta\delta} = 0;\ \ I_{22} = -\frac{\delta^2 l}{\delta\delta^2} = \frac{m}{\delta^2}$$

$$I_{13} = I_{31} = -\frac{\delta^2 l}{\delta\gamma\delta\beta} = -e^{r_n^{-\beta}} r_n^{-\beta} log r_n;\quad I_{23} = I_{32} = -\frac{\delta^2 l}{\delta\delta\delta\beta} = -e^{s_m^{-\beta}} s_m^{-\beta} log r_m$$

$$\begin{aligned} I_{33} \quad = \quad & -\frac{\delta^2 l}{\delta\beta^2} = \frac{m+n}{\beta^2} - \gamma(log r_n)^2 r_n^{-\beta} e^{r_n^{-\beta}} \left(1 + r_n^{-\beta}\right) + \sum_{i=1}^{n} r_i^{-\beta} (log r_i)^2 \\ & -\delta(log s_m)^2 s_m^{-\beta} e^{s_m^{-\beta}} \left(1 + s_m^{-\beta}\right) + \sum_{i=1}^{m} s_i^{-\beta} (log s_i)^2 \end{aligned}$$

As $n \to \infty$ and $m \to \infty$, by the asymptotic properties of MLE, $\hat{R}$ is asymptotically normal with mean R and asymptotic variance

$$\sigma_R^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\delta R}{\delta\lambda_i} \frac{\delta R}{\delta\lambda_j} I_{ij}^{-1}$$

where $\lambda_1 = \gamma,\ \lambda_2 = \delta,\ \lambda_3 = \beta$ and $I_{ij}^{-1}$ is the (i,j)th element of the inverse of the $I(\gamma, \delta, \beta)$ (Rao[49]).

Since

$$\frac{\delta R}{\delta \gamma} = \frac{\delta}{(\delta+\gamma)^2}\ ,\ \ \frac{\delta R}{\delta \delta} = \frac{-\gamma}{(\delta+\gamma)^2}\ ,\ \ \frac{\delta R}{\delta \beta} = 0,$$

we have

$$\sigma_R^2 = \frac{1}{(\delta+\gamma)^4}\left[\delta^2 I_{11}^{-1} + \gamma^2 I_{22}^{-1} - 2\gamma\delta I_{12}^{-1}\right].$$

Now to compute the asymptotic confidence interval of $R$, the variance $\sigma_R^2$ is to be estimated. The estimate of $\sigma_R^2$ say $\hat{\sigma}_R^2$ can be obtained by replacing $(\gamma, \delta, \beta)$ involved in $\hat{\sigma}_R^2$ by their corresponding MLE's. Therefore the asymptotic $100(1-\alpha)\%$ confidence interval of $R$ is given by

$$\left(\hat{R} - z_{1-\frac{\alpha}{2}}\hat{\sigma}_R^2,\ \ \hat{R} + z_{1-\frac{\alpha}{2}}\hat{\sigma}_R^2\right) \tag{3.17}$$

where $z_\alpha$ is the $\alpha$ quantile of the standard normal distribution.

### 3.2.4 Bootstrap confidence intervals

In this subsection we consider some confidence intervals based on the parametric bootstrap methods. We propose to use the following method to generate parametric bootstrap samples of $R$ as suggested by Effron and Tibshirani[50].

Step 1: Compute $\hat{\gamma}, \hat{\delta}$,and $\hat{R}$ the MLE's of $\gamma, \delta$ and $R$ based on the original two samples of lower records. $\underset{\sim}{r} = (r_1, \ldots, r_n)$, $\underset{\sim}{s} = (s_1, \ldots, s_m)$.

Step 2: Generate independent bootstrap lower record samples $\underset{\sim}{r}^* = (r_1^*, \ldots, r_n^*)$ and $\underset{\sim}{s}^* = (s_1^*, \ldots, s_m^*)$ from inverse Chen distribution with the parameters $\hat{\gamma}, \hat{\beta}$ and $\hat{\delta}, \hat{\beta}$ respectively. By using these data we compute the bootstrap estimates say $\hat{\gamma}^*, \hat{\delta}^*, \hat{\beta}^*$ and $\hat{R}^*$.

Step 3: Repeat **step2.** B times to obtain a set of bootstrap samples of $R$ say $\hat{R}_1^*, \ldots, \hat{R}_B^*$.

Using the above bootstrap samples of R, we obtain three different bootstrap confidence intervals of $R$ as follows:

1. **Standard normal interval**: The simplest $100(1-\alpha)\%$ bootstrap interval is the standard normal interval

$$\left(\hat{R} - z_{1-\frac{\alpha}{2}} \hat{se}_{boot} \ , \ \hat{R} + z_{1-\frac{\alpha}{2}} \hat{se}_{boot}\right). \tag{3.18}$$

where $\hat{se}_{boot}$ is the bootstrap estimate of the standard error based on $\hat{R}_1^*, \ldots, \hat{R}_B^*$.

2. **Basic pivotal interval**: The $100(1-\alpha)\%$ basic pivotal confidence interval is

$$\left(2\hat{R} - \hat{r}^*_{(1-\frac{\alpha}{2})B} \ , \ 2\hat{R} - \hat{r}^*_{(\frac{\alpha}{2})B}\right) \tag{3.19}$$

where $\hat{r}^*_\beta$ is the $\beta$ quantile of $\hat{R}_1^*, \ldots, \hat{R}_B^*$.

3. **Percentile interval**: The $100(1-\alpha)\%$ bootstrap percentile interval is defined by

$$\left(\hat{R}_{(\frac{\alpha}{2})B} \ , \ \hat{R}_{(1-\frac{\alpha}{2})B}\right) \tag{3.20}$$

where the confidence limits are $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of the bootstrap sample $\hat{R}_1^*,\ldots,\hat{R}_B^*$.

## 3.3 Bayesian inference

In this section we deal with the Bayesian method for making inference based on lower record values. We consider two cases, one with known shape parameter $\beta$ and the other in which it is unknown.

### 3.3.1 Known shape parameter $\beta$

When $\beta$ is known we assume two priors, namely conjugate prior and non-informative prior .

The likelihood function of $\gamma$ and $\delta$ suggest that the conjugate family of prior distributions for $\gamma$ and $\delta$ to be gamma family of distributions.

$$\begin{aligned} \pi(\gamma) &= \frac{b_1^{a_1}}{\Gamma(a_1)}\gamma^{a_1-1}e^{-b_1\gamma}, \;\; \gamma>0, \;\; a_1,b_1>0 \\ \pi(\delta) &= \frac{b_2^{a_2}}{\Gamma(a_2)}\delta^{a_2-1}e^{-b_2\delta}, \;\; \delta>0, \;\; a_2,b_2>0 \end{aligned} \tag{3.21}$$

where $a_1,b_1$ and $a_2,b_2$ are the parameters of the prior distributions of $\gamma$ and $\delta$ respectively. Using these prior distributions and likelihood functions in (3.5), the posterior distributions of $\gamma$ and $\delta$ are obtained respectively as

$$\begin{aligned} \gamma|\underset{\sim}{r} \sim \quad & Gamma(n+a_1,\ e^{r_n^{-\beta}}+b_1-1) \\ \text{and} \quad \delta|\underset{\sim}{s} \sim \quad & Gamma(m+a_2,\ e^{s_m^{-\beta}}+b_2-1). \end{aligned} \tag{3.22}$$

Since the priors $\gamma$ and $\delta$ are independent, then using standard transformation technique and after some manipulations the posterior pdf of $R$ will be

$$f_R(r) = C\frac{r^{n+a_1-1}(1-r)^{m+a_2-1}}{\left[r\left(e^{r_n^{-\beta}}+b_1-1\right)+(1-r)\left(e^{s_m^{-\beta}}+b_2-1\right)\right]^{n+m+a_1+a_2}}$$

where

$$C = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)}\left(e^{r_n^{-\beta}}+b_1-1\right)^{n+a_1}\left(e^{s_m^{-\beta}}+b_2-1\right)^{m+a_2}.$$

Under squared error loss function, the Bayes estimate of $R$ is the expected value of $R$. This expected value contains an integral which is not obtainable in a closed form. Therefore we can use the approximate method of Lindley [19], to find the approximate Bayes estimator $\bar{R_B}$ relative to squared error loss function. By the approximate method of Lindley the Bayes estimator for $u(\theta)$ for a prior $v(\theta)$ is given by

$$\begin{aligned} \frac{\int_\theta u(\theta)v(\theta)\exp^{L(\theta)}\,d\theta}{\int_\theta v(\theta)\exp^{L(\theta)}\,d\theta} &= [u^* + \frac{1}{2}(u_{11}\sigma_{11}+u_{22}\sigma_{22})+\rho_1 u_1\sigma_{11}+\rho_2 u_2\sigma_{22} \\ &+ \frac{1}{2}[\sigma_{11}\sigma_{22}(u_1L_{12}+u_2L_{21})+u_1\sigma_{11}^2 L_{30} \\ &+ u_2\sigma_{22}^2 L_{03}]+O(\frac{1}{n^2})]_{at\,\hat{\theta}} \end{aligned} \tag{3.23}$$

where

$$u(\theta)=\frac{\gamma}{\gamma+\delta};\;\; v(\theta)=\pi(\gamma)\pi(\delta);\;\; \rho=logv(\theta)=\log C+(a_1-1)\log\gamma-b_1\gamma+(a_2-1)\log\delta-b_2\delta.$$

$u^*$ is the MLE of $u(\theta)$ and $L(\theta)$ is the logarithm of likelihood function, $C$ is independent of $\gamma$ and $\delta$.

Further

$$u_1=\frac{\delta u}{\delta\gamma}=\frac{\delta}{(\delta+\gamma)^2};\;\; u_2=\frac{\delta u}{\delta\delta}=\frac{-\gamma}{(\delta+\gamma)^2};\;\; u_{11}=\frac{\delta^2 u}{\delta\gamma^2}=\frac{-2\delta}{(\delta+\gamma)^3};\;\; u_{22}=\frac{\delta^2 u}{\delta\delta^2}=\frac{2\gamma}{(\delta+\gamma)^3}$$

$$\rho_1=\frac{\delta\rho}{\delta\gamma}=\frac{a_1-1}{\gamma}-b_1;\;\; \rho_2=\frac{\delta\rho}{\delta\delta}=\frac{a_2-1}{\delta}-b_2$$

$$\sigma=[-L_{ij}]^{-1}\;\; where\;\; L_{ij}=\left[\frac{\delta^2 L}{\delta\theta_i\delta\theta_j}\right]$$

$\sigma=\begin{bmatrix}\frac{\gamma^2}{n} & 0\\ 0 & \frac{\delta^2}{m}\end{bmatrix};\;\; L_{30}=\frac{\delta^3 L}{\delta\gamma^3}=\frac{2n}{\gamma^3};\;\; L_{03}=\frac{\delta^3 L}{\delta\delta^3}=\frac{2m}{\delta^3}.$

Substituting in eqn (3.23) we get the Bayes estimator as

$$\bar{R_B}=\hat{R}\left[1+\left(1-\hat{R}\right)\left(\frac{1-\hat{R}}{m}-\frac{\hat{R}}{n}+\frac{(a_1-b_1\hat{\gamma})}{n}-\frac{(a_2-b_2\hat{\delta})}{m}\right)\right]. \tag{3.24}$$

Again, it follows from (3.22) that

$$2(e^{r_n^{-\beta}}+b_1-1)(\gamma|\underset{\sim}{r})\sim\chi^2_{2(n+a_1)}$$

and

$$2(e^{s_m^{-\beta}}+b_2-1)(\delta|\underset{\sim}{s})\sim\chi^2_{2(m+a_2)}.$$

Hence $\pi\left(R|\underset{\sim}{r},\underset{\sim}{s}\right)$, the posterior distribution of $R$, is equal to that of $(1+AF)^{-1}$ where

$$F \sim F_{2(m+a_2),2(n+a_1)}, \quad A = \frac{(m+a_2)(e^{r_n^{-\beta}}+b_1-1)}{(n+a_1)(e^{s_m^{-\beta}}+b_2-1)}.$$

Therefore a Bayesian $(1-\alpha)\%$ confidence interval for $R$ is given by

$$\left((AF_{1-\frac{\alpha}{2},2(m+a_2),2(n+a_1)}+1)^{-1},\ (AF_{\frac{\alpha}{2},2(m+a_2),2(n+a_1)}+1)^{-1}\right). \tag{3.25}$$

When we are ignorant about the parameter Jeffreys non informative prior is used. We assume

$$\begin{aligned} \pi(\gamma) \propto & \ \frac{1}{\gamma}, \quad \gamma > 0. \\ \pi(\delta) \propto & \ \frac{1}{\delta}, \quad \delta > 0. \end{aligned} \tag{3.26}$$

Using the priors and the likelihood function(3.5), the posterior distributions of $\beta_1$ and $\beta_2$ are obtained as

$$\begin{aligned} \gamma|\underset{\sim}{r} &\sim Gamma\left(n\ ,\ e^{r_n^{-\beta}}-1\right). \\ \delta|\underset{\sim}{s} &\sim Gamma\left(m\ ,\ e^{s_m^{-\beta}}-1\right). \end{aligned} \tag{3.27}$$

Since $\gamma$ and $\delta$ are independent, then using standard transformation techniques and after some manipulations the posterior pdf of R is given by

$$f_R(r) = C\frac{(1-r)^{m-1}r^{n-1}}{\left[r\left(e^{r_n^{-\beta}}-1\right)+(1-r)\left(e^{s_m^{-\beta}}-1\right)\right]^{n+m}}, \quad 0<r<1$$

where $C = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\left(e^{r_n^{-\beta}}-1\right)^n\left(e^{s_m^{-\beta}}-1\right)^m$.

Under squared error loss function, the Bayes estimator of $R$ is the expected value of $R$. This expected value contains an integral which can't be obtained in a simple closed form. Therefore, using the approximate method of Lindley [19], we can find the approximate Bayes estimator $\bar{R_B}$ relative to squared error loss function. By the approximate method of Lindley the Bayes estimator for $u(\theta)$ for a prior $v(\theta)$ is given by

$$\begin{aligned}\frac{\int_\theta u(\theta)v(\theta)\exp^{L(\theta)}d\theta}{\int_\theta v(\theta)\exp^{L(\theta)}d\theta} &= [u^* + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + \rho_1 u_1 \sigma_{11} + \rho_2 u_2 \sigma_{22} \\ &+ \frac{1}{2}[\sigma_{11}\sigma_{22}(u_1 L_{12} + u_2 L_{21}) + u_1 \sigma_{11}^2 L_{30} \\ &+ u_2 \sigma_{22}^2 L_{03}] + O(\frac{1}{n^2})]_{at\hat{\theta}} \end{aligned} \tag{3.28}$$

where

$$u(\theta) = \frac{\gamma}{\delta+\gamma};\ \ v(\theta) \propto \frac{1}{\gamma\delta};\ \ \rho = \log v(\theta) \propto \log\left(\frac{1}{\gamma\delta}\right),$$

$u^*$ is the MLE of $u(\theta)$ and $L(\theta)$ is the logarithm of likelihood function.

Further

$$u_1 = \frac{\delta u}{\delta\gamma} = \frac{\delta}{(\delta+\gamma)^2};\ \ u_2 = \frac{\delta u}{\delta\delta} = \frac{-\gamma}{(\delta+\gamma)^2};\ \ u_{11} = \frac{\delta^2 u}{\delta\gamma^2} = \frac{-2\delta}{(\delta+\gamma)^3};\ \ u_{22} = \frac{\delta^2 u}{\delta\delta^2} = \frac{2\gamma}{(\delta+\gamma)^3},$$

$$\rho_1 = \frac{\delta\rho}{\delta\gamma} = \frac{-1}{\gamma};\ \ \rho_2 = \frac{\delta\rho}{\delta\delta} = \frac{-1}{\delta},$$

$$\sigma = [-L_{ij}]^{-1}\ \ where\ \ L_{ij} = \left[\frac{\delta^2 L}{\delta\theta_i\delta\theta_j}\right],$$

$$\sigma = \begin{bmatrix} \frac{\gamma^2}{n} & 0 \\ 0 & \frac{\delta^2}{m} \end{bmatrix};\ \ L_{30} = \frac{\delta^3 L}{\delta\gamma^3} = \frac{2n}{\gamma^3};\ \ L_{03} = \frac{\delta^3 L}{\delta\delta^3} = \frac{2m}{\delta^3}.$$

Substituting in (3.28) we get the Bayes estimator as

$$\bar{R_B} = \hat{R} + \hat{R}(1-\hat{R})\left[\frac{(1-\hat{R})}{m} - \frac{\hat{R}}{n}\right].$$

Further more, it follows from (3.27) that

$$2(e^{r_n^{-\beta}}-1)\left(\gamma|\underset{\sim}{r}\right) \sim \chi^2_{2n}$$

and

$$2(e^{s_m^{-\beta}}-1)\left(\delta|\underset{\sim}{s}\right) \sim \chi^2_{2m}.$$

It follows that $\pi(R|\underset{\sim}{r},\underset{\sim}{s})$, the posterior distribution of $R$, is equal to that of $(1+AW)^{-1}$ where

$$W \sim F_{2m,2n}, \quad A = \frac{m(e^{r_n^{-\beta}}-1)}{n(e^{s_m^{-\beta}}-1)}.$$

Therefore a Bayesian $(1-\alpha)\%$ credible interval for $R$ is given by

$$\left(\left(AF_{1-\frac{\alpha}{2},2m,2n}+1\right)^{-1}, \left(AF_{\frac{\alpha}{2},2m,2n}+1\right)^{-1}\right). \tag{3.29}$$

### 3.3.2 Unknown shape parameter $\beta$

We assume that all the parameters $\gamma,\delta$ and $\beta$ are unknown and have independent gamma priors $(a_i,b_i),\ i=1,2,3$ respectively. That is

$$\begin{aligned}
\pi(\gamma) &= \frac{b_1^{a_1}}{\Gamma(a_1)}\gamma^{a_1-1}e^{-b_1\gamma},\ \ \gamma>0,\ \ a_1,b_1>0.\\
\pi(\delta) &= \frac{b_2^{a_2}}{\Gamma(a_2)}\delta^{a_2-1}e^{-b_2\delta},\ \ \delta>0,\ \ a_2,b_2>0.\\
\pi(\beta) &= \frac{b_3^{a_3}}{\Gamma(a_3)}\beta^{a_3-1}e^{-b_3\beta},\ \ \beta>0,\ \ a_3,b_3>0.
\end{aligned}$$

The joint posterior distribution of $(\gamma,\delta,\beta)$ is given by

$$\pi(\gamma,\delta,\beta|\underset{\sim}{r},\underset{\sim}{s}) = \frac{L(\vec{r},\vec{s};\gamma,\delta,\beta)\,\pi(\gamma)\pi(\delta)\pi(\beta)}{\int L\left(\underset{\sim}{r},\underset{\sim}{s};\gamma,\delta,\beta\right)\pi(\gamma)\pi(\delta)\pi(\beta)d\gamma d\delta d\beta} \tag{3.30}$$

where the numerator is

$$\left(\frac{b_1^{a_1}\gamma^{n+a_1-1}e^{-(e^{r_n^{-\beta}}+b_1-1)\gamma}}{\Gamma(a_1)}\right)\left(\frac{b_2^{a_2}\delta^{m+a_2-1}e^{-(e^{s_m^{-\beta}}+b_2-1)\delta}}{\Gamma(a_2)}\right)$$
$$\left(\frac{b_3^{a_3}\beta^{m+n+a_3-1}e^{-b_3\beta}}{\Gamma(a_3)}\right)\left(\prod_{i=1}^{n}r_i^{-(\beta+1)}e^{r_i^{-\beta}}\right)\left(\prod_{i=1}^{m}s_i^{-(\beta+1)}e^{s_i^{-\beta}}\right)$$

Clearly it is not possible to obtain an explicit Bayes estimate of the model parameters. For this we have to use a simulation technique to compute the Bayes estimate of $R$ and the corresponding credible interval. We use Gibbs sampling technique which uses the posterior distributions of each parameter conditional on others ( Gelfand & Smith [51]). The conditional posterior distributions of $\gamma,\delta$ and $\beta$ can be obtained as follows.

$$\left(\gamma|\delta,\beta,\underset{\sim}{r},\underset{\sim}{s}\right) \sim Gamma\left(n+a_1,\ e^{r_n^{-\beta}}+b_1-1\right)$$
$$\left(\delta|\gamma,\beta,\underset{\sim}{r},\underset{\sim}{s}\right) \sim Gamma\left(m+a_2,\ e^{s_m^{-\beta}}+b_2-1\right)$$

and

$$\pi(\beta|\gamma,\delta,\underset{\sim}{r},\underset{\sim}{s}) \propto \beta^{m+n+a_3-1}e^{-b_3\beta-\gamma e^{r_n^{-\beta}}-\delta e^{s_m^{-\beta}}}\left(\prod_{i=1}^{n}r_i^{-(\beta+1)}e^{r_i^{-\beta}}\right)\left(\prod_{i=1}^{m}s_i^{-(\beta+1)}e^{s_i^{-\beta}}\right) \tag{3.31}$$

Though it is easy to generate samples of $\gamma$ and $\delta$ from gamma distributions, it is not possible to sample directly from the conditional posterior distribution for $\beta$. Hence we use MCMC technique. To do this Metropolis-Hastings algorithm (Metropolis et al. [20] and Hastings [21]) with q(.) as a proposal distribution is applied. Therefore the method of Gibbs sampling along with M-H algorithm is described as follows.

**Step1**. Start with $\beta^{(0)} = \hat{\beta}$ as an initial guess and set t = 1.

**Step2**. Generate $\gamma^{(t)}$ from Gamma$\left(n + a_1, e^{r_n^{-\beta}} + b_1 - 1\right)$

**Step3**. Generate $\delta^{(t)}$ from Gamma$\left(m + a_2, e^{s_m^{-\beta}} + b_2 - 1\right)$

**Step4**. Using Metropolis-Hastings method, generate $\beta^{(t)}$ from $\pi(\beta|\gamma, \delta, \vec{r}, \vec{s})$ with the proposal distribution as $q(\beta) \propto N\left(\beta^{(t-1)}, 0.25\right)$, $(\beta > 0)$

**Step5**. Compute $R^{(t)} = \frac{\gamma^{(t)}}{(\delta^{(t)} + \gamma^{(t)})}$

**Step6**. set t=1

**Step7**. Repeat steps 2-6 $N$ times.

Now the approximate posterior mean of $R$ is given by

$$\hat{E}(R|\vec{r}, \vec{s}) = \frac{1}{N-M} \sum_{t=M+1}^{N} R^{(t)} \tag{3.32}$$

where $M$ is the burn-in period (that is the number of iterations before the stationary distribution is achieved).

Based on $N$ and $R^{(t)}$ values, using the method proposed by Chen and Shao [24], a $100(1-\alpha)\%$ HPD credible interval can be constructed as

$$\left(R_{[\frac{\alpha}{2}N]}, R_{[(1-\frac{\alpha}{2})N]}\right) \tag{3.33}$$

where $R_{[\frac{\alpha}{2}N]}$ and $R_{[(1-\frac{\alpha}{2})N]}$ are the $[\frac{\alpha}{2}N]$th smallest integer and the $[(1-\frac{\alpha}{2})N]$ th smallest integer of $\{R^{(t)}, t = M+1, M+2, \ldots, N\}$, respectively.

## 3.4 A Simulation study

In this section a Monte Carlo simulation study is conducted to investigate and compare the performance of point estimators and confidence intervals presented in this chapter. The performance of MLE's

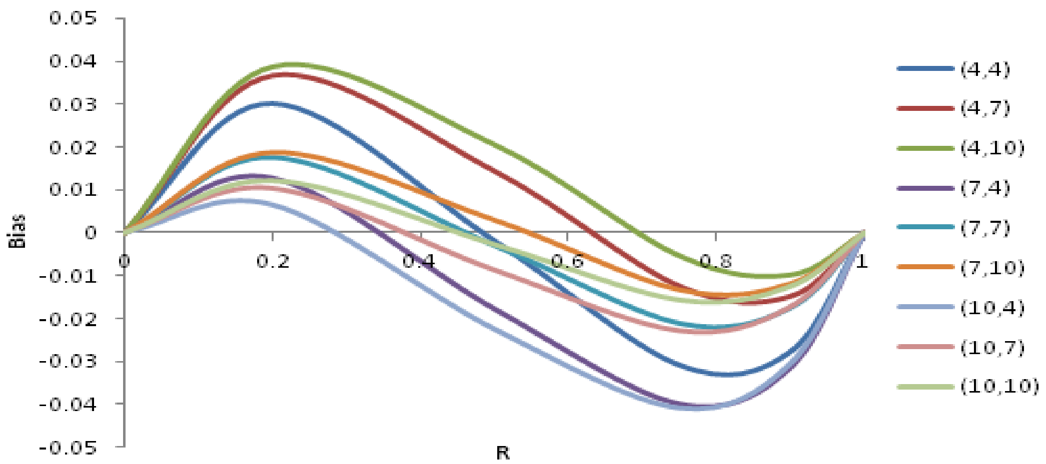


Fig. 3.2 Curve of bias of $\hat{R}$ against $R$

and Bayes estimators is compared in terms of their biases and mean squared errors (MSE's). We also compare different confidence intervals in terms of their coverage probability and expected length. We consider two cases; when the parameter $\beta$ known and $\beta$ unknown separately. We use the parameter values $(\gamma, \delta) = (4,1), (2,2), (1,3), (1,9)$ and $\beta = 2$. Therefore $R_{exact} = 0.2, 0.5, 0.75, 0.9$. To compute the Bayes estimators and HPD credible intervals, we consider two priors as follows:

1. Conjugate Prior : $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0.5$

Table 3.1 Average estimates (AVR), biases and MSE's of the estimators of $R$.

| (n,m) | R | MLE | | | Bayes conjugate | | | Bayes non-inform | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | AVR | bias | MSE | AVR | bias | MSE | AVR | bias | MSE |
| (3,3) | | 0.23 | 0.03 | 0.0216 | 0.3311 | 0.1311 | 0.0298 | 0.2529 | 0.0529 | 0.0216 |
| (3,5) | | 0.2366 | 0.0366 | 0.0192 | 0.2808 | 0.0808 | 0.0159 | 0.2449 | 0.0449 | 0.0176 |
| (3,7) | | 0.2389 | 0.0389 | 0.0181 | 0.2594 | 0.0594 | 0.0117 | 0.2406 | 0.0406 | 0.0158 |
| (5,3) | | 0.2127 | 0.0127 | 0.0146 | 0.3298 | 0.1298 | 0.0284 | 0.2427 | 0.0427 | 0.0166 |
| (5,5) | 0.2 | 0.2175 | 0.0175 | 0.0121 | 0.2807 | 0.0807 | 0.0157 | 0.2331 | 0.0331 | 0.0125 |
| (5,7) | | 0.2188 | 0.0188 | 0.0107 | 0.2597 | 0.0597 | 0.0114 | 0.228 | 0.028 | 0.0105 |
| (7,3) | | 0.2064 | 0.0064 | 0.0126 | 0.3275 | 0.1275 | 0.0261 | 0.2395 | 0.0395 | 0.0149 |
| (7,5) | | 0.2102 | 0.0102 | 0.0094 | 0.2787 | 0.0787 | 0.014 | 0.2286 | 0.0286 | 0.0094 |
| (7,7) | | 0.2122 | 0.0122 | 0.0084 | 0.2582 | 0.0582 | 0.0104 | 0.2241 | 0.0241 | 0.0087 |
| (3,3) | | 0.4981 | -0.0019 | 0.0358 | 0.4982 | -0.0018 | 0.0101 | 0.4983 | -0.0016 | 0.0283 |
| (3,5) | | 0.5144 | 0.0144 | 0.03 | 0.4658 | -0.0342 | 0.012 | 0.4986 | -0.0014 | 0.0246 |
| (3,7) | | 0.5207 | 0.0207 | 0.0277 | 0.4539 | -0.046 | 0.0126 | 0.4979 | -0.0021 | 0.0229 |
| (5,3) | | 0.4823 | -0.0177 | 0.0305 | 0.5317 | 0.0317 | 0.0121 | 0.4982 | -0.0018 | 0.0249 |
| (5,5) | 0.5 | 0.497 | -0.0029 | 0.0234 | 0.4979 | -0.002 | 0.0117 | 0.4972 | -0.0027 | 0.0199 |
| (5,7) | | 0.503 | 0.003 | 0.02 | 0.4855 | -0.0144 | 0.0112 | 0.4962 | -0.0037 | 0.0174 |
| (7,3) | | 0.4774 | -0.0226 | 0.0276 | 0.5448 | 0.0448 | 0.0125 | 0.5003 | 0.0003 | 0.0229 |
| (7,5) | | 0.4911 | -0.0088 | 0.02 | 0.5103 | 0.0103 | 0.011 | 0.4982 | -0.0018 | 0.0173 |
| (7,7) | | 0.4973 | -0.0027 | 0.0172 | 0.4979 | 0.0021 | 0.0106 | 0.4974 | -0.0026 | 0.0152 |
| (3,3) | | 0.7191 | -0.0308 | 0.0264 | 0.6401 | -0.1098 | 0.0247 | 0.6986 | -0.0514 | 0.0247 |
| (3,5) | | 0.7376 | -0.0124 | 0.0194 | 0.6391 | -0.1108 | 0.0246 | 0.7026 | -0.0424 | 0.0202 |
| (3,7) | | 0.7447 | -0.0053 | 0.0167 | 0.6407 | -0.1093 | 0.0231 | 0.7111 | -0.038 | 0.0182 |
| (5,3) | | 0.7102 | -0.0398 | 0.0247 | 0.6829 | -0.0671 | 0.0156 | 0.7054 | -0.0445 | 0.0214 |
| (5,5) | 0.75 | 0.7289 | -0.021 | 0.0161 | 0.6791 | -0.0709 | 0.0158 | 0.7146 | -0.0354 | 0.0156 |
| (5,7) | | 0.7367 | -0.0133 | 0.0128 | 0.68 | -0.0609 | 0.0144 | 0.7185 | -0.0315 | 0.0131 |
| (7,3) | | 0.7089 | -0.041 | 0.0224 | 0.7016 | -0.0484 | 0.0121 | 0.7111 | -0.0389 | 0.0188 |
| (7,5) | | 0.7269 | -0.023 | 0.0141 | 0.6968 | -0.0531 | 0.0123 | 0.7195 | -0.0304 | 0.0133 |
| (7,7) | | 0.7343 | -0.0157 | 0.0114 | 0.6976 | -0.0524 | 0.0114 | 0.7232 | -0.0267 | 0.0112 |
| (3,3) | | 0.8723 | -0.0277 | 0.0103 | 0.7162 | -0.1838 | 0.048 | 0.8507 | -0.0493 | 0.0126 |
| (3,5) | | 0.8852 | -0.0148 | 0.0066 | 0.7281 | -0.1719 | 0.0404 | 0.8614 | -0.0386 | 0.0091 |
| (3,7) | | 0.8902 | -0.0098 | 0.0051 | 0.7347 | -0.1652 | 0.0356 | 0.8657 | -0.0342 | 0.0075 |
| (5,3) | | 0.8686 | -0.0314 | 0.0099 | 0.7864 | -0.1136 | 0.0208 | 0.8576 | -0.0423 | 0.0104 |
| (5,5) | 0.9 | 0.8826 | -0.0174 | 0.0053 | 0.7914 | -0.1085 | 0.0184 | 0.8688 | -0.0311 | 0.0064 |
| (5,7) | | 0.8881 | -0.0119 | 0.0038 | 0.796 | 0.1039 | 0.016 | 0.8735 | -0.0264 | 0.0048 |
| (7,3) | | 0.8693 | -0.0307 | 0.0088 | 0.8171 | -0.0829 | 0.0124 | 0.8629 | -0.0371 | 0.0087 |
| (7,5) | | 0.8826 | -0.0174 | 0.0046 | 0.8195 | -0.0805 | 0.0115 | 0.8734 | -0.0266 | 0.0051 |
| (7,7) | | 0.8875 | -0.0125 | 0.0034 | 0.8232 | -0.0768 | 0.01 | 0.8775 | -0.0225 | 0.004 |

Table 3.2 Expected lengths(EL) and coverage probability (CP) of the confidence intervals with $(1-\alpha)=0.95$.

| (n,m) | R | MLE | | Bayes conjugate | | Bayes non-inform | |
|---|---|---|---|---|---|---|---|
| | | EL | CP | EL | CP | EL | CP |
| (3,3) | | 0.5218 | 0.953 | 0.3181 | 0.854 | 0.5218 | 0.953 |
| (3,5) | | 0.4583 | 0.946 | 0.3062 | 0.876 | 0.4583 | 0.946 |
| (3,7) | | 0.4292 | 0.951 | 0.2979 | 0.98 | 0.4292 | 0.951 |
| (5,3) | | 0.4831 | 0.951 | 0.5252 | 0.961 | 0.4831 | 0.951 |
| (5,5) | 0.2 | 0.4064 | 0.952 | 0.4337 | 0.97 | 0.4064 | 0.952 |
| (5,7) | | 0.371 | 0.948 | 0.3888 | 0.971 | 0.371 | 0.948 |
| (7,3) | | 0.4659 | 0.949 | 0.5073 | 0.94 | 0.4659 | 0.949 |
| (7,5) | | 0.3823 | 0.946 | 0.4107 | 0.959 | 0.3823 | 0.946 |
| (7,7) | | 0.343 | 0.948 | 0.3626 | 0.961 | 0.343 | 0.948 |
| | | | | | | | |
| (3,3) | | 0.6452 | 0.953 | 0.6352 | 0.995 | 0.6452 | 0.953 |
| (3,5) | | 0.5981 | 0.946 | 0.5856 | 0.983 | 0.5981 | 0.946 |
| (3,7) | | 0.5752 | 0.951 | 0.5602 | 0.979 | 0.5752 | 0.951 |
| (5,3) | | 0.5974 | 0.951 | 0.5855 | 0.989 | 0.5974 | 0.951 |
| (5,5) | 0.5 | 0.5369 | 0.952 | 0.5273 | 0.985 | 0.5369 | 0.952 |
| (5,7) | | 0.5067 | 0.948 | 0.4969 | 0.983 | 0.5067 | 0.948 |
| (7,3) | | 0.5754 | 0.949 | 0.5605 | 0.983 | 0.5754 | 0.949 |
| (7,5) | | 0.5069 | 0.946 | 0.4973 | 0.982 | 0.5069 | 0.946 |
| (7,7) | | 0.4703 | 0.948 | 0.4624 | 0.977 | 0.4703 | 0.948 |
| | | | | | | | |
| (3,3) | | 0.5669 | 0.953 | 0.5841 | 0.978 | 0.5669 | 0.953 |
| (3,5) | | 0.5267 | 0.946 | 0.5456 | 0.964 | 0.5267 | 0.946 |
| (3,7) | | 0.508 | 0.951 | 0.5263 | 0.959 | 0.508 | 0.951 |
| (5,3) | | 0.5045 | 0.951 | 0.5105 | 0.981 | 0.5045 | 0.951 |
| (5,5) | 0.75 | 0.4535 | 0.952 | 0.4644 | 0.967 | 0.4535 | 0.952 |
| (5,7) | | 0.4283 | 0.948 | 0.4402 | 0.962 | 0.4283 | 0.948 |
| (7,3) | | 0.4758 | 0.949 | 0.4742 | 0.983 | 0.4758 | 0.949 |
| (7,5) | | 0.4182 | 0.946 | 0.4237 | 0.97 | 0.4182 | 0.946 |
| (7,7) | | 0.3879 | 0.948 | 0.3956 | 0.962 | 0.3879 | 0.948 |
| | | | | | | | |
| (3,3) | | 0.3824 | 0.953 | 0.5063 | 0.903 | 0.3824 | 0.953 |
| (3,5) | | 0.3451 | 0.946 | 0.4709 | 0.863 | 0.3451 | 0.946 |
| (3,7) | | 0.3289 | 0.951 | 0.4535 | 0.818 | 0.3289 | 0.951 |
| (5,3) | | 0.3191 | 0.951 | 0.3976 | 0.941 | 0.3191 | 0.951 |
| (5,5) | 0.9 | 0.2756 | 0.952 | 0.3579 | 0.907 | 0.2756 | 0.952 |
| (5,7) | | 0.2553 | 0.948 | 0.337 | 0.874 | 0.2553 | 0.948 |
| (7,3) | | 0.2908 | 0.949 | 0.3438 | 0.963 | 0.2908 | 0.949 |
| (7,5) | | 0.2446 | 0.946 | 0.3029 | 0.928 | 0.2446 | 0.946 |
| (7,7) | | 0.2226 | 0.948 | 0.2804 | 0.897 | 0.2226 | 0.948 |

Table 3.3 Average estimates (AVR), biases and MSE's of the estimators of $R$.

| (n,m) | R | MLE | | | Bayes | | |
|---|---|---|---|---|---|---|---|
| | | AVR | bias | MSE | AVR | bias | MSE |
| (3,3) | | 0.1724 | -0.0276 | 0.031 | 0.3337 | 0.1337 | 0.0482 |
| (3,5) | | 0.2056 | 0.0056 | 0.032 | 0.3154 | 0.1154 | 0.0342 |
| (3,7) | | 0.2286 | 0.0286 | 0.0327 | 0.3066 | 0.1066 | 0.0288 |
| (5,3) | | 0.1628 | -0.0374 | 0.0201 | 0.3713 | 0.1713 | 0.0753 |
| (5,5) | 0.2 | 0.1722 | -0.0278 | 0.0169 | 0.1673 | -0.0326 | 0.0331 |
| (5,7) | | 0.1825 | -0.0175 | 0.0156 | 0.1818 | -0.0182 | 0.0358 |
| (7,3) | | 0.1949 | -0.0052 | 0.0211 | 0.2792 | 0.0792 | 0.0749 |
| (7,5) | | 0.1762 | -0.0238 | 0.0142 | 0.2195 | 0.0195 | 0.0431 |
| (7,7) | | 0.1759 | -0.0241 | 0.0119 | 0.1202 | -0.0798 | 0.0381 |
| (3,3) | | 0.4993 | -0.0007 | 0.074 | 0.5031 | 0.0031 | 0.3222 |
| (3,5) | | 0.5661 | 0.0661 | 0.0602 | 0.5182 | 0.0182 | 0.0355 |
| (3,7) | | 0.5821 | 0.0821 | 0.0513 | 0.5899 | 0.0899 | 0.058 |
| (5,3) | | 0.4334 | -0.0666 | 0.06 | 0.4342 | -0.0658 | 0.0378 |
| (5,5) | 0.5 | 0.496 | -0.0039 | 0.0445 | 0.5951 | 0.0951 | 0.0321 |
| (5,7) | | 0.5281 | 0.0281 | 0.0363 | 0.4862 | -0.0138 | 0.0202 |
| (7,3) | | 0.4136 | -0.0864 | 0.0523 | 0.4142 | -0.0858 | 0.0477 |
| (7,5) | | 0.4659 | -0.034 | 0.0365 | 0.4118 | -0.0882 | 0.0299 |
| (7,7) | | 0.4978 | -0.0022 | 0.0296 | 0.6846 | 0.1846 | 0.0641 |
| (3,3) | | 0.7738 | 0.0238 | 0.0428 | 0.8458 | 0.0958 | 0.0259 |
| (3,5) | | 0.7869 | 0.0369 | 0.028 | 0.7569 | 0.0069 | 0.0519 |
| (3,7) | | 0.7639 | 0.0139 | 0.0248 | 0.8309 | 0.0809 | 0.0774 |
| (5,3) | | 0.7316 | -0.0183 | 0.0443 | 0.723 | -0.0269 | 0.0225 |
| (5,5) | 0.75 | 0.7752 | 0.0252 | 0.0236 | 0.8786 | 0.1286 | 0.0449 |
| (5,7) | | 0.7719 | 0.0219 | 0.0179 | 0.8821 | 0.1321 | 0.0451 |
| (7,3) | | 0.7072 | -0.0428 | 0.0413 | 0.6373 | -0.1127 | 0.0338 |
| (7,5) | | 0.7575 | 0.0075 | 0.0215 | 0.8497 | 0.0997 | 0.038 |
| (7,7) | | 0.7649 | 0.0149 | 0.0159 | 0.6568 | -0.0932 | 0.0452 |
| (3,3) | | 0.9105 | 0.0105 | 0.0118 | 0.9737 | 0.0737 | 0.0108 |
| (3,5) | | 0.9107 | 0.0107 | 0.0107 | 0.9958 | 0.0958 | 0.0094 |
| (3,7) | | 0.8867 | -0.0133 | 0.016 | 0.929 | 0.029 | 0.0288 |
| (5,3) | | 0.8972 | -0.0028 | 0.0117 | 0.9306 | 0.0306 | 0.0119 |
| (5,5) | 0.9 | 0.9144 | 0.0144 | 0.0067 | 0.9586 | 0.0586 | 0.0085 |
| (5,7) | | 0.9091 | 0.0091 | 0.0068 | 0.9994 | 0.0994 | 0.0099 |
| (7,3) | | 0.8924 | -0.0076 | 0.0113 | 0.9716 | 0.0716 | 0.0088 |
| (7,5) | | 0.9123 | 0.0123 | 0.0056 | 0.9974 | 0.0974 | 0.0096 |
| (7,7) | | 0.9121 | 0.0121 | 0.0049 | 0.9001 | 0.0001 | 0.0214 |

Table 3.4 Expected lengths (EL) and coverage probability (CP) of the asymptotic confidence interval with $(1-\alpha)=0.95$.

| (n,m) | R | Asymptotic con_intl | |
|---|---|---|---|
| | | EL | CP |
| (3,3) | | 0.3596 | 0.572 |
| (3,5) | | 0.3758 | 0.631 |
| (3,7) | | 0.3908 | 0.691 |
| (5,3) | | 0.3362 | 0.611 |
| (5,5) | 0.2 | 0.3133 | 0.649 |
| (5,7) | | 0.3075 | 0.697 |
| (7,3) | | 0.3674 | 0.683 |
| (7,5) | | 0.3018 | 0.691 |
| (7,7) | | 0.2798 | 0.688 |
| (3,3) | | 0.5631 | 0.618 |
| (3,5) | | 0.5434 | 0.662 |
| (3,7) | | 0.5381 | 0.701 |
| (5,3) | | 0.5438 | 0.664 |
| (5,5) | 0.5 | 0.509 | 0.709 |
| (5,7) | | 0.4905 | 0.739 |
| (7,3) | | 0.5349 | 0.678 |
| (7,5) | | 0.4901 | 0.74 |
| (7,7) | | 0.4618 | 0.774 |
| (3,3) | | 0.4251 | 0.58 |
| (3,5) | | 0.4036 | 0.641 |
| (3,7) | | 0.4214 | 0.712 |
| (5,3) | | 0.4362 | 0.657 |
| (5,5) | 0.75 | 0.375 | 0.674 |
| (5,7) | | 0.3639 | 0.723 |
| (7,3) | | 0.4534 | 0.712 |
| (7,5) | | 0.3724 | 0.725 |
| (7,7) | | 0.3438 | 0.747 |
| (3,3) | | 0.2231 | 0.535 |
| (3,5) | | 0.2024 | 0.545 |
| (3,7) | | 0.2289 | 0.613 |
| (5,3) | | 0.2304 | 0.637 |
| (5,5) | 0.9 | 0.1779 | 0.586 |
| (5,7) | | 0.1741 | 0.603 |
| (7,3) | | 0.2293 | 0.666 |
| (7,5) | | 0.171 | 0.63 |
| (7,7) | | 0.1579 | 0.635 |

2. Non-informative Prior : Jeffreys prior

We report all the results based on 2000 replications.

**Case I**: In this case, we obtained the average estimates, biases and MSE's of the MLE and the approximate Bayes estimator of $R$ based on the two priors. We also obtained the coverage probability and expected length for the confidence intervals by using the maximum likelihood and Bayes methods. The results are reported in Tables 3.1 and 3.2.

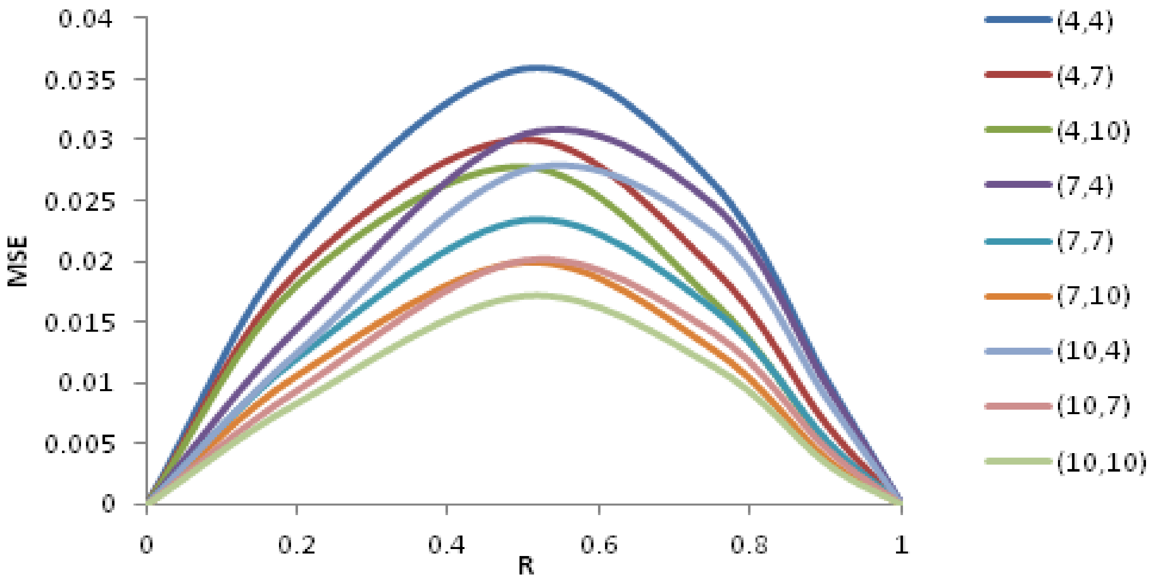


Fig. 3.3 Curve of MSE of $\hat{R}$ against $R$

**Case II**: In this case, first we obtained the MLE of $\beta$ and then computed the average estimates, biases and MSE's of the MLE and the approximate Bayes estimator of $R$ based on conjugate prior. We also computed the expected lengths of the asymptotic confidence interval.

The results are reported in Tables 3.3 and 3.4. Based on our simulation results, it is observed that as the sample sizes $(n,m)$ increase the MSE's decrease. Thus the consistency property of all the methods is satisfied. Increasing the sample size on either variable results in shorter intervals. It is observed that the length of the interval is maximized when $R = 0.5$ and gets shorter and shorter as we move away to smaller and larger values of $R$. For the Bayes estimators based on conjugate prior, the estimators and credible intervals are sensitive to the assumed values of hyper parameters.

From Table 3.2, it is also observed that the results of the both interval based on MLE and the interval based on non-informative prior are the same. Further the confidence intervals based on ML method have the coverage probability (CP) independent of the value of $R$. The CP for the confidence interval based on conjugate prior increases as $R$ increases from 0.2 to 0.5 and then decreases.
The results given in Table 3.3 showed a decrease of MSE as the sample size increases satisfying the consistency property of the proposed estimators. From Table 3.4 it is observed that expected length of the asymptotic confidence intervals is maximum when $R = 0.5$ and it decreases as we move away to smaller or larger values.

## 3.5 Data analysis

In this section we analyze a real data to illustrate the use of our proposed estimation methods. The data from Crowder [45] give the lifetimes of the steel specimens tested at two different stress levels.
DataSet 1: (38 stress level): $100, 90, 59, 80, 128, 117, 177, 98, 158, 107, 125, 118$

DataSet 2: (37 stress level): $141, 143, 98, 122, 110, 132, 194, 155, 104, 83, 125, 165, 146, 100$

We fit the inverse Chen distribution to the two data sets separately. The estimated parameters (based on ml methods), Kolmogorov-Smirinov(K-S) distances between the fitted and the empirical distribution functions and corresponding p-values are presented in Table 3.5. From the Table it is

Table 3.5 K-S distances and p-values

| Data set | Shape parameter1 | Shape parameter2 | K-S distance | p-value |
|---|---|---|---|---|
| 1 | 10697998 | 3.5936 | 0.1604 | 0.6253 |
| 2 | 32610674 | 3.5936 | 0.0721 | 0.9997 |

clear that inverse Chen distribution with common second shape parameter fits quite well to both data. For the above data we observe the lower record values as follows

$$r : 100, 90, 59; \quad s : 141, 98, 83.$$

We consider the following two cases:

**Case 1:** Based on these record values and assuming $\hat{\beta} = 3.5936$ we obtain the MLE's of $\gamma$ and $\delta$ from (3.10) as 6931942 and 23633076 respectively. Therefore the MLE of $R$ becomes $\hat{R} = 0.2268$. The corresponding 95% confidence interval based on (3.13) is $(0.0479, 0.6306)$. By assuming Jeffreys prior the Bayes estimator is obtained as 0.2587 and the corresponding 95% confidence interval based on (3.29) is equal to $(0.0479, 0.6306)$.

**Case II:** When $\beta$ is unknown, we obtain the MLE's of $\beta, \gamma$ and $\delta$ from (3.14) as 3.6453, 8558704 and 29698643 respectively. Thus the MLE of $R$ is obtained as $\hat{R}$ = 0.2237. The asymptotic 95% confidence interval of $R$ from (3.17) is obtained as (0, 0.5016). Based on 1000 bootstrap samples, the 95% bootstrap confidence intervals from (3.18) to (3.20) are obtained as (0, 0.5316), (0, 0.4327) and (0.0146, 0.6618) respectively. We use the hyperparameters $a_1 = a_2 = a_3 = 0.5,\ b_1 = b_2 = b_3 = 1$. In this case we obtain the approximate Bayes estimator of $R$, $\bar{R}_B = 0.2506$. Also the 95% HPD credible interval from (3.33) is obtained as $(0.0139, 0.5720)$.

The simulated values of $R$ and histogram of $R$ generated by the algorithm of Gibbs sampling are plotted in Fig.3.4.

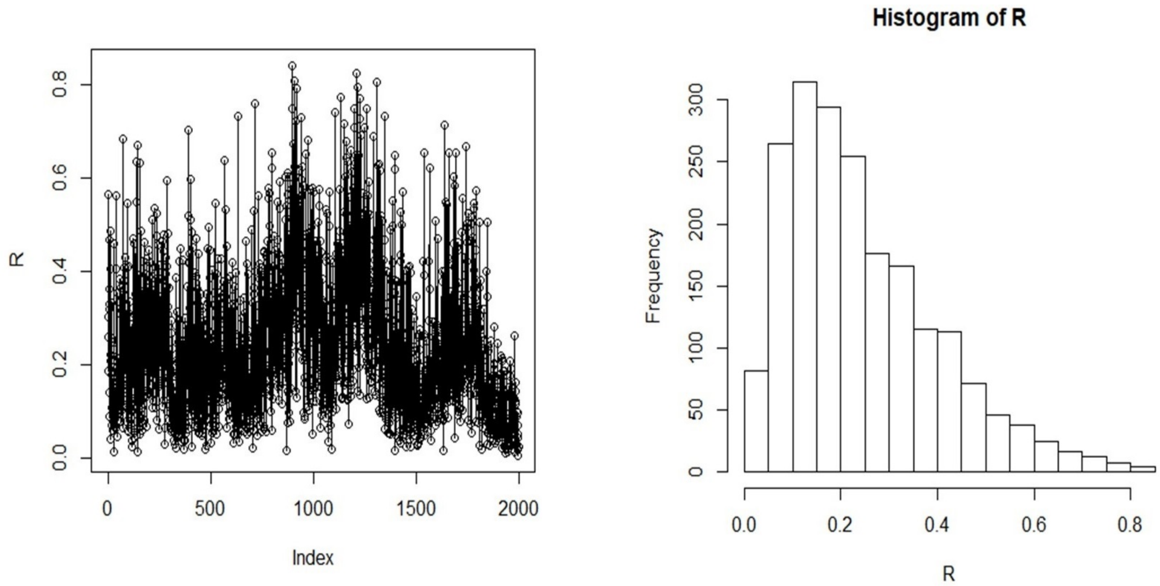


Fig. 3.4 Simulated values of $R$ and Histogram of $R$

## 3.6 Exponentiated inverse Chen distribution

We can define an exponentiated family of distributions by adding a parameter as the exponent to the cdf of the given distribution. Gupta et al. [52] introduced exponentiated exponential (EE) distribution as a generalization of exponential distribution. Nadarajah [53] introduced exponentiated Gumbel (GE) distribution. By adding a parameter to the c.d.f of inverse Chen distribution, we can define exponentiated inverse Chen distribution. The cdf of inverse Chen distribution is given by

$$F(x) = e^{\gamma(1-e^{x^{-\beta}})}, \qquad x > 0, \gamma, \beta > 0 \tag{3.34}$$

where $\gamma$ and $\beta$ are two shape parameters. The cdf of an exponentiated inverse-Chen distribution is defined as

$$F(x) = [e^{\gamma(1-e^{x^{-\beta}})}]^{\alpha}, \qquad x > 0, \alpha, \beta, \gamma > 0. \tag{3.35}$$

The corresponding pdf is given by

$$f(x) = \alpha\beta\gamma x^{-(\beta+1)} e^{x^{-\beta}} [e^{\gamma(1-e^{x^{-\beta}})}]^{\alpha}. \tag{3.36}$$

### 3.6.1 Likelihood inference

Let

$$\begin{aligned} &X \sim EICD(\alpha_1, \beta, \gamma) \qquad \text{and} \\ &Y \sim EICD(\alpha_2, \beta, \gamma) \end{aligned} \tag{3.37}$$

Then

$$\begin{aligned} R &= P(X > Y) \\ &= \int_0^{\infty}\int_{-\infty}^{x} \alpha_1\beta\gamma x^{-(\beta+1)} [e^{\gamma(1-e^{x^{-\beta}})}]^{\alpha_1} \alpha_2\beta\gamma y^{-(\beta+1)} [e^{\gamma(1-e^{y^{-\beta}})}]^{\alpha_2} dxdy \\ &= \frac{\alpha_1}{\alpha_1+\alpha_2} \end{aligned} \tag{3.38}$$

We would like to estimate $R = P(X > Y)$ based on lower records. Let $\underset{\sim}{r} = (r_1, r_2, ..., r_n)$ be a set of lower records from the distribution of $X$ and let $\underset{\sim}{s} = (s_1, s_2, ..., s_n)$ be an independent set of lower records from the distribution of $Y$. The likelihood function based on records is given by

$$\begin{aligned} L(\alpha_1, \alpha_2, \beta, \gamma) &= f(r_n)\prod_{i=1}^{n-1}\frac{f(r_i)}{1-F(r_i)} f(s_i)\prod_{i=1}^{m-1}\frac{f(s_i)}{1-F(s_i)} \\ &= \alpha_1^n \gamma^n \beta^n [e^{\gamma(1-e^{r_n^{-\beta}})}]^{\alpha_1} \prod_{i=1}^{n} r_i^{-(\beta+1)} e^{r_i^{-\beta}} \\ &\times \alpha_2^m \gamma^m \beta^m [e^{\gamma(1-e^{s_m^{-\beta}})}]^{\alpha_2} \prod_{i=1}^{m} s_i^{-(\beta+1)} e^{s_i^{-\beta}}. \end{aligned} \tag{3.39}$$

The loglikelihood is given by

$$
\begin{aligned}
l = & \, n\log\alpha_1 + n\log\gamma + n\log\beta + \alpha_1\gamma(1-e^{r_n^{-\beta}}) + \sum_{i=1}^{n}[-(\beta+1)\log r_i + r_i^{-\beta}] \\
& + m\log\alpha_2 + m\log\gamma + m\log\beta + \alpha_2\gamma(1-e^{s_m^{-\beta}}) + \sum_{i=1}^{m}[-(\beta+1)\log s_i + s_i^{-\beta}]. \qquad (3.40)
\end{aligned}
$$

When $\gamma$ and $\beta$ are known,

$$
\frac{\partial l}{\partial \alpha_1} = \frac{n}{\alpha_1} + \gamma(1-e^{r_n^{-\beta}}) = 0 \Rightarrow \text{the MLE of } \alpha_1, \;\; \hat{\alpha}_1 = \frac{-n}{\gamma(1-e^{r_n^{-\beta}})}. \qquad (3.41)
$$

Similarly, the MLE of $\alpha_2$

$$
\hat{\alpha}_2 = \frac{-m}{\gamma(1-e^{s_m^{-\beta}})}. \qquad (3.42)
$$

Therefore the MLE of $R$ is given by

$$
\hat{R} = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\alpha}_2}. \qquad (3.43)
$$

When $\beta$ is unknown and $\gamma$ is known

$$
\begin{aligned}
\frac{\partial l}{\partial \beta} = & \, \frac{m+n}{\beta} - \frac{n e^{r_n^{-\beta}} r_n^{-\beta} \log r_n}{1-e^{r_n^{-\beta}}} - \sum_{i=1}^{n} \log r_i (1 + r_i^{-\beta}) \\
& - \frac{m e^{s_m^{-\beta}} s_m^{-\beta} \log s_m}{1-e^{s_m^{-\beta}}} - \sum_{i=1}^{m} \log s_i (1 + s_i^{-\beta}). \qquad (3.44)
\end{aligned}
$$

Hence $\hat{\beta}$, the MLE of $\beta$, can be obtained by solving the non-linear equation of the form

$$
\beta = h(\beta) \qquad (3.45)
$$

where

$$h(\beta) = (m+n)[\frac{ne^{r_n^{-\beta}} r_n^{-\beta} \log r_n}{1-e^{r_n^{-\beta}}} + \sum_{i=1}^{n} \log r_i (1+r_i^{-\beta})$$
$$+ \frac{me^{s_m^{-\beta}} s_m^{-\beta} \log s_m}{1-e^{s_m^{-\beta}}} + \sum_{i=1}^{m} \log s_i (1+s_i^{-\beta})]^{-1}. \quad (3.46)$$

## 3.6.2 Bayesian inference when $\beta$ is known

$\alpha_1$ and $\alpha_2$ are assumed to have prior gamma distributions:

$$\alpha_1 \sim G(a_1, b_1) \quad ; \quad \alpha_2 \sim G(a_2, b_2) \quad (3.47)$$

Using the likelihood and priors we get the joint posterior whose numerator is given by

$$\gamma^{n+n} \beta^{m+n} \alpha_1^{n+a_1-1} \alpha_2^{m+a_2-1} e^{-(b_1-\gamma(1-e^{r_n^{-\beta}}))\alpha_1}$$
$$\times e^{-(b_2-\gamma(1-e^{s_m^{-\beta}}))\alpha_2} \prod_{i=1}^{n} e^{r_i^{-\beta}} \prod_{i=1}^{m} e^{s_i^{-\beta}} \left( \prod_{i=1}^{n} r_i \prod_{i=1}^{m} s_i \right)^{-(\beta+1)} \quad (3.48)$$

from which we get the posterior distribution of $\alpha_1$ and $\alpha_2$ as

$$(\alpha_1 | \underset{\sim}{r}) \sim G(n+a_1, \quad b_1 - \gamma(1-e^{r_n^{-\beta}}))$$
$$(\alpha_2 | \underset{\sim}{s}) \sim G(m+a_2, \quad b_2 - \gamma(1-e^{s_m^{-\beta}})) \quad (3.49)$$

By transformation $r = \frac{\alpha_1}{\alpha_1+\alpha_2}; \; t = \alpha_1$

$$f(r,t) = \Pi(\alpha_1/\underset{\sim}{r})\Pi(\alpha_2/\underset{\sim}{s})|J,| \qquad |J| = \frac{t}{r^2}$$

$$f(r,t) = \frac{A^{n+a_1} B^{m+a_2}}{\Gamma(n+a_1)\Gamma(m+a_2)} e^{-At} t^{n+a_1-1} [t\left(\frac{1-r}{r}\right)]^{m+a_2-1} e^{-Bt\left(\frac{1-r}{r}\right)} \frac{t}{r^2}$$

$$= \frac{A^{n+a_1} B^{m+a_2}}{\Gamma(n+a_1)\Gamma(m+a_2)} e^{-(A+B\left(\frac{1-r}{r}\right))t} t^{n+m+a_1+a_2-1} \frac{(1-r)^{m+a_2-1}}{r^{m+a_2-1}} \tag{3.50}$$

$$f(r) = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)} \frac{A^{n+a_1} B^{m+a_2} r^{n+a_1-1} (1-r)^{m+a_2-1}}{(Ar+B(1-r))^{n+m+a_1+a_2}}, \tag{3.51}$$

where

$$A = b_1 - \gamma(1 - e^{r_n^{-\beta}}); \qquad B = b_2 - \gamma(1 - e^{s_m^{-\beta}}).$$

To find the Bayes estimator, we use Lindley's method which is obtained as

$$\bar{R}_B = \hat{R} + \hat{R}(1-\hat{R}) \left[ \frac{1-R}{m} - \frac{R}{n} + \frac{(a_1 - b_1\alpha_1)}{n} - \frac{(a_2 - b_2\alpha_2)}{m} \right] \tag{3.52}$$

From 3.49 we get

$$2(n+a_1)A\alpha_1 \sim \chi^2(2(n+a_1))$$

$$2(m+a_2)B\alpha_2 \sim \chi^2(2(m+a_2)). \tag{3.53}$$

Hence

$$\frac{2(m+a_2)B\alpha_2/2(m+a_2)}{2(n+a_1)A\alpha_1/2(n+a_1)} \sim F(2(m+a_2), 2(n+a_1)). \tag{3.54}$$

Therefore, a $100(1-\alpha)\%$ CI for $R$ is given by

$$\left[ \left(1 + \frac{A}{B} F_{1-\alpha/2}\right)^{-1}, \left(1 + \frac{A}{B} F_{\alpha/2}\right)^{-1} \right]. \tag{3.55}$$

### 3.6.3 Bayesian inference when $\beta$ is unknown

$\alpha_1$ & $\alpha_2$ are assumed to have gamma priors

$$\alpha_1 \sim G(a_1, b_1); \qquad \alpha_2 \sim G(a_2, b_2) \tag{3.56}$$

$\beta$ is assumed to have Jeffreys non-informative prior

$$\pi(\beta) \propto 1/\beta, \qquad \beta > 0. \tag{3.57}$$

Then the joint posterior distribution whose numerator is given by

$$\gamma^{m+n}\beta^{m+n-1}\alpha_1^{n+a_1-1}\alpha_2^{m+a_2-1}e^{-(b1-\gamma(1-e^{r_n^{-\beta}}))\alpha_1}$$
$$\times e^{-(b_2-\gamma(1-e^{s_m^{-\beta}}))\alpha_2}\prod_{i=1}^{n}e^{r_i^{-\beta}}\prod_{i=1}^{m}e^{s_i^{-\beta}}\left[\prod_{i=1}^{n}r_i\prod_{i=1}^{m}s_i\right]^{-(\beta+1)}. \tag{3.58}$$

It is not possible to obtain explicit expressions for the posterior distributions. So, we apply Gibbs sampling which uses the posterior distributions of each parameters conditional upon the others. The conditional posterior distributions are

$$(\alpha_1|\alpha_2,\beta) \sim G(n+a_1, A) \tag{3.59}$$

$$(\alpha_2|\alpha_1,\beta) \sim G(m+a_2, B) \tag{3.60}$$

$$\text{and} \quad \pi(\beta) \propto \beta^{m+n-1}e^{-\gamma\alpha_1 e^{r_n^{-\beta}}}e^{-\gamma\alpha_2 e^{s_m^{-\beta}}}\left[\prod_{i=1}^{n}r_i e^{r_i}\prod_{i=1}^{m}s_i e^{s_i}\right]^{-\beta} \tag{3.61}$$

We can easily generate sample of $\alpha_1$ & $\alpha_2$ from their posterior distributions. To generate samples from the posterior distribution of $\beta$, we use MCMC method which uses Metropoles-Hasting algorithm with $q(.)$ as proposal density. The algorithm is given as follows

Step 1: Start with $\beta^{(0)}=\hat{\beta}$ as an initial guess and set $t = 1$.

Step 2: Generate $\alpha_1^{(t)}$ from Gamma$(n+a_1, A)$

Step 3: Generate $\alpha_2^{(t)}$ from Gamma$(m+a_2, B)$

Step 4: Using Metropoles-Hasting method, generate $\beta^{(t)}$ from $\pi(\beta|\alpha_1, \alpha_2)$ with the proposal distribution as $q(\beta) \propto N(\beta^{(t-1)}, 0.25) \qquad (\beta > 0)$

Step 5: Compute $R^{(t)} = \frac{\alpha_1^{(t)}}{\alpha_1^{(t)}+\alpha_2^{(t)}}$

Step 6: Set $t = t+1$

Step 7: Repeat steps 2-6, N times

Now the approximate posterior mean gives the Bayes estimator given by

$$\hat{E}(R|\underset{\sim}{r}, \underset{\sim}{s}) = \frac{1}{N-M} \sum_{t=M+1}^{N} R^{(t)}, \tag{3.62}$$

where $M$ is the burn in period.

### 3.6.4 Numerical example

In this section we present the analysis of a data generated from exponentiated inverse Chen distribution to illustrate the proposed methods. First we generate 20 observations from EICD$(\gamma = 4, \alpha = 1, \beta = 2)$ and 20 observations from EICD$(\gamma = 4, \alpha = 3, \beta = 2)$ reported in Table 3.6.

Table 3.6 Exponentiated inverse Chen data

| x | y | x | y | x | y | x | y |
|---|---|---|---|---|---|---|---|
| 1.6 | 1.24 | 3.61 | 1.42 | 1.81 | 1.33 | 1.54 | 2.51 |
| 1.94 | 2.69 | 1.41 | 1.51 | 2.56 | 1.21 | 5.11 | 1.23 |
| 1.84 | 1.22 | 23.46 | 3.61 | 5.43 | 2.53 | 3.77 | 1.81 |
| 1.37 | 1.46 | 4.55 | 1.57 | 1.13 | 8.54 | 3.22 | 0.86 |
| 1.22 | 1.08 | 21.76 | 1.2 | 3.41 | 1.09 | 1.91 | 1.22 |

The lower records from these two sets of data are

$$r : 1.60, 1.37, 1.22, 1.13.$$

$$s : 1.24, 1.22, 1.08, 0.86.$$

It is assumed that $\gamma$ and $\beta$ are known and are equal to 4 & 2 respectively. Then the MLE of $\alpha_1$ and $\alpha_2$ are 1.1883 and 2.8656 respectively. Hence the MLE of $R$ is 0.2931 . The Bayes estimator of $R$ using the conjugate prior is computed as 0.3154.

When $\beta$ is assumed to be unknown the MLE of $\beta$ is computed to be 3.3034. The MLE's of $\alpha_1$ and $\alpha_2$ are obtained as 0.9499 and 4.1852 respectively. Hence the MLE of $R$ is 0.1849.

## 3.7 Conclusion

This chapter deals with the estimation of the stress-strength reliability $R = P(X > Y)$ based on lower record values where $X$ and $Y$ are independent random variables from an inverse Chen distribution with the same second shape parameter but different first shape parameters. The maximum likelihood and Bayesian methods is used for estimation of $R$ when the shape parameter is known and unknown separately. To investigate different proposed methods, a Monte Carlo simulation study is conducted. A real data analysis is also done. All computations are done in R software. From the simulation results, it is observed that the MSE's of the estimators for all the methods decrease as the sample size increase. Thus the consistency property of the estimators is found to be satisfied. It is also observed that the interval based on MLE is maximized when $R = 0.5$ and it becomes shorter and shorter as we move away to smaller and larger values. Increasing the sample size on either variable also results in shorter interval. We also observe that the Bayesian credible interval performs well when the true value of $R$ is close to both 0 and 1. Further the inference of $R$ for exponentiated inverse Chen distribution is discussed. A numerical example is also presented to illustrate the proposed methods.

# Chapter 4

# Estimation of stress-strength reliability for inverse Weibull distribution based on lower records

## 4.1 Introduction

The convenient structure of the distribution function has made the Weibull distribution to be used very effectively in studying various life-time data. We note that the hazard function of Weibull distribution is decreasing or increasing depending on the shape parameter. But this distribution cannot be used when the data has a non-monotone hazard function and the inverse Weibull may be an appropriate model (Kundu et al. [54]).

If $T$ has a Weibull distribution with a pdf,

$$f(t) = \frac{\alpha}{\theta} t^{\alpha-1} e^{\frac{-t^{\alpha}}{\theta}}, \ \ t > 0.$$

where $\alpha > 0$ is the shape parameter and $\theta > 0$ is the scale parameter, then the random variable $X = \frac{1}{T}$ has an inverse Weibull distribution with pdf

$$f(x) = \frac{\alpha}{\theta} x^{-\alpha-1} e^{\frac{-x^{-\alpha}}{\theta}}, \quad x > 0, \alpha > 0, \theta > 0.$$

The cdf is given by

$$F(x) = e^{\frac{-x^{-\alpha}}{\theta}}, \quad x > 0, \alpha > 0, \theta > 0. \tag{4.1}$$

Moreover the inverse Weibull distribution plays some important role in other areas such as describing

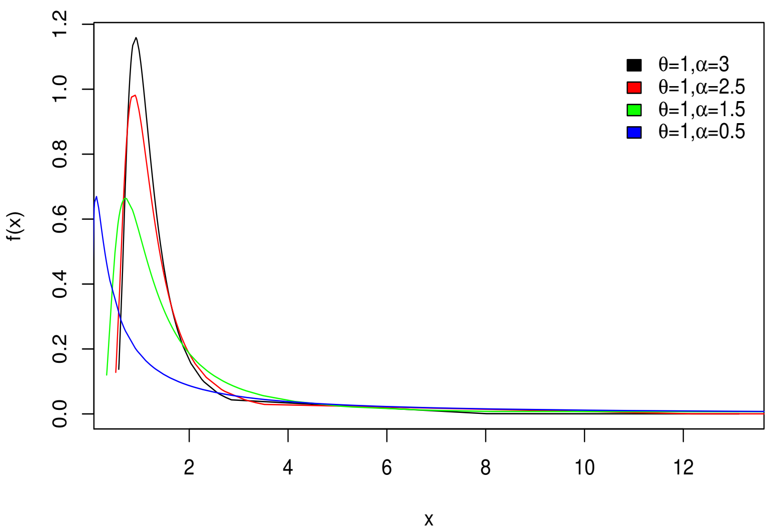


Fig. 4.1 pdf of inverse Weibull distribution

the degradation phenomena of components, describing the context of a load-strength relationship for a component and providing a good fit to survival data (Kundu et al.[54]). It is a lifetime distribution which can be used in the reliability engineering. This distribution has the ability to model failure rates which is common in reliability and biological studies (Kundu and Howalder [55]). Estimation of stress-strength reliability for inverse-Weibull distribution under progressive Type II censoring scheme was considered by Abhimanyu et al. [56]. Inference on stress-strength model for this distribution was done by Li and Hao [57] using random samples. Here we propose to study the inference of stress-strength model using lower record values from inverse Weibull distribution.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution of $X \sim IW(\alpha, \theta_1)$ and $Y_1, Y_2, \ldots, Y_n$ be an independent random sample from the distribution of $Y \sim IW(\alpha, \theta_2)$. Then,

$$R = P(X < Y) = \int_0^\infty F_X(z|\theta) f_Y(z|\theta) dz = \frac{\theta_1}{\theta_1 + \theta_2}.$$

## 4.2 Likelihood inference

Let $\underset{\sim}{r} = (r_1, r_2, \ldots, r_n)$ be a set of lower records from $X \sim IW(\alpha, \theta_1)$ and $\underset{\sim}{s} = (s_1, s_2, \ldots, s_n)$ be an independent set of lower records from $Y \sim IW(\alpha, \theta_2)$. The joint likelihood function and log joint likelihood function based on $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are given respectively by

$$L(\alpha, \theta_1, \theta_2, \underset{\sim}{r}, \underset{\sim}{s}) = \frac{\alpha^n}{\theta_1^n} e^{-\frac{r_n^{-\alpha}}{\theta_1}} \prod_{i=1}^n r_i^{-\alpha-1} \frac{\alpha^n}{\theta_2^n} e^{-\frac{s_m^{-\alpha}}{\theta_2}} \prod_{i=1}^m s_i^{-\alpha-1}. \tag{4.2}$$

$$\begin{aligned} l(\alpha, \theta_1, \theta_2, \underset{\sim}{r}, \underset{\sim}{s}) &= n\log\alpha - n\log\theta_1 - \frac{r_n^{-\alpha}}{\theta_1} + \sum_{i=1}^n -(\alpha+1)\log r_i + m\log\alpha - m\log\theta_2 - \frac{s_m^{-\alpha}}{\theta_2} \\ &+ \sum_{i=1}^m -(\alpha+1)\log s_i. \end{aligned} \tag{4.3}$$

$$\frac{\delta l}{\delta \theta_1} = 0 \quad \Rightarrow \quad \frac{-n}{\theta_1} + \frac{r_n^{-\alpha}}{\theta_1^2} = 0. \tag{4.4}$$

$$\frac{\delta l}{\delta \theta_2} = 0 \quad \Rightarrow \quad \frac{-m}{\theta_2} + \frac{s_m^{-\alpha}}{\theta_2^2} = 0. \tag{4.5}$$

$$\frac{\delta l}{\delta \alpha} = \frac{(m+n)}{\alpha} + n\log r_n + m\log s_m - \left(\sum_{i=1}^{n} \log r_i + \sum_{i=1}^{m} \log s_i\right) = 0 \tag{4.6}$$

### 4.2.1 When shape parameter $\alpha$ is known

Under the assumption that the shape parameter $\alpha$ is known, the MLE's of $\theta_1$ and $\theta_2$ respectively can be obtained from (4.4) and (4.5) as

$$\hat{\theta}_1 = \frac{r_n^{-\alpha}}{n} \ ; \quad \hat{\theta}_2 = \frac{s_m^{-\alpha}}{m} \tag{4.7}$$

Then the MLE of R is given by

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}.$$

In order to study the distribution of $\hat{R}$ one should first obtained the distribution of

$$\hat{\theta}_1 = \frac{r_n^{-\alpha}}{n}.$$

The pdf of $R_n$ is given by

$$\begin{aligned} f_{R_n}(r_n) &= \frac{1}{(n-1)!} f(r_n)\left[-\log F(r_n)\right]^{n-1} \\ &= \frac{1}{(n-1)!}\left(\frac{r_n^{-\alpha}}{\theta_1}\right)^{n-1} \frac{\alpha}{\theta_1} r_n^{-(\alpha+1)} e^{\frac{-r_n^{-\alpha}}{\theta_1}}, \ r_n > 0. \end{aligned} \tag{4.8}$$

Therefore the pdf of $Z_1 = \hat{\theta}_1$ is given by

$$f_{Z_1}(z_1) = \frac{1}{(n-1)!}\left(\frac{n}{\theta_1}\right)^n z_1^{n-1} e^{\frac{-nz_1}{\theta_1}}, \;\; z_1 > 0. \tag{4.9}$$

This is identified as the gamma distribution. ie

$$Z_1 \sim G\left(n, \; \frac{n}{\theta_1}\right).$$

Similarly, for $Z_2 = \hat{\theta}_2$, we can obtain

$$Z_2 \sim G\left(m, \; \frac{m}{\theta_2}\right).$$

Therefore we can find the pdf of

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2} = \frac{Z_1}{Z_1 + Z_2} = \frac{1}{1 + \frac{Z_2}{Z_1}}.$$

We have

$$\frac{nz_1}{\theta_1} \sim G(n, 1); \quad \frac{mz_2}{\theta_2} \sim G(m, 1).$$

Hence

$$\frac{2nz_1}{\theta_1} \; \sim \; \chi^2_{2n}; \quad \frac{2mz_2}{\theta_2} \; \sim \; \chi^2_{2m}.$$

Clearly by the independence of two $\chi^2$ random variables, we have

$$\frac{2nz_1/2n\theta_1}{2mz_2/2m\theta_2} = \frac{\theta_2 z_1}{\theta_1 z_2} = \frac{1-R}{R} \sim F(2n,2m).$$

This fact leads us to the construction of the following $100(1-\gamma)\%$ confidence interval for $R$ as follows.

$$\left(\left(1+\frac{\hat{\theta}_2 F_{1-\frac{\gamma}{2},2n,2m}}{\hat{\theta}_1}\right)^{-1}, \left(1+\frac{\hat{\theta}_2 F_{\frac{\gamma}{2},2n,2m}}{\hat{\theta}_1}\right)^{-1}\right). \tag{4.10}$$

### 4.2.2 When shape parameter $\alpha$ is unknown

In this subsection we discuss likelihood inference of $R$ when all of the parameters $\theta_1, \theta_2$ and $\alpha$ are unknown. In this case by using (4.4) and (4.5) we obtain

$$\hat{\theta}_1 = \frac{r_n^{-\hat{\alpha}}}{n} \; ; \quad \hat{\theta}_2 = \frac{s_m^{-\hat{\alpha}}}{m} \tag{4.11}$$

where $\hat{\alpha}$ is the MLE of the parameter $\alpha$ which is obtained by solving the following equation.

$$\frac{(m+n)}{\alpha} + n\log r_n + m\log s_m - \left(\sum_{i=1}^{n}\log r_i + \sum_{i=1}^{m}\log s_i\right) = 0. \tag{4.12}$$

Therefore $\hat{\alpha}$ can be obtained as a solution of the equation

$$\hat{\alpha} = (m+n)\left[\sum_{i=1}^{n}\log r_i + \sum_{i=1}^{m}\log s_i - n\log r_n - m\log s_m\right]^{-1}. \tag{4.13}$$

Once we obtain $\hat{\alpha}$, $\hat{\theta}_1$ and $\hat{\theta}_2$ can be deduced from (4.11) and therefore the MLE of $R$ is computed to be

$$\hat{R} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}.$$

## 4.3 Bayesian inference

In this section we deal with the Bayesian method for making inference of $R$ based on lower record values. We consider two cases (i) The shape parameter $\alpha$ is known and (ii) $\alpha$ is unknown.

### 4.3.1 Known shape parameter $\alpha$

When $\alpha$ is known we assume two priors; namely conjugate prior and non-informative prior.

**Conjugate prior:** We assume conjugate prior of inverse gamma distribution to each of the parameters.

$$\theta_1 \sim InvG(a_1,\ b_1),\ a_1 > 0, b_1 > 0.$$

$$\theta_2 \sim InvG(a_2,\ b_2),\ a_2 > 0, b_2 > 0.$$

Using these priors and the likelihood function (4.2), we get the joint posterior density

$$\Pi(\theta_1, \theta_2 | \underset{\sim}{r}, \underset{\sim}{s}) = \frac{L(\underset{\sim}{r}, \underset{\sim}{s}; \theta_1, \theta_2)\pi(\theta_1)\pi(\theta_2)}{\int L(\underset{\sim}{r}, \underset{\sim}{s}; \theta_1, \theta_2)\pi(\theta_1)\pi(\theta_2) d\theta_1 d\theta 2},$$

where the numerator is given by

$$\frac{b_1^{a_1} e^{\frac{-(r_n^{-\alpha}+b_1)}{\theta_1}}}{\Gamma(a_1)\theta_1^{n+a_1+1}} \frac{b_2^{a_2} e^{\frac{-(s_m^{-\alpha}+b_2)}{\theta_2}}}{\Gamma(a_2)\theta_2^{m+a_2+1}} \alpha^{n+m} \left(\prod_{i=1}^{n} r_i \prod_{i=1}^{m} s_i\right)^{-(\alpha+1)}.$$

The posterior distribution of $\theta_1$ and $\theta_2$ are obtained to be

$$\begin{aligned} \theta_1 | \underset{\sim}{r} \sim & \quad InvG(n + a_1,\ r_n^{-\alpha} + b_1) \\ \theta_2 | \underset{\sim}{s} \sim & \quad InvG(m + a_2,\ s_m^{-\alpha} + b_2). \end{aligned} \tag{4.14}$$

Since the priors $\theta_1$ and $\theta_2$ are independent, then using standard transformation technique and after some manipulations the posterior pdf of $R$ will be

$$f_R(r) = C \frac{r^{m+a_2-1}(1-r)^{n+a_1-1}}{[(1-r)(r_n^{-\alpha} + b_1) + r(s_m^{-\alpha} + b_2)]^{n+m+a_1+a_2}}$$

where

$$C = \frac{\Gamma(n+m+a_1+a_2)}{\Gamma(n+a_1)\Gamma(m+a_2)} \left(r_n^{-\alpha} + b_1\right)^{n+a_1} \left(s_m^{-\alpha} + b_2\right)^{m+a_2}.$$

Under squared error loss function, the Bayes estimate of $R$ is the expected value of $R$ and this can be derived as

$$\begin{aligned} \hat{R}_B &= \int_0^1 r f_R(r) dr = \frac{m + a_2}{n + m + a_1 + a_2} \left( \frac{s_m^{-\alpha} + b_2}{r_n^{-\alpha} + b_1} \right)^{m+a_2} \\ &\times \ F_{2.1}\left( n + m + a_1 + a_2, m + a_2 + 1, n + m + a_1 + a_2, 1 - \frac{s_m^{-\alpha} + b_2}{r_n^{-\alpha} + b_1} \right). \end{aligned} \tag{4.15}$$

By the properties of the inverted gamma distribution and its relation with gamma distribution, we have from (4.14)

$$\frac{(r_n^{-\alpha} + b_1)}{\theta_1} \sim Gamma(n + a_1,\ 1)$$

$$\frac{(s_m^{-\alpha} + b_2)}{\theta_2} \sim Gamma(m + a_2,\ 1).$$

Thus

$$\frac{2(r_n^{-\alpha}+b_1)}{\theta_1} \sim \chi^2\left(2(n+a_1)\right)$$

$$\frac{2(s_m^{-\alpha}+b_2)}{\theta_2} \sim \chi^2\left(2(m+a_2)\right),$$

from which we get an $F$ variable,

$$F = \frac{2(r_n^{-\alpha}+b_1)/2(n+a_1)\theta_1}{2(s_m^{-\alpha}+b_2)/2(m+a_2)\theta_2} \sim F\left(2(n+a_1),\ 2(m+a_2)\right).$$

The distribution of $R$ is equal to the distribution of $\frac{1}{1+AF}$ where

$$F \sim F\left(2(n+a_1), 2(m+a_2)\right).$$

Hence a $100(1-\gamma)\%$ credible interval for $R$ is

$$\left((1+AF_{1-\frac{\gamma}{2}})^{-1},\ (1+AF_{\frac{\gamma}{2}})^{-1})\right) \tag{4.16}$$

where $A = \frac{s_m^{-\alpha}+b_2}{r_n^{-\alpha}+b_1}$.

**Non-informative prior**

We use Jeffreys non informative prior.

$$\pi(\theta_1) \propto \frac{1}{\theta_1}, \quad \theta_1 > 0;$$

$$\pi(\theta_2) \propto \frac{1}{\theta_2}, \quad \theta_2 > 0.$$

Then the posterior distributions of $\theta_1$ and $\theta_2$ are obtained as

$$\begin{aligned} \theta_1|\underset{\sim}{r} &\sim invG\left(n\ ,\ r_n^{-\alpha}\right) \\ \theta_2|\underset{\sim}{s} &\sim invG\left(m\ ,\ s_m^{-\alpha}\right). \end{aligned} \tag{4.17}$$

Using transformation the posterior pdf of R is obtained as

$$f_R(r) = C\frac{(1-r)^{n-1}r^{m-1}}{[rs_m^{-\alpha}+(1-r)r_n^{-\alpha}]^{n+m}}, \quad 0 < r < 1$$

where $C = \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}\left(\frac{s_m^{-\alpha}}{r_n^{-\alpha}}\right)^m$.

Then the Bayes estimator $\bar{R_B}$ relative to squared error loss function is given by the expected value of $f_R(r)$.

$$\begin{aligned} \bar{R_B} &= \int_0^1 r f_R(r)dr = \frac{m}{n+m}\left(\frac{s_m^{-\alpha}}{r_n^{-\alpha}}\right)^m \\ &\times \ F_{2.1}\left(n+m,\ m+1,\ n+m+1,\ 1-\frac{s_m^{-\alpha}}{r_n^{-\alpha}}\right). \end{aligned} \tag{4.18}$$

By the properties of inverse gamma and its relation with gamma distribution from (4.17) we have

$$\frac{r_n^{-\alpha}}{\theta_1} \sim Gamma(n,\ 1)$$

$$\frac{s_m^{-\alpha}}{\theta_2} \sim Gamma(m,\ 1).$$

Thus

$$\frac{2r_n^{-\alpha}}{\theta_1} \sim \chi^2\,(2n)$$

$$\frac{2s_m^{-\alpha}}{\theta_2} \sim \chi^2(2m),$$

from which we get an $F$ variable,

$$F = \frac{2r_n^{-\alpha}/2n\theta_1}{2s_m^{-\alpha}/2m\theta_2} \sim F(2n, 2m)).$$

The distribution of $R$ is equal to the distribution of $\frac{1}{1+AF}$ where

$$F \sim F(2n, 2m).$$

Hence a Bayesian $100(1-\gamma)\%$ credible interval for $R$ is given by

$$\left(\left(1+AF_{1-\frac{\gamma}{2},2n,2m}\right)^{-1}, \left(1+AF_{\frac{\gamma}{2},2n,2m}\right)^{-1}\right) \tag{4.19}$$

where $A = \frac{ns_m^{-\alpha}}{mr_n^{-\alpha}}$.

### 4.3.2 Unknown shape parameter $\alpha$

We assume conjugate priors of inverse gamma distribution for $\theta_1$ and $\theta_2$. $\alpha$ is assumed to follow gamma distribution. That is

$$\begin{aligned}
\theta_1 &\sim InvG(a_1, b_1) \\
\theta_2 &\sim InvG(a_2, b_2) \\
\alpha &\sim G(a_3, b_3)
\end{aligned}$$

Using these priors and the likelihood function, we get the joint posterior distribution

$$\Pi\left(\theta_1,\theta_2,\alpha;\underset{\sim}{r},\underset{\sim}{s}\right)=\frac{L\left(\theta_1,\theta_2,\alpha;\underset{\sim}{r},\underset{\sim}{s}\right)\pi(\theta_1)\pi(\theta_2)\pi(\alpha)}{\int L\left(\theta_1,\theta_2,\alpha;\underset{\sim}{r},\underset{\sim}{s}\right)\pi(\theta_1)\pi(\theta_2)\pi(\alpha)d\theta_1 d\theta_2 d\alpha}$$

where the numerator is

$$\frac{b_1^{a_1}}{\Gamma a_1}\frac{e^{-(r_n^{-\alpha}+b_1)/\theta_1}}{\theta_1^{n+a_1+1}}\frac{b_2^{a_2}}{\Gamma a_2}\frac{e^{-(s_m^{-\alpha}+b_2)/\theta_2}}{\theta_2^{m+a_2+1}}\frac{b_3^{a_3}}{\Gamma a_3}\alpha^{n+m+a_3-1}e^{-(b_3+\sum_{i=1}^{n}\log r_i+\sum_{i=1}^{m}\log s_i)\alpha}\left(\prod_{i=1}^{n}r_i\prod_{i=1}^{m}s_i\right)^{-1}.$$

To compute the Bayes estimate of $R$, we use Gibbs sampling method which uses the posterior distribution of each parameter conditional on others( Gelfand et al. [51]). The conditional posterior distributions of $\theta_1,\theta_2$ and $\alpha$ can be obtained as follows.

$$(\theta_1|\theta_2,\alpha,\underset{\sim}{r},\underset{\sim}{s}) \sim InvG\left(n+a_1,\ r_n^{-\alpha}+b_1\right) \quad (4.20)$$

$$(\theta_2|\theta_1,\alpha,\underset{\sim}{r},\underset{\sim}{s}) \sim InvG\left(m+a_2,\ s_m^{-\alpha}+b_2\right) \quad (4.21)$$

$$\Pi(\alpha|\theta_1,\theta_2,\underset{\sim}{r},\underset{\sim}{s}) \propto e^{-(r_n^{-\alpha}+b_1)/\theta_1}e^{-(s_m^{-\alpha}+b_2)/\theta_2}\alpha^{n+m+a_3-1}e^{-(b_3+\sum_{i=1}^{n}\log r_i+\sum_{i=1}^{m}\log s_i)\alpha}. \quad (4.22)$$

Though it is easy to generate samples of $\theta_1$ and $\theta_2$ from gamma distributions, it is not possible to sample directly from the conditional posterior distribution of $\alpha$. So we use MCMC technique in which Metropolis Hastings algorithm with $q(.)$ as a proposal distribution is applied (Metropolis et al. [20] and Hastings [21]).

Now the approximate posterior mean of $R$ is given by

$$\hat{E}(R|\underset{\sim}{r},\underset{\sim}{s})=\frac{1}{N-M}\sum_{t=M+1}^{N}R^{(t)} \quad (4.23)$$

where $R^{(t)}$ is the value of $R$ for the $i^{th}$ iteration, $N$ is the number of iterations and $M$ is the burn-in period (that is the number of iterations before the stationary distribution is achieved).

Based on $N$ and $R^{(t)}$ values, using the method proposed by Chen and Shao [24], a $100(1-\gamma)\%$ HPD credible interval can be constructed as

$$\left(R_{[\frac{\gamma}{2}N]}, R_{[(1-\frac{\gamma}{2})N]}\right) \tag{4.24}$$

where $R_{[\frac{\gamma}{2}N]}$ and $R_{[(1-\frac{\gamma}{2})N]}$ are the $[\frac{\gamma}{2}N]$th smallest integer and the $[(1-\frac{\gamma}{2})N]$ th smallest integer of $\{R^{(t)}, t = M+1, M+2, \ldots, N\}$, respectively.

## 4.4 Numerical example

In this section we present the analysis of a data generated from inverse Weibull distribution to illustrate the proposed methods. First we generated 20 observations from invWeibul$(\theta = 1, \alpha = 2)$ and 20 observations from invWeibul$(\theta = 2, \alpha = 2)$ and is reported in Table 4.1.

The lower records from these two sets of data are

$$r : 1.0760, 0.7941, 0.6939$$

$$s : 1.5888, 0.4499, 0.4246.$$

It is assumed that $\alpha$ is known and is equal to 2. Then the MLE of $R$ is 0.3545 and the corresponding confidence interval is (0.086250, 0.761735). The Bayes estimator of $R$ using the conjugate prior is computed as 0.37414 and the corresponding confidence interval is (0.086255, 0.761733). For

Table 4.1 Inverse Weibull data

| x | y | x | y | x | y | x | y |
|---|---|---|---|---|---|---|---|
| 1.076 | 0.8353 | 1.1207 | 0.6607 | 1.692 | 1.4757 | 0.6939 | 0.5143 |
| 0.7941 | 0.5369 | 5.8659 | 1.9002 | 1.7987 | 1.4649 | 1.3549 | 0.7207 |
| 3.0363 | 1.0283 | 3.4299 | 0.7631 | 1.2163 | 0.8523 | 3.937 | 0.6536 |
| 1.0379 | 0.6617 | 3.1152 | 1.3921 | 1.1658 | 0.5734 | 2.2469 | 0.6148 |
| 2.4776 | 0.5411 | 1.1378 | 1.1653 | 1.3286 | 1.6536 | 6.8603 | 0.895 |

non-informative prior the Bayes estimator is 0.37413 and the corresponding confidence interval is (0.08625, 0.761735).

When $\alpha$ is assumed to be unknown the mle is computed to be 0.1636 and the Bayes estimator is obtained as 0.2604. The HPD interval is computed as (0.0045, 0.6298).

## 4.5 Conclusion

This chapter deals with the estimation of the stress-strength reliability $R = P(X < Y)$ based on lower record values where $X$ and $Y$ are independent random variables from inverse Weibull distribution with the same shape parameter but different scale parameters. The maximum likelihood estimator, Bayes estimator and the corresponding confidence intervals of $R$ are obtained when the shape parameter is known. When the shape parameter is unknown maximum likelihood estimator, Bayes estimator using MCMC method and the HPD interval are found. A numerical example is done using $R$ software to illustrate the proposed methods presented in this chapter.

# Chapter 5

# Summary and final conclusions

This thesis deals with estimation of stress strength reliability for some distributions based on records. In reliability engineering, if the stress experienced by a system is greater than the strength of the system, the system fails. Then the probability of not failing is called stress-strength reliability. This probability can be expressed by P(X<Y) or P(X>Y) according to the representation of X and Y. This probability has applications in industry, medical science, life testing etc. In life testing when $X$ and $Y$ represent the life lengths of two devices, $P(X < Y)$ gives the probability that the device with life length $X$ fails before the other. In medical science if $X$ and $Y$ represents the remission times of two treatments $P(X < Y)$ represents the probability that one is better than other. Moreover it can be used as a general measure of difference between two populations under various conditions and situations.

Estimation of stress-strength reliability has been considered for many common distributions like exponential, Weibull, Chen, Rayleigh etc. based on random samples as well as record values. Here we present the estimation procedures for stress-strength reliability for some distributions namely Pareto

Type I, Type II and Type IV, inverse Chen, exponentiated inverse Chen and inverse Weibull based on record values.

In real life we observe lower records as well as upper records. For example in the case of temperature and rainfall we note down lower and upper records where as in sports for events like long jump, high jump and shot put upper records are reported; but for race items lower records are reported.

Upper records were used for estimation of stress-strength parameter in Chen distribution by Tarvirdizade and Ahmadpour [42] and for Burr Type XII distribution by Nadar et al. [43] and in the present work for Pareto distribution while inference on stress strength reliability in Burr Type X [58], inverse Chen, and inverse Weibull in the present work were considered using lower records. Analytical tractability is essentially the basis of choosing upper or lower records for estimation in different distributions. We can select appropriate distributions according as the data is lower records or upper records. The work done is presented as follows.

Chapter 1 is an introductory chapter with a brief description of the stress-strength reliability, some distributional results of record values, and the theory of methods used for estimation of $R$, the stress-strength parameter followed by a review of the literature.

Chapter 2 describes the estimation of $R = P(X > Y)$ based on upper record values when X and Y are independent Pareto Type I random variables with same scale parameter and different shape parameters. In manufacturing industry when we have to compare between the life lengths of two products with same guarantee period it can be well described by Pareto Type I distribution where the guarantee period is represented by the scale parameter. Maximum likelihood and Bayesian methods are used for the estimation of $R$ when the scale parameter is known. The corresponding confidence intervals are also found. From the simulation study we observe that the point estimators are consistent.

The interval estimators perform well in terms of expected length. A real data analysis is done to illustrate the methods proposed. Further we have presented the likelihood inference of $R$ based on upper records from Pareto Type II and Type IV distributions. Numerical examples are presented to illustrate the methods. A proposed future work is the estimation of $R$ in the case of exponentiated Pareto Type I distribution based on records.

Chapter 3 deals with the estimation of $R = P(X > Y)$ when $X$ and $Y$ are from inverse Chen distribution with different first shape parameters and same second shape parameter based on lower record values. The MLE, Bayes estimator and the corresponding confidence intervals of $R$ are derived when the second shape parameter $\beta$ is known. When $\beta$ is unknown MLE, asymptotic confidence interval, Bayes estimator using MCMC method, HPD interval and bootstrap intervals are obtained. The results of the simulation study give good performance for the estimators. A real data analysis is presented for illustrative purpose. As an extension to inverse Chen distribution the exponentiated inverse Chen distribution is introduced. Likelihood inference and Bayesian inference of $R = P(X > Y)$ is considered when the second shape parameter $beta$ is known as well as unknown. The work can be extended to the estimation of this parameter based on lower records when $X$ and $Y$ are from inverse Chen distribution with same first shape parameters and different second shape parameters.

Chapter 4 considers inverse Weibull distribution with same shape parameter and different scale parameters for estimation of $R = P(X < Y)$ based on lower record values. When the shape parameter is known MLE, Bayes estimator and the corresponding confidence intervals are derived. For unknown shape parameter MLE, Bayes estimator using MCMC method and HPD interval are obtained. To illustrate the methods proposed a numerical example is presented. This study can be extended to the estimation of $R$ for inverse Weibull distribution with same scale parameter and different shape parameters.

All the models considered in this thesis have explicit form for distribution function. This was used to express the likelihood function directly. However there are a number of important probability models in which we have only explicit density functions. In these cases likelihood functions will consist of product of integrals. Obtaining MLE's in these situations will be far more challenging. One may use E M algorithm to find MLE. Bootstrap procedure may be used for doing further statistical inference. The same is true for mixture models.

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# Appendix A

# R Codes for Pareto distribution

## A.1 R Code of simulation for estimation of R for Pareto distribution

```
asim<-2000

beta1hat<-rep(NA,asim)

beta2hat<-rep(NA,asim)

rhat<-rep(NA,asim)

nrec<-rep(NA,asim)

mrec<-rep(NA,asim)

rbays0<-rep(NA,asim)

rbays1<-rep(NA,asim)

ll<-rep(NA,asim)
```

```
ul<-rep(NA,asim)

ln<-rep(NA,asim)

ll0<-rep(NA,asim)

ul0<-rep(NA,asim)

ln0<-rep(NA,asim)

A0<-rep(NA,asim)

llb0<-rep(NA,asim)

ulb0<-rep(NA,asim)

lnb0<-rep(NA,asim)

llb1<-rep(NA,asim)

ulb1<-rep(NA,asim)

lnb1<-rep(NA,asim)

for(j in 1:asim){

set.seed(j)

n=10

alpha=1

beta1=1

x=rep(0,n)

u=runif(1)

x[1]=alpha/(u^(1/beta1))

for(i in 2:n){

u=runif(1)

x[i]=x[i-1]/(u^(1/beta1))
```

```
    }
m=10
alpha=1
beta2=9
y=rep(0,m)
u=runif(1)
y[1]=alpha/(u^(1/beta2))
for(i in 2:m){
u=runif(1)
y[i]=y[i-1]/(u^(1/beta2))
    }
nrec[j]<-x[n]
mrec[j]<-y[m]
beta1hat[j]<-n/(log(nrec[j]/alpha))
beta2hat[j]<-m/(log(mrec[j]/alpha))
rhat[j]<-beta2hat[j]/(beta1hat[j]+beta2hat[j])
gamma10<-0.5
theta10<-0.5
gamma20<-0.5
theta20<-0.5
rbays0[j]<-rhat[j]+rhat[j]*(1-rhat[j])*(((1-rhat[j])/n)-(rhat[j]/m))
rbays1[j]<-rhat[j]+rhat[j]*(1-rhat[j])*(((1-rhat[j]-
gamma10+(beta1hat[j]*theta10))/n)+((gamma20-rhat[j]-
```

```
(beta2hat[j]*theta20))/m))
f1<-qf(0.025,2*m,2*n)
ll[j]<-(1+(beta1hat[j]/(beta2hat[j]*f1)))^-1
f2<-qf(0.975,2*m,2*n)
ul[j]<-(1+(beta1hat[j]/(beta2hat[j]*f2)))^-1
ln[j]<-ul[j]-ll[j]
f10<-qf(0.05,2*m,2*n)
ll0[j]<-(1+(beta1hat[j]/(beta2hat[j]*f10)))^-1
f20<-qf(0.95,2*m,2*n)
ul0[j]<-(1+(beta1hat[j]/(beta2hat[j]*f20)))^-1
ln0[j]<-ul0[j]-ll0[j]
A0[j]<-((n+gamma10)*(theta20+log(mrec[j]/alpha)))/((m+gamma20)*
(theta10+log(nrec[j]/alpha)))
f1b0<-qf(0.975,2*(n+gamma10),2*(m+gamma20))
f2b0<-qf(0.025,2*(n+gamma10),2*(m+gamma20))
llb0[j]<-(1+A0[j]*f1b0)^-1
ulb0[j]<-(1+A0[j]*f2b0)^-1
lnb0[j]<-ulb0[j]-llb0[j]
f1b1<-qf(0.95,2*(n+gamma10),2*(m+gamma20))
f2b1<-qf(0.05,2*(n+gamma10),2*(m+gamma20))
llb1[j]<-(1+A0[j]*f1b1)^-1
ulb1[j]<-(1+A0[j]*f2b1)^-1
lnb1[j]<-ulb1[j]-llb1[j]
```

```
}
avrg<-mean(rhat)
rexact<-beta2/(beta1+beta2)
bias<-avrg-rexact
mser<-mean((rhat-rexact)^2)
avrg
bias
mser
avrgb0<-mean(rbays0)
biasb0<-avrgb0-rexact
mseb0<-mean((rbays0-rexact)^2)
avrgb0
biasb0
mseb0
avrgb1<-mean(rbays1)
biasb1<-avrgb1-rexact
mseb1<-mean((rbays1-rexact)^2)
avrgb1
biasb1
mseb1
Eln<-mean(ln)
Eln
y0<-ifelse(ll<rexact&rexact<ul,1,0)
```

```
p0<-length(subset(y0,y0==1))

p0

cp0<-p0/asim

cp0

Eln1<-mean(ln0)

Eln1

y1<-ifelse(ll0<rexact&rexact<ul0,1,0)

p1<-length(subset(y1,y1==1))

p1

cp1<-p1/asim

cp1

Elnb0<-mean(lnb0)

Elnb0

yb0<-ifelse(llb0<rexact&rexact<ulb0,1,0)

pb1<-length(subset(yb0,yb0==1))

pb1

cpb1<-pb1/asim

cpb1

Elnb1<-mean(lnb1)

Elnb1

yb1<-ifelse(llb1<rexact&rexact<ulb1,1,0)

pb2<-length(subset(yb1,yb1==1))

pb2
```

```
cpb2<-pb2/asim

cpb2
```

## A.2 R code of data analysis for Pareto distribution

```
x<-c(60,51,83,140,109,106,119,76,68,67)

y<-c(100,90,59,80,128,117,177,98,158,107)

r<-c(60,83,140)

s<-c(100,128,177)

n=3

m=3

alpha=60

nrec<-140

mrec<-177

beta1hat<-n/log(nrec/alpha)

beta2hat<-m/log(mrec/alpha)

rhat<-beta2hat/(beta1hat+beta2hat)

rhat

f1<-qf(0.025,6,6)

f2<-qf(0.975,6,6)

ll=(1+(beta1hat/(beta2hat*f1)))^-1

ul=(1+(beta1hat/(beta2hat*f2)))^-1

ll

ul
```

```
gamma10<-0.5

theta10<-0.5

gamma20<-0.5

theta20<-0.5

rbays0<-rhat+rhat*(1-rhat)*(((1-rhat-gamma10+ <-

(beta1hat*theta10))/n)+((gamma20-rhat-(beta2hat*theta20))/m))

rbays1<-rhat+rhat*(1-rhat)*(((1-rhat)/n)-(rhat/m))

rbays0

rbays1

A1<-(n+gamma1)*(theta2+log(mrec/alpha))/(m+gamma2)*

(theta1+log(nrec/alpha))

f1b<-qf(0.975,2*(n+gamma1),2*(m+gamma2))

f2b<-qf(0.025,2*(n+gamma1),2*(m+gamma2))

llb1<-(1+A1*f1b)^-1

ulb1<-(1+A1*f2b)^-1

llb1

ulb1

A2<-(n/m)*((log(mrec/alpha))/(log(nrec/alpha)))

llb2<-(1+A2/f1)^-1

ulb2<-(1+A2/f2)^-1

llb2

ulb2
```

# Appendix B

# R codes for inverse Chen distribution

## B.1 R Code of simulation for estimation of R for inverse Chen distribution

### B.1.1 Shape parameter beta known

```
asim=2000

gammahat<-rep(NA,asim)

deltahat<-rep(NA,asim)

rhat<-rep(NA,asim)

nrec<-rep(NA,asim)

mrec<-rep(NA,asim)

rbays0<-rep(NA,asim)
```

```
rbays1<-rep(NA,asim)

ll<-rep(NA,asim)

ul<-rep(NA,asim)

ln<-rep(NA,asim)

A0<-rep(NA,asim)

llb0<-rep(NA,asim)

ulb0<-rep(NA,asim)

lnb0<-rep(NA,asim)

A1<-rep(NA,asim)

llb1<-rep(NA,asim)

ulb1<-rep(NA,asim)

lnb1<-rep(NA,asim)

for(j in 1:asim){

set.seed(j)

n=3

gamma=1

beta=2

x=rep(0,n)

u=runif(1)

x[1]=(log(1-(log(u)/gamma)))^-(1/beta)

for(i in 2:n){

u=runif(1)

x[i]=(log(1-((log(u)+gamma*(1-exp((x[i-1])^-beta)))/gamma)))^-(1/beta)}
```

```
m=3

delta=4

beta=2

y=rep(0,m)

u=runif(1)

y[1]=(log(1-(log(u)/delta)))^-(1/beta)

for(i in 2:m){

u=runif(1)

y[i]=(log(1-((log(u)+delta*(1-exp((y[i-1])^-beta)))/delta)))^-(1/beta)

 }

nrec[j]<-x[n]

mrec[j]<-y[m]

gammahat[j]<--n/(1-exp(nrec[j]^-beta))

deltahat[j]<--m/(1-exp(mrec[j]^-beta))

rhat[j]<-gammahat[j]/(deltahat[j]+gammahat[j])

a1=0.5

b1=0.5

a2=0.5

b2=0.5

rbays0[j]<-rhat[j]*(1+(1-rhat[j])*(((1-rhat[j])/m)-(rhat[j]/n)+

((a1-(b1*gammahat[j]))/n)-((a2-(b2*deltahat[j]))/m)))

rbays1[j]<-rhat[j]*(1+(1-rhat[j])*(((1-rhat[j])/m)-(rhat[j]/n)))

f1=qf(0.025,2*n,2*m)
```

```
f2=qf(0.975,2*n,2*m)

ll[j]=(1+(deltahat[j]/(gammahat[j]*f1)))^-1

ul[j]=(1+(deltahat[j]/(gammahat[j]*f2)))^-1

ln[j]=ul[j]-ll[j]

A0[j]=((m+a2)*(exp(nrec[j]^-beta)+b1-1))/((n+a1)* <-

(exp(mrec[j]^-beta)+b2-1))

f1b0=qf(0.025,2*(n+a1),2*(m+a2))

f2b0=qf(0.975,2*(n+a1),2*(m+a2))

llb0[j]=(1+(A0[j]/f1b0))^-1

ulb0[j]=(1+(A0[j]/f2b0))^-1

lnb0[j]=ulb0[j]-llb0[j]

A1[j]=(m*(exp(nrec[j]^-beta)-1))/(n*(exp(mrec[j]^-beta)-1))

f1b1=qf(0.025,2*n,2*m)

f2b1=qf(0.975,2*n,2*m)

llb1[j]=(1+(A1[j]/f1b1))^-1

ulb1[j]=(1+(A1[j]/f2b1))^-1

lnb1[j]=ulb1[j]-llb1[j]

}

avrg<-mean(rhat)

rexact=gamma/(delta+gamma)

bias<-avrg-rexact

mser<-mean((rhat-rexact)^2)

avrg
```

```
bias

mser

avrgb0<-mean(rbays0)

biasb0<-avrgb0-rexact

mseb0<-mean((rbays0-rexact)^2)

avrgb0

biasb0

mseb0

avrgb1<-mean(rbays1)

biasb1<-avrgb1-rexact

mseb1<-mean((rbays1-rexact)^2)

avrgb1

biasb1

mseb1

Eln<-mean(ln)

Eln

y<-ifelse(ll<rexact&rexact<ul,1,0)

p<-length(subset(y,y==1))

p

cp<-p/asim

cp

Elnb0<-mean(lnb0)

Elnb0
```

```
yb0<-ifelse(llb0<rexact&rexact<ulb0,1,0)

pb0<-length(subset(yb0,yb0==1))

pb0

cpb0<-pb0/asim

cpb0

Elnb1<-mean(lnb1)

Elnb1

yb1<-ifelse(llb1<rexact&rexact<ulb1,1,0)

pb1<-length(subset(yb1,yb1==1))

pb1

cpb1<-pb1/asim

cpb1
```

### B.1.2 Shape parameter beta unknown

```
asim=2000

betahat<-rep(NA,asim)

betahat1<-rep(NA,asim)

gammahat<-rep(NA,asim)

deltahat<-rep(NA,asim)

rhat<-rep(NA,asim)

nrec<-rep(NA,asim)

mrec<-rep(NA,asim)
```

```
sigma<-rep(NA,asim)

lla<-rep(NA,asim)

ula<-rep(NA,asim)

lna<-rep(NA,asim)

for(j in 1:asim){

set.seed(j)

n=3

gamma=1

beta=3

x=rep(0,n)

u=runif(1)

x[1]=(log(1-(log(u)/gamma)))^-(1/beta)

for(i in 2:n){

u=runif(1)

x[i]=(log(1-((log(u)+gamma*(1-exp((x[i-1])^-beta)))/gamma)))^-(1/beta)

              }

m=3

delta=4

beta=3

y=rep(0,m)

u=runif(1)

y[1]=(log(1-(log(u)/delta)))^-(1/beta)

for(i in 2:m){
```

```
u=runif(1)

y[i]=(log(1-((log(u)+delta*(1-exp((y[i-1])^-beta)))/delta)))^-(1/beta)

          }

nrec[j]<-x[n]

mrec[j]<-y[m]

Inputs = 0.1

nleqn=function(beta){

n=length(x)

m=length(y)

h=(m+n)*((n*(x[n]^-beta)*exp(x[n]^-beta)* <-

log(x[n])/(1-exp(x[n]^-beta)))+(m*(y[m]^-beta)*exp(y[m]^-beta)*

log(y[m])/(1-exp(y[m]^-beta)))+sum(log(x)*(1+(x^-beta)))+

sum(log(y)*(1+(y^-beta))))^-1

}

A = FixedPoint(nleqn, Inputs, Method = "Aitken", Dampening = 0.5)

A

B = FixedPoint(nleqn, Inputs, Method = "Anderson", Dampening = 1.0)

B

betahat[j]<-A$FixedPoint

gammahat[j]<--n/(1-exp(nrec[j]^-betahat[j]))

deltahat[j]<--m/(1-exp(mrec[j]^-betahat[j]))

rhat[j]<-gammahat[j]/(deltahat[j]+gammahat[j])

}
```

```
rhat1<-rhat[!is.na(rhat)]

avrg<-mean(rhat1)

rexact=gamma/(delta+gamma)

bias<-avrg-rexact

mser<-mean((rhat1-rexact)^2)

avrg

bias

mser

gammahat1<-gammahat[!is.na(gammahat)]

deltahat1<-deltahat[!is.na(deltahat)]

sigma<-sqrt(((gammahat1*deltahat1)^2/(gammahat1+deltahat1)^4)*

((1/n)+(1/m)))

z=qnorm(0.975)

z

lla=rhat1-(z*sigma)

ula=rhat1+(z*sigma)

lna=ula-lla

Elna<-mean(lna)

Elna

ya<-ifelse(lla<rexact&rexact<ula,1,0)

pa<-length(subset(ya,ya==1))

pa

cpa<-pa/length(rhat1)
```

```
cpa
```

## B.2 R code for MCMC simulation for inverse Chen

```
posterior.density=function(param,a3,b3,n,m,nrec,mrec){

param=c(gamma,delta,beta)

n=length(x)

m=length(y)

nrec=x[n]

mrec=y[m]

loglike.beta=(n+m+b3-1)*log(param[3])-a3*param[3]-param[1]*

exp(nrec^-param[3])-param[2]*exp(mrec^-param[3])+

sum(-(param[3]+1)*log(x)+(x^-param[3]))+

sum(-(param[3]+1)*log(y)+(y^-param[3]))

exp(loglike.beta)

}

n=3

gamma=1

beta=2

x=rep(0,n)

u=runif(1)

x[1]=(log(1-(log(u)/gamma)))^-(1/beta)

for(i in 2:n){
```

```
u=runif(1)

x[i]=(log(1-((log(u)+gamma*(1-exp((x[i-1])^-beta)))/gamma)))^-(1/beta)

              }

m=3

delta=4

beta=2

y=rep(0,m)

u=runif(1)

y[1]=(log(1-(log(u)/delta)))^-(1/beta)

for(i in 2:m){

u=runif(1)

y[i]=(log(1-((log(u)+delta*(1-exp((y[i-1])^-beta)))/delta)))^-(1/beta)

              }

Inputs = 0.1

nleqn=function(beta){

n=length(r)

m=length(s)

h=(m+n)*((n*(x[n]^-beta)*exp(x[n]^-beta)*

log(x[n])/(1-exp(x[n]^-beta)))+(m*(y[m]^-beta)*exp(y[m]^-beta)*

log(y[m])/(1-exp(y[m]^-beta)))+sum(log(x)*(1+(x^-beta)))+

sum(log(y)*(1+(y^-beta))))^-1

}

A = FixedPoint(nleqn, Inputs, Method = "Aitken", Dampening = 0.5)
```

```
A

B = FixedPoint(nleqn, Inputs, Method = "Anderson", Dampening = 1.0)

B

betahat=FixedPoint$A

gammahat=-n/(1-exp(x[n]^-betahat))

deltahat=-m/(1-exp(y[m]^-betahat))

gammahat

deltahat

rhat=gammahat/(deltahat+gammahat)

rhat

nrec=x[n]

mrec=y[m]

nmc=2000

samples<-array (dim = c(3 ,nmc), dimnames =

list( c("gamma", "delta","beta"),NULL))

samples[3,1]=3.6453

betahat=samples[3,1]

a1=0.5

b1=0.5

a2=0.5

b2=0.5

a3=0.5

b3=0.5
```

```
gammahat<-rgamma(1,(n+a1),exp(nrec^-betahat)+b1-1)

deltahat<-rgamma(1,(m+a2),exp(mrec^-betahat)+b2-1)

samples[,1] = c(gammahat,deltahat,betahat)

for ( i in 2:nmc ) {

samples[,i]=samples[,i-1]

for ( j in rownames(samples)) {

proposal = samples[3,i]

proposal = proposal + rnorm (n = 1 , mean = 0, sd = 0.1)

if (proposal<0){

next

}

new.likelihood = posterior.density (param =c(samples[c(1:2),i],

proposal),a3,b3,n,m,nrec,mrec)

old.likelihood = posterior.density (param =samples[,i],a3,b3,n,m,nrec,mrec)

likelihood.ratio = new.likelihood / old.likelihood

if (runif(1) < likelihood.ratio) {

samples[3,i]= proposal

}else {

samples[3,i] = samples[3,i - 1]

}

beta=samples[3,i]

gamma<-rgamma(1,(n+a1),exp(nrec^-beta)+b1-1)

delta<-rgamma(1,(m+a2),exp(mrec^-beta)+b2-1)
```

```
samples[,i] = c(gamma,delta,beta)

}

}

R<-c(NA,nmc)

for(i in 1:nmc){

R[i]<-samples[1,i]/(samples[1,i]+samples[2,i])

}

R1=R[c(501:2000)]

Rb<-mean(R1)

Rb
```

## B.3 R code of data analysis for inverse Chen distribution, $\beta$ known

```
Inputs=0.1

nleqn=function(beta){

n=length(x)

m=length(y)

h=(m+n)*((sum(n*(x^-beta)*exp(x^-beta)*log(x))/sum(1-exp(x^-beta)))+

(sum(m*(y^-beta)*exp(y^-beta)*log(y))/sum(1-exp(y^-beta)))+

sum(log(x)*(1+(x^-beta)))+sum(log(y)*(1+(y^-beta))))^-1

}
```

```
x<-c(100,90,59,80,128,117,177,98,158,107,125,118,99,186,66,132,

97,87,69,109)

y<-c(141,143,98,122,110,132,194,155,104,83,125,165,146,100,

318,136,200,201,251,111)

B = FixedPoint(nleqn, Inputs, Method = "Anderson", Dampening = 1.0)

B

A = FixedPoint(nleqn, Inputs, Method = "Aitken", Dampening = 0.5)

A

beta=3.5936

n=length(x)

gammahat=-n/sum(1-exp(x^-beta))

gammahat

cdf=function(x){exp(gammahat*(1-exp(x^-beta)))

}

ks.test(x,cdf)

beta=3.5936

m=length(y)

deltahat=-m/sum(1-exp(y^-beta))

deltahat

cdf=function(y){exp(deltahat*(1-exp(y^-beta)))

}

ks.test(y,cdf)

r=c(100,90,59)
```

```
s=c(141,98,83)

n=3

m=3

beta=3.5936

gammahat=-n/(1-exp(r[n]^-beta))

deltahat=-m/(1-exp(s[m]^-beta))

gammahat

deltahat

rhat=gammahat/(deltahat+gammahat)

rhat

a1=2

b1=3

a2=2

b2=3

rb1=rhat*(1+(1-rhat)*(((1-rhat)/m)-(rhat/n)+

((a1-(b1*gammahat))/n)-((a2-(b2*deltahat))/m)))

rb1

rb2=rhat*(1+(1-rhat)*(((1-rhat)/m)-(rhat/n)))

rb2

f1=qf(0.025,6,6)

ll=(1+(deltahat/(gammahat*f1)))^-1

ll

f2=qf(0.975,6,6)
```

```
ul=(1+(deltahat/(gammahat*f2)))^-1

ul

A1=((m+a2)*(exp(r[n]^-beta)+b1-1))/((n+a1)*(exp(s[m]^-beta)+b2-1))

f1b=qf(0.025,7,7)

llb=(1+(A1/f1b))^-1

llb

f2b=qf(0.975,7,7)

ulb=(1+(A1/f2b))^-1

ulb

A2=(m*(exp(r[n]^-beta)-1))/(n*(exp(s[m]^-beta)-1))

llb1=(1+A2/f1)^-1

llb1

ulb1=(1+A2/f2)^-1

ulb1
```

## B.4 R code of data analysis for inverse Chen distribution, $\beta$ unknown

```
Inputs = 0.1

nleqn=function(beta){

n=length(r)

m=length(s)

h=(m+n)*((n*(r[n]^-beta)*exp(r[n]^-beta)*
```

```
log(r[n])/(1-exp(r[n]^-beta)))+(m*(s[m]^-beta)*
exp(s[m]^-beta)*log(s[m])/(1-exp(s[m]^-beta)))+sum(log(r)*
(1+(r^-beta)))+sum(log(s)*(1+(s^-beta))))^-1
}
r=c(100,90,59)
s=c(141,98,83)
A = FixedPoint(nleqn, Inputs, Method = "Aitken", Dampening = 0.5)
A
B = FixedPoint(nleqn, Inputs, Method = "Anderson", Dampening = 1.0)
B
betahat=3.6453
gammahat=-n/(1-exp(r[n]^-betahat))
deltahat=-m/(1-exp(s[m]^-betahat))
gammahat
deltahat
rhat=gammahat/(deltahat+gammahat)
rhat
n=3
m=3
sigsqr=((gammahat*deltahat)^2/(gammahat+deltahat)^4)*((1/n)+(1/m))
sigma=sqrt(sigsqr)
sigma
z=qnorm(0.975)
```

```
z

lla=rhat-(z*sigma)

lla

ula=rhat+(z*sigma)

ula
```

## B.5 R Code for MCMC - real data from inverse Chen

```
posterior.density=function(param,a3,b3,n,m,nrec,mrec){

param=c(gamma,delta,beta)

n=length(r)

m=length(s)

nrec=r[n]

mrec=s[m]

loglike.beta=(n+m+a3-1)*log(param[3])-b3*param[3]-param[1]*

exp(nrec^-param[3])-param[2]*exp(mrec^-param[3])+

sum((-(param[3]+1)*log(r))+(r^-param[3]))+

sum((-(param[3]+1)*log(s))+(s^-param[3]))

exp(loglike.beta)

}

nmc=2000

r<-c(100,90,59)

s<-c(141,98,83)

samples<-array (dim = c(3 ,nmc), dimnames =
```

```
list( c("gamma", "delta","beta"),NULL))

samples[3,1]=3.6453

betahat=samples[3,1]

a1=2

b1=3

a2=2

b2=3

a3=2

b3=3

gammahat<-rgamma(1,(n+a1),exp(nrec^-betahat)+b1-1)

deltahat<-rgamma(1,(m+a2),exp(mrec^-betahat)+b2-1)

samples[,1] = c(gammahat,deltahat,betahat)

for ( i in 2:nmc ) {

samples[,i]=samples[,i-1]

for ( j in rownames(samples)) {

proposal = samples[3,i] + rnorm (n = 1 , mean = 0, sd = 0.1)

if (proposal<0){

next

}

new.likelihood = posterior.density (param =c(samples[c(1:2),i],

proposal),a3,b3,n,m,nrec,mrec)

old.likelihood = posterior.density

(param =samples[,i],a3,b3,n,m,nrec,mrec)
```

```
likelihood.ratio = new.likelihood / old.likelihood

if (runif(1)<likelihood.ratio){

samples[3,i]= proposal

}else{

sample[3,i]=sample[3,i-1]

}

beta=samples[3,i]

gamma<-rgamma(1,(n+a1),exp(nrec^-beta)+b1-1)

delta<-rgamma(1,(m+a2),exp(mrec^-beta)+b2-1)

samples[,i]=c(gamma,delta,beta)

}

}

R<-c(NA,nmc)

for(i in 1:nmc){

R[i]<-samples[1,i]/(samples[1,i]+samples[2,i])

}

R1=R[c(501:2000)]

Rb<-mean(R1)

Rb

hdi(R1, credMass = 0.95)
```

# LIST OF PUBLICATIONS

## Papers published:

1. *Estimation of Stress-Strength Reliability for the Pareto Distribution Based on Upper Record Values*,

   Juvairiyya R. M., and Anilkumar, P. Statistica, ISSN 1973-2201, 78(4), 397-409 (2018).

## Conference papers presented:

1. *Estimation of Stress-Strength Reliability for Pareto Distribution Based on Upper Record Values*,

   International Conference on Statistics for Twenty-first Century-2017 (ICSTC-2017) organized by the Department of Statistics, University of Kerala, Trivandrum during 14-16, December 2017.

2. *Estimation of Stress-Strength Reliability Based on Lower Record Values from Inverse Chen Distribution*,

   International Conference on Computer Age Statistics in the Era of Big and High Dimensional Data organized by the Department of Statistics, Savitribai Phule Pune University during January 3-5, 2019.