SOME INTEGRAL TRANSFORMS OF HYPERFUNCTIONS AND THEIR PROPERTIES

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This is to certify that the thesis entitled ***Some Integral Transforms of Hyperfunctions and Their Properties***, submitted by part-time research scholar Ms. Deepthi A. N., Department of Mathematics, Sree Narayana College, Nattika, Thrissur, to the University of Calicut, in partial fulfilment of requirement for the degree of Doctor of Philosophy in Mathematics, is a bonafide record of research work undertaken by her in the Centre for Research in Mathematical Science, St.Joseph's College, Irinjalakuda, under my supervision during the period 2009-2019 and that no part thereof has been presented before for any other degree.

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DECLARATION

I hereby declare that this thesis entitled ***Some Integral Transforms of Hyperfunctions and Their Properties***, is the record of bonafide research I carried out in the Centre for Research in Mathematical Sciences, St.Joseph's College (Autonomous), Irinjalakuda, under the guidance of Dr. Mangalambal N. R., Research Supervisor, Centre for Research in Mathematical Sciences, Department of Mathematics, St.Joseph's College (Autonomous), Irinjalakuda.

I further declare that this thesis, or any part thereof, has not previously formed the basis for the award of any other degree, diploma, associateship, fellowship or any other similar title of recognition.

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ABSTRACT

Hyperfunctions are one of the generalisations of generalized function, introduced by Mikio Sato. Urs Graf applied Laplace transform, Fourier transform, Hilbert transform, Mellin transform and Hankel transform to the hyperfunctions. In this study Weierstrass transform, Stieltjes transform, $\mathfrak{L}_2$ transform, Fourier-Laplace transform, Laplace-Stieltjes transform and Fourier-Stieltjes transform have been developed for hyperfunctions and some properties of these transforms have also been investigated. Abelian -Tauberian theorem is proved for Laplace transform, Stieltjes transform and Laplace-Stieltjes transform of hyperfunctions. The sufficient condition for the existence of Two Dimensional Laplace transform of hyperfunctions in two variables with separable defining function is investigated. Some order theoretic properties of the linear space of hyperfunctions and norm convergence of the sequence of hyperfunctions are also studied. The concept of completely monotonic hyperfunction has been developed. A partial differential equation involving hyperfunction has been solved using Weierstrass transform of hyperfunction.

Keywords: Hyperfunction, Laplace Transform,Fourier Transform,Weierstrass Transform, Stieltjes Transform, $\mathfrak{L}_2$ Transform, Fourier-Laplace Transform, Laplace-Stieltjes Transform, Fourier-Stieltjes Transform, Two dimensional Laplace Transform, Ordered linear space, Norm convergence, Completely monotone functions.

Contents

Notations Used

$\mathbb{C}$: Complex plane

$\mathbb{C}_+$: the upper half plane of the complex plane $\mathbb{C}$

$\mathbb{C}_-$: the lower half plane of the complex plane $\mathbb{C}$

$N(I)$: Complex neighbourhood of I

$N_+(I)$: upper half complex neighbourhood of I

$N_-(I)$: lower half complex neighbourhood of I

$\mathfrak{O}(N(I))$: Ring of holomorphic functions in the complex neighbourhood $N(I)$ of I

$\mathfrak{O}(N(I)\backslash I)$: Ring of holomorphic functions in $N(I)\backslash I$

$F_+(z)$: upper component of $F(z)$

$F_-(z)$: lower component of $F(z)$

$[F(z)]$: Equivalence class of $F(z)$

$\mathfrak{B}(I)$: Set of all hyperfunctions defined on the interval I

$\mathcal{A}(I)$: Ring of all real analytic functions on I

$\delta(x)$: Delta function

$u(x)$: Unit step function

$D^n f(x)$: n^{th} derivative of the hyperfunction $f(x)$

$f_e(x)$: Even hyperfunction

$f_o(x)$: Odd hyperfunction

$\overline{f(x)}$: Conjugate of the hyperfunction $f(x)$

$\delta_N(x-a)$:Generalised delta function of order N at a

$f(x,\alpha)$: Hyperfunction $f(x)$ depending on a continuous parameter α

$\mathfrak{B}_{\mathfrak{O}}(J)$: Set of all holomorphic hyperfunctions on J

$\mathbb{E}$: Linear space of entire hyperfunctions

$\mathfrak{B}^{exp}(I)$: Linear space of exponentially bounded hyperfunctions on the real interval I

$\mathfrak{B}_{bv}^{e+}$: Set of all non-decreasing, non-negative, real valued, holomorphic, measurable, exponentially bounded hyperfunctions of bounded variation defined on the closed subset $I \subset [0,\infty)$

$\mathfrak{B}_{\mathbb{R}}(I_1 \times I_2)$: Set of all real valued hyperfunction in $I_1 \times I_2$

$\mathfrak{B}_{\mathbb{R}}^{exp}(I_1 \times I_2)$: Set of all hyperfunctions in $\mathfrak{B}_{\mathbb{R}}(I_1 \times I_2)$ having bounded exponential growth

$\mathfrak{B}_{\mathbb{R}}(I)$: Set of all real valued hyperfunction in I

$\mathfrak{B}_B^M(I)$: Linear subspace of $\mathfrak{B}_{\mathbb{R}}(I)$ of hyperfunctions of bounded exponential growth and has a complex measurable function as defining function

$\mathfrak{B}_B^K(I)$: Set of all exponentially bounded hyperfunctions with compact support on I

Abbreviations Used

supp f : support of the function f

$ker f$: kernel of f

$im f$: image of f

sing supp$f(x)$: singular support of $f(x)$

sing spec$f(x)$: singular spectrum of $f(x)$

sgn : sign function

$\mathcal{R}s$: real part of s

Ims : imaginary part of s

$\mathfrak{L}$: Laplace transform

$\mathfrak{F}$: Fourier transform

$\mathfrak{W}_t$: Generalised Weierstrass transform

Res : Residue

$\mathcal{S}$: Stieltjes transform

$\mathfrak{L}_2$: L_2 transform

$S_P(\gamma)$: Support function

$\mathfrak{F}\mathfrak{L}$: Fourier-Laplace transform

$\mathfrak{L}_S$: Laplace-Stieltjes transform

$\mathfrak{F}_S$: Fourier-Stieltjes transform

Introduction

History

The basic idea of delta function came in the beginning of 19^{th} century. It was mainly used by Poisson, Fourier and Cauchy in the mathematical modelling of physical situations. Kirchhoff and Heaviside [28][35], first tried to give the mathematical interpretation of delta function. Kirchhoff used the concept of delta function in the fundamental solution of wave equation. Heaviside[45] used it in Operational Calculus. Also he expressed delta function as the derivative of unit step function. Heaviside expressed unit step function as

$$Y(x-c) = \begin{cases} 0, & x < c; \\ 1, & x > c. \end{cases}$$

which is not differentiable at $x = c$.

In 1926, Paul Dirac[17] introduced the notation of delta function and established some properties of it in his works on quantum field theory. The delta function is defined as

$$\delta(x-c) = 0 \;\; \forall \;\; x \neq c$$

$$\int_a^b \delta(x-c)dx = \begin{cases} 1, & c \in (a,b); \\ 0, & \text{otherwise.} \end{cases}$$

Physicists like Pauli, Heisenberg and Jordan widely used it for the development of quantum field theory. Even though the uses of Dirac delta function by Physicists increased, the classical mathematics fails to explain delta function mathematically, because using classical integration theory, the integral of a function which is zero everywhere except exactly at one point, has the integral value zero. Many mathematicians started to think on how to define such singular functions mathematically. They called such functions (functions like delta function) as 'distributions' or 'generalized functions'.

In 1932, Bochner S, developed the theory for such situations and in 1935, Sobolev S.L.[54] gave the definition for distributions in terms of functionals.

Later in 1945, French mathematician Laurent Schwartz [51][52] developed, further theory of distributions in an efficient way using the concept of test function space. The book by L.Schwartz, "Theory of Distributions" became a foundation for further development of distribution theory. Then onwards the theory of generalized functions began to develop intensively. Mathematicians started to investigate this area because of its wide applications in Mathematical Physics mainly for finding the solutions of partial differential equations, boundary value problems and initial value problems that appeared in physical problems.

In 1954, Schwartz[50] published a paper, showing the impossibility of product of two arbitrary distributions. But, some physical problems required product of distributions. Hence mathematicians tried to define product of distributions by various methods. J.F.Colombeau [11] proposed a theory to solve this impossibility through the introduction of the concept of new generalized functions.

Physics problems involving generalized functions expressed in terms of partial differential equations or differential equations can be effectively solved using the application of integral transforms to it. A.H. Zemanian [64] applied several integral transforms to distributions by the method of adjoints in his book. Also V.S. Vladimirov [58],[57] developed many properties of integral transforms of generalized functions and proved Tauberian theorems for the Laplace transform of generalized functions.

Introduction

Mikio Sato[48], a Japanese mathematician, developed hyperfunctions, which is more generalised than the concept of generalized functions. Sato applied classical complex analysis function theory to generalize notion of function of a real variable. Sato expressed the concept of generalized function in a less abstract way than that by L.Schwartz. Schwartz defined a generalized function as the limit of sequences of ordinary functions using the notion of equivalence classes. Sato defined a function of a real variable as the difference of the boundary values of a complex function, which is holomorphic and call it a hyperfunction. Hyperfunctions are always infinitely differentiable. The set of all distributions form a subspace of the linear space of hyperfunctions.

Mathematicians like A. Kaneko[32], Mitsuo Morimoto[40] and Isao Imai[30] did immense work based on this hyperfunction theory. Isao Imai applied Sato's hyperfunction theory in a non trivial concrete way and showed the computational power of hyperfunction in his book 'Applied Hyperfunction theory'. Urs Graf[25] further developed Imai's work and published a book 'Introduction to Hyperfunctions and Their Integral Transforms'. He applied Laplace transforms, Fourier transforms, Hilbert transforms, Mellin transforms and Hankel transforms to hyperfunctions and used them to solve integral equations. He focused more on the practical approach than the theoretical way.

Research Motivation

Urs Graf's approach to hyperfunction is the main motivation for this present work titled "Some integral transforms of hyperfunctions and their properties". As a continuation of the development of application of integral transforms on hyperfunctions by Urs Graf, in the background of no other study in this regard, in this thesis we have applied the integral transforms such as Weierstrass transform, Stieltjes transform, $\mathfrak{L}_2$-transform, Fourier-Laplace transform, Laplace-Stieltjes transform, Fourier-Stieltjes transform and Two dimensional Laplace transform to hyperfunctions, and have proved some Abelian - Tauberian theorems for the integral transforms of hyperfunctions. In fact the notion of order theoretic and convergence aspects of hyperfunctions have been obtained through this present study.

The content of the thesis are organised in the following way.

Chapter 1 contains some preliminary definitions and results necessary to understand the study taken up in the thesis. Definition of Hyperfunctions and its properties are included in it.

In chapter 2, the integral transforms like Weierstrass transform, Stieltjes transform and $\mathfrak{L}_2$ transform are defined for hyperfunctions. Some properties of these transforms are proved.

Weierstrass transform is applied to a class of hyperfunctions having bounded exponential growth. We have established some properties of this background using the relation connecting Weierstrass and Laplace transforms of hyperfunctions.

The following are the main results obtained.

If $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth then,

- $$\mathfrak{W}_{\mathfrak{t}}[f(-x)](-s) = -\mathfrak{W}_{\mathfrak{t}}[f(x)](s)$$

- for $t > 0$,
$$\mathfrak{W}_{\mathfrak{t}}[\overline{f(x)}](s) = \overline{\mathfrak{W}_{\mathfrak{t}}[f(x)](\overline{s})}$$

- for $f(x)$ a real analytic hyperfunction and constant c
$$\mathfrak{W}_{\mathfrak{t}}[f(x+c)](s) = \mathfrak{W}_{\mathfrak{t}}[f(x)](s+c), \ \ if \ \ \sigma_{-}(f) < \mathcal{R}(s) < \sigma_{+}(f).$$

- The relation connecting Weierstrass and Laplace transform of hyperfunction is
$$\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathfrak{L}[f(x)](\frac{-s}{2t})$$

- $$\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f^n(x)](s) = (\frac{-s}{2t})^n \mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s)$$

- The inverse Weierstrass transform is defined.

  If $\tilde{f}(s) = \mathfrak{W}_{\mathfrak{t}}[f(x)](s)$ is the Weierstrass transform of $f(x) = [F(z)]$, a hyperfunction of bounded exponential growth then
$$f(x) = [F(z)] = \lim_{t \to 1} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s+iz)^2}{4t}} \tilde{f}(is) ds$$

There exist different methods for defining Stieltjes transform of distributions. Here the Stieltjes transform of a hyperfunction is defined in the following way.
For a hyperfunction $f(x) = [F(z)]$ of bounded exponential growth the Stieltjes transform is defined by

$$\tilde{f}(t) = \mathcal{S}[f(x)](t) = \int_0^\infty \frac{f(x)}{x+t}dx = \int_0^\infty \frac{F(z)}{z+t}dz.$$

The following results are obtained

- The relation connecting Stieltjes and Laplace transform of hyperfunction is

$$\mathcal{S}[f(x)](t) = \mathfrak{L}[\mathfrak{L}[f(x)](s)](t)$$

  if $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ and $\sigma_-(f) < \mathcal{R}t < \sigma_+(f)$

- If $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are any two hyperfunctions having bounded exponential growth with the Stieltjes transform $\tilde{f}(t) = \mathcal{S}[f(x)](t)$ and $\tilde{g}(t) = \mathcal{S}[g(x)](t)$ respectively, then

$$\int_0^\infty f(x)\tilde{g}(x)dx = \int_0^\infty g(t)\tilde{f}(t)dt$$

- If $f(x) = [F(z)]$ is a real valued hyperfunction with bounded exponential growth and $f(x) > 0$ for all $x > 0$ then the Stieltjes transform $\tilde{f}(t) = \mathcal{S}[f(x)](t) > 0$ for all $t > 0$

- If $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, $\hat{f}(s) =$

$\mathfrak{L}[f(x)](s)$ exists, then

$$\mathcal{S}[f^n(x)](t) = (-1)^n \frac{d^n}{dt^n}(\mathfrak{L}[\hat{f}(s)](t)),$$

for $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ and $\sigma_-(\hat{f}) < \mathcal{R}t < \sigma_+(\hat{f})$

- If $f(x) = [F(z)]$ and $g(y) = [G(z)]$ are two hyperfunctions of bounded exponential growth. If support $f(x)$ is a compact subset of $(0, \infty)$ then

$$\int_0^\infty \mathfrak{L}[f(x)](s)\mathfrak{L}[g(y)](s)ds = \int_0^\infty g(y)\mathcal{S}[f(x)](y)dy$$

A.Aghili, A.Ansari, A.Sadghi, David Brown, John Maceli, Osman Yurekli, Scott Wilson etc[1],[10],[3],[62],[61] developed $\mathfrak{L}_2$ transform of ordinary functions and solved differential and integral equations by applying $\mathfrak{L}_2$ transform. The $\mathfrak{L}_2$ transform of hyperfunction is defined by imposing the convergence criterion for the integral of $\mathfrak{L}_2$ transform.

If $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$, the $\mathfrak{L}_2$ transform of $f(x)$ is defined as

$$\mathfrak{L}_2[f(x)](s) = \int_0^\infty xe^{-x^2s^2} f(x)dx = \int_0^\infty ze^{-z^2s^2} F(z)dz,$$

Found that the following results exists for $\mathfrak{L}_2$ transform of hyperfunctions also.

- If $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$, then

$$\mathfrak{L}_2[f(x)](s) = \frac{1}{2}\mathfrak{L}[f(\sqrt{x})](s^2)$$

- If $f(x) = [F(z)]$ is a holomorphic hyperfunction having bounded exponential growth, $f'(x) = [F'(z)]$ is also holomorphic and $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$, then

$$\lim_{x \to 0} f(x) = \lim_{s \to \infty} 2s^2 \mathfrak{L}_2[f(x)](s)$$

$$Also, \quad \lim_{x \to \infty} f(x) = \lim_{s \to 0} 2s^2 \mathfrak{L}_2[f(x)](s)$$

- If $L(\sqrt{s})$ is a holomorphic function of s (by assuming $s = 0$ having no branch point) having finite number of poles which lies to the left side of the line $\mathcal{R}s = a$ and all $F(z) \in [F(z)]$ has a common strip of convergence, $\mathfrak{L}_2[f(x)](s) = L(s)$, then

$$f(x) = \mathfrak{L}_2^{-1}(L(s)) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2L(\sqrt{s}) e^{s^2 x} ds$$

In chapter 3, combined transforms such as Fourier-Laplace, Laplace- Stieltjes and Fourier-Stieltjes transforms and some of their properties are developed.

Mitusuo Morimoto in his book[40] mentioned about the Fourier-Laplace transform of an entire function of exponential type. Laplace and Fourier transforms are defined for hyperfunctions. The existence of the combined Fourier-Laplace transform of hyperfunctions is studied using the convergence criteria for Fourier and Laplace transform of hyperfunction. We have defined entire hyperfunctions of exponential type and a norm for such hyperfunctions.

For a convex compact set P subset of $\mathbb{C}$ and for the hyperfunction $f(x) = [F(Z)]$ defined $\| f(x) \|_{(P)}$ by

$$\| f(x) \|_{(P)} = sup\{|F(z)| e^{-S_P(z)} : z \in \mathbb{C}, F(z) \in [F(z)]\}$$

For a convex compact set P, we let

$$\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P) = \{f(x) \in \mathbb{E} : \| f(x) \|_{(P)} < \infty\}$$

and proved that it is a Banach space with respect to $\| f(x) \|_{(P)}$. Fourier-Laplace transform is defined for hyperfunction in $\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$.

Laplace-Stieltjes transform is defined for Hyperfunctions with defining function having bounded variation property. The relation connecting Laplace-Stieltjes transform and Laplace transform of hyperfunction is proved. The main results obtained are as follows.

- If $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ then

$$\mathfrak{L}_{\mathcal{S}}[f(x)](s) = s\mathfrak{L}[f(x)](s)$$

- If $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$, then

$$\mathfrak{L}_{\mathcal{S}}[f^{'}(x)](s) = s\mathfrak{L}_{\mathcal{S}}[f(x)](s) - sf(0) - f^{'}(0)$$

- If $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ then for any $a \in \mathbb{C}$,

$$\mathfrak{L}_{\mathcal{S}}[e^{ax}f(x)](s) = \mathfrak{L}[f^{'}(x)](s-a) + a\mathfrak{L}[f(x)](s-a),$$

where $\sigma_-(f) + \mathcal{R}(a) < \mathcal{R}(s) < \sigma_+(f) + \mathcal{R}(a)$

- Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth having Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$ with $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ then

$$\mathfrak{L}_{\mathcal{S}}[f^n(x)](s) = s^n \mathfrak{L}_{\mathcal{S}}[f(x)](s),$$

if the strip of convergence is same.

We could define inversion formula for the Laplace-Stieltjes transform of hyperfunction. Using the connection between Fourier and Laplace transform and by a change of variable we have obtained Fourier-Stieltjes transform of hyperfunctions.

In Chapter 4, we proved some Abelian -Tauberian type theorems for some integral transforms of hyperfunctions. Tauberian theory was first developed by Norbert Wiener [63] in 1932. Various types of Abelian Tauberian theorems are proved by many mathematicians for integral transforms. Using Wiener's Tauberian theorem, Shikao Ikehara proved a Tauberian theorem for Dirichlet series, which is known as Wiener Ikehara Theorem. In 1980, using contour integration, Newmann invented new method to prove Tauberian theorems. Korevaar further developed Newmann's method[36].

In this study, first we have proved Abelian Tauberian theorem for the integral of Laplace transform for hyperfunction of bounded exponential growth using the Abelian Tauberian theorem for Laplace transform of measure functions [20].

Secondly, we have proved Abelian -Tauberian theorem for Stieltjes transform of hyperfunction and Abelian type and Tauberian type theorem for Laplace-Stieltjes transform of hyperfunction separately.

The main theorems are

- (Abelian -Tauberian Theorem for Laplace Transform of Hyperfunctions)

  Let $f(x) = [F(z)]$ be a measurable, holomorphic hyperfunction on $(0, \infty)$ having compact support and bounded exponential growth. If the Laplace transform $\hat{f}(s) = [f(x)](s)$ is bounded for $s > 0$ then the following conditions are equivalent.

  (a) $\dfrac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \dfrac{1}{s^{\alpha+1}}$ as $q \to 0$

  (b) $\dfrac{f(px)}{f(p)} \to x^{\alpha}$ as $p \to \infty$

  Also

  $$\mathcal{L}[f(x)](q) \sim f(p)\alpha!, \ \ \alpha \ \geqslant \ 0 \text{ is an integer}$$

- (Abelian -Tauberian Theorem for Stieltjes Transform of Hyperfunctions)

  Let $f(x) = [F(z)]$ be a holomorphic, measurable, non decreasing hyperfunction of bounded exponential growth with compact support contained in $(0, \infty)$ such that the Stieltjes transform $\tilde{f}(t) = S[f(x)](t) = \int_0^\infty \frac{f(x)}{x+t}dx = \int_0^\infty \frac{F(z)}{z+t}dz$ exists for all $t > 0$. Let ρ be a number with $0 \ \leqslant \ \rho < 1$, then the following statements are equivalent

  $$\tilde{f}(t) \sim Ct^{\rho-1} \ \ as \ \ t \to \infty$$

$$f(x) \sim \frac{C}{\Gamma(1+\rho)\Gamma(1-\rho)} x^{\rho} \ \ as \ \ x \to \infty$$

- (Abelian theorem for Laplace-Stieltjes Transformation of Hyperfunctions)

  For $f(x) \in \mathfrak{B}_{bv}^{e+}(I)$,if
  $$\lim_{x\to\infty} \frac{f(x)}{x^n} = \frac{M}{n!}$$
  then
  $$\lim_{s\to 0} s^n f^*(s) = M,$$
  where n is a non-negative number and M is a constant.

- (Tauberian theorem for Laplace-Stieltjes Transformation of Hyperfunctions)

  Let $f(x) \in \mathfrak{B}_{bv}^{e+}(I)$ with Laplace-Stieltjes transform
  $$f^*(s) = \int_0^{\infty} e^{-sx} df(x),$$
  which converges for some $\mathcal{R}(s) > 0$ and
  $$\lim_{s\to 0} s^n f^*(s) = M,$$
  for some constant M and $n > 0$, then
  $$\lim_{x\to\infty} \frac{f(x)}{x^n} = \frac{M}{n!}$$

In Chapter 5, Two-dimensional Laplace transform of hyperfunctions is defined to hyperfunction in two variables. We defined it for hyperfunctions with a separable defining function. As for the ordinary functions some operational properties of the two dimensional Laplace transform of hyperfunction are established. In addition two dimensional inverse transform of Laplace hyperfunction is defined.

The main definitions and results are

- If I_1 and I_2 are open intervals in $\mathbb{R}$ and $N(I_i)$ is a complex neighbourhood of I_i (i.e.$N(I_i)$ contains I_i as a closed subset) for $i = 1, 2$ then the open set $N(I_1) \times N(I_2)$ in $\mathbb{C}^2$ is called a *complex neighbourhood* of $I_1 \times I_2$, if $I_1 \times I_2$ is a closed subset of $N(I_1) \times N(I_2)$.

- Two functions $F(z_1, z_2)$ and $G(z_1, z_2)$ in $\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))$ are *equivalent*, if for $(z_1, z_2) \in (N_1(I_1) \times N_1(I_2)) \cap (N_2(I_1) \times N_2(I_2))$,

$$G(z_1, z_2) = F(z_1, z_2) + \phi_1(z_1, z_2) + \phi_2(z_1, z_2)$$

  with $\phi_1(z_1, z_2) \in \mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $\phi_2(z_1, z_2) \in \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$. Here $N_1(I_1) \times N_1(I_2)$ and $N_2(I_1) \times N_2(I_2)$ are the complex neighbourhoods of $I_1 \times I_2$ of $F(z_1, z_2)$ and $G(z_1, z_2)$ respectively. We denoted it by $F(z_1, z_2) \sim G(z_1, z_2)$

  The relation $\sim$ is an equivalence relation in $\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))$.

- $$\mathfrak{F}(I_1 \times I_2) = \mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))$$

  An equivalence class of functions $F(z_1, z_2) \in \mathfrak{F}(I_1 \times I_2)$ defines a hyperfunction

$f(x,y)$ on $I_1 \times I_2$. It is denoted by

$$f(x,y) = [F(z_1, z_2)]$$

$F(z_1, z_2)$ is called defining or generating function of the hyperfunction $f(x,y)$. The set of all hyperfunctions on the set $I_1 \times I_2$ is denoted by $\mathfrak{B}(I_1 \times I_2)$. Then as a quotient space,

$$\mathfrak{B}(I_1 \times I_2) := \frac{\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))}{\mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2)) + \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))}$$

- The value of a hyperfunction $f(x,y) = [F(x,y)]$ at a point $(x,y) \in I_1 \times I_2$ is defined as

$$f(x,y) = \lim_{\epsilon \to 0+} \{F(x + i\epsilon, y + i\epsilon) - F(x - i\epsilon, y - i\epsilon)\},$$

  provided the limit exists.

- A point $(x,y) \in I_1 \times I_2$ is called a *regular* point of the hyperfunction $f(x,y) = [F(z_1, z_2)]$ if $\lim_{\epsilon \to 0+} \{F(x+i\epsilon, y+i\epsilon) - F(x-i\epsilon, y-i\epsilon)\}$ exists. A point $(x,y) \in I_1 \times I_2$ is called a *singular* point if it is not a regular point. Hence at a regular point the hyperfunction $f(x,y)$ has a value as an ordinary function.

- For $f(x,y) = [F(z_1, z_2)], g(x,y) = [G(z_1, z_2)] \in \mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ and $c \in \mathbb{C}$ defined addition and scalar multiplication as

$$f(x,y) + g(x,y) = [F(z_1, z_2) + G(z_1, z_2)],$$

$$cf(x,y) = [cF(z_1, z_2)]$$

Then $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is a linear space.

- A hyperfunction $f(x,y) = [F(z_1,z_2)]$ is called holomorphic on $I_1 \times I_2$ if the defining function $[F(z_1,z_2)]$ is holomorphic on $U = N(I_1) \times N(I_2)$. i.e

  (i) For each point $a = (a_1,a_2) \in U \subset \mathbb{C}^2$, $F(z_1,z_2)$ has a convergent power series expansion on U ,

$$F(z_1,z_2) = \Sigma c_{n_1,n_2}(z_1 - a_1)^{n_1}(z_2 - a_2)^{n_2}$$

  OR

  (ii) If $F(z_1,z_2)$ is continuous on U and for each variable $z_j, j = 1,2$, $F(z_1,z_2)$ is holomorphic, (i.e.$\frac{\partial F}{\partial \bar{z_1}^n} = 0$ and $\frac{\partial F}{\partial \bar{z_2}^n} = 0$ by the generalisation of Cauchy-Riemann equations)

- For a hyperfunction $f(x,y) = [F(z_1,z_2)]$ in $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$, $sing\ supp f(x,y) \subset supp f(x,y)$

- A hyperfunction $f(x,y) = [F(z_1,z_2)]$ on $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is said to be of bounded exponential growth if there exist real constants $M > 0, \sigma^{'}, \sigma^{''}$ such that

$$|F(z_1,z_2)| < Me^{\sigma^{'}\mathcal{R}z_1 + \sigma^{''}\mathcal{R}z_2} < \infty$$

  on every compact subset of $N(I_1) \times N(I_2)$ and for every equivalent defining functions. $\mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ denotes the set of all hyperfunction in $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ having bounded exponential growth.

- A hyperfunction $f(x,y) = [F(z_1,z_2)]$ on $\mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ is said to be separable if

$$F(z_1,z_2) = F_1(z_1)F_2(z_2),$$

where $F_1(z_1) \in \dfrac{\mathfrak{O}(N(I_1)\backslash I_1)}{\mathcal{A}(I_1)}$ and $F_2(z_2) \in \dfrac{\mathfrak{O}(N(I_2)\backslash I_2)}{\mathcal{A}(I_2)}$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunction with $I_1 \subset [0,\infty)$ and $I_1 \subset [0,\infty)$ the two dimensional Laplace transform of $f(x,y)$ is defined as

$$\hat{f}(u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v) = \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}f(x,y)dxdy,$$

where u and v are complex numbers.

- The image function $\hat{f}(u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$ is a holomorphic function

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunction with $I_1 \subset [0,\infty)$ and $I_1 \subset [0,\infty)$ then

$$\mathfrak{L}_y\mathfrak{L}_x[\overline{f(x,y)}](u,v) = \overline{\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](\bar{u},\bar{v})}$$

- If $f(x,y) = [F(z_1,z_2)], g(x,y) = [G(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ are two separable hyperfunctions with $I_1 \subset [0,\infty)$ and $I_1 \subset [0,\infty)$ then

$$\mathfrak{L}_y\mathfrak{L}_x[f(x,y) + g(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v) + \mathfrak{L}_y\mathfrak{L}_x[g(x,y)](u,v)$$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and c be a constant then

$$\mathfrak{L}_y\mathfrak{L}_x[cf(x,y)](u,v) = c\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and a and

b are two constants then

$$\mathfrak{L}_y\mathfrak{L}_x[e^{ax+by}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u-a,v-b)$$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and a and b are two constants then

$$\mathfrak{L}_y\mathfrak{L}_x[e^{ax}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u-a,v)$$

$$\mathfrak{L}_y\mathfrak{L}_x[e^{by}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v-b)$$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and a and b are two non zero real constants then

$$\mathfrak{L}_y\mathfrak{L}_x[f(ax,by)](u,v) = \frac{1}{ab}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](\frac{u}{a},\frac{v}{b})$$

- If $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and a and b are two constants then

$$\mathfrak{L}_y\mathfrak{L}_x[f(x+a,y+b)](u,v) = e^{au+bv}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$$

- If $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunction, for positive integers m and n,

$$\mathfrak{L}_y\mathfrak{L}_x[x^m y^n f(x,y)](u,v) = (-1)^{m+n}(\frac{d^m}{du^m}(\hat{f}_1(u)))(\frac{d^n}{dv^n}(\hat{f}_2(v))),$$

Where $\hat{f}_1(u) = \int_0^\infty e^{-uz_1}F_1(z_1)dz_1$ and $\hat{f}_2(v) = \int_0^\infty e^{-vz_2}F_2(z_2)dz_2$

- If $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and $\hat{f}_1(u) = \int_0^\infty e^{-uz_1}F_1(z_1)dz_1,\ \hat{f}_2(v) = \int_0^\infty e^{-vz_2}F_2(z_2)dz_2$ then

  (a)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial x}f(x,y)](u,v) = u\hat{f}_1(u)\hat{f}_2(v)$

  (b)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial y}f(x,y)](u,v) = v\hat{f}_1(u)\hat{f}_2(v)$

  (c)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x^2}f(x,y)](u,v) = u^2\hat{f}_1(u)\hat{f}_2(v)$

  (d)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y^2}f(x,y)](u,v) = v^2\hat{f}_1(u)\hat{f}_2(v)$

  (e)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x\partial y}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y\partial x}f(x,y)](u,v) = uv\hat{f}_1(u)\hat{f}_2(v)$

- Inverse of Two Dimensional Laplace Transform of Hyperfunctions

  If $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunction with two dimensional Laplace transform $\hat{f}(u,v)$ the inverse transform is defined by

$$f(x,y) = \int_0^\infty e^{vy}\int_0^\infty e^{ux}\hat{f}(u,v)dudv$$

In chapter 6, some order theoretic properties of hyperfunctions are investigated. Order relation and norm convergence in the linear space of hyperfunctions are studied. The concept of completely monotone hyperfunctions is defined and some of their properties are proved.

H. H. Schaefer[49] and Anthony L. Peressini[43] studied the properties of ordered topological vector spaces. Using the concept of positive cones an order relation was introduced. Here we are mainly considering hyperfunctions having bounded exponential growth and having defining function a complex measurable holomorphic function. An order relation is introduced among these types of hyperfunctions using the defining functions of hyperfunctions. A cone is defined in the linear space of hyperfunctions and some properties of this cone are studied. An inductive limit topology is defined on this ordered space and compares it with the order topology on the space.

Stevan Pilipovic and Bogoljub Stankovic [44] discussed the convergence in the space of Fourier Hyperfunctions. Using these ideas we defined norm to a subclass of hyperfunctions and introduced the concept of norm convergence to that subclass of the linear space of hyperfunctions

The main results are

- If $f(x), g(x) \in \mathfrak{B}_{\mathbb{R}}(I)$ where $f(x) = [F(z)]$, $g(x) = [G(z)]$ defined the order relation $' \leqslant'$ by $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in I$. In terms of defining functions $(F(x+i0) - F(x-i0)) \leqslant (G(x+i0) - G(x-i0)))$ for all $F(z) \in [F(z)]$ and $G(z) \in [G(z)]$.

  The relation $' \leqslant'$ is a partial order on $\mathfrak{B}_{\mathbb{R}}(I)$.

  With this order relation $\mathfrak{B}_{\mathbb{R}}(I)$ is an ordered linear space.

- If $\bar{N}(0,n) = \{z \in \mathcal{C} : |z| \leqslant n\}$, i.e. $\bar{N}(0,n)$ is a closed complex neighbourhood of 0 and N a complex neighbourhood of I. For $n = 1, 2, ...$ defined

$$K_n = \bar{N}(0,n) \cap \{z : |z - w| \geqslant \frac{1}{n}, \forall w \in \mathbb{C} \backslash N\}$$

  Then $\{K_n\}$ has the following properties

  i) K_n is compact

  ii)$K_n \subseteq K_{n+1}$

  iii)If $K \subseteq N$ is compact then $K \subseteq K_n$ for sufficiently large n.

On each K_n and $f(x) = [F(Z)] \in \mathfrak{B}(I)$ defined

$$\beta_{K_n,m}(f(x)) = \sup\{|\frac{d^m}{dz^m}F(z)| : \forall F(z) \in [F(z)], z \in K_n\}, m = 0, 1, 2, ...$$

$\mathfrak{B}^M_{B,K_n,m}(I)$ denotes the subspace of $\mathfrak{B}^M_B(I)$, consisting of all hyperfunctions with support contained in K_n. Then $\{\beta_{K_n,m}\}_{m=0}^{\infty}$ is a multinorm on $\mathfrak{B}^M_{B,K_n,m}(I)$. The defined set of multinorms generates a topology $\tau_{K_n,m}$ on $\mathfrak{B}^M_{B,K_n,m}(I)$.
$\mathfrak{B}^M_B(I)$ assigns the inductive limit topology τ when K_n varies over all compact sets $K_1, K_2, ...$

- $\mathfrak{B}^M_B(I)$ is an ordered topological linear space.

- The Cone, P of $\mathfrak{B}^M_B(I)$ is, when $\mathfrak{B}^M_B(I)$ restricted to the set of all non-negative hyperfunctions in $\mathfrak{B}_{\mathbb{R}}(I)$. The Positive Cone in $\mathfrak{B}^M_B(I)$ is $P + iP$ which is denoted as $\mathcal{P}$

- The cone $\mathcal{P}$ in $\mathfrak{B}^M_B(I)$ has the following properties:
  i)$\mathcal{P} + \mathcal{P} \subset \mathcal{P}$
  ii)$c\mathcal{P} \subset \mathcal{P}$ for every real number $c > 0$
  iii)$\mathcal{P} \cap -\mathcal{P} = \{[0]\}$.

- $\mathcal{P}$ is a convex set in $\mathfrak{B}^M_B(I)$.

- For $f(x)$, $g(x) \in \mathfrak{B}^M_B(I)$ with $f \leqslant g$, defined the *order interval* between f and g by

$$[f, g] = \{h(x) \in \mathfrak{B}^M_B(I) : f(x) \leqslant h(x) \leqslant g(x)\}$$

A subset E of $\mathfrak{B}_B^M(I)$ is *order bounded* if there exists $f(x),\ g(x) \in \mathfrak{B}_B^M(I)$such that $E \subset [f, g]$

- The cone $\mathcal{P}$ in $\mathfrak{B}_B^M(I)$ is generating.
- For $E \subset \mathfrak{B}_B^M(I)$ the *full hull*$[E]$ of E is defined as

$$[E] = \{h(x) \in \mathfrak{B}_B^M(I) : f(x) \leqslant h(x) \leqslant g(x), f(x), g(x) \in E\}$$

- The cone $\mathcal{P}$ in $\mathfrak{B}_B^M(I)$ is normal
- Every order bounded subset of $\mathfrak{B}_B^M(I)$ is τ bounded
- If $\mathcal{P}$ is a normal cone in $\mathfrak{B}_B^M(I)$ then $\mathcal{P} \cap \mathfrak{B}_{B,K_n,m}^M(I)$ is a normal cone in $\mathfrak{B}_{B,K_n,m}^M(I)$
- $\mathfrak{B}_B^K(I)$ is a sub family of hyperfunctions having bounded exponential growth with compact support on $I \subset \mathbb{R}$

  For $f(x) = [F(z)] \in \mathfrak{B}_B^K(I)$ the function $||.||_K$ is defined as

$$||f||_K = \sup_{K \subset I}\{|G(x+i0) - G(x-i0)| : G(z) \in [F(z)], x \in K, K \textit{ is compact subset of } I\}$$

  Then $\mathfrak{B}_B^K(I)$ is a normed linear space.

- A sequence $f_n(x) = [F_n(x)]$ is a Cauchy sequence in $\mathfrak{B}_B^K(I)$ if $\forall\ \epsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying the condition $||f_n - f_m||_K < \epsilon$ for $n, m \geqslant n_0$.
- $\mathfrak{B}_B^K(I)$ is a Banach space
- $\mathfrak{B}_B^K(I)$ is separable.

- If $f(x) = [F(z)],\ g(x) = [G(z)] \in \mathfrak{B}_B^K(I)$ are non negative, real valued measurable hyperfunctions and if $f(x) \leqslant g(x)$ *i.e.* $F(z) \leqslant G(z)$ and it holds for every functions in the equivalence classes of $F(z)$ and $G(z)$) then

$$\int f(x)dx \leqslant \int g(x)dx$$

- If $f_k(x) = [F_k(z)], k = 1, 2, 3, ...$ be a sequence of measurable hyperfunctions in $\mathfrak{B}_B^K(I)$ and $f(x) = \lim f_k(x), f(x) = [F(z)]$ then

$$\int ||f_k(x) - f(x)||_K dx \to 0 \ \ as \ \ k \to \infty$$

in the sense of hyperfunctions.

- A positive real valued hyperfunction $f(x) = [F(Z)]$ defined on $(0, \infty)$ is called a completely monotone hyperfunction if it satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0, \forall x > 0, n = 0, 1, 2, ...$$

- A positive real valued hyperfunction $f(x) = [F(z)]$ defined on $(0, \infty)$ is a completely monotone hyperfunction if there exists a positive valued hyperfunction $g(x) = [G(z)]$ on $(0, \infty)$ with bounded exponential growth such that

$$f(s) = \mathcal{L}[g(x)](s), \forall s > 0$$

- If $f(x) = [F(x)]$ and $g(x) = [G(x)]$ be two completely monotone hyperfunctions then $f(x)g(x)$ is a completely monotone hyperfunction whenever the product is defined and $f(s) = \mathcal{L}[h(x)](s)$ and $g(x) = \mathcal{L}[j(x)](s)$, where $h(x) = [H(z)]$

and $j(x) = [J(z)]$ are two hyperfunctions, $s > 0$

- If $f(x)$ be a completely monotone hyperfunction and $g(x)$ be a positive valued hyperfunction defined on $(0, \infty)$ such that $g\prime(x)$ is a completely monotone hyperfunction then $f \circ g$ is also a completely monotone hyperfunction.

In chapter 7, solved an initial value problem involving hyperfunction by applying Weierstrass transform of hyperfunctions.

The last section consists of further possibilities of the present work.
All hyperfunction integral involved in this study are integrated over curves by taking suitable curves in the region of integration.

Chapter 1

Preliminaries

In this chapter some preliminary ideas on Hyperfunctions, Laplace transform of hyperfunctions, Fourier transform of hyperfunctions, Weierstrass transform, Stieltjes transform, Ordered linear space, Topological vector space, Ordered topological vector space necessary for the coming chapters are included.

1.1. Introduction to Hyperfunctions

Let the upper half plane and lower half-plane of the complex plane $\mathbb{C}$ be denoted by

$$\mathbb{C}_+ = \{z \in \mathbb{C} : Iz > 0\},$$

$$\mathbb{C}_- = \{z \in \mathbb{C} : Iz < 0\}$$

respectively.

Definition 1.1.1. [25] For an open interval I of the real line $\mathbb{R}$, the open subset $N(I) \subset \mathbb{C}$ is called a complex neighbourhood of I, if I is a closed subset of $N(I)$.

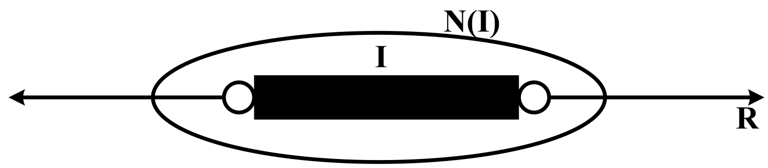

Figure 1.1: Complex neighbourhood $N(I)$

In the above figure the two endpoints of the interval I do not belong to $N(I)$ since the subset $N(I)\backslash I$ is open in $N(I)$. Because $N(I)$ is open in $\mathbb{C}$, the subset $N(I)\backslash I$ is also open in $\mathbb{C}$. The intersection of two or a finite number of complex neighbourhoods of I, is again a complex neighbourhood of I.

Definition 1.1.2. [25] For any complex neighbourhood $N(I)$ of I the two open sets

$$N_+(I) = N(I) \cap \mathbb{C}_+ \text{ and } N_-(I) = N(I) \cap \mathbb{C}_-$$

are called upper half-neighbourhood and lower half-neighbourhood of I, respectively.

Definition 1.1.3. [25] $\mathfrak{O}(N(I))$ denotes the ring of all holomophic functions in the complex neighbourhood $N(I)$ of I.

$\mathfrak{O}(N(I)\backslash I)$ denotes the ring of holomophic functions in $N(I)\backslash I$. For a given interval I a function $F(z) \in \mathfrak{O}(N(I)\backslash I)$ can be written as

$$F(z) = \begin{cases} F_+(z) & \text{for } z \in N_+(I), \\ F_-(z) & \text{for } z \in N_-(I) \end{cases}$$

where $F_+(z) \in \mathfrak{O}(N_+(I))$ and $F_-(z) \in \mathfrak{O}(N_-(I))$ are called upper and lower component of $F(z)$ respectively.

Remark. [25] In general the upper and lower component of $F(z)$ need not be related to each other, i.e., they may be independent holomorphic functions. If the upper and lower components are analytic continuations from each other we call $F(z)$ a global analytic function on $N(I)$ and we can write

$$F_+(z) = F_-(z) = F(z)$$

Definition 1.1.4. [25] Two functions $F(z)$ and $G(z)$ in $\mathfrak{O}(N(I)\backslash I)$ are equivalent, denoted by $F(z) \sim G(z)$ if for $z \in N_1(I) \cap N_2(I)$,

$$G(z) = F(z) + \phi(z),$$

with $\phi(z) \in \mathfrak{O}(N(I))$, i.e., $F(z)$ and $G(z)$ differ by a holomorphic function on $N(I)$. Here $N_1(I)$ and $N_2(I)$ are complex neighbourhoods of I of $F(z)$ and $G(z)$ respectively.

Definition 1.1.5. [25] An equivalence class of functions $F(z) \in \mathfrak{O}(N(I)\backslash I)$ defines a hyperfunction $f(x)$ on I, which is denoted by $f(x) = [F(z)]$. If the upper and the lower component of $F(z)$ should be emphasized, we also use the more explicit notation $f(x) = [F_+(z), F_-(z)]$. The function

$$F(z) = \begin{cases} F_+(z), & z \in N_+(I), \\ F_-(z), & z \in N_-(I) \end{cases}$$

is called a defining or generating function of the hyperfunction $f(x)$.

The set of all hyperfunctions defined on the interval I is denoted by $\mathfrak{B}(I)$.

$$\mathfrak{B}(I) = \mathfrak{O}(N(I)\backslash I)/\mathfrak{O}(N(I))$$

i.e. the quotient space of all functions holomorphic in a complex neighbourhood $N(I)\backslash I$ over the space of all holomorphic functions in $N(I)$.

Remark. [25] There is no reason to prefer a particular choice of neighbourhood $N(I)$. Indeed, if $N_1(I)$ is another complex neighbourhood of I such that $N(I) \supset N_1(I)$, then $\mathfrak{O}(N_1(I)\backslash I)/\mathfrak{O}(N_1(I))$ works as well. This shows that what is essential to the definition of hyperfunctions is the behaviour of the defining functions in a narrow vicinity of I. Now in the above definition let $N(I)$ become narrower and narrower. Intuitively we then write as the inductive limit

$$\mathfrak{B}(I) = \varinjlim_{N(I)\supset I} \mathfrak{O}(N(I)\backslash I)/\mathfrak{O}(N(I))$$

and the definition of the space of hyperfunctions has become independent of any particular complex neighbourhood of I.
Using the intuitive idea of rendering the complex neighbourhoods narrower and narrower around I, leads to another consequence.

A real analytic function $\phi(x)$ on I is defined by the fact that $\phi(x)$ can analytically be continued to a full neighbourhood $U \supset I$, i.e. we then have $\phi(z) \in \mathfrak{O}(U)$. For any complex neighbourhood $N(I)$ containing U we may then write

$$\mathfrak{B}(I) = \mathfrak{O}(N(I)\backslash I)/\mathcal{A}(I),$$

where $\mathcal{A}(I)$ denotes the ring of all real analytic functions on I. Thus a hyperfunction $f(x) \in \mathfrak{B}(I)$, denoted by $f(x) = [F(z)]$ is determined by a defining function $F(z)$ which is holomorphic in an adjacent(small) neighbourhood above and below the interval I, but is only determined upto a real analytic function on I.

Definition 1.1.6. [25] If

$$F(x+i0) - F(x-i0) = \lim_{\epsilon\to 0^+} \{F_+(x+i\epsilon) - F_-(x-i\epsilon)\}$$

exists for a point $x_0 \in I_0 \subset I$ then x_0 is called a regular point of the hyperfunction. At regular points, the given function $F(z) \in \mathfrak{O}(N(I)\backslash I)$ defines an ordinary function $x \mapsto f(x)$. Where the function value $f(x)$ is given by $f(x) = F(x+i0) - F(x-i0)$, i.e., the ordinary function is given by the difference of the boundary values of the two holomorphic functions $F_+(z)$ and $F_-(z)$

At a regular point, a hyperfunction $f(x)$ has a function value as an ordinary function. The value of a hyperfunction $f(x) = [F(z)]$ at a regular point x is

$$f(x) = F(x+i0) - F(x-i0) = \lim_{\epsilon\to 0^+} \{F_+(x+i\epsilon) - F_-(x-i\epsilon)\}$$

Definition 1.1.7. [25] The set of all points $I\backslash I_0$, consisting of real points where one or both of the limits $F_+(x+i0)$ and $F_-(x-i0)$ do not exists is called the singular points of the hyperfunction.

Examples 1.1.8. *[25] A given ordinary function may have more than one hyperfunction representation*

The ordinary constant function $x \mapsto 1, x \in I_0 = \mathbb{R}$ represented by the hyperfunction $f(x)$ has three defining functions.

$$1_+(z) = \begin{cases} 1, & \text{if } Iz > 0; \\ 0, & \text{if } Iz < 0. \end{cases}$$

$$1(z) = \begin{cases} \frac{1}{2}, & \text{if } Iz > 0; \\ -\frac{1}{2}, & \text{if } Iz < 0. \end{cases}$$

$$1_-(z) = \begin{cases} 0, & \text{if } Iz > 0; \\ -1, & \text{if } Iz < 0. \end{cases}$$

Since $1_+(z) \sim 1(z) \sim 1_-(z)$,

$$\begin{aligned} f(x) = 1 &= [1_+(z)] = [1(z)] = [1_-(z)] \\ &= [1, 0] = [1/2, -1/2] = [0, -1] \end{aligned}$$

Remark. [25] Using the previous example the notation $f(x) = [F_+(z), F_-(z)]$ can be change to $f(x) = [F(z)]$ by defining

$$F(z) = 1_+(z)F_+(z) - 1_-(z)F_-(z)$$

Definition 1.1.9. [25] Any real analytic function $\phi(x) \in \mathcal{A}(\mathbb{R})$, interpreted as a hyperfunction again denoted by $\phi(x)$

$$\begin{aligned} \phi(x) &= [\phi(z), 0] = [\phi(z)1_+(z)] \\ &= [\phi(z)/2, -\phi(z)/2] = [\phi(z)1(z)] \\ &= [0, -\phi(z)] = [\phi(z)1_-(z)] \end{aligned}$$

Also

$$\phi(x) = [F(z)],\ F(z) = \frac{\phi(z)}{2}\{1_+(z) + 1_-(z)\}$$

Examples 1.1.10. *[25] Dirac delta function at* $x = 0$ *is represented in terms of hyperfunction as*

$$\delta(x) = [\frac{-1}{2\pi i z}].$$

Here the defining function is

$$F(z) = \frac{-1}{2\pi i z}$$

$F(z)$ *is defined except at* $z = 0$*. At* $z = 0$*,* $F(z)$ *has an isolated singularity, which is a pole of order* 1*. For every real number* $x \neq 0$*, the limit* $\lim_{\epsilon \to 0^+} \{F_+(x + i\epsilon) - F_-(x - i\epsilon)\}$ *exists and equal to* 0

Remark. [25] Let $f(x) = [F_+(z), F_-(z)]$ be a specified hyperfunction. The sequence of ordinary functions

$$f_n(x) = \lim_{n \to \infty} \{F_+(x + \frac{i}{n}) - F_-(x - \frac{i}{n})\}$$

is always defined for sufficiently large n. The family of these functions yields, for increasing n, an intuitive picture of the hyperfunction $f(x)$.

For the Dirac's impulse hyperfunction $\delta(x) = [-\frac{1}{2\pi i z}]$ we have

$$\begin{aligned}
\delta_n(x) &= \lim_{n \to \infty} \{F_+(x + \frac{i}{n}) - F_-(x - \frac{i}{n})\} \\
&= \lim_{n \to \infty} \{-\frac{1}{2\pi i(x + \frac{i}{n})} + \frac{1}{2\pi i(x - \frac{i}{n})}\} \\
&= \frac{n}{\pi(1 + n^2 x^2)}
\end{aligned}$$

Examples 1.1.11. *[25] The representation of ordinary unit-step function or Heaviside function $Y(x)$ as hyperfunction is*

$$u(x) = [-\frac{1}{2\pi i} log(-z)]$$

which is vanishing on the negative part of the real axis, and has the constant value 1 on the positive part. In this case we consider only the principal branch. Hence with $F(z) = -\frac{1}{2\pi i} log(-z)$, *for every real number* $x \neq 0$,

$$\begin{aligned} u(x) &= \lim_{\epsilon \to 0+} \{F_+(x + i\epsilon) - F_-(x - i\epsilon)\} \\ &= \begin{cases} 0 & if\ x < 0 \\ 1 & if\ x > 0 \end{cases} \end{aligned}$$

For $x = 0$ *this hyperfunction has no value, it is a singular point of the hyperfunction.*

Remark. [25] By a real analytic function $\phi(x)$ we mean a function which is holomorphic in a full neighbourhood of the entire real axis, i.e., $\phi(x) \in \mathcal{A}(\mathbb{R})$. The function e^x is real analytic, for it can be analytically continued to the entire function e^z holomorphic in the entire complex plane. The same holds for functions such as $\sin x, \cos x$, polynomials and all rational functions having no poles on the real axis.

Definition 1.1.12. [25] The hyperfunction $f(x) = [\phi(z)]$, where $\phi(x) \in \mathcal{A}(\mathbb{R})$ is any real analytic function, represents the zero hyperfunction. We denote the zero hyperfunction by 0 since it can be identified with the ordinary zero function.

Definition 1.1.13. [25] For $f(x),\ g(x) \in \mathfrak{B}(I)$ with $f(x) = [F(z)],\ g(x) = [G(z)]$,

$$f(x) + g(x) = [F(z) + G(z)]$$

Also for any complex constant c,

$$cf(x) = c[F(z)] = [cF(z)]$$

Proposition 1.1.14. *[25]* $\mathfrak{B}(I)$ *is a linear space.*

Definition 1.1.15. [25] If $f(x) = [F(z)] \in \mathfrak{B}(I)$ is a hyperfunction and $\phi(x) \in \mathcal{A}(I)$ is a real analytic function on I, the product is defined as

$$\phi(x)f(x) = [\phi(z)F(z)]$$

The product is again a hyperfunction, i.e. $\phi(x)f(x) \in \mathfrak{B}(I)$

Remark. [25] We can multiply hyperfunctions with polynomials and functions such as $sinx, cosx, e^x$ and so on and the result is again a hyperfunction.

Definition 1.1.16. [25] Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction then the hyperfunction $f(ax+b)$, with $a, b \in \mathbb{R}$ is defined by

$$f(ax+b) = \begin{cases} [F_+(az+b), F_-(az+b)] & \text{if } a > 0 \\ [-F_-(az+b), -F_+(az+b)] & \text{if } a < 0 \end{cases}$$

For the special case where the upper and lower components of the defining function are function elements of the same global analytic function. i.e., $f(x) = [F(z)] = [F(z), F(z)]$ then the above definition can be written as

$$f(ax+b) = [sgn(a)F(az+b)]$$

Where $sgn(a)$ denotes the *sign* function of a, it has value -1 for $a < 0$ and value 1 for $a > 0$

Definition 1.1.17. [25] For any given hyperfunction $f(x) = [F_+(z), F_-(z)]$ its *derivative* in the sense of hyperfunction is defined and denoted as

$$Df(x) = f^{'}(x) = [\frac{dF_+}{dz}, \frac{dF_-}{dz}]$$

$$D^n f(x) = f^{(n)}(x) = [\frac{d^n F_+}{dz^n}, \frac{d^n F_-}{dz^n}]$$

Proposition 1.1.18. *[25] Hyperfunctions are always infinitely differentiable.*

Examples 1.1.19. *[25] The derivative of unit- step hyperfunction is Dirac's delta function*

$$\begin{aligned} u^{'}(x) &= [\frac{d}{dz}(-\frac{1}{2\pi i}log(-z))] \\ &= [-\frac{1}{2\pi i}\frac{-1}{(-z)}] \\ &= [-\frac{1}{2\pi i}] \\ &= \delta(x) \end{aligned}$$

Also,

$$u^{'}(x-a) = \delta(x-a)$$

Proposition 1.1.20. *[25] For any hyperfunction $f(x)$ and any real analytic function $\phi(x)$,*

$$D(\phi(x)f(x)) = \phi^{'}(x)f(x) + \phi(x)f^{'}(x)$$

Remark. [25] Generally product of two hyperfunctions cannot be defined without some restrictions.

Definition 1.1.21. [25] For a given hyperfunction $f(x) = [F_+(z), F_-(z)]$, $f(-x)$ is defined by

$$f(-x) = [-F_-(z), -F_+(z)]$$

Definition 1.1.22. [25] If $f(-x) = f(x)$, the hyperfunction $f(x) = [F_+(z), F_-(z)]$ is said to be an even hyperfunction. If $f(-x) = -f(x)$, it is called an odd hyperfunction.

Remark. [25] If the upper and lower component of the defining function are restrictions of one global analytic function, i.e., $f(x) = [F(z), F(z)] = [F(z)]$, an odd defining function defines an even hyperfunction and an even defining function defines an odd one.

Proposition 1.1.23. *[25] Any hyperfunction $f(x) = [F_+(z), F_-(z)]$ can be decomposed into an even and an odd hyperfunction*

$$f(x) = f_e(x) + f_o(x)$$

where $f_e(x)$ is even and $f_o(x)$ is odd.

Definition 1.1.24. [25] If $f(x) = [F_+(z), F_-(z)]$ is a given hyperfunction, the complex conjugate hyperfunction of $f(x)$ is defined and denoted by

$$\overline{f(x)} = [-\overline{F_-(\overline{z})}, -\overline{F_+(\overline{z})}]$$

Definition 1.1.25. [25] A hyperfunction $f(x) = [F_+(z), F_-(z)]$ is real, if $\overline{f(x)} = f(x)$ and is pure imaginary if $\overline{f(x)} = -f(x)$

Definition 1.1.26. [25] A linear combination with arbitrary coefficients of Dirac's impulses and their derivatives at $x = a$,

$$\delta_{(N)}(x-a) = \Sigma_{j=0}^{N} c_j \delta^{(j)}(x-a),\ N \in \mathbb{N}$$

is called a generalized delta-hyperfunction of order N at a.

Proposition 1.1.27. *[25] If the hyperfunction $f_1(x)$ is any particular solution of equation*

$$(x-a)^m \psi(x) f(x) = h(x),$$

$\psi(x) \neq 0$, and real analytic, $h(x)$ is a given hyperfunction then the hyperfunction

$$f(x) = f_1(x) + \delta_{(m-1)}(x-a) \text{ is also a solution}$$

Definition 1.1.28. [25] A hyperfunction $f(x)$ is called holomorphic at $x = a$, if the lower and upper component of the defining function can analytically be continued to a full (two- dimensional) neighbourhood of the real point a i.e. the upper/ lower component can analytically be continued across a into the lower/upper half-plane.

Definition 1.1.29. [25] Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction, holomorphic at both end points of the finite interval $[a, b]$, then the (definite)integral of $f(x)$ over $[a, b]$ is defined and denoted by

$$\int_a^b f(x)dx = \int_{\gamma_{a,b}^+} F_+(z)dz - \int_{\gamma_{a,b}^-} F_-(z)dz = -\oint_{(a,b)} F(z)dz$$

where the contour $\gamma_{a,b}^+$ runs in N_+ from a to b above the real axis, and the contour $\gamma_{a,b}^-$ is in N_- from a to b below the real axis.

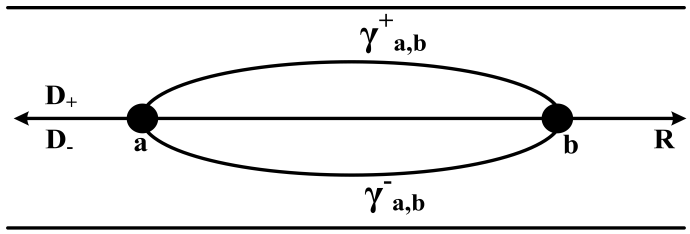

Figure 1.2: Contours $\gamma^+_{a,b}$ and $\gamma^-_{a,b}$

Examples 1.1.30. *[25]*

$$\int_{-\infty}^{\infty} \delta(x)dx = -\oint \frac{-1}{2\pi i z}dz = 1$$

Proposition 1.1.31. *[25] Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction, holomorphic at the finite real points a and b. Then,*

$$\int_a^b f^{'}(x)dx = f(b) - f(a)$$

1.2. Analytic Properties of Hyperfunctions

Definition 1.2.1. [25] Consider hyperfunctions depending on a continuous parameter α or an integral parameter k. The continuous parameter α varies in some open region Ω of the complex plane and α_0 is a limit point of Ω. In the case of integral parameter k may vary in $\mathbb{N}$ or $\mathbb{Z}$. Hence,

$$f(x, \alpha) = [F(z, \alpha)],\ \alpha \in \Omega;\ \ f_k(x) = [F_k(z)],\ k \in \mathbb{N} \text{ or } k \in \mathbb{Z}.$$

We say that a family of holomorphic functions $F(z, \alpha)$, or a sequence of holomorphic functions $F_k(z)$ defined on a common domain $N \subset \mathbb{C}$ *converges uniformly in the interior of* N to $F(z)$ as $\alpha \to \alpha_0$, or $k \to \infty$, respectively if $F(z, \alpha)$ or $F_k(z)$ converges uniformly to $F(z)$ in every compact sub domain of N. This uniform convergence in the interior of N is also called *compact convergence in N*.

Definition 1.2.2. [25] Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction and $f(x, \alpha) = [F(z, \alpha)]$ a family of hyperfunctions depending on the parameter α. Assume that for every α an equivalent defining function $G(z, \alpha)$ of $F(z, \alpha)$ exists, such that $G_+(z, \alpha)$ and $G_-(z, \alpha)$ converge uniformly in the interior of $N_+(I)$ and $N_-(I)$ to $F_+(z)$ and $F_-(z)$, respectively. Then we write

$$f(x) = \lim_{\alpha \to \alpha_0} f(x, \alpha)$$

and say that the family of hyperfunctions $f(x, \alpha)$ converges in the sense of hyperfunctions to $f(x)$.

Definition 1.2.3. [25] Let $f(x) = [F_+(z), F_-(z)]$ be a hyperfunction defined on I such that, for every k, equivalent defining functions $G_k(z)$ of $F_k(z)$ exists, such that $G_{k+}(z)$ and $G_{k-}(z)$ are uniformly convergent in the interior of $N_+(I)$ and $N_-(I)$ to $F_+(z)$ and $F_-(z)$ respectively. Then we write

$$f(x) = \lim_{k \to \infty} f_k(x),$$

and say that the sequence of hyperfunctions $f_k(x)$ converges in the sense of hyperfunctions to $f(x)$.

Definition 1.2.4. [25] We write

$$f(x) = \Sigma_{k=0}^{\infty} f_k(x)$$

if the sequence of partial sums converges in the sense of hyperfunctions to $f(x)$.

Proposition 1.2.5. *[25] If a limit in the sense of hyperfunctions exists, it is unique.*

Definition 1.2.6. [25] A full (two-dimensional) neighbourhood of a real point $a \in I$ is a subset of the form $\{z \in \mathbb{C} : |z - a| < \epsilon, \epsilon > 0\}$
A real neighbourhood of a is a subset of the form $\{x \in \mathbb{R} : |x - a| < \epsilon, \epsilon > 0\}$

Definition 1.2.7. [25] Let the hyperfunction $f(x) = [F(z)]$ be specified on the interval I. i.e., there is a complex neighbourhood N containing I such that $F(z) \in \mathfrak{O}(N \backslash I)$. Then we say that the hyperfunction $f(x) = [F(z)]$ is holomorphic at $x = a \in I$, if the upper and the lower component $F_+(z)$ and $F_-(z)$ can analytically be continued across the real axis to a full neighbourhood of a. A hyperfunction is called holomorphic or analytic in an open interval $J = (a, b) \subset I$, denoted by $f(x) \in \mathfrak{B}_{\mathfrak{O}}(J)$, if it is holomorphic at all $x \in (a, b)$.
This definition implies that if $f(x)$ is a holomorphic hyperfunction at $x = a$, there exists a real neighbourhood of a where $f(x)$ is holomorphic.

Definition 1.2.8. [25] A hyperfunction $f(x) = [F(z)]$ is entire if the upper and lower component of the defining function $F(z)$ are both entire functions.

Definition 1.2.9. [25] A hyperfunction $f(x) = [F(z)]$ is called meromorphic if the upper and lower component of the defining function $F(z)$ are both meromorphic functions (having poles on the real axis)

Definition 1.2.10. [25] Let $f(x)$ is a hyperfunction having a compact support $[a, b]$. Then it's defining function $F(z) \in \mathfrak{O}(N \backslash [a, b])$. It's upper and lower components are analytic continuations from each other i.e. we have $F_+(z) = F_-(z) = F(z)$. Such a hyperfunction is called a *perfect hyperfunction*.

Definition 1.2.11. [25] Standard defining function of a perfect hyperfunction $f(x)$ is defined as

$$\tilde{F}(z) = -\frac{1}{2\pi i} \oint_{(a,b)} \frac{F(t)}{t-z} dt$$

Definition 1.2.12. [25] The hyperfunction $f(x) = [F_+(z), F_-(z)]$ is micro-analytic from above at $x = a \in I$, if the upper component $F_+(z)$ can analytically be continued across the real axis to a full neighbourhood of a. Similarly, $f(x)$ is micro-analytic from below at $x = a \in I$, if the lower component $F_-(z)$ can analytically be continued across the real axis to a full neighbourhood of a.

Definition 1.2.13. [25] Let Σ_0 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is vanishing. Its complement $K_0 = \mathbb{R}\backslash\Sigma_0$ is said to be the support of the hyperfunction $f(x)$ denoted by $supp f(x)$.

Definition 1.2.14. [25] Let Σ_1 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is holomorphic. Its complement $K_1 = \mathbb{R}\backslash\Sigma_1$ is said to be the singular support of the hyperfunction $f(x)$ denoted by $sing\ supp f(x)$.

Definition 1.2.15. [25] Let Σ_2 be the largest open subset of the real line where the hyperfunction $f(x) = [F(z)]$ is micro-analytic (from above or from below). Its complement $K_2 = \mathbb{R}\backslash\Sigma_2$ is said to be the singular spectrum of the hyperfunction $f(x)$ denoted by *sing spec f(x).*

Proposition 1.2.16. *[25] For a hyperfunction $f(x)$ we have*

$$sing\ spec\ f(x) \subset\ sing\ supp\ f(x) \subset\ supp\ f(x)$$

Proposition 1.2.17. *[25] (Theorem of identity for hyperfunctions) If two hyperfunctions $f_1(x)$ and $f_2(x)$ defined on an open interval I have the same singularities, and* *if there is an open subinterval* $(a,b) \subset I$ *where* $f_1(x) = f_2(x)$*, then* $f_1(x) = f_2(x)$ *holds on* I

Proposition 1.2.18. *[25] (Theorem of analytic continuation of hyperfunctions) Let* $f_1(x)$ *and* $f_2(x)$ *be hyperfunctions defined on* (a_1, b_1) *and* (a_2, b_2) *with a non-void overlap* $S = (a_1, b_1) \cap (a_2, b_2) \neq \phi$*. If for any* $\delta > 0$

(i) $f_1(x)$ *is holomorphic in* $(a_1 + \delta, b_1 - \delta)$*, and* $f_2(x)$ *is holomorphic in* $(a_2 + \delta, b_2 - \delta)$

(ii) they are equal in the overlap S

then there exists a unique hyperfunction $f(x)$ *such that*

$$f(x) = \begin{cases} f_1(x), & x \in (a_1, b_1), \\ f_2(x), & x \in (a_2, b_2). \end{cases}$$

which is called analytic continuation of $f_1(x)$ *to* $f_2(x)$*. The given* $f_1(x)$ *and* $f_2(x)$ *are said to be the analytic continuation of each other.*

Definition 1.2.19. [25] Product of hyperfunctions in the case of disjoint singular supports:

We assume that the two hyperfunctions $f(x) = [F(z)] = [F_+(z), F_-(z)]$ and $g(x) = [G(z)] = [G_+(z), G_-(z)]$ satisfy the condition *sing supp*$f(x) \cap$ *sing supp*$g(x) = \phi$. Thus, for any $x \in \mathbb{R}$ at least one of the two hyperfunctions is holomorphic at x.

For definiteness, let I be a real interval where $f(x)$ is holomorphic. i.e., $f(x) \in \mathfrak{B}_{\mathfrak{O}}(I)$ then

$$\begin{aligned} f(x).g(x) &= \{F_+(z) - F_-(z)\}[G(z)] \\ &= [\{F_+(z) - F_-(z)\}G_+(z), \{F_+(z) - F_-(z)\}G_-(z)] \end{aligned}$$

where the order of the factors is important owing to the lack of symmetry of the situation. In the above formula the left factor is the holomorphic hyperfunction. In the event the right factor is the assumed holomorphic hyperfunction, we have

$$\begin{aligned} f(x).g(x) &= \{G_+(z) - G_-(z)\}[F(z)] \\ &= [\{G_+(z) - G_-(z)\}F_+(z), \{G_+(z) - G_-(z)\}F_-(z)] \end{aligned}$$

Proposition 1.2.20. *[25] If $f(x),\ g(x) \in \mathfrak{B}_{\mathfrak{O}}(I)$, i.e., both factors $f(x)$ and $g(x)$ are holomorphic hyperfunctions on I, then the defined product becomes commutative, i.e., $f(x).g(x) = g(x).f(x)$*

1.3. Laplace Transform of Hyperfunctions

Definition 1.3.1. [25] Let $t \mapsto f(t)$ be an ordinary function defined on the entire real axis, and s a complex variable. The two-sided Laplace transform is defined by

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt$$

provided that the improper integral is convergent for some s.

Definition 1.3.2. [25] Consider open sets $J = (a, 0) \cup (0, b)$ with some $a < 0$ and some $b > 0$ and compact subsets $K = \left[a', a''\right] \cup \left[b', b''\right]$ with $a < a' \leqslant a'' < 0$ and $0 < b' \leqslant b'' < b$. Also consider the following open neighbourhoods $[-\delta, \infty) + iJ$ and $(-\infty, \delta] + iJ$ of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively for some $\delta > 0$

Introduce the subclass $\mathfrak{O}(\mathbb{R}_+)$ of hyperfunctions $f(x) = [F(z)]$ on $\mathbb{R}$ satisfying

(i) The support $suppf(x)$ is contained in $[0, \infty)$

(ii) Either the support $suppf(x)$ is bounded on the right by a finite number $\beta > 0$ or we demand that among all equivalent defining functions, there is one, $F(z)$ defined in $[-\delta, \infty) + iJ$ such that for any compact subset $K \subset J$ there exist some real constants $M' > 0$ and σ' such that $|F(z)| \leqslant M' e^{\sigma' \Re z}$ holds uniformly for all $z \in [0, \infty) + iK$

Because $supp f(x) \subset \mathbb{R}_+$ and since the singular support *sing suppf* is a subset of the support, we have *sing suppf* $\subset \mathbb{R}_+$. Therefore $f(x)$ is a holomorphic hyperfunction for all $x < 0$. Moreover, the fact that $F_+(x + i0) - F_-(x - i0) = 0$ for all $x < 0$ shows that $F(z)$ is real analytic on the negative part of the real axis. Hence $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ implies that $\chi_{(-\epsilon,\infty)} f(x) = f(x)$ for any $\epsilon > 0$.

We call the subclass of hyperfunctions $\mathfrak{O}(\mathbb{R}_+)$ the class of *rightsided originals.*

In the case of an unbounded support $supp f(x)$, let $\sigma_- = \inf \sigma'$ be the greatest lower bound of all σ' where the infimum is taken over all σ' and all equivalent defining functions satisfying (ii). This number $\sigma_- = \sigma_-(f)$ is called the growth index of $f(x) \in \mathbb{R}_+$. It has the following properties

(i)$\sigma_- \leqslant \sigma'$

(ii) For every $\epsilon > 0$ there is a σ' with $\sigma_- \leqslant \sigma' \leqslant \sigma_- + \epsilon$ and an equivalent defining function $F(z)$ such that $|F(z)| \leqslant M' e^{\sigma' \Re z}$ uniformly for all $z \in [0, \infty) + iK$.

In the case of a bounded support $supp f(x)$, we set $\sigma_-(f) = -\infty$

Definition 1.3.3. [25] The Laplace transform of a right-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_+)$ is defined by

$$\hat{f}(s) = \mathcal{L}[f(x)](s) = -\int_{\infty}^{(0+)} e^{-sz} F(z) dz.$$

Proposition 1.3.4. *[25] The image function $\hat{f}(s)$ of $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ is holomorphic in the right half-plane $\mathcal{R}s > \sigma_-(f)$*

Definition 1.3.5. [25] Similarly, we introduce the class $\mathfrak{O}(\mathbb{R}_-)$ of hyperfunctions specified by

(i) The support $suppf(x)$ is contained in $\mathbb{R}_- = (-\infty, 0]$

(ii) Either the support $suppf(x)$ is bounded on the left by a finite number $\alpha < 0$, or we demand that among all equivalent defining functions there is one, denoted by $F(z)$ and defined in $(-\infty, \delta] + iJ$ such that for any compact subset $K \subset J$ there are some real constants $M'' > 0$ and σ'' such that $|F(z)| \leqslant M'' e^{\sigma'' \Re z}$ holds uniformly for $z \in (-\infty, 0] + iK$.

The set $\mathfrak{O}(\mathbb{R}_-)$ is said to be the class of left-sided originals.

In the case of an unbounded support let $\sigma_+ = \sup \sigma''$ be the least upper bound of all σ'', where the supremum is taken over all σ'' and all equivalent defining functions satisfying (ii). The number $\sigma_+ = \sigma_+(f)$ is called the growth index of $f(x) \in \mathfrak{O}(\mathbb{R}_-)$. It has the properties

(i) $\sigma'' \leqslant \sigma_+$.

(ii) For every $\epsilon > 0$ there is a σ'' such that $\sigma_+ - \epsilon \leqslant \sigma'' \leqslant \sigma_+$ and a defining function $F(z)$ such that $|F(z)| \leqslant M'' e^{\sigma'' \Re z}$ uniformly for all $z \in (-\infty, 0] + iK$.

If the support $suppf(x)$ is bounded, we set $\sigma_+(f) = +\infty$

Definition 1.3.6. [25] The Laplace transform of a left-sided original $f(x) = [F(z)] \in \mathfrak{O}(\mathbb{R}_-)$ is defined by

$$\hat{f}(s) = \mathcal{L}[f(x)](s) = -\int_{-\infty}^{(0+)} e^{-sz} F(z) dz.$$

Proposition 1.3.7. *[25] The image function $\hat{f}(s)$ of $f(x) \in \mathfrak{O}(\mathbb{R}_-)$ is holomorphic in the left half-plane $\mathcal{R}s < \sigma_+(f)$*

Examples 1.3.8. *[25] $\delta(x) \in \mathfrak{O}(\mathbb{R}_-) \cap \mathfrak{O}(\mathbb{R}_+)$ and $\sigma_-(\delta) = -\infty$ and $\sigma_+(\delta) = +\infty$. For $f(x) = u(x)e^{-x}$ we have $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ and $\sigma_-(f) = -1$. Similarly, for $f(x) = u(-x)e^{x}$ we have $f(x) \in \mathfrak{O}(\mathbb{R}_-)$ and $\sigma_+(f) = +1$. Let $g(x)$ be any polynomial, then $h_2(x) = u(x)g(x) \in \mathfrak{O}(\mathbb{R}_+)$ with $\sigma_-(h_2) = 0$, also $h_1(x) = u(-x)g(x) \in \mathfrak{O}(\mathbb{R}_-)$ with $\sigma_+(h_1) = 0$.*

Remark. [25] With a left-sided original $g(x) \in \mathfrak{O}(\mathbb{R}_-)$ with growth index $\sigma_+(g)$ and a right-sided original $f(x) \in \mathfrak{O}(\mathbb{R}_+)$ with growth index $\sigma_-(f)$ we form the hyperfunction $h(x) = g(x) + f(x)$ whose support is now the entire real axis. If $\widehat{g}(s) = \mathcal{L}[g(x)](s)$, $\mathcal{R}s < \sigma_+(g)$ and $\widehat{f}(s) = \mathcal{L}[f(x)](s)$, $\mathcal{R}s > \sigma_-(f)$ we may add the two image functions, provided they have a common strip of convergence, i.e. $\sigma_-(f) < \sigma_+(g)$ holds.

Definition 1.3.9. [25] With $g(x) \in \mathfrak{O}(\mathbb{R}_-)$, $f(x) \in \mathfrak{O}(\mathbb{R}_+)$, $h(x) = g(x) + f(x)$,

$$\mathcal{L}[h(x)](s) = \widehat{g(x)}(s) + \widehat{f(x)}(s),\ \sigma_-(f) < \Re s < \sigma_+(g),$$

provided $\sigma_-(f) < \sigma_+(g)$.

Definition 1.3.10. [25] Hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_+)$ are said to be of bounded exponential growth as $x \to \infty$ and hyperfunctions of the subclass $\mathfrak{O}(\mathbb{R}_-)$ are said to be of bounded exponential growth as $x \to -\infty$.

Definition 1.3.11. [25] An ordinary function $f(x)$ is called of bounded exponential growth as $x \to \infty$, if there are some real constants $M' > 0$ and σ' such that $|f(x)| \leqslant M'e^{\sigma' x}$ for sufficiently large x. It is called of bounded exponential growth as $x \to -\infty$, if there are some real constants $M'' > 0$ and σ'' such that $|f(x)| \leqslant M''e^{\sigma'' x}$, for sufficiently negative large x.

Definition 1.3.12. [25] A function or a hyperfunction is of *bounded exponential growth*, if it is of bounded exponential growth for $x \to -\infty$ as well as for $x \to \infty$. Thus a hyperfunction or ordinary function $f(x)$ has a Laplace transform, if it is of bounded exponential growth, and if $\sigma_-(f) < \sigma_+(f)$

Proposition 1.3.13. *[25] If $f(x) = [F(z)]$ is a hyperfunction of bounded exponential growth which is holomorphic at $x = c$, then*

$$-\int_{-\infty}^{(c+)} e^{-sz} F(z)dz = \int_{-\infty}^{c} e^{-sx} f(x)dx,$$

$$-\int_{\infty}^{(c+)} e^{-sz} F(z)dz = \int_{c}^{\infty} e^{-sx} f(x)dx$$

thus,

$$-\{\int_{-\infty}^{(c+)} e^{-sz} F(z)dz + \int_{\infty}^{(c+)} e^{-sz} F(z)dz\} = \int_{-\infty}^{\infty} e^{-sx} f(x)dx$$

Proposition 1.3.14. *[25] Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth with an arbitrary support and holomorphic at some point $x = c$. If in addition*

$\sigma_- = \sigma_-(\chi_{(0,\infty)} f(x)) < \sigma_+ = \sigma_+(\chi_{(-\infty,0)} f(x))$, *then its Laplace transform is given by*

$$\begin{aligned} \mathcal{L}[f(x)](s) &= \mathcal{L}[\chi_{(-\infty,c)} f(x)](s) + \mathcal{L}[\chi_{(c,\infty)} f(x)](s) \\ &= \int_{-\infty}^{c} e^{-sx} f(x)dx + \int_{c}^{\infty} e^{-sx} f(x)dx \\ &= \int_{-\infty}^{\infty} e^{-sx} f(x)dx \end{aligned}$$

Proposition 1.3.15. *[25] If $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, having arbitrary support and holomorphic at x_i, $-\infty < x_1 < x_2 < \ldots < x_n < \infty$ and such that $\mathfrak{L}[f(x)](s) = \hat{f}(s)$ has the strip of convergence $\sigma_- < \mathcal{R}s < \sigma_+$ then its Laplace transform is*

$$\mathfrak{L}[f(x)](s) = \int_{-\infty}^{x_1} e^{-sx} f(x)dx + \Sigma_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} e^{-sx} f(x)dx + \int_{x_n}^{\infty} e^{-sx} f(x)dx$$

Proposition 1.3.16. *[25] If $f(x) = [F(z)]$ be a hyperfunction with an arbitrary support and which is holomorphic at $x = 0$, $f_1(x) = \chi_{(-\infty,0)} f(x) \in \mathfrak{O}(\mathbb{R}_-)$ and $f_2(x) = \chi_{(0,\infty)} f(x) \in \mathfrak{O}(\mathbb{R}_+)$ with $\sigma_- = \sigma_-(f_2(x))$ and $\sigma_+ = \sigma_+(f_1(x))$, $\sigma_- < \sigma_+$ then*

$$\mathfrak{L}[f(x)](s) = \mathfrak{L}[f_1(-x)](-s) + \mathfrak{L}[f_2(x)](s)$$

with $-\sigma_+ < \mathcal{R}s, \sigma_- < \mathcal{R}s$

Definition 1.3.17. [25] (Canonical splitting)

A given hyperfunction $f(x) = [F(z)]$ can be split into a sum of two hyperfunctions $f_1(x) = [F_1(z)] \in \mathfrak{O}(\mathbb{R}_-)$ and $f_2(x) = [F_2(z)] \in \mathfrak{O}(\mathbb{R}_+)$ such that $f(x) = f_1(x) + f_2(x)$, $F_1(z) + F_2(z) \sim F(z)$ and $\sigma_-(f_2) < \sigma_+(f_1)$. This is achieved by taking if 0 is not an element in *sing supp*(f), by forming the projections $f_1(x) = \chi_{(-\infty,0)} f(x)$ and $f_2(x) = \chi_{(0,\infty)} f(x)$. So if the Laplace transform $\hat{f}(s)$ of $f(x)$ exists with strip of convergence $\sigma_-(f_2) < \mathcal{R}s < \sigma_+(f_1)$ and $(\hat{f}_1)(s) = \mathcal{L}[f_1(x)](s)$ and $(\hat{f}_2)(s) = \mathcal{L}[f_2(x)](s)$ then $\hat{f}(s) = (\hat{f}_1)(s) + (\hat{f}_2)(s)$.

But the decomposition $f(x) = f_1(x) + f_2(x)$ need not be unique and is only determined upto hyperfunctions concentrated at the point $x = 0$. Regardless of the fact that the Laplace transform of a hyperfunction concentrated at a point is not zero, this arbitrariness generally does not harm for the Laplace transforms.

Definition 1.3.18. [25] **Convolution of hyperfunctions :** Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ be two given hyperfunctions, their convolution is

$$h(x) = f(x) * g(x) = [H(z)]$$

if exists, where

$$H_+(z) = \int_{-\infty}^{\infty} F_+(z-t)g(t)dt, \ \mathcal{I}z > 0$$

$$H_-(z) = \int_{-\infty}^{\infty} F_-(z-t)g(t)dt, \ \mathcal{I}z < 0$$

Proposition 1.3.19. *[25] Let the Laplace transforms $\hat{f}(s)$ and $\hat{g}(s)$ of the hyperfunctions $f(x)$ and $g(x)$ have a non-void common strip of convergence $\sigma_- < \mathcal{R}s < \sigma_+$, and assume further that the convolution $h(x) = (f * g)(x)$ exists and has the Laplace transform $\hat{h}(s)$, then*

$$\hat{h}(s) = \hat{f}(s)\hat{g}(s).$$

1.4. Fourier Transform of Hyperfunctions

Definition 1.4.1. [25] Let $f(x) = f_1(x) + f_2(x)$ be a hyperfunction with $f_1(x) = [F_1(z)] \in \mathfrak{O}(\mathbb{R}_-), f_2(x) = [F_2(z)] \in \mathfrak{O}(\mathbb{R}_+)$. Moreover, assume $-\sigma_+(f_1) \leqslant 0 \leqslant -\sigma_-(f_2)$ holds, then, the Fourier transform of $f(x)$ is defined as being the hyperfunction $\hat{f}(\omega) = \mathfrak{F}[f(x)](\omega) = [H_+(\zeta), H_-(\zeta)]$, $\omega \in \mathbb{R}$ where the two components of the defining function are

$$H_+(\zeta) = \mathfrak{L}[f_1(x)](i\zeta) = -\int_{-\infty}^{0+} e^{-i\zeta z} F_1(z)dz,$$

$$H_-(\zeta) = -\mathfrak{L}[f_2(x)](i\zeta) = \int_{\infty}^{0+} e^{-i\zeta z} F_2(z) dz$$

1.5. Some Preliminary Definitions

Definition 1.5.1. [36] A function $h : (0, \infty) \to (0, \infty)$ is said to be slowly varying at infinity if

$$\lim_{p \to \infty} \frac{h(px)}{h(p)} = 1, \ \forall \ x > 0$$

Definition 1.5.2. [25] The conventional generalized Weierstrass transform of $f(x)$ with parameter t is defined by

$$g(s,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x) dx, \ t > 0$$

Definition 1.5.3. [2] Stieltjes transform of a function $f(t)$ is defined as

$$\rho\{f(t), y\} = \int_0^{\infty} \frac{f(t)dt}{t+y}$$

Definition 1.5.4. [10] $\mathfrak{L}_2$-transform of $f(x)$ is

$$\mathfrak{L}_2\{f(x); y\} = \int_0^{\infty} x e^{-x^2 y^2} f(x) dx$$

Definition 1.5.5. [4] The two dimensional Laplace transform of an ordinary function $f(x,t)$ is defined as

$$L_t L_x f(x,t)(p,s) = \bar{F}(p,s) = \int_0^{\infty} e^{-st} \int_0^{\infty} e^{-px} f(x,t) dx dt \tag{1.1}$$

whenever the integral converges, where $x, t > 0$ are two independent variables and p, s

are complex numbers.

If $f(x,t)$ is a continuous function in $[0,\infty)$ and $\sup\limits_{x>0, t>0} \dfrac{|f(x,t)|}{e^{ax+bt}} < \infty$ for some $a, b \in \mathbb{R}$ then equation (1.1) exists $\forall\, p > a$ and $s > b$.

The inverse Laplace integral is

$$f(x,t) = \frac{1}{(2\pi i)^2} \int_{c-i\infty}^{c+i\infty} \int_{d-i\infty}^{d+i\infty} \bar{F}(p,s) e^{px+st} ds dp$$

where c and d are real constants, $c > a$ and $d > b$

Definition 1.5.6. [40] The space of hyperfunctions in two variables is defined as

$$\mathfrak{B}(\mathbb{R}^2) \cong \frac{\mathfrak{O}((\mathbb{C}\backslash\mathbb{R}) \times (\mathbb{C}\backslash\mathbb{R}))}{\mathfrak{O}((\mathbb{C}\backslash\mathbb{R}) \times \mathbb{C}) \times \mathfrak{O}(\mathbb{C} \times (\mathbb{C}\backslash\mathbb{R}))}$$

Definition 1.5.7. [43] An ordered vector space is a real vector space E equipped with a transitive, reflexive, antisymmetric relation $\leqslant$ satisfying the following conditions:

- If x, y, z are elements of E and $x \leqslant y$, then $x + z \leqslant y + z$
- If x, y are elements of E and α is a positive real number, then $x \leqslant y$ implies $\alpha x \leqslant \alpha y$

Definition 1.5.8. [43] The Positive cone (or simply the cone) K in an ordered vector space E is defined by $K = \{x \in E : x \geqslant 0\}$, where 0 denotes the zero element in E.

Definition 1.5.9. [43] A topological vector space $E(\tau)$ is a vector space E equipped with a topology τ for which the operations of addition and scalar multiplication in E are jointly continuous

Definition 1.5.10. [43] An ordered vector space which is also a topological vector space is called an ordered topological vector space.

Definition 1.5.11. [43] If A is a subset of a vector space E ordered by a cone K, the full hull $[A]$ of A is defined by

$$[A] = \{z \in E : x \leqslant z \leqslant y \text{ for } x \in A, y \in A\}$$

Definition 1.5.12. [43] Suppose that $E(\tau)$ is an ordered topological vector space and that K is the positive cone in $E(\tau)$. K is normal for the topology τ if there is a neighbourhood basis of 0 for τ consisting of full sets.

1.6. Supporting Theorems

Note: In the following theorems $s(\nu)$ denotes the Stieltjes transform of the function $F(t)$

Theorem 1.6.1. *[36] Let $s(\nu)$ vanish for $\nu < 0$, be a non decreasing, continuous from the right and such that*

$$F(t) = \int_{0-}^{\infty-} e^{-t\nu} ds(\nu) = \int_{0-}^{\infty} e^{-t\nu} ds(\nu)$$

exists for $t > 0$. Suppose that for some constant $\alpha > 0$,

$$F(t) \sim \frac{A}{t^{\alpha}} \;\; as \;\; t \to 0 \;\; [or \;\; as \;\; t \to \infty]$$

Then

$$s(u) \sim \frac{A}{\Gamma(\alpha+1)} \;\; as \;\; u \to \infty \;\; [or \;\; as \;\; u \to 0, respectively].$$

Theorem 1.6.2. *[36] Let $s(\nu)$ vanish for $\nu < 0$, be non decreasing, continuous from the right and such that the Stieltjes transform $g(x) = \int_{0-}^{\infty-} \frac{ds(\nu)}{x+\nu}$ exists for $x > 0$. Suppose that for some number $\alpha \in [0, 1)$,*

$$g(x) \sim Ax^{\alpha-1} as \;\; x \to \infty$$

Then

$$s(u) \sim \frac{A}{\Gamma(1+\alpha)\Gamma(1-\alpha)} u^{\alpha} \;\; as \;\; u \to \infty.$$

Theorem 1.6.3. *[36] Let $s(\nu)$ vanish for $\nu < 0$, be non decreasing, continuous from the right and such that the Stieltjes transform $F_\rho(x) = \int_{0-}^{\infty} \frac{ds(\nu)}{(x+\nu)^\rho}$ exists for every $x > 0$. Let $L(x)$ be slowly varying and $0 \leqslant \alpha < \rho$. Then for $x \to \infty$,*

$$s(x) \sim Ax^{\alpha} L(x) \;\; if \;\; and \;\; only \;\; if$$

$$F_\rho(x) \sim A \frac{\Gamma(\alpha+1)\Gamma(\rho-\alpha)}{\Gamma(\rho)} x^{\alpha-\rho} L(x)$$

Chapter 2

Weierstrass, Stieltjes and $\mathfrak{L}_2$ Transforms of Hyperfunctions

In this chapter integral transforms like Weierstrass transform, Stieltjes transform and $\mathfrak{L}_2$-transform are defined for hyperfunctions. Some operational and theoretic properties are established.

2.1. Weierstrass Transform of Hyperfunctions

The Weierstrass transform of Sato's hyperfunctions and some of its properties are studied here using the concept of defining function of hyperfunctions and the Laplace transform of hyperfunctions.

In this study we consider hyperfunctions of bounded exponential growth.

Definition 2.1.1. Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth. The generalized Weierstrass transform of $f(x)$ with parameter $t > 0$ is defined by

$$\mathfrak{W}_t[f(x)](s) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(z)dz,$$

if it exists in the sense of hyperfunction.

Proposition 2.1.2. *The image function $\mathfrak{W}_t[f(x)](s)$ is a holomorphic function.*

Proof. The Kernel of the transformation $e^{\frac{-(s-z)^2}{4t}}$ is holomorphic for all $t > 0$. Hence the integrand becomes a well defined hyperfunction whose integral is a number depending on s holomorphically. □

Proposition 2.1.3. *The Weierstrass transform of hyperfunction is injective.*

Proof. Suppose that $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are two hyperfunctions of bounded exponential growth with $\mathfrak{W}_t[f(x)](s) = \mathfrak{W}_t[g(x)](s)$

$$\begin{aligned}
\mathfrak{W}_t[f(x)](s) = \mathfrak{W}_t[g(x)](s) \quad &\Rightarrow \quad \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} g(x)dx \\
&\Rightarrow \quad \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(z)dz = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} G(z)dz \\
&\Rightarrow \quad \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} (F(z) - G(z))dz = 0 \\
&\Rightarrow \quad F(z) - G(z) = 0 \\
&\Rightarrow \quad [F(z)] = [G(z)] \\
&\Rightarrow \quad f(x) = g(x)
\end{aligned}$$

□

Remark. The following proposition establishes the relation between Weierstrass transform and Laplace transform of a hyperfunction of bounded exponential growth.

Proposition 2.1.4. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth. If $\sigma_-(f) < \mathcal{R}(\frac{-s}{2t}) < \sigma_+(f)$ then*

$$\mathfrak{W}_t[f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathcal{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}), t > 0$$

Proof.

$$\begin{aligned}
\mathfrak{W}_t[f(x)](s) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx \\
&= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{(\frac{-s^2}{4t}+\frac{xs}{2t}-\frac{x^2}{4t})} f(x)dx \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(-s)x}{2t}}(e^{\frac{-x^2}{4t}} f(x))dx \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathcal{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) \text{ if } \sigma_-(f) < \mathcal{R}(\frac{-s}{2t}) < \sigma_+(f),\ t > 0
\end{aligned}$$

□

Proposition 2.1.5. *Let $f(x) = [F(z)]$ is a hyperfunction, holomorphic at $x = c$ having bounded exponential growth with an arbitrary support, $t > 0$, satisfies $\sigma_- = \sigma_-(\chi_{(0,\infty)}e^{\frac{-x^2}{4t}}f(x)) < \sigma_+ = \sigma_+(\chi_{(-\infty,0)}e^{\frac{-x^2}{4t}}f(x))$ then*

$$\mathfrak{W}_t[f(x)](s) = \mathfrak{W}_t[\chi_{(-\infty,c)}f(x)](s) + \mathfrak{W}_t[\chi_{(c,\infty)}f(x)](s)$$

Proof. If $\sigma_- = \sigma_-(\chi_{(0,\infty)}e^{\frac{-x^2}{4t}}f(x)) < \sigma_+ = \sigma_+(\chi_{(-\infty,0)}e^{\frac{-x^2}{4t}}f(x))$ then by 1.3.14 we have

$$\mathfrak{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) = \mathfrak{L}[\chi_{(-\infty,c)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) + \mathfrak{L}[\chi_{(c,\infty)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t})$$

Applying this result and the proposition 2.1.4 we get

$$\begin{aligned}
\mathfrak{W}_{\mathfrak{t}}[f(x)](s) &= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathcal{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}[\mathfrak{L}[\chi_{(-\infty,c)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) + \mathfrak{L}[\chi_{(c,\infty)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t})] \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathfrak{L}[\chi_{(-\infty,c)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) + \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathfrak{L}[\chi_{(c,\infty)}e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) \\
&= \mathfrak{W}_{\mathfrak{t}}[\chi_{(-\infty,c)}f(x)](s) + \mathfrak{W}_{\mathfrak{t}}[\chi_{(c,\infty)}f(x)](s)
\end{aligned}$$

□

Proposition 2.1.6. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, holomorphic at x_i, $-\infty < x_1 < x_2 <...< x_n < \infty$, with an arbitrary support and such that $\mathfrak{L}(f(x)) = \hat{f}(s)$ has the strip of convergence $\sigma_-(f) < \mathcal{R}(\frac{-s}{2t}) < \sigma_+(f)$, for $t > 0$, then the Weierstrass transform*

$$\mathfrak{W}_{\mathfrak{t}}[f(x)](s) = \frac{1}{\sqrt{4\pi t}}[\int_{-\infty}^{x_1} e^{\frac{-(s-x)^2}{4t}}f(x)dx + \int_{x_1}^{x_2} e^{\frac{-(s-x)^2}{4t}}f(x)dx + ... + \int_{x_n}^{\infty} e^{\frac{-(s-x)^2}{4t}}f(x)dx]$$

Proof. By proposition 1.3.15

$$\mathfrak{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t}) = \int_{-\infty}^{x_1} e^{-(\frac{-s}{2t})x}e^{\frac{-x^2}{4t}}f(x)dx + \Sigma_{k=1}^{n-1}\int_{x_k}^{x_{k+1}} e^{-(\frac{-s}{2t})x}e^{\frac{-x^2}{4t}}f(x)dx + \int_{x_n}^{\infty} e^{-(\frac{-s}{2t})x}e^{\frac{-x^2}{4t}}f(x)dx$$

Then using proposition 2.1.4,

$$\mathfrak{W}_{\mathfrak{t}}[f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathcal{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{-s}{2t})$$

$$
\begin{aligned}
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}[\int_{-\infty}^{x_1} e^{-(\frac{-s}{2t})x} e^{\frac{-x^2}{4t}} f(x)dx + \\
&\Sigma_{k=1}^{n-1} \int_{x_k}^{x_{k+1}} e^{-(\frac{-s}{2t})x} e^{\frac{-x^2}{4t}} f(x)dx + \int_{x_n}^{\infty} e^{-(\frac{-s}{2t})x} e^{\frac{-x^2}{4t}} f(x)dx]
\end{aligned}
$$

$$
= \frac{1}{\sqrt{4\pi t}}[\int_{-\infty}^{x_1} e^{\frac{-(s-x)^2}{4t}} f(x)dx + \int_{x_1}^{x_2} e^{\frac{-(s-x)^2}{4t}} f(x)dx + ... + \int_{x_n}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx]
$$

□

Proposition 2.1.7. *Let* $f(x) = [F(z)]$ *is a hyperfunction with bounded exponential growth. Then*

$$
\mathfrak{W}_{\mathfrak{t}}[f(-x)](-s) = -\mathfrak{W}_{\mathfrak{t}}[f(x)](s)
$$

Proof.

$$
\begin{aligned}
\mathfrak{W}_{\mathfrak{t}}[f(-x)](-s) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(-s-x)^2}{4t}} f(-x)dx \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(-s-z)^2}{4t}} F(-z)dz
\end{aligned}
$$

By putting $-z = \zeta$, we have

$$
\begin{aligned}
\mathfrak{W}_{\mathfrak{t}}[f(-x)](-s) &= -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(-s+\zeta)^2}{4t}} F(\zeta)d\zeta \\
&= -\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-\zeta)^2}{4t}} F(\zeta)d\zeta \\
&= -\mathfrak{W}_{\mathfrak{t}}[f(x)](s)
\end{aligned}
$$

□

Proposition 2.1.8. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth, holomorphic at $x = 0$ having an arbitrary support. Let*

$$f_1(x) = \chi_{(-\infty,0)} e^{\frac{-x^2}{4t}} f(x) \in \mathfrak{O}(\mathbb{R}_-) \text{ and } f_2(x) = \chi_{(0,\infty)} e^{\frac{-x^2}{4t}} f(x) \in \mathfrak{O}(\mathbb{R}_+)$$

with $\sigma_- = \sigma_-(f_2(x))$ and $\sigma_+ = \sigma_+(f_1(x))$. If $\sigma_-(f) < \sigma_+(f)$ then the Weierstrass transform

$$\mathfrak{W}_t[f(x)](s) = \mathfrak{W}_t[f_1(-x)](-s) + \mathfrak{W}_t[f_2(x)](s) \text{ with } -\sigma_+(f) < \mathcal{R}(\frac{s}{2t}), \sigma_- < \mathcal{R}(\frac{-s}{2t}), t > 0$$

Proof. By proposition 1.3.16 we have

$$\mathcal{L}[e^{\frac{-x^2}{4t}} f(x)](\frac{-s}{2t}) = \mathcal{L}[e^{\frac{-x^2}{4t}} f_1(-x)](\frac{s}{2t}) + \mathcal{L}[e^{\frac{-x^2}{4t}} f_2(x)](\frac{-s}{2t})$$

Using proposition 2.1.4 we get,

$$\begin{aligned}
\mathfrak{W}_t[f(x)](s) &= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f(x)](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} [\mathcal{L}[e^{\frac{-x^2}{4t}} f_1(-x)](\frac{s}{2t}) + \mathcal{L}[e^{\frac{-x^2}{4t}} f_2(x)](\frac{-s}{2t})] \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f_1(-x)](\frac{s}{2t}) + \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f_2(x)](\frac{-s}{2t}) \\
&= \mathfrak{W}_t[f_1(-x)](-s) + \mathfrak{W}_t[f_2(x)](s)
\end{aligned}$$

□

Proposition 2.1.9. *If $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth, then for $t > 0$,*

$$\mathfrak{W}_t[\overline{f(x)}](s) = \overline{\mathfrak{W}_t[f(x)](\overline{s})}$$

Proof. By Proposition 3.7[25], $\mathfrak{L}[\overline{f(x)}](s) = \overline{\mathfrak{L}[f(x)(\overline{s})]}$. Then

$$\mathfrak{L}[e^{\frac{-x^2}{4t}}\overline{f(x)}](\frac{-s}{2t}) = \overline{\mathfrak{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{\overline{-s}}{2t})}$$

Hence

$$\begin{aligned}
\mathfrak{W}_t[\overline{f(x)}](s) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}}\overline{f(x)}dx \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathfrak{L}[e^{\frac{-x^2}{4t}}\overline{f(x)}](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\overline{\mathfrak{L}[e^{\frac{-x^2}{4t}}f(x)](\frac{\overline{-s}}{2t})} \\
&= \overline{\mathfrak{W}_t[f(x)](\overline{s})}
\end{aligned}$$

□

Examples 2.1.10. *Weierstrass transform of Dirac's delta function as a hyperfunction*

$$\begin{aligned}
\mathfrak{W}_t[\delta(x)](s) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}}\delta(x)dx \\
&= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}}\frac{-1}{2\pi i z}dz \\
&= \frac{-1}{2\pi i\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}}\frac{1}{z}dz \\
&= \frac{-1}{2\pi i\sqrt{4\pi t}}Res_{z=0}[\frac{e^{\frac{-(s-z)^2}{4t}}}{z}] \\
&= \frac{-1}{2\pi i\sqrt{4\pi t}}[e^{\frac{-(s-z)^2}{4t}}]_{z=0} \\
&= \frac{-1}{2\pi i\sqrt{4\pi t}}e^{\frac{-s^2}{4t}}
\end{aligned}$$

Remark. The classical Laplace transform of a negative integral power does not exist, but the Laplace transform of negative integral power of a hyperfunction exist. Hence we can find Weierstrass transform of the negative integral power of a hyperfunction using the relation connecting Weierstrass and Laplace transform. Also the classical Laplace transform of a non-integral power of unit step function $u(t)t^{\alpha}$ exists only for $\alpha > -1$, but the Laplace transform of that hyperfunction exists for any non-integral power α

2.1.1 Operational Properties

Proposition 2.1.11. *Let $f_1(x) = [F_1(z)]$ and $f_2(x) = [F_2(z)]$ are any two hyperfunctions having bounded exponential growth with Weierstrass transforms $\mathfrak{W}_t[f_1(x)](s)$, $\mathfrak{W}_t[f_2(x)](s)$. If the two image functions have a non empty intersection then for constants c_1, c_2,*

$$\mathfrak{W}_t[c_1 f_1(x) + c_2 f_2(x)](s) = c_1 \mathfrak{W}_t[f_1(x)](s) + c_2 \mathfrak{W}_t[f_2(x)](s),$$

where s belongs to the common strip of convergence.

Proof.

$$\begin{aligned}
\mathfrak{W}_t[c_1 f_1(x) + c_2 f_2(x)](s) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} (c_1 f_1(x) + c_2 f_2(x)) dx \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} (c_1 F_1(z) + c_2 F_2(z)) dz \\
&= \frac{1}{\sqrt{4\pi t}} (\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} c_1 F_1(z) dz + \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} c_2 F_2(z) dz)
\end{aligned}$$

$$= \frac{1}{\sqrt{4\pi t}}(\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} c_1 f_1(x)dx + \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} c_2 f_2(x)dx)$$

$$= c_1 \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f_1(x)dx + c_2 \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f_2(x)dx$$

$$= c_1 \mathfrak{W}_{\mathfrak{t}}[f_1(x)](s) + c_2 \mathfrak{W}_{\mathfrak{t}}[f_1(x)](s)$$

□

Proposition 2.1.12. *If the hyperfunction $f(x) = [F(z)]$ has the Weierstrass transform $\mathfrak{W}_{\mathfrak{t}}[f(x)](s)$ and has the canonical splitting $f(x) = f_1(x) + f_2(x)$, then*

$$\mathfrak{W}_{\mathfrak{t}}[f(x)](s) = \mathfrak{W}_{\mathfrak{t}}[f_1(x)](s) + \mathfrak{W}_{\mathfrak{t}}[f_2(x)](s),$$

if $\sigma_-(f_2) < \mathcal{R}(\frac{-s}{2t}) < \sigma_+(f_1)$

Proof. If the hyperfunction $f(x) = [F(z)]$ has the canonical splitting, $f(x) = f_1(x) + f_2(x)$ then

$$\mathfrak{L}[f(x)](s) = \mathfrak{L}[f_1(x)](s) + \mathfrak{L}[f_2(x)](s), \text{ if } \sigma_-(f_2) < \mathcal{R}s < \sigma_+(f_1)$$

Hence

$$\mathfrak{W}_{\mathfrak{t}}[f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f(x)](\frac{-s}{2t})$$

$$= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \{\mathcal{L}[e^{\frac{-x^2}{4t}} f_1(x)](\frac{-s}{2t}) + \mathcal{L}[e^{\frac{-x^2}{4t}} f_2(x)](\frac{-s}{2t})\}$$

$$= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f_1(x)](\frac{-s}{2t}) + \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathcal{L}[e^{\frac{-x^2}{4t}} f_2(x)](\frac{-s}{2t})$$

$$= \mathfrak{W}_{\mathfrak{t}}[f_1(x)](s) + \mathfrak{W}_{\mathfrak{t}}[f_2(x)](s), \text{ if} \sigma_-(f_2) < \mathcal{R}(\frac{-s}{2t}) < \sigma_+(f_1)$$

□

Proposition 2.1.13. *Let $f(x) = [F(z)]$ is a real analytic hyperfunction having bounded exponential growth and c be a constant, then*

$$\mathfrak{W}_{\mathfrak{t}}[f(x+c)](s) = \mathfrak{W}_{\mathfrak{t}}[f(x)](s+c),$$

if $\sigma_{-}(f) < \mathcal{R}(s) < \sigma_{+}(f)$.

Proof.

$$\begin{aligned}\mathfrak{W}_{\mathfrak{t}}[f(x+c)](s) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x+c)dx \\ &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} F(z+c)dz\end{aligned}$$

By putting $z + c = \zeta$, we have

$$\begin{aligned}\mathfrak{W}_{\mathfrak{t}}[f(x+c)](s) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s+c-\zeta)^2}{4t}} F(\zeta)d\zeta \\ &= \mathfrak{W}_{\mathfrak{t}}[f(x)](s+c)\end{aligned}$$

□

Proposition 2.1.14. *Let $f(x) = [F(z)]$ is a hyperfunction of bounded exponential growth and the Weierstrass transform $\mathfrak{W}_{\mathfrak{t}}[f(x)](s)$ exists. Then*

$$\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s) = \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}}\mathfrak{L}[f(x)](\frac{-s}{2t})$$

Proof.

$$\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} e^{\frac{x^2}{4t}} f(x)dx$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} e^{\frac{z^2}{4t}} F(z)dz \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t} + \frac{z^2}{4t}} F(z)dz \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t} + \frac{sz}{2t}} F(z)dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(\frac{-s}{2t})z} F(z)dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathfrak{L}[f(x)](\frac{-s}{2t})
\end{aligned}
$$

□

Proposition 2.1.15. *Let $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth with Weierstrass transform $\mathfrak{W}_{\mathfrak{t}}[f(x)](s)$. Then*

$$
\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f^n(x)](s) = (\frac{-s}{2t})^n \mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s)
$$

Proof.

$$
\begin{aligned}
\mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f^n(x)](s) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} e^{\frac{z^2}{4t}} F^n(z)dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(\frac{-s}{2t})z} F^n(z)dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathfrak{L}[f^n(x)](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} (\frac{-s}{2t})^n \mathfrak{L}[f(x)](\frac{-s}{2t}) \\
&= (\frac{-s}{2t})^n \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathfrak{L}[f(x)](\frac{-s}{2t}) \\
&= (\frac{-s}{2t})^n \mathfrak{W}_{\mathfrak{t}}[e^{\frac{x^2}{4t}} f(x)](s)
\end{aligned}
$$

□

Proposition 2.1.16. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth. Then*

$$\mathfrak{W}_t[e^{\frac{x^2}{4t}} x^n f(x)](s) = (-1)^n e^{\frac{-s^2}{4t}} \frac{d^n}{ds^n}[e^{\frac{s^2}{4t}} \mathfrak{W}_t[e^{\frac{x^2}{4t}} f(x)](s)]$$

Proof.

$$\begin{aligned}
\mathfrak{W}_t[e^{\frac{x^2}{4t}} x^n f(x)](s) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-z)^2}{4t}} e^{\frac{z^2}{4t}} z^n F(z) dz \\
&= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t} + \frac{sz}{2t}} z^n F(z) dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-(\frac{-s}{2t})z} z^n F(z) dz \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} \mathfrak{L}[x^n f(x)](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} (-1)^n \frac{d^n}{ds^n} \mathfrak{L}[f(x)](\frac{-s}{2t}) \\
&= \frac{e^{\frac{-s^2}{4t}}}{\sqrt{4\pi t}} (-1)^n \frac{d^n}{ds^n} (\mathfrak{W}_t[e^{\frac{x^2}{4t}} f(x)](s) \sqrt{4\pi t} e^{\frac{s^2}{4t}}) \\
&= (-1)^n e^{\frac{-s^2}{4t}} \frac{d^n}{ds^n}[e^{\frac{s^2}{4t}} \mathfrak{W}_t[e^{\frac{x^2}{4t}} f(x)](s)]
\end{aligned}$$

□

2.1.2 Inverse Weierstrass Transform of Hyperfunctions

In [33] V.Karunakarn and T.Venugopal gave inversion formula for Weierstrass Transform for a class of generalized functions. In this study, the inversion formula for Weierstrass transform of Hyperfunction can be defined as follows:

Definition 2.1.17. If $\tilde{f}(s) = \mathfrak{W}_t[f(x)](s)$ is the Weierstrass transform of $f(x) = [F(z)]$, a hyperfunction of bounded exponential growth then

$$f(x) = [F(z)] = \lim_{t \to 1} \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s+iz)^2}{4t}} \tilde{f}(is) ds$$

2.2. Stieltjes Transform of Hyperfunctions

We define Stieltjes transform for hyperfunctions having bounded exponential growth. Some properties of Stieltjes transform of hyperfunctions are proved.

2.2.1 Stieltjes Transformation of Hyperfunctions

Definition 2.2.1. For a hyperfunction $f(x) = [F(z)]$ of bounded exponential growth the Stieltjes transform is defined by

$$\tilde{f}(t) = \mathcal{S}[f(x)](t) = \int_0^{\infty} \frac{f(x)}{x+t} dx = \int_0^{\infty} \frac{F(z)}{z+t} dz, \text{ where } t \in \mathbb{R}$$

if it exists.

Remark. As for the normal functions, the Stieltjes transform of hyperfunction is also the second iterate of Laplace transform of hyperfunction if it is of bounded exponential growth.

Proposition 2.2.2. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth then*

$$\mathcal{S}[f(x)](t) = \mathfrak{L}[\mathfrak{L}[f(x)](s)](t)$$

if $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ and $\sigma_-(f) < \mathcal{R}t < \sigma_+(f)$

Proof. Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth and $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ and $\sigma_-(f) < \mathcal{R}t < \sigma_+(f)$.

$$\begin{aligned}
\mathfrak{L}[\mathfrak{L}[f(x)](s)](t) &= \int_0^\infty e^{-st}\mathfrak{L}[f(x)](s)ds \\
&= \int_0^\infty e^{-ts}[\int_0^\infty e^{-sx}f(x)dx]ds \\
&= \int_0^\infty [\int_0^\infty e^{-sx-st}ds]f(x)dx \\
&= \int_0^\infty [\int_0^\infty e^{-(x+t)s}ds]f(x)dx \\
&= \int_0^\infty [\int_0^\infty e^{-(z+t)\zeta}d\zeta]F(z)dz \\
&= \int_0^\infty \frac{F(z)}{z+t}dz \\
&= \int_0^\infty \frac{f(x)}{x+t}dx \\
&= \mathcal{S}[f(x)](t)
\end{aligned}$$

□

Examples 2.2.3. *Consider the delta function as a hyperfunction* $\delta(x) = [\frac{-1}{2\pi i z}]$

$$\mathfrak{L}[\delta(x)](s) = 1,$$

$$\mathfrak{L}[\mathfrak{L}[\delta(x)](s)](t) = \int_0^\infty e^{-ts}\mathfrak{L}[\delta(x)](s)ds = \int_0^\infty e^{-ts}ds = \frac{1}{t}$$

Also

$$\mathcal{S}[\delta(x)](t) = \frac{1}{t}$$

Proposition 2.2.4. *Let* $f(x) = [F(z)]$ *is a hyperfunction with bounded exponential growth then the Stieltjes transform* $\tilde{f}(t) = \mathcal{S}[f(x)](t)$ *is a holomorphic function*

Proof. Since the image of Laplace transform of a hyperfunction is a holomorphic function the double Laplace transform is again a holomorphic function. Hence $\mathcal{S}[f(x)](t)$ is a holomorphic function of t. □

Proposition 2.2.5. *Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are any two hyperfunctions having bounded exponential growth with the Stieltjes transform $\tilde{f}(t) = \mathcal{S}[f(x)](t)$ and $\tilde{g}(t) = \mathcal{S}[g(x)](t)$ respectively, then*

$$\int_0^\infty f(x)\tilde{g}(x)dx = \int_0^\infty g(t)\tilde{f}(t)dt$$

Proof.

$$\begin{aligned}
\int f(x)\tilde{g}(x)dx &= \int_0^\infty f(x)(\int_0^\infty \frac{g(t)}{t+x}dt)dx \\
&= \int_0^\infty g(t)(\int_0^\infty \frac{f(x)}{x+t}dx)dt \\
&= \int_0^\infty g(t)\tilde{f}(t)dt
\end{aligned}$$

□

Proposition 2.2.6. *Let $f(x) = [F(z)]$ is a real valued hyperfunction with bounded exponential growth and $f(x) > 0$ for all $x > 0$ then the Stieltjes transform*

$$\tilde{f}(t) = \mathcal{S}[f(x)](t) > 0$$

for all $t > 0$

Proof. Since the integral of a positive valued function in $(0, \infty)$ is always positive we have $\tilde{f}(t) = \mathcal{S}[f(x)](t) > 0$ for all $t > 0$. □

2.2.2 Operational Properties

Proposition 2.2.7. *Let $f(x) = [F(z)]$ is a hyperfunction with bounded exponential growth and $c > 0$ be a constant then*

$$\mathcal{S}[f(cx)](t) = c\mathcal{S}[f(x)](ct)$$

Proof.

$$\begin{aligned} S[f(cx)](t) &= \int_0^\infty \frac{f(cx)}{x+t}dx \\ &= \int_0^\infty \frac{F(cz)}{z+t}dz \\ &= \int_0^\infty \frac{F(\zeta)}{\frac{\zeta}{c}+t}d\zeta, \text{by letting } \zeta = cz \\ &= c\int_0^\infty \frac{F(\zeta)}{\zeta+ct}d\zeta \\ &= c\int_0^\infty \frac{f(x)}{x+ct}dx \\ &= c\mathcal{S}[f(x)](ct) \end{aligned}$$

□

Proposition 2.2.8. *Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are two hyperfunctions having bounded exponential growth then*

$$\mathcal{S}[f(x) + g(x)](t) = \mathcal{S}[f(x)](t) + \mathcal{S}[g(x)](t)$$

Proof.

$$\mathcal{S}[f(x) + g(x)](t) = \int_0^\infty \frac{(f(x) + g(x))}{x+t}dx$$

$$
\begin{aligned}
&= \int_0^\infty \frac{(F(z)+G(z))}{z+t}dz \\
&= \int_0^\infty \frac{F(z)}{z+t}dz + \int_0^\infty \frac{G(z)}{z+t}dz \\
&= \mathcal{S}[f(x)](t) + \mathcal{S}[g(x)](t)
\end{aligned}
$$

□

Proposition 2.2.9. *Let* $f(x) = [F(z)]$ *is a hyperfunction with bounded exponential growth and* $c > 0$ *be a constant then*

$$
\mathcal{S}[f(x+c)](t) = \mathcal{S}[f(x)](t-c)
$$

Proof.

$$
\begin{aligned}
\mathcal{S}[f(x+c)](t) &= \int_0^\infty \frac{f(x+c)}{x+t}dx \\
&= \int_0^\infty \frac{F(z+c)}{z+t}dz \\
&= \int_0^\infty \frac{F(\zeta)}{\zeta - c + t}d\zeta, \text{ by putting } z+c=\zeta \\
&= \int_0^\infty \frac{f(x)}{x+(t-c)}dx \\
&= \mathcal{S}[f(x)](t-c)
\end{aligned}
$$

□

Proposition 2.2.10. *Let* $f(x) = [F(z)]$ *is a hyperfunction with bounded exponential growth,* $\hat{f}(s) = \mathfrak{L}[f(x)](s)$ *exists and* c *be a constant then*

$$
\mathcal{S}[e^{cx}f(x)](t) = \mathfrak{L}[\hat{f}(s-c)](t)
$$

if $\sigma_-(f) + \mathcal{R}c < \mathcal{R}s < \sigma_+(f) + \mathcal{R}c$ *and* $\sigma_-(\hat{f}) < \mathcal{R}t < \sigma_+(\hat{f})$

Proof. Suppose that $\sigma_-(f) + \mathcal{R}c < \mathcal{R}s < \sigma_+(f) + \mathcal{R}c$ and $\sigma_-(\hat{f}) < \mathcal{R}t < \sigma_+(\hat{f})$. Then

$$\begin{aligned} \mathcal{S}[e^{cx} f(x)](t) &= \mathfrak{L}[\mathfrak{L}[e^{cx} f(x)](s)](t) \\ &= \mathfrak{L}[\hat{f}(s-c)](t) \end{aligned}$$

□

Proposition 2.2.11. *Let* $f(x) = [F(z)]$ *is a hyperfunction with bounded exponential growth,* $\hat{f}(s) = \mathfrak{L}[f(x)](s)$ *exists, then*

$$\mathcal{S}[f^{(n)}(x)](t) = (-1)^n \frac{d^n}{dt^n}(\mathfrak{L}[\hat{f}(s)](t))$$

if $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ *and* $\sigma_-(\hat{f}) < \mathcal{R}t < \sigma_+(\hat{f})$

Proof. Suppose that $\sigma_-(f) < \mathcal{R}s < \sigma_+(f)$ and $\sigma_-(\hat{f}) < \mathcal{R}t < \sigma_+(\hat{f})$. Then

$$\begin{aligned} \mathcal{S}[f^{(n)}(x)](t) &= \mathfrak{L}[\mathfrak{L}[f^{(n)}(x)](s)](t) \\ &= \mathfrak{L}[s^n \hat{f}(s)](t) \\ &= (-1)^n \frac{d^n}{dt^n}(\mathfrak{L}[\hat{f}(s)](t)) \end{aligned}$$

□

Proposition 2.2.12. *Let* $f(x) = [F(z)]$ *and* $g(y) = [G(z)]$ *are two hyperfunctions of bounded exponential growth. If support* $f(x)$ *is a compact subset of* $(0, \infty)$ *then*

$$\int_0^\infty \mathfrak{L}[f(x)](s)\mathfrak{L}[g(y)](s)ds = \int_0^\infty g(y)\mathcal{S}[f(x)](y)dy$$

Proof.

$$\begin{aligned}
\int_0^\infty \mathfrak{L}[f(x)](s)\mathfrak{L}[g(y)](s)ds &= \int_0^\infty \mathfrak{L}[f(x)](s)(\int_0^\infty e^{-sy}g(y)dy)ds \\
&= \int_0^\infty g(y)(\int_0^\infty e^{-sy}\mathfrak{L}[f(x)](s)ds)dy \\
&= \int_0^\infty g(y)\mathfrak{L}[\mathfrak{L}[f(x)](s)](y)dy \\
&= \int_0^\infty g(y)\mathcal{S}[f(x)](y)dy
\end{aligned}$$

□

Proposition 2.2.13. *Let $f(x) = [F(z)]$ and $g(y) = [G(z)]$ are two hyperfunctions of bounded exponential growth and support $f(x)$ is a compact subset of $(0, \infty)$. If the Laplace transforms $\mathfrak{L}[f(x)](s)$ and $\mathfrak{L}[g(y)](s)$ of $f(x)$ and $g(x)$ have a common strip of convergence, then*

$$\int_0^\infty \mathfrak{L}[(f * g)(x)](s)ds = \int_0^\infty g(y)\mathcal{S}[f(x)](y)dy$$

Proof. If the Laplace transforms $\mathfrak{L}[f(x)](s)$ and $\mathfrak{L}[g(y)](s)$ of $f(x)$ and $g(x)$ have a common strip of convergence, then $\mathfrak{L}[f(x)](s)\mathfrak{L}[g(y)](s) = \mathfrak{L}[(f * g)(x)](s)$
Then using previous proposition we have,

$$\begin{aligned}
\int_0^\infty \mathfrak{L}[(f * g)(x)](s)ds &= \int_0^\infty \mathfrak{L}[f(x)](s)\mathfrak{L}[g(y)](s)ds \\
&= \int_0^\infty g(y)\mathcal{S}[f(x)](y)dy
\end{aligned}$$

□

2.2.3 Inverse Stieltjes Transform of Hyperfunctions

A complex inversion formula for Stieltjes transform is defined by A. Aghili and A. Ansari in [2]. In this study we see that the inverse formula exists for Stieltjes transform of hyperfunction.

Definition 2.2.14. If $f(x)$ is a hyperfunction of bounded exponential growth, with Stieltjes transform $\tilde{f}(t)$ then

$$f(x) = S^{-1}[\tilde{f}(t)] = \frac{1}{2\pi}\int_{a-i\infty}^{a+i\infty}(\int_{b-i\infty}^{b+i\infty}\tilde{f}(t)e^{ty}dt)e^{xy}dy$$

Where a and b are two positive constants chosen suitably.
The above integral is an improper integral which can be evaluated as a contour integral. It can be integrated along vertical lines $t = b + i\beta$ and $y = a + i\alpha$ in the complex plane.

If $f(x)$ is a hyperfunction of bounded exponential growth inverse Stieltjes transform of a hyperfunction can also be defined in terms of the inverse Laplace transform as

$$f(x) = S^{-1}[\tilde{f}(t)] = \mathfrak{L}^{-1}[\mathfrak{L}^{-1}[\tilde{f}(t)](y)](x)$$

2.3. $\mathfrak{L}_2-$ Transform of Hyperfunctions

O.Yurekli and I.Sadek in 1991 introduced the Laplace transform like $\mathfrak{L}_2$-transform for solving partial differential equations and integral equations. Here we define $\mathfrak{L}_2$ transform for Hyperfunctions having bounded exponential growth. Since the $\mathfrak{L}_2$ -transform integral converges for hyperfunctions of bounded exponential growth.

2.3.1 $\mathfrak{L}_2$ Transform of Hyperfunctions

Definition 2.3.1. Let $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth, the $\mathfrak{L}_2$ transform of $f(x)$ is defined as

$$\mathfrak{L}_2[f(x)](s) = \int_0^\infty xe^{-x^2s^2} f(x)dx = \int_0^\infty ze^{-z^2s^2} F(z)dz,$$

if $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$

Proposition 2.3.2. *The image function $\mathfrak{L}_2[f(x)](s)$ is holomorphic*

Proof. Since the kernel of the transform is a holomorphic function and the defining function $F(z)$ is holomorphic, $\mathfrak{L}_2[f(x)](s)$ is holomorphic. □

As for ordinary function the relation connecting $\mathfrak{L}_2$ transform and Laplace transform of hyperfunction exists.

Proposition 2.3.3. *If $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$, then*

$$\mathfrak{L}_2[f(x)](s) = \frac{1}{2}\mathfrak{L}[f(\sqrt{x})](s^2)$$

Proof. Let $f(x) = [F(z)]$ is a hyperfunction having bounded exponential growth and

$\sigma_{-}(f) < \mathcal{R}(s^2) < \sigma_{+}(f)$.

$$\begin{aligned}
\mathfrak{L}_2[f(x)](s) &= \int_0^\infty x e^{-x^2 s^2} f(x) dx \\
&= \int_0^\infty z e^{-z^2 s^2} F(z) dz \\
&= \frac{1}{2}\int_0^\infty e^{-\zeta s^2} F(\sqrt{\zeta}) d\zeta, \text{ by putting } z^2 = \zeta \\
&= \frac{1}{2}\mathfrak{L}[f(\sqrt{x})](s^2)
\end{aligned}$$

□

Examples 2.3.4.

$$\begin{aligned}
\mathfrak{L}_2[\delta(x)](s) &= \int_0^\infty x e^{-x^2 s^2} \delta(x) dx \\
&= \int_0^\infty z e^{-z^2 s^2} (\frac{-1}{2\pi i z}) dz \\
&= \frac{-1}{2\pi i}\int_0^\infty e^{-z^2 s^2} dz \\
&= \frac{-1}{4\pi i s}\int_0^\infty u^{\frac{1}{2}-1} e^{-u} du, \; \textit{by putting } z^2\zeta^2 = u \\
&= \frac{-1}{4\pi i s}\Gamma(\frac{1}{2}) \\
&= \frac{i}{4\sqrt{\pi}} s
\end{aligned}$$

Examples 2.3.5.

$$\begin{aligned}
\mathfrak{L}_2[x^m](s) &= \int_0^\infty x e^{-x^2 s^2} x^m dx \\
&= \int_0^\infty z e^{-z^2 s^2} z^m dz \\
&= \frac{\Gamma(\frac{m}{2}+1)}{2 s^{m+2}}
\end{aligned}$$

Remark. We show that the initial value and final value type theorem also exist for $\mathfrak{L}_2$-transform of hyperfunctions

Proposition 2.3.6. *If $f(x) = [F(z)]$ is a holomorphic hyperfunction having bounded exponential growth, $f'(x) = [F'(z)]$ is also holomorphic and $\sigma_-(f) < \mathcal{R}(s^2) < \sigma_+(f)$, then*

$$\lim_{x\to 0} f(x) = \lim_{s\to\infty} 2s^2 \mathfrak{L}_2[f(x)](s)$$

$$\textit{Also,} \quad \lim_{x\to\infty} f(x) = \lim_{s\to 0} 2s^2 \mathfrak{L}_2[f(x)](s)$$

Proof. We have

$$\begin{aligned} \int_0^\infty e^{-s^2x^2} f'(x)dx &= \int_0^\infty e^{-s^2z^2} F'(z)dz \\ &= 2s^2 \mathfrak{L}_2[f(x)](s) - f(0) \end{aligned}$$

But

$$\lim_{s\to\infty} \int_0^\infty e^{-s^2x^2} f'(x)dx = 0$$

Therefore

$$\lim_{s\to\infty} 2s^2 \mathfrak{L}_2[f(x)](s) - f(0) = 0$$

Since $f(x)$ is holomorphic,

$$f(0) = \lim_{x\to 0} f(x)$$

Thus,

$$\lim_{s\to\infty} 2s^2 \mathfrak{L}_2[f(x)](s) = f(0) = \lim_{x\to 0} f(x)$$

Letting $s \to 0$ in $\int_0^\infty e^{-s^2x^2} f'(x)dx$ we have

$$\begin{aligned} \lim_{s\to 0}\int_0^\infty e^{-s^2x^2} f'(x)dx &= \int_0^\infty f'(x)dx \\ &= \int_0^\infty F'(z)dz \\ &= \lim_{x\to\infty} f(x) - f(0) \end{aligned}$$

Thus

$$\lim_{s\to 0}(2s^2\mathfrak{L}_2[f(x)](s) - f(0)) = \lim_{x\to\infty} f(x) - f(0)$$

$$i.e. \lim_{s\to 0} 2s^2\mathfrak{L}_2[f(x)](s) = \lim_{x\to\infty} f(x)$$

□

Proposition 2.3.7. *If $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are hyperfunctions of bounded exponential growth and $\sigma_-(f+g) < \mathcal{R}(s^2) < \sigma_+(f+g)$ then*

$$\mathfrak{L}_2[f(x) + g(x)](s) = \mathfrak{L}_2[f(x)](s) + \mathfrak{L}_2[g(x)](s),$$

Proof.

$$\begin{aligned} \mathfrak{L}_2[f(x) + g(x)](s) &= \int_0^\infty xe^{-x^2s^2}(f(x) + g(x))dx \\ &= \int_0^\infty ze^{-z^2s^2}(F(z) + G(z))dz \\ &= \int_0^\infty ze^{-z^2s^2}F(z)dz + \int_0^\infty ze^{-z^2s^2}F(z)dz \\ &= \mathfrak{L}_2[f(x)](s) + \mathfrak{L}_2[g(x)](s) \end{aligned}$$

□

Proposition 2.3.8. *Let $f(x) = [F(z)]$ is a hyperfunctions of bounded exponential growth and c be a non zero constant. If $\sigma_-(f) < \mathcal{R}(\frac{s^2}{c^2}) < \sigma_+(f)$ then*

$$\mathfrak{L}_2[f(cx)](s) = \frac{1}{c^2}\mathfrak{L}_2[f(x)](\frac{s}{c})$$

Proof.

$$\begin{aligned}
\mathfrak{L}_2[f(cx)](s) &= \int_0^\infty x e^{-x^2 s^2} f(cx)dx \\
&= \int_0^\infty z e^{-z^2 s^2} F(cz)dz \\
&= \frac{1}{c^2}\int_0^\infty \zeta e^{-\zeta^2 (\frac{s}{c})^2} F(\zeta)d\zeta, \text{ by putting } cz = \zeta \\
&= \frac{1}{c^2}\mathfrak{L}_2[f(x)](\frac{s}{c})
\end{aligned}$$

□

2.3.2 Inverse $\mathfrak{L}_2$ Transform of Hyperfunctions

A. Aghili, A. Ansari and A. Sedghi defined a complex inversion formula for $\mathfrak{L}_2$ transform in [3]. The inverse formula for $\mathfrak{L}_2$- transform of hyperfunctions exist only when the inverse Laplace transform of $\mathfrak{L}[f(\sqrt{x})](s^2)$ exist.

Definition 2.3.9. If $L(\sqrt{s})$ is a holomorphic function of s (by assuming $s = 0$ having no branch point) having finite number of poles which lies to the left side of the line $\mathcal{R}s = a$ and all $F(z) \in [F(z)]$ has a common strip of convergence, $\mathfrak{L}_2[f(x)](s) = L(s)$, then

$$f(x) = \mathfrak{L}_2{}^{-1}(L(s)) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} 2L(\sqrt{s})e^{s^2 x}ds$$

The above complex integral can be evaluated using the residue method.

Chapter 3

Fourier-Laplace, Laplace-Stieltjes and Fourier-Stieltjes Transforms of Hyperfunctions

In this chapter combined transforms like Fourier-Laplace transform, Laplace-Stieltjes transform and Fourier-Stieltjes transform of hyperfunctions are studied.

3.1. Fourier-Laplace Transform of Hyperfunctions

Laplace and Fourier transforms exist for hyperfunctions. Here we have studied the existence of Fourier- Laplace transform of entire hyperfunction with exponential type defining function.

Mitusuo Morimoto in his book[40]has mentioned about the Fourier-Laplace transform of an entire function of exponential type. The existence of the combined Fourier-Laplace transform of hyperfunctions is studied using the convergence criteria for Fourier and Laplace transform of hyperfunction.

Definition 3.1.1. A hyperfunction $f(x) = [F(z)] \in \mathfrak{B}(\mathbb{R})$ is called an *entire hyperfunction* if the defining function of the hyperfunction $F(z)$ is an entire function in $\mathbb{C}$.

Definition 3.1.2. An entire hyperfunction $f(x) = [F(z)] \in \mathfrak{B}(\mathbb{R})$ is called an *entire hyperfunction of exponential type* if there exists $M \geqslant 0$ and $n > 0$ such that $|F(z)| \leqslant Me^{n|z|}$ for all $F(z) \in [F(z)]$.

Definition 3.1.3. For a convex compact set P subset of $\mathbb{C}$ *support function* S_P of P is defined by

$$S_P(\gamma) = \sup\{Re(z\gamma) : z \in P\}, \gamma \in \mathbb{C}$$

Note:Let $\mathbb{E}$ denotes the linear space of entire hyperfunctions of exponential type with compact support.

Definition 3.1.4. For a convex compact set P subset of $\mathbb{C}$ and for the hyperfunction $f(x) = [F(Z)]$ define $\| f(x) \|_{(P)}$ by

$$\| f(x) \|_{(P)} = sup\{|F(z)|e^{-S_P(z)} : z \in \mathbb{C}, F(z) \in [F(z)]\}$$

Definition 3.1.5. For a convex compact set P define

$$\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P) = \{f(x) \in \mathbb{E} : \| f(x) \|_{(P)} < \infty\}$$

Proposition 3.1.6. *$\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ is a Banach space with respect to the norm $\| f(x) \|_{(P)}$.*

Proof. Clearly $\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ is a normed linear space.
Suppose that sequence $(f_n(x))$ is a Cauchy sequence in $\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$.

Then $\| f_n(x) \|_{(P)} < \infty, \forall n = 1, 2, 3...$

Let $f_n(x) = [F_n(z)]$, for $n = 1, 2, 3...$

Let $\epsilon > 0$

Since $(f_n(x))$ is a Cauchy sequence in $\mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\| f_n(x) = f_m(x) \|_{(P)} < \epsilon$ for all $n, m \geqslant N$.

i.e. $\sup |F_n(z) - F_m(z)| e^{-S_P(z)} < \epsilon$ for all $n, m \geqslant N$.

So we get $(F_n(z))$ is a Cauchy sequence in $\mathbb{C}$. But $\mathbb{C}$ is complete.

Hence $F_n(z) \to F(z)$ as $n \to \infty$.

Let $f(x) = [F(z)]$. Then $f(x) \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ and $f_n(x) \to f(x)$ with respect to $\| f(x) \|_{(P)}$ in the sense of convergence of hyperfunctions. □

Definition 3.1.7. For $\gamma \in \mathbb{C}$, with $|\gamma| = 1$ define

$$V(\gamma) = \{z \in \mathbb{C} : S_P(\gamma) < Re(z\gamma)\}$$

Then $V(\gamma)$ is an open half plane in $\mathbb{C}$.

For a given $f(x) = [F(z)] \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ and $\eta > 0$ there exists $M_\eta \geqslant 0$ such that

$$|F(\gamma)| \leqslant M_\eta e^{S_P(\gamma) + \eta|\gamma|} \ \ for \ \ \gamma \in \mathbb{C}.$$

Hence if $f(x) \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$, $\sigma_+(f)$ and $\sigma_-(f)$ always exists.

Remark. We are going to define Fourier-Laplace transform for entire hyperfunctions of exponential type.

Definition 3.1.8. Let $f(x) = f_1(x) + f_2(x) \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ be an entire hyperfunction of exponential type with $f_1(x) = [F_1(z)] \in \mathfrak{O}(\mathbb{R}_-), f_2(x) = [F_2(z)] \in \mathfrak{O}(\mathbb{R}_+)$ and if

$-\sigma_+(f_1) \leqslant 0 \leqslant -\sigma_-(f_2)$ then Fourier-Laplace transform of $f(x)$ is defined as

$$\mathfrak{FL}[f(x)](z,\gamma) = [G_+(z,\gamma), G_-(z,\gamma)]$$

where

$$G_+(z,\gamma) = \frac{-1}{2\pi i}\int_{\mathbb{R}_-\gamma} e^{-z\tau}F_1(\tau)d\tau = \frac{-1}{2\pi i}\int_0^{-\infty} e^{-zt\gamma}F_1(t\gamma)\gamma dt, \quad \mathbb{R}_-\gamma = \{t\gamma : t < 0\},$$

and

$$G_-(z,\gamma) = \frac{-1}{2\pi i}\int_{\mathbb{R}_+\gamma} e^{-z\tau}F_2(\tau)d\tau = \frac{-1}{2\pi i}\int_0^{\infty} e^{-zt\gamma}F_2(t\gamma)\gamma dt, \;\; \mathbb{R}_+\gamma = \{t\gamma : t > 0\},$$

for $z \in V(\gamma)$

Proposition 3.1.9. *For a fixed γ the function $h(z) = \mathfrak{FL}[f(x)](z,\gamma)$ is holomorphic on $V(\gamma)$*

Proof. Let $f(x) = [F(z)]$.

For $\eta > 0$, take

$$V_\eta'(\gamma) = \{z \in \mathbb{C} : S_P(\gamma) + \eta \leqslant Re(z\gamma)\}$$

Then for $z \in V_\eta'(\gamma)$ we have

$$|F(t\gamma)e^{-zt\gamma}\gamma| \leqslant M_{\frac{\eta}{2}} e^{\frac{-\eta t}{2}}$$

Hence the integral in the above definition converges absolutely and uniformly $\forall z \in V_\eta'$

But $V(\gamma) = \bigcup V_\eta'(\gamma)$.

Therefore $\mathfrak{FL}[f(x)](z,\gamma)$ is holomorphic on $V(\gamma)$. $\square$

Proposition 3.1.10. *For* $z \in V(\gamma) \cap V(\zeta)$, $\mathfrak{FL}[f(x)](z,\gamma) = \mathfrak{FL}[f(x)](z,\zeta)$

Proof. Applying Cauchy's integral theorem for the Fourier-Laplace integral of hyperfunctions we get the required result. □

Proposition 3.1.11. *If* $f(x) = [F(z)] \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$ *is a measurable hyperfunction then there exists a unique* $G(Z) \in \mathfrak{O}(\mathbb{C}\backslash P)$ *such that* $G(z) = \mathfrak{FL}[f(x)](z,\gamma)$ *on* $V(\gamma)$, *where* $G(z)$ *vanishes at infinity.*

Proof. Since P is convex and compact, $\bigcup_{|\gamma|=1} V(\gamma) = \mathbb{C}\backslash P$.
Then using proposition $3.1.9, 3.1.10$ and Lebesgue convergence theorem the result follows. □

Proposition 3.1.12. *For* $f_1(x), f_2(x) \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$,
$\mathfrak{FL}[c_1 f_1(x) + c_2 f_2(x)](z,\gamma) = c_1 \mathfrak{FL}[f_1(x)](z,\gamma) + c_2 \mathfrak{FL}[f_2(x)](z,\gamma)$ *provided the two integrals on the left has a common strip of convergence.*

Proof. Follows from definition. □

Proposition 3.1.13. *Let* $f(x) = [F(z)] \in \mathcal{E}_{\mathcal{B}}(\mathbb{C}, P)$, *holomorphic at* $x = d$ *and* $-\sigma_+ = \sigma_+(\chi_{(-\infty,0)} f(x)) < 0 < -\sigma_-(f) = \sigma_-(\chi_{(0,\infty)} f(x))$, *then*

$$\mathfrak{FL}[f(x)](z,\gamma) = \mathfrak{FL}[\chi_{(-\infty,d)} f(x)](z,\gamma) + \mathfrak{FL}[\chi_{(d,\infty)} f(x)](z,\gamma)$$

Proof. Follows from proposition 1.3.14. □

3.2. Laplace-Stieltjes Transform of Hyperfunctions

We define Laplace-Stieltjes transform for Hyperfunctions with defining function having bounded variation property.

3.2.1 Laplace-Stieltjes Transform of Hyperfunctions

Definition 3.2.1. A hyperfunction $f(x) = [F(z)] \in \mathfrak{B}(I)$, where I is a closed subset of $\mathbb{R}$ is said to be of bounded variation if all the functions $G(z) \in [F(z)]$ are of bounded variation (i.e. $ReG(z)$ and $ImG(z)$ are real functions of bounded variation)

Definition 3.2.2. Let $f(x) = [F(z)] \in \mathfrak{B}(I)$, where $I \subset [0, \infty)$ is a closed , be a measurable, non decreasing, exponentially bounded hyperfunction and of bounded variation. For the complex variable s the Laplace-Stieltjes transform is defined as

$$\mathfrak{L}_{\mathcal{S}}[f(x)](s) = f^*(s) = \int_0^\infty e^{-sx} df(x) = \int_0^\infty e^{-sz} dF(z),$$

provided the integral converges at some point s_0. Then it converges $\forall s$ with $\mathcal{R}s > \mathcal{R}s_0$ (Here the integral is Stieltjes Integral)

Proposition 3.2.3. $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$ *is strictly positive.*

Proof. If $f(x) > 0$, clearly $\mathfrak{L}_{\mathcal{S}}[f(x)](s) > 0$. □

Examples 3.2.4. *Consider the hyperfunction* $f(x) = x^n$. *In terms of the defining function* $f(x) = [F(z)]$, *where* $F(z) = z^n$. *Then*

$$\begin{aligned} f^*(s) &= \mathfrak{L}_{\mathcal{S}}[f(x)](s) \\ &= \int_0^\infty e^{-sx} d(x^n) \\ &= \int_0^\infty e^{-sz} d(z^n) \\ &= \int_0^\infty e^{-sz} n z^{n-1} dz \\ &= \frac{n!}{s^n} \end{aligned}$$

Remark. Next proposition is the relation connecting Laplace-Stieltjes transform and the Laplace transform of a hyperfunction

Proposition 3.2.5. *Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$. Then*

$$\mathfrak{L}_{\mathcal{S}}[f(x)](s) = s\mathfrak{L}[f(x)](s)$$

Proof. Suppose that $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$.

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[f(x)](s) &= \int_0^\infty e^{-sx} df(x) \\
&= \int_0^\infty e^{-sz} dF(z) \\
&= \int_0^\infty e^{-sz} F'(z) dz \\
&= \mathfrak{L}[f'(x)](s) \\
&= s\mathfrak{L}[f(x)](s)
\end{aligned}$$

□

Proposition 3.2.6. *Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ be a hyperfunction of bounded exponential growth with Laplace-Stieltjes transforms $f^*(s)$ and $g^*(s)$ respectively. Then*

$$\mathfrak{L}_{\mathcal{S}}[f(x) + g(x)](s) = \mathfrak{L}_{\mathcal{S}}[f(x)](s) + \mathfrak{L}_{\mathcal{S}}[g(x)](s)$$

Also for some constant c,

$$\mathfrak{L}_{\mathcal{S}}[cf(x)](s) = c\mathfrak{L}_{\mathcal{S}}[f(x)](s)$$

Proof. Suppose that $f(x) = [F(z)]$ and $g(x) = [G(z)]$ be a hyperfunction of bounded exponential growth with Laplace-Stieltjes transforms $f^*(s)$ and $g^*(s)$ respectively.

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[f(x) + g(x)](s) &= \int_0^\infty e^{-sx} d(f(x) + g(x)) \\
&= \int_0^\infty e^{-sz} d(F(z) + G(z)) \\
&= \int_0^\infty e^{-sz} (F'(z) + G'(z)) dz \\
&= \int_0^\infty e^{-sz} F'(z) dz + \int_0^\infty e^{-sz} G'(z) dz \\
&= \mathfrak{L}_{\mathcal{S}}[f(x)](s) + \mathfrak{L}_{\mathcal{S}}[g(x)](s)
\end{aligned}$$

Also for any constant c,

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[cf(x)](s) &= \int_0^\infty e^{-sx} d(cf(x)) \\
&= \int_0^\infty e^{-sz} d(cF(z)) \\
&= \int_0^\infty e^{-sz} cF'(z) dz \\
&= c\int_0^\infty e^{-sz} F'(z) dz \\
&= c\mathfrak{L}_{\mathcal{S}}[f(x)](s)
\end{aligned}$$

□

Proposition 3.2.7. *Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$. Then*

$$\mathfrak{L}_{\mathcal{S}}[f'(x)](s) = s\mathfrak{L}_{\mathcal{S}}[f(x)](s) - sf(0) - f'(0)$$

Proof. Suppose that $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$ with Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$.

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[f'(x)](s) &= \int_0^\infty e^{-sx} d(f'(x)) \\
&= \int_0^\infty e^{-sz} d(F'(z)) \\
&= \int_0^\infty e^{-sz} F''(z) dz \\
&= \mathfrak{L}[f''(x)](s) \\
&= s^2 \mathfrak{L}[f(x)](s) - sf(0) - f'(0) \\
&= s\mathfrak{L}_{\mathcal{S}}[f(x)](s) - sf(0) - f'(0)
\end{aligned}$$

□

Proposition 3.2.8. *Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$. Then for any $a \in \mathbb{C}$,*

$$\mathfrak{L}_{\mathcal{S}}[e^{ax} f(x)](s) = \mathfrak{L}[f'(x)](s-a) + a\mathfrak{L}[f(x)](s-a),$$

where $\sigma_-(f) + \mathcal{R}(a) < \mathcal{R}(s) < \sigma_+(f) + \mathcal{R}(a)$

Proof. Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth and $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$.

Also let $a \in \mathbb{C}$ be a constant with $\sigma_-(f) + \mathcal{R}(a) < \mathcal{R}(s) < \sigma_+(f) + \mathcal{R}(a)$

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[e^{ax} f(x)](s) &= \int_0^\infty e^{-sx} d(e^{ax} f(x)) \\
&= \int_0^\infty e^{-sz} d(e^{az} F(z)) \\
&= \int_0^\infty e^{-sz} (e^{az} F'(z) + ae^{az} F(z)) dz
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty e^{-sz}e^{az}F^{'}(z)dz + a\int_0^\infty e^{-sz}e^{az}F(z)dz \\
&= \int_0^\infty e^{-(s-a)z}F^{'}(z)dz + a\int_0^\infty e^{-(s-a)z}F(z)dz \\
&= \mathfrak{L}[f^{'}(x)](s-a) + a\mathfrak{L}[f(x)](s-a)
\end{aligned}
$$

□

Proposition 3.2.9. *Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth having Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$ with $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$. Then*

$$\mathfrak{L}_{\mathcal{S}}[f^n(x)](s) = s^n\mathfrak{L}_{\mathcal{S}}[f(x)](s),$$

if the strip of convergence is same.

Proof. Let $f(x) = [F(z)]$ be a hyperfunction of bounded exponential growth having Laplace-Stieltjes transform $\mathfrak{L}_{\mathcal{S}}[f(x)](s)$ with $\sigma_-(f) < \mathcal{R}(s) < \sigma_+(f)$.

$$
\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[f^n(x)](s) &= \int_0^\infty e^{-sx}d(f^n(x)) \\
&= \int_0^\infty e^{-sz}d(F^n(z)) \\
&= \int_0^\infty e^{-sz}F^{n+1}(z)dz \\
&= \mathfrak{L}[f^{n+1}(x)](s) \\
&= s^{n+1}\mathfrak{L}[f(x)](s) \\
&= s^n\mathfrak{L}_{\mathcal{S}}[f(x)](s)
\end{aligned}
$$

□

Remark. We prove the existence Laplace-Stieltjes transform for convolution of hyperfunctions

Proposition 3.2.10. *Let $f(x) = [F(z)]$ and $h(x) = [H(z)]$ be two hyperfunctions having bounded exponential growth with compact support $[a, b]$ and $[c, d]$ respectively. If $f^*(s) = \mathfrak{L}_{\mathcal{S}}[f(x)](s)$ and $h^*(s) = \mathfrak{L}_{\mathcal{S}}[h(x)](s)$ exists then the Laplace-Stieltjes transform of the convolution $f(x) * h(x)$ exists.*

$$\mathfrak{L}_{\mathcal{S}}[f(x) * h(x)](s) = \frac{1}{s}\mathfrak{L}_{\mathcal{S}}[f(x)](s)\mathfrak{L}_{\mathcal{S}}[h(x)](s)$$

Proof. Suppose that $f(x) = [F(z)]$ and $h(x) = [H(z)]$ be two hyperfunctions having bounded exponential growth with compact support $[a, b]$ and $[c, d]$ respectively. Then the convolution $f(x) * h(x)$ of $f(x)$ and $h(x)$ is exists as a hyperfunction with compact support contained in $[a + c, b + d]$.

In general,

$$f(x) * h(x) = \int_0^\infty F(z - \tau)H(\tau)d\tau$$

Hence

$$\begin{aligned}
\mathfrak{L}_{\mathcal{S}}[f(x) * h(x)](s) &= \mathfrak{L}_{\mathcal{S}}[\int_0^\infty F(z - \tau)H(\tau)d\tau](s) \\
&= s\mathfrak{L}[\int_0^\infty F(z - \tau)H(\tau)d\tau](s) \\
&= s\int_0^\infty e^{-sz}(\int_0^\infty F(z - \tau)H(\tau)d\tau)dz \\
&= s\int_0^\infty H(\tau)d\tau \int_0^\infty e^{-sz}F(z - \tau)dz
\end{aligned}$$

Putting $z - \tau = \rho$ we get

$$\mathfrak{L}_{\mathcal{S}}[f(x) * h(x)](s) = s\int_0^\infty H(\tau)d\tau \int_0^\infty e^{-s(\tau+\rho)}F(\rho)d\rho$$

$$
\begin{aligned}
&= s\int_0^\infty e^{-s\tau}H(\tau)d\tau\int_0^\infty e^{-s\rho}F(\rho)d\rho \\
&= s.\mathfrak{L}[F(x)](s).\mathfrak{L}[h(x)](s) \\
&= s.\frac{1}{s}\mathfrak{L}_{\mathcal{S}}[f(x)](s).\frac{1}{s}\mathfrak{L}_{\mathcal{S}}[h(x)](s) \\
&= \frac{1}{s}\mathfrak{L}_{\mathcal{S}}[f(x)](s)\mathfrak{L}_{\mathcal{S}}[h(x)](s)
\end{aligned}
$$

□

3.2.2 Inversion formula for Laplace-Stieltjes Transform of Hyperfunction

D. Salltz[46] defined an inversion formula for the Laplace-Stieltjes transform of ordinary functions. The inversion formula for Laplace-Stieltjes transform of a hyperfunction we defined as follows.

Definition 3.2.11. The inversion formula for the Laplace-Stieltjes transform of the hyperfunction of bounded exponential growth is defined as

$$f(x) = \lim_{b\to\infty}\int_{a-ib}^{a+ib}\frac{e^{sx}}{s}f^*(s)ds$$

where $a > 0$ is greater than the radius of convergence.

3.3. Fourier- Stieltjes Transform of Hyperfunctions

Using the relation connecting Fourier transform and Laplace transform of hyperfunctions, the Fourier-Stieltjes transform of hyperfunction can be defined as follows.

Definition 3.3.1. Let $f(x) = [F(z)] \in \mathfrak{B}(I)$, where $I \subset [0,\infty)$ is a closed , be a measurable, non decreasing, exponentially bounded hyperfunction and of bounded variation. For the complex variable s the Fourier-Stieltjes transform is defined as

$$\mathfrak{F}_{\mathcal{S}}[f(x)](s) = \mathfrak{L}_{\mathcal{S}}[f(x)](is), s \in \mathbb{R}$$

The operational properties of Fourier-Stieltjes transform of hyperfunctions can be established using the operational properties of Laplace-Stieltjes transform of hyperfunctions.

Chapter 4

Abelian - Tauberian Theorems for Some Integral Transforms of Hyperfunctions

In this chapter we have proved Abelian - Tauberian theorem for Laplace transform of Hyperfunctions, Abelian - Tauberian theorem for Stieltjes transform of Hyperfunctions and Abelian - Tauberian theorem for Laplace-Stieltjes transform of Hyperfunctions.

4.1. Abelian -Tauberian theorem for Laplace Transform of Hyperfunctions

We first develops the background for deriving the Continuity theorem of Hyperfunctions which then leads to the Abelian -Tauberian theorem for Hyperfunctions.

4.1.1 Measurable Hyperfunctions

Definition 4.1.1. A hyperfunction $f(x) = [F(Z)] = [F_+(z), F_-(z)]$ is said to be a measurable hyperfunction if the defining function $F(z) \in [F(z)]$ are all complex Lebesgue measurable functions.

Remark. We consider sequence of hyperfunctions $(f_n(x)) = ([F_n(z)])$, where the sequence of defining functions $(F_n(z))$ are defined on a common domain $N \subset \mathbb{C}$ for the following theorems.

Lemma 4.1.2. *Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth. Then*

$$\liminf \int f_n(x)dx \geqslant \int \liminf f_n(x)dx$$

Proof. Applying Fatou's lemma for measurable functions to the sequence of defining functions of $(f_n(x))$ we have

$$\liminf \int F_n(z)dz \geqslant \int \liminf F_n(z)dz$$

Also it holds for every $G_n(z) \in [F_n(z)]$.

Hence the result follows. □

Theorem 4.1.3. *Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth. If $(f_n(x))$ is monotonic increasing and $(f_n(x)) \to f(x)$, where $f(x) = [F(x)]$ then*

$$\int f(x)dx = \lim \int f_n(x)dx$$

Proof. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Then by Lemma (4.1.2) we have

$$\begin{aligned}
\int f(x)dx &= \int \lim f_n(x)dx \\
&= \int \liminf f_n(x)dx \\
&= \int \liminf F_n(z)dz \\
&\leqslant \liminf \int F_n(z)dz \\
&= \liminf \int f_n(x)dx
\end{aligned}$$

Since $(f_n(x))$ is monotonic increasing and $(f_n(x)) \to f(x)$ in the sense of hyperfunction we have $f_n(x) \leqslant f(x)$.

Hence

$$\int f_n(x)dx \leqslant \int f(x)dx$$

Then

$$\limsup \int f_n(x)dx \leqslant \int f(x)dx$$

So

$$\int f(x)dx \leqslant \liminf \int f_n(x)dx \leqslant \limsup \int f_n(x)dx \leqslant \int f(x)dx$$

Thus

$$\int f(x)dx = \lim \int f_n(x)dx$$

□

Theorem 4.1.4. *Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth. If $|f_n(x)| \leqslant g(x)$, where $g(x) = [G(z)]$ is a real valued hyperfunction* *and*

$$\lim_{n\to\infty} f_n(x) = f(x), f(x) = [F(z)]$$

then $f(x)$ *is integrable and*

$$\lim \int f_n(x)dx = \int f(x)dx$$

Proof. Applying Dominated convergence theorem for measurable functions to the sequence $(F_n(z))$ of defining functions of $(f_n(x))$ we have $F(z)$ is integrable and

$$\lim \int F_n(z)dz = \int F(z)dz$$

Then using the convergence in the sense of hyperfunctions we get $f(x)$ is integrable and

$$\lim \int f_n(x)dx = \int f(x)dx.$$

□

Theorem 4.1.5. *Let* $(f_n(x)) = ([F_n(z)])$ *be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth, defined on* $(0, \infty)$. *If* $|f_n(x)| \leqslant P$ *and* $\lim_{n\to\infty} f_n(x) = f(x), f(x) = [F(z)]$ *then*

$$\lim \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx$$

Proof. Follows from Bounded convergence theorem for real valued measurable functions and using the convergence in the sense of hyperfunctions. □

4.1.2 Continuity Theorem for Hyperfunction

Lemma 4.1.6. *Let $f(x) = [F(z)]$ and $g(x) = [G(z)]$ are two holomorphic hyperfunctions of bounded exponential growth with Laplace transforms $\hat{f}(s) = \mathcal{L}[f(x)](s)$ and $\hat{g}(s) = \mathcal{L}[g(x)](s)$. If they have a common vertical strip of convergence then $\hat{f}(s) = \hat{g}(s)$ implies $f(x) = g(x)$*

Proof. Suppose that $\hat{f}(s) = \hat{g}(s)$.

$$\begin{aligned}
\hat{f}(s) = \hat{g}(s) \quad &\Rightarrow \quad \mathcal{L}[f(x)](s) = \mathcal{L}[g(x)](s) \\
&\Rightarrow \quad \int_0^\infty e^{-sz}F(z)dz = \int_0^\infty e^{-sz}F(z)dz \\
&\Rightarrow \quad \int_0^\infty e^{-sz}(F(z) - G(Z))dz = 0 \\
&\Rightarrow \quad F(z) - G(z) = 0 \\
&\Rightarrow \quad [F(z)] = [G(z)] \\
&\Rightarrow \quad f(x) = g(x)
\end{aligned}$$

□

Theorem 4.1.7. *(Continuity theorem for Hyperfunctions)*
Let $(f_n(x)) = ([F_n(z)])$ be a sequence of non-negative, real valued, holomorphic, measurable hyperfunctions with compact support and having bounded exponential growth, defined on $(0, \infty)$.
(a) Let $f(x) = [F(z)]$ be a measurable hyperfunction with support contained in $(0, \infty)$ such that $f_n(x) \to f(x)$ for all points x at which f_n's and f are holomorphic. If there exists $t \geqslant 0$ such that $\sup_{n \geqslant 1} \mathcal{L}[f_n(x)](t) < \infty$ then

$$\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$$

as $n \to \infty$ for all $s > t$

(b) Suppose there exists $t \geqslant 0$ such that $\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$ as $n \to \infty$ for all $s > t$ then

$$f_n(x) \to f(x)$$

for all points x at which f_n's and f are holomorphic if the Laplace transforms of f_n's and f have a common vertical strip of convergence.

Proof. (a) Let

$$M = \sup_{n \geqslant 1} \mathcal{L}[f_n(x)](t) < \infty.$$

Then for any $s > t$ and $x \in (0, \infty)$,

$$\int_0^\infty e^{-sx} f_n(x)dx \to \int_0^\infty e^{-sx} f(x)dx$$

by proposition 4.1.5.

Let $s > t$ and $\epsilon > 0$ such that f is holomorphic at $y \in (0, \infty)$ with $Me^{-(s-t)y} \leqslant \epsilon$.

$$\begin{aligned}
\int_0^y e^{-sx} f_n(x)dx &\leqslant \mathcal{L}[f_n(x)](s) \\
&\leqslant \int_0^y e^{-sx} f_n(x)dx + e^{-(s-t)y} \int_y^\infty e^{-tx} f_n(x)dx \\
&\leqslant \int_0^y e^{-sx} f_n(x)dx + \epsilon
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^y e^{-sx} f(x)dx &\leqslant \liminf_{n \to \infty} \mathcal{L}[f_n(x)](s) \\
&\leqslant \limsup_{n \to \infty} \mathcal{L}[f_n(x)](s)
\end{aligned}$$

$$\leqslant \int_0^y e^{-sx} f(x)dx + \epsilon$$

Letting $y \to \infty$ along holomorphic points of $f(x) = [F(z)]$

$$\begin{aligned}
\int_0^\infty e^{-sx} f(x)dx &\leqslant \liminf_{n\to\infty} \mathcal{L}[f_n(x)](s) \\
&\leqslant \limsup_{n\to\infty} \mathcal{L}[f_n(x)](s) \\
&\leqslant \int_0^\infty e^{-sx} f(x)dx + \epsilon
\end{aligned}$$

$$i.e. \mathcal{L}[f(x)](s) \leqslant \liminf_{n\to\infty} \mathcal{L}[f_n(x)](s) \leqslant \limsup_{n\to\infty} \mathcal{L}[f_n(x)](s) \leqslant \mathcal{L}[f(x)](s) + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s), \ \ as \ \ n \to \infty \ \ for \ \ all \ \ s > t$$

(b) Suppose that

$$\mathcal{L}[f_n(x)](s) \to \mathcal{L}[f(x)](s)$$

as $n \to \infty$ for all $s > t$ and the Laplace transforms of f_n's and f have a common vertical strip of convergence. By Lemma (4.1.6) and proposition (4.1.5)

$$\begin{aligned}
f_n(x) &= \int_0^\infty e^{sx} \mathcal{L}[f_n(x)](s) ds \\
&\to \int_0^\infty e^{sx} \mathcal{L}[f(x)](s) ds \\
&= f(x)
\end{aligned}$$

□

4.1.3 Abelian -Tauberian Theorem for Laplace Transform of Hyperfunctions

We now prove Abelian -Tauberian theorem for the Laplace transform of hyperfunctions. To avoid formulas consisting of reciprocals we introduce two positive variables p and q such that $pq = 1$. Then $q \to 0$ when $p \to \infty$

Theorem 4.1.8. *Let $f(x) = [F(z)]$ be a measurable, holomorphic hyperfunction on $(0, \infty)$ having compact support and bounded exponential growth. If the Laplace transform $\hat{f}(s) = [f(x)](s)$ is bounded for $s > 0$ then the following conditions are equivalent.*

(a) $\dfrac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \dfrac{1}{s^{\alpha+1}}$ *as* $q \to 0$

(b) $\dfrac{f(px)}{f(p)} \to x^{\alpha}$ *as* $p \to \infty$

$$\textit{Also } \mathcal{L}[f(x)](q) \sim f(p)\alpha!, \alpha \geqslant 0 \textit{ is an integer}$$

Proof. **(a)⇒ (b)**

Suppose

$$\frac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \frac{1}{s^{\alpha+1}} \ \ as \ \ q \to 0.$$

Then by Continuity theorem for hyperfunctions

$$\frac{f(px)}{\mathcal{L}[f(x)](q)} \to \frac{x^{\alpha}}{\alpha!}$$

Letting $x = 1$, we have

$$\frac{f(p)}{\mathcal{L}[f(x)](q)} \to \frac{1}{\alpha!}.......(1)$$

Hence

$$\frac{\alpha! f(p)}{\mathcal{L}[f(x)](q)} \to 1 \ \ as \ \ p \to \infty.$$

Thus

$$\mathcal{L}[f(x)](q) \sim f(p)\alpha!.......(2)$$

Substituting (2) in (1)

$$\frac{f(px)}{f(p)\alpha!} \to \frac{x^{\alpha}}{\alpha!} \ \ \text{as} \ \ p \to \infty,$$

$$i.e. \frac{f(px)}{f(p)} \to x^{\alpha}, \ \ \text{as} \ \ p \to \infty$$

(b)⇒ (a)

Suppose

$$\frac{f(px)}{f(p)} \to x^{\alpha} \ \ as \ \ p \to \infty$$

Then by again by Continuity theorem for hyperfunctions,

$$\frac{\mathcal{L}[f(x)](qs)}{f(p)} \to \frac{\alpha!}{s^{\alpha+1}}........(3)$$

But

$$f(p) \sim \frac{\mathcal{L}[f(x)](q)}{\alpha!}........(4)$$

Substituting (4) in (3)

$$\frac{\mathcal{L}[f(x)](qs)\alpha!}{\mathcal{L}[f(x)](q)} \to \frac{\alpha!}{s^{\alpha+1}},$$

$$i.e. \frac{\mathcal{L}[f(x)](qs)}{\mathcal{L}[f(x)](q)} \to \frac{1}{s^{\alpha+1}} \ \ as \ \ q \to 0.$$

□

We can express the above theorem in terms of slowly varying function also.

Theorem 4.1.9. *Let $f(x) = [F(z)]$ be a measurable, holomorphic hyperfunction on $(0, \infty)$ having compact support and bounded exponential growth. If the Laplace transform $\hat{f}(s) = [f(x)](s)$ is bounded for $s > 0$ then the following conditions are equivalent.*

(a) $\mathcal{L}[f(x)](s) \sim \frac{1}{s^{\alpha+1}} h(\frac{1}{s})$ *as* $s \to 0+$

(b)$f(x) \sim \dfrac{x^{\alpha+1}}{\alpha!} h(x)$ *as* $x \to \infty$

where $h : (0, \infty) \to (0, \infty)$ *is a slowly varying function at infinity and* $\alpha \geqslant 0$ *is an integer*

Proof. **(a)⇒ (b)**

Suppose

$$\mathcal{L}[f(x)](s) \sim \frac{1}{s^{\alpha+1}} h(\frac{1}{s}) \ \ as \ \ s \to 0+ .$$

Then

$$\begin{aligned}\frac{\mathcal{L}[f(x)](\frac{s}{t})}{\mathcal{L}[f(x)](\frac{1}{t})} &\sim \frac{1}{s^{\alpha+1}} \frac{h(\frac{t}{s})}{h(t)} \\ &\sim \frac{1}{s^{\alpha+1}} \ \ as \ \ t \to \infty\end{aligned}$$

Using previous theorem and putting $q = \frac{1}{t}$ we have

$$\begin{aligned}f(t) &\sim \frac{\mathcal{L}[f(x)](\frac{1}{t})}{\alpha!} \\ &\sim \frac{t^{\alpha+1}}{\alpha!} h(t) \ \ as \ \ t \to \infty\end{aligned}$$

Similarly we can prove **(b)⇒ (a)** □

4.2. Abelian -Tauberian Theorem for Stieltjes Transform of Hyperfunctions

In the case of hyperfunctions we prove an Abelian -Tauberian type theorem for Stieltjes transform of hyperfunction by imposing certain additional conditions in Karamata's[36] Abelian -Tauberian theorem for the Stieltjes transform.

Proposition 4.2.1. *Let* $f(x) = [F(z)]$ *be a holomorphic, measurable, non decreasing hyperfunction of bounded exponential growth with compact support contained in* $(0, \infty)$

such that the Stieltjes transform

$$\tilde{f}(t) = S[f(x)](t) = \int_0^\infty \frac{f(x)}{x+t}dx = \int_0^\infty \frac{F(z)}{z+t}dz$$

exists for all $t > 0$. Let ρ be a number with $0 \leqslant \rho < 1$. Then the following statements are equivalent

$$\tilde{f}(t) \sim Ct^{\rho-1} \;\; as \;\; t \to \infty$$

$$f(x) \sim \frac{C}{\Gamma(1+\rho)\Gamma(1-\rho)} x^{\rho} \;\; as \;\; x \to \infty$$

Proof. Suppose that

$$\tilde{f}(t) \sim Ct^{\rho-1} \;\; as \;\; t \to \infty$$

Using the result

$$\int_0^\infty e^{-(x+t)u} du = \frac{1}{x+t}$$

we have

$$\begin{aligned} \tilde{f}(t) &= \int_0^\infty \frac{f(x)}{x+t}dx \\ &= \int_0^\infty (\int_0^\infty e^{-(x+t)u}du) f(x)dx \\ &= \int_0^\infty e^{-tu} g(u)du, \text{ where } g(u) = \int_0^\infty e^{-xu} f(x)dx \end{aligned}$$

Since $g \geqslant 0$, the integral of g will be non-decreasing. So by theorem 1.6.1 we get

$$\int_0^x g(\nu)d\nu \sim \frac{c}{\Gamma(-\rho+1+1)} x^{-\rho+1} \;\; as \;\; x \to 0$$

$$i.e. \int_0^x g(\nu)d\nu \sim \frac{c}{\Gamma(2-\rho)} x^{-\rho+1} \;\; as \;\; x \to 0$$

Differentiating with respect to x,

$$g(x) \sim \frac{C}{\Gamma(2-\rho)}(1-\rho)x^{1-\rho-1} \;\; as \;\; x \to 0$$

$$i.e. g(x) \sim \frac{C}{\Gamma(1-\rho)}x^{-\rho} \;\; as \;\; x \to 0$$

Applying theorem 1.6.1, we have

$$f(x) \sim \frac{C}{\Gamma(1+\rho)\Gamma(1-\rho)}x^{\rho} \;\; as \;\; x \to \infty$$

Conversely suppose that

$$f(x) \sim \frac{C}{\Gamma(1+\rho)\Gamma(1-\rho)}x^{\rho} \;\; as \;\; x \to \infty$$

Then by letting $L(x) = 1$ in theorem 1.6.3 we get

$$\tilde{f}(t) \sim Ct^{\rho-1} \;\; as \;\; t \to \infty$$

□

4.3. Abelian -Tauberian Theorem for Laplace-Stieltjes Transform of Hyperfunctions

Let $\mathfrak{B}^{e+}_{bv}(I)$ denote the set of all non-decreasing, non-negative, real valued, holomorphic, measurable, exponentially bounded hyperfunction of bounded variation defined on the closed subset $I \subset [0, \infty)$

Lemma 4.3.1. *Let $f(x) = [F(z)]$, $g(x) = [G(z)]$ are in $\mathfrak{B}^{e+}_{bv}(I)$ with Laplace-Stieltjes transforms $f^*(s) = \mathfrak{L}_{\mathcal{S}}[f(x)](s)$ and $g^*(s) = \mathfrak{L}_{\mathcal{S}}[g(x)](s)$. If they have a common vertical strip of convergence and if $f^*(s) = g^*(s)$ then $f(x) = g(x)$*

Proof. Suppose that $f^*(s) = g^*(s)$.

$$\begin{aligned}
f^*(s) = g^*(s) &\Rightarrow \mathfrak{L}_{\mathcal{S}}[f(x)](s) = \mathfrak{L}_{\mathcal{S}}[g(x)](s) \\
&\Rightarrow \int_0^\infty e^{-sx} df(x) = \int_0^\infty e^{-sx} dg(x) \\
&\Rightarrow \int_0^\infty e^{-sz} dF(z) = \int_0^\infty e^{-sz} dG(z) \\
&\Rightarrow \int_0^\infty e^{-sz} d(F(z) - G(z)) = 0 \\
&\Rightarrow d(F(z) - G(z)) = 0 \\
&\Rightarrow [F^{'}(z)] = [G^{'}(z)] \\
&\Rightarrow [F(z)] = [G(z)] \\
&\Rightarrow f(x) = g(x)
\end{aligned}$$

□

Theorem 4.3.2. *Let $(f_n(x)) = ([F_n(z)])$ be a sequence of hyperfunction in $\mathfrak{B}^{e+}_{bv}(I)$ with compact support.*

(a) Let $f(x) = [F(z)]$ be a measurable hyperfunction with support contained in $(0, \infty)$ such that $f_n(x) \to f(x)$ for all points x at which f_n's and f are holomorphic. If there exists $t \geqslant 0$ such that $\sup_{n \geqslant 1} \mathfrak{L}_{\mathcal{S}}[f_n(x)](t) < \infty$ then $\mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[f(x)](s)$ as $n \to \infty$ for all $s > t$

(b) Suppose there exists $t \geqslant 0$ such that $\mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[f(x)](s)$ as $n \to \infty$

for all $s > t$ then $f_n(x) \to f(x)$ for all points x at which f_n's and f are holomorphic, if the Laplace-Stieltjes transforms of f_n's and f have a common vertical strip of convergence.

Proof. (a) Let

$$A = \sup_{n \geqslant 1} \mathfrak{L}_{\mathcal{S}}[f_n(x)](t) < \infty.$$

Then for any $s > t$ and $x \in (0, \infty)$,

$$\int_0^\infty e^{-sx} df_n(x) \to \int_0^\infty e^{-sx} df(x)$$

by proposition (4.1.4).

Let $s > t$ and $\epsilon > 0$ such that f is holomorphic at $y \in (0, \infty)$ with $Ae^{-(s-t)y} \leqslant \epsilon$.

$$\begin{aligned}
\int_0^y e^{-sx} df_n(x) &\leqslant \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \\
&\leqslant \int_0^y e^{-sx} df_n(x) + e^{-(s-t)y} \int_y^\infty e^{-tx} df_n(x) \\
&\leqslant \int_0^y e^{-sx} df_n(x) + \epsilon
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^y e^{-sx} df(x) &\leqslant \liminf_{n \to \infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \\
&\leqslant \limsup_{n \to \infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \\
&\leqslant \int_0^y e^{-sx} df(x) + \epsilon
\end{aligned}$$

Letting $y \to \infty$ along holomorphic points of $f(x) = [F(z)]$

$$\begin{aligned}\int_0^\infty e^{-sx} df(x) &\leqslant \liminf_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \\ &\leqslant \limsup_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \\ &\leqslant \int_0^\infty e^{-sx} df(x) + \epsilon\end{aligned}$$

$$i.e. \mathfrak{L}_{\mathcal{S}}[f(x)](s) \leqslant \liminf_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \leqslant \limsup_{n\to\infty} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \leqslant \mathfrak{L}_{\mathcal{S}}[f(x)](s) + \epsilon$$

Since $\epsilon > 0$ is arbitrary,

$$\mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[f(x)](s)$$

as $n \to \infty$ for all $s > t$

(b) Suppose that

$$\mathfrak{L}_{\mathcal{S}}[f_n(x)](s) \to \mathfrak{L}_{\mathcal{S}}[f(x)](s)$$

as $n \to \infty$ for all $s > t$ and the Laplace-Stieltjes transforms of f_n's and f have a common vertical strip of convergence.

By previous Lemma and proposition (4.1.4)

$$\begin{aligned}f_n(x) &= \int_0^\infty e^{sx} \mathfrak{L}_{\mathcal{S}}[f_n(x)](s) ds \\ &\to \int_0^\infty e^{sx} \mathfrak{L}_{\mathcal{S}}[f(x)](s) ds \\ &= f(x)\end{aligned}$$

□

Proposition 4.3.3. *(Abelian theorem for Laplace-Stieltjes Transformation of Hyperfunctions)*

For $f(x) \in \mathfrak{B}_{bv}^{e+}(I)$, *if*

$$\lim_{x\to\infty} \frac{f(x)}{x^n} = \frac{M}{n!}$$

then

$$\lim_{s\to 0} s^n f^*(s) = M,$$

where n *is a non-negative number and* M *is a constant.*

Proof. Let $f(x) \in \mathfrak{B}_{bv}^{e+}(I)$, n is a non-negative number and M be a constant. Suppose that

$$\lim_{x\to\infty} \frac{f(x)}{x^n} = \frac{M}{n!}$$

Then

$$f(x) \to \frac{Mx^n}{n!} \;\; as \;\; x \to \infty$$

Hence by the previous proposition 4.3.2

$$f^*(s) \to \frac{M}{n!} . \frac{n!}{s^n} \;\; as \;\; s \to 0$$

$$i.e. s^n f^*(s) \to M \;\; as \;\; s \to 0$$

Hence $\lim_{s\to 0} s^n f^*(s) = M$ □

Proposition 4.3.4. *(Tauberian theorem for Laplace-Stieltjes Transformation of Hyperfunctions)*

Let $f(x) \in \mathfrak{B}_{bv}^{e+}(I)$ *with Laplace-Stieltjes transform* $f^*(s) = \int_0^\infty e^{-sx} df(x)$, *which converges for some* $\mathcal{R}(s) > 0$ *and* $\lim_{s\to 0} s^n f^*(s) = M$, *for some constant* M *and* $n > 0$.

Then

$$\lim_{x\to\infty}\frac{f(x)}{x^n}=\frac{M}{n!}$$

Proof. Let $f(x)\in\mathfrak{B}^{e+}_{bv}(I)$ with Laplace-Stieltjes transform $f^*(s)=\int_0^\infty e^{-sx}df(x)$, which converges for some $\mathcal{R}(s)>0$ and $\lim_{s\to 0}s^n f^*(s)=M$, for some constant M and $n>0$.

$$i.e.s^n f^*(s)\to M\ \ as\ \ s\to 0$$

$$i.e.\frac{f^*(s)}{M}\to\frac{1}{s^n}\ \ as\ \ s\to 0$$

Then by proposition 4.3.2 we have

$$\frac{f(x)}{M}\to\frac{x^n}{n!}\ \ as\ \ x\to\infty$$

$$\frac{f(x)}{x^n}\to\frac{M}{n!}\ \ as\ \ x\to\infty$$

Hence

$$\lim_{x\to\infty}\frac{f(x)}{x^n}=\frac{M}{n!}$$

□

Chapter 5

Two Dimensional Laplace Transform of Hyperfunctions

Urs Graf applied Laplace transform to Sato's hyperfunctions. In this chapter we have applied two dimensional Laplace transform to hyperfunctions in two variables. We have established some properties of this transform and defined the inverse transform.

5.1. Hyperfunctions in Two Variables

Definition 5.1.1. Let I_1 and I_2 are open intervals in $\mathbb{R}$ and $N(I_i)$ is a complex neighbourhood of I_i (i.e.$N(I_i)$ contains I_i as a closed subset) for $i = 1, 2$ then the open set $N(I_1) \times N(I_2)$ in $\mathbb{C}^2$ is called a *complex neighbourhood* of $I_1 \times I_2$, if $I_1 \times I_2$ is a closed subset of $N(I_1) \times N(I_2)$.

Definition 5.1.2. Two functions $F(z_1, z_2)$ and $G(z_1, z_2)$ in $\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))$ are *equivalent*, if for $(z_1, z_2) \in (N_1(I_1) \times N_1(I_2)) \cap (N_2(I_1) \times N_2(I_2))$,

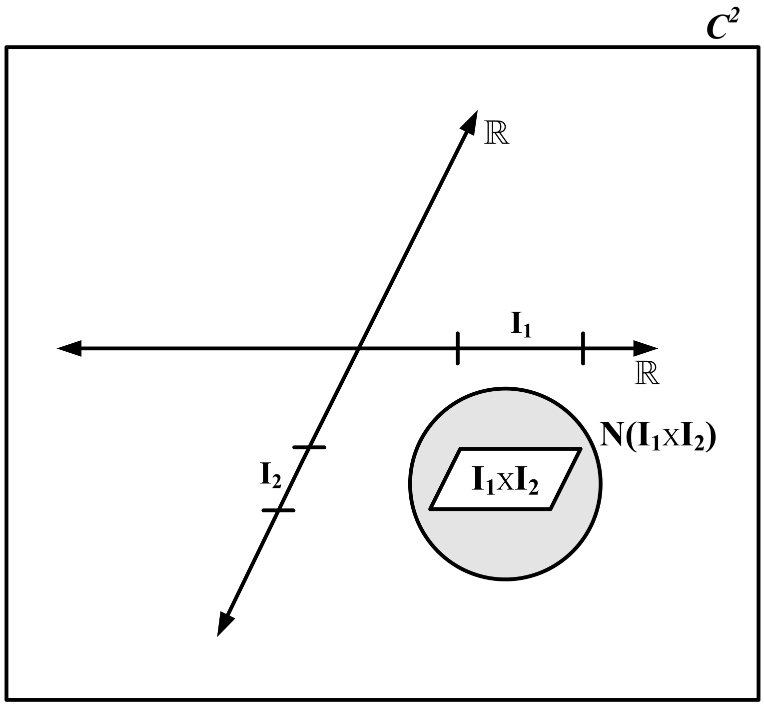

Figure 5.1: Complex neighbourhood $N(I_1 X I_2)$

$$G(z_1, z_2) = F(z_1, z_2) + \phi_1(z_1, z_2) + \phi_2(z_1, z_2)$$

with $\phi_1(z_1, z_2) \in \mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $\phi_2(z_1, z_2) \in \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$. Here $N_1(I_1) \times N_1(I_2)$ and $N_2(I_1) \times N_2(I_2)$ are the complex neighbourhoods of $I_1 \times I_2$ of $F(z_1, z_2)$ and $G(z_1, z_2)$ respectively. We denoted it by $F(z_1, z_2) \sim G(z_1, z_2)$

Proposition 5.1.3. *The relation $\sim$ defined above is an equivalence relation*

Proof. Since the zero function is a holomorphic function in $\mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $\mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$ we have $F(z_1, z_2) \sim F(z_1, z_2)$. Hence $\sim$ is reflexive.
Suppose

$$G(z_1, z_2) = F(z_1, z_2) + \phi_1(z_1, z_2) + \phi_2(z_1, z_2).$$

If $\phi_1(z_1, z_2) \in \mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $\phi_2(z_1, z_2) \in \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$ then

$-\phi_1(z_1, z_2) \in \mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $-\phi_2(z_1, z_2) \in \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$.

Hence

$$F(z_1, z_2) = G(z_1, z_2) - \phi_1(z_1, z_2) - \phi_2(z_1, z_2).$$

Thus $F(z_1, z_2) \sim G(z_1, z_2) \Rightarrow G(z_1, z_2) \sim F(z_1, z_2)$.

Therefore $\sim$ is symmetric.

Suppose that $F(z_1, z_2) \sim G(z_1, z_2)$ and $G(z_1, z_2) \sim H(z_1, z_2)$.Then

$$G(z_1, z_2) = F(z_1, z_2) + \phi_1(z_1, z_2) + \phi_2(z_1, z_2),$$

$$H(z_1, z_2) = G(z_1, z_2) + \phi_1^{'}(z_1, z_2) + \phi_2^{'}(z_1, z_2)$$

where $\phi_1(z_1, z_2), \phi_1^{'}(z_1, z_2)$ are in $\mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2))$ and $\phi_2(z_1, z_2), \phi_2^{'}(z_1, z_2)$ are in $\mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$.

Since the sum of two holomorphic functions is again holomorphic,

$$\phi_1(z_1, z_2) + \phi_1^{'}(z_1, z_2) \in \mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2)),$$

$$\phi_2(z_1, z_2) + \phi_2^{'}(z_1, z_2) \in \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))$$

Hence

$$\begin{aligned} H(z_1, z_2) &= G(z_1, z_2) + \phi_1^{'}(z_1, z_2) + \phi_2^{'}(z_1, z_2) \\ &= F(z_1, z_2) + \phi_1(z_1, z_2) + \phi_2(z_1, z_2) + \phi_1^{'}(z_1, z_2) + \phi_2^{'}(z_1, z_2) \\ &= F(z_1, z_2) + (\phi_1(z_1, z_2) + \phi_1^{'}(z_1, z_2)) + (\phi_2(z_1, z_2) + \phi_2^{'}(z_1, z_2)) \end{aligned}$$

So $F(z_1, z_2) \sim H(z_1, z_2)$.

Hence $\sim$ is transitive.

Therefore $\sim$ is an equivalence relation. $\square$

Definition 5.1.4. Define

$$\mathfrak{F}(I_1 \times I_2) = \mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))$$

Definition 5.1.5. An equivalence class of functions $F(z_1, z_2) \in \mathfrak{F}(I_1 \times I_2)$ defines a hyperfunction $f(x, y)$ on $I_1 \times I_2$. It is denoted by

$$f(x, y) = [F(z_1, z_2)]$$

$F(z_1, z_2)$ is called defining or generating function of the hyperfunction $f(x, y)$.

The set of all hyperfunctions on the set $I_1 \times I_2$ is denoted by $\mathfrak{B}(I_1 \times I_2)$. Then as a quotient space,

$$\mathfrak{B}(I_1 \times I_2) := \frac{\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))}{\mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2)) + \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))}$$

There is no importance in the choice of neighbourhood $N(I_1) \times N(I_2)$. If $N^{'}(I_1) \times N^{'}(I_2)$ is any other complex neighbourhood of $I_1 \times I_2$ such that $N^{'}(I_1) \times N^{'}(I_2) \subset N(I_1) \times N(I_2)$. Then

$$\frac{\mathfrak{O}((N^{'}(I_1)\backslash I_1) \times (N_{'}(I_2)\backslash I_2))}{\mathfrak{O}((N'(I_1)\backslash I_1) \times N'(I_2)) + \mathfrak{O}(N'(I_1) \times (N'(I_2)\backslash I_2))}$$

works as well. This means that the behaviour of the hyperfunction depends on the defining function in a small neighbourhood of $I_1 \times I_2$ in $\mathbb{C}^2$.

As an inductive limit

$$\mathfrak{B}(I_1 \times I_2) := \varinjlim_{N(I_1)\times N(I_2)\supset I_1\times I_2} \frac{\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))}{\mathfrak{O}((N(I_1)\backslash I_1) \times N(I_2)) + \mathfrak{O}(N(I_1) \times (N(I_2)\backslash I_2))}$$

By making complex neighbourhoods smaller and smaller around $I_1 \times I_2$ leads to the following concept:

A real analytic function $\phi(x, y)$ on $I_1 \times I_2$ is defined with the understanding that $\phi(x, y)$ can analytically be continued to a full complex neighbourhood Ω in $\mathbb{C}^2$, $\Omega \supset I_1 \times I_2$. So if any complex neighbourhood $N(I_1) \times N(I_2)$ containing Ω. Then $\phi(x, y) \in \mathfrak{O}(\Omega)$. Let

$$\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2) := \frac{\mathfrak{O}((N(I_1)\backslash I_1) \times (N(I_2)\backslash I_2))}{\mathcal{A}(I_1 \times I_2)}$$

Where $\mathcal{A}(I_1 \times I_2)$ denotes the ring of real analytic functions on $I_1 \times I_2$.

Thus as in the case of hyperfunction of one variable here also $f(x, y) = [F(z_1, z_2)] \in \mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is determined by the defining function $F(z_1, z_2)$, holomorphic in an adjacent complex neighbourhood of $I_1 \times I_2$. It is determined only upto a real analytic function on $I_1 \times I_2$.

Note: Now by a hyperfunction we mean a hyperfunction in $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$

Definition 5.1.6. The value of a hyperfunction $f(x, y) = [F(x, y)]$ at a point $(x, y) \in I_1 \times I_2$ is defined as

$$f(x, y) = \lim_{\epsilon \to 0+} \{F(x + i\epsilon, y + i\epsilon) - F(x - i\epsilon, y - i\epsilon)\},$$

provided the limit exists.

The above definition is well defined.

Consider $G(z_1, z_2) \in [F(z_1, z_2)]$. Then

$$\lim_{\epsilon \to 0+} \{G(x+i\epsilon, y+i\epsilon) - G(x-i\epsilon, y-i\epsilon)\}$$

$$\begin{aligned}
&= \lim_{\epsilon \to 0+} \{F(x+i\epsilon, y+i\epsilon) + \phi(x+i\epsilon, y+i\epsilon) - F(x-i\epsilon, y-i\epsilon)\} - \phi(x-i\epsilon, y-i\epsilon) \\
&= \lim_{\epsilon \to 0+} \{F(x+i\epsilon, y+i\epsilon) - F(x-i\epsilon, y-i\epsilon)\} + \lim_{\epsilon \to 0+} \{\phi(x+i\epsilon, y+i\epsilon) - \phi(x-i\epsilon, y-i\epsilon)\} \\
&= \lim_{\epsilon \to 0+} \{F(x+i\epsilon, y+i\epsilon) - F(x-i\epsilon, y-i\epsilon)\} + 0 \\
&= f(x,y), where \;\; \phi(z_1, z_2) \in \mathcal{A}(I_1 \times I_2)
\end{aligned}$$

Definition 5.1.7. A point $(x, y) \in I_1 \times I_2$ is called a *regular* point of the hyperfunction $f(x, y) = [F(z_1, z_2)]$ if $\lim_{\epsilon \to 0+} \{F(x+i\epsilon, y+i\epsilon) - F(x-i\epsilon, y-i\epsilon)\}$ exists. A point $(x, y) \in I_1 \times I_2$ is called a *singular* point if it is not a regular point.

Hence at a regular point the hyperfunction $f(x, y)$ has a value as an ordinary function.

Examples 5.1.8. *The Dirac's delta function in two dimension is $\delta(x, y) = 0$ for $x \neq 0$ and $y \neq 0$, and has value 1 otherwise.*

$$\int\int \delta(x,y)dxdy = 1$$

Then $\delta(x, y) = \delta(x)\delta(y)$.

In terms of the defining function

$$\delta(x,y) = [\frac{-1}{4\pi^2 z_1 z_2}]$$

Then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \frac{-1}{4\pi^2}\int \frac{1}{z_1}dz_1 \int \frac{1}{z_2}dz_2 = 1,$$

considering z_1, z_2 as two independent variables.

Definition 5.1.9. The hyperfunction $f(x,y) = [\phi(z_1,z_2)]$, $\phi(x,y) \in \mathcal{A}(\mathbb{R}^2)$, represents the *zero hyperfunction* in two variables. $\phi(x,y) \in [O]$, where O denotes the zero function.

Definition 5.1.10. For $f(x,y) = [F(z_1,z_2)], g(x,y) = [G(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ and $c \in \mathbb{C}$ define

$$f(x,y) + g(x,y) = [F(z_1,z_2) + G(z_1,z_2)],$$

$$cf(x,y) = [cF(z_1,z_2)]$$

We can prove the definition of addition and scalar multiplication on $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is well defined by taking representatives from the corresponding equivalence classes. Let $F'(z_1,z_2) \in [F(z_1,z_2)]$ and $G'(z_1,z_2) \in [G(z_1,z_2)]$. Then

$$F'(z_1,z_2) = F(z_1,z_2) + \phi_1(z_1,z_1) \text{ and}$$

$$G'(z_1,z_2) = G(z_1,z_2) + \phi_2(z_1,z_1)$$

Then

$$\begin{aligned}
[F'(z_1,z_2) + G'(z_1,z_2)] &= [F(z_1,z_2) + \phi_1(z_1,z_1) + G(z_1,z_2) + \phi_2(z_1,z_1)] \\
&= [F(z_1,z_2) + G(z_1,z_2) + \phi_1(z_1,z_1) + \phi_2(z_1,z_1)] \\
&= [F(z_1,z_2) + G(z_1,z_2) + O] \\
&= [F(z_1,z_2) + G(z_1,z_2)] \\
&= f(x,y) + g(x,y)
\end{aligned}$$

Also

$$\begin{aligned}[cF'(z_1, z_2)] &= [c(F(z_1, z_2) + \phi_1(z_1, z_1))] \\ &= [cF(z_1, z_2) + c\phi_1(z_1, z_1)] \\ &= [cF(z_1, z_2) + O] \\ &= [cF(z_1, z_2)] \\ &= cf(x, y)\end{aligned}$$

Proposition 5.1.11. *$\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is a linear space.*

Proof. With the addition and scalar multiplication defined on $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ as above all the properties required for a linear space can be verified. □

Definition 5.1.12. A hyperfunction $f(x, y) = [F(z_1, z_2)]$ is called holomorphic on $I_1 \times I_2$ if the defining function $[F(z_1, z_2)]$ is holomorphic on $U = N(I_1) \times N(I_2)$. i.e
(i) For each point $a = (a_1, a_2) \in U \subset \mathbb{C}^2$, $F(z_1, z_2)$ has a convergent power series expansion on U ,

$$F(z_1, z_2) = \Sigma c_{n_1,n_2}(z_1 - a_1)^{n_1}(z_2 - a_2)^{n_2}$$

OR

(ii) If $F(z_1, z_2)$ is continuous on U and for each variable z_j,$j = 1, 2$, $F(z_1, z_2)$ is holomorphic, (i.e.$\dfrac{\partial F}{\partial \bar{z}_1{}^n} = 0$ and $\dfrac{\partial F}{\partial \bar{z}_2{}^n} = 0$ by the generalisation of Cauchy-Riemann equations)

Definition 5.1.13. Let S_0 be the largest open subset of $\mathbb{R}^2$ where the hyperfunction $f(x, y) = [F(z_1, z_2)]$ has zero value. Then the support of the hyperfunction $f(x, y)$ is $\mathbb{R}^2 \backslash S_0$. It is denoted by $supp f(x, y)$

Definition 5.1.14. Let S_1 be the largest open subset of $\mathbb{R}^2$ where the hyperfunction $f(x, y) = [F(z_1, z_2)]$ is holomorphic. Then the singular support of the hyperfunction $f(x, y)$ is $\mathbb{R}^2 \backslash S_1$. It is denoted by *sing supp* $f(x, y)$

Proposition 5.1.15. *For a hyperfunction* $f(x, y) = [F(z_1, z_2)]$, *sing supp* $f(x, y) \subset$ *supp* $f(x, y)$

Proof. Since $S_0 \subset S_1$, we have $(S_1)^C \subset (S_0)^C$

Hence *sing supp* $f(x, y) \subset$ *supp* $f(x, y)$ □

5.2. Two Dimensional Laplace Transform of Hyperfunctions

Definition 5.2.1. We say that a hyperfunction $f(x, y) = [F(z_1, z_2)]$ on $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ is said to be of bounded exponential growth if there exist real constants $M > 0, \sigma', \sigma''$ such that

$$|F(z_1, z_2)| < Me^{\sigma' \mathcal{R} z_1 + \sigma'' \mathcal{R} z_2} < \infty$$

on every compact subset of $N(I_1) \times N(I_2)$ and for every equivalent defining functions. Let $\mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ denotes the set of all hyperfunction in $\mathfrak{B}_{\mathcal{R}}(I_1 \times I_2)$ having bounded exponential growth.

Definition 5.2.2. A hyperfunction $f(x, y) = [F(z_1, z_2)]$ on $\mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ is said to be separable if

$$F(z_1, z_2) = F_1(z_1)F_2(z_2),$$

where $F_1(z_1) \in \dfrac{\mathfrak{O}(N(I_1)\backslash I_1)}{\mathcal{A}(I_1)}$ and $F_2(z_2) \in \dfrac{\mathfrak{O}(N(I_2)\backslash I_2)}{\mathcal{A}(I_2)}$

Now we define the two dimensional Laplace transform of hyperfunction in two variables.

Definition 5.2.3. Let $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ be a separable hyperfunction with $I_1 \subset [0,\infty)$ and $I_1 \subset [0,\infty)$. The two dimensional Laplace transform of $f(x,y)$ is defined as

$$\hat{f}(u,v) = \mathfrak{L}_y \mathfrak{L}_x [f(x,y)](u,v) = \int_0^\infty e^{-vy} \int_0^\infty e^{-ux} f(x,y) dx dy,$$

where u and v are complex numbers.

The above integral can be evaluated in the following way.

$$\begin{aligned}
\hat{f}(u,v) &= \mathfrak{L}_y \mathfrak{L}_x [f(x,y)](u,v) \\
&= \int_0^\infty e^{-vy} \int_0^\infty e^{-ux} f(x,y) dx dy \\
&= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} F(z_1,z_2) dz_1 dz_2 \\
&= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} F_1(z_1) F_2(z_2) dz_1 dz_2 \\
&= (\int_0^\infty e^{-uz_1} F_1(z_1) dz_1)(\int_0^\infty e^{-vz_2} F_2(z_2) dz_2),
\end{aligned}$$

exists for all $\mathcal{R}u > \sigma^{'}$, $\mathcal{R}v > \sigma^{''}$

Examples 5.2.4.

$$\mathfrak{L}_y\mathfrak{L}_x[\delta(x,y)](u,v) = \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\delta(x,y)dxdy = 1$$

Proposition 5.2.5. *The image function $\hat{f}(u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$ is a holomorphic function*

Proof. We have

$$\hat{f}(u,v) = (\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2)$$

Since $f(x,y)$ is hyperfunction of bounded exponential growth $F_1(z_1)$ and $F_1(z_1)$ are bounded exponential holomorphic functions in one variable. Hence the Laplace transform $F_1(z_1)$ and $F_1(z_1)$ are holomorphic. Also the product of two holomorphic function is holomorphic. Therefore $\hat{f}(u,v)$ is a holomorphic function. □

Proposition 5.2.6. *Let $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ be a separable hyperfunction with $I_1 \subset [0,\infty)$ and $I_1 \subset [0,\infty)$ then*

$$\mathfrak{L}_y\mathfrak{L}_x[\overline{f(x,y)}](u,v) = \overline{\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](\bar{u},\bar{v})}$$

Proof.

$$\begin{aligned} \mathfrak{L}_y\mathfrak{L}_x[\overline{f(x,y)}](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\overline{f(x,y)}dxdy \\ &= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\overline{F(z_1,z_2)}dz_1dz_2 \end{aligned}$$

$$= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} \overline{F_1(z_1)} \ \overline{F_2(z_2)} dz_1 dz_2$$

$$= \int_0^\infty e^{-uz_1} \overline{F_1(z_1)} dz_1 \int_0^\infty e^{-vz_2} \overline{F_2(z_2)} dz_2$$

$$= \overline{\int_0^\infty e^{-\bar{u}z_1} F_1(z_1) dz_1} \ \overline{\int_0^\infty e^{-\bar{v}z_2} F_2(z_2) dz_2},$$

Since by proposition 3.7 [25] $\mathfrak{L}[\overline{f(x)}](s) = \overline{\mathfrak{L}[f(x)](\bar{s})}$

$$= \overline{\int_0^\infty e^{-\bar{v}z_2} \int_0^\infty e^{-\bar{u}z_1} F_1(z_1) F_2(z_2) dz_1 dz_2}$$

$$= \overline{\int_0^\infty e^{-\bar{v}y} \int_0^\infty e^{-\bar{u}x} f(x,y) dx dy}$$

$$= \overline{\mathfrak{L}_y \mathfrak{L}_x [f(x,y)](\bar{u}, \bar{v})}$$

□

5.3. Operational Properties

Proposition 5.3.1. *Let* $f(x,y) = [F(z_1,z_2)], g(x,y) = [G(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ *are two separable hyperfunctions with* $I_1 \subset [0,\infty)$ *and* $I_1 \subset [0,\infty)$ *then*

$$\mathfrak{L}_y \mathfrak{L}_x [f(x,y) + g(x,y)](u,v) = \mathfrak{L}_y \mathfrak{L}_x [f(x,y)](u,v) + \mathfrak{L}_y \mathfrak{L}_x [g(x,y)](u,v)$$

Proof.

$$\begin{aligned}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)+g(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}(f(x,y)+g(x,y))dxdy \\ &= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}(F(z_1,z_2)+G(z_1,z_2))dz_1dz_2 \\ &= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F(z_1,z_2)dz_1dz_2 \\ &\quad +\int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}G(z_1,z_2)dz_1dz_2 \\ &= \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)+\mathfrak{L}_y\mathfrak{L}_x[g(x,y)](u,v)\end{aligned}$$

□

Proposition 5.3.2. *Let $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and c be a constant then*

$$\mathfrak{L}_y\mathfrak{L}_x[cf(x,y)](u,v) = c\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$$

Proof.

$$\begin{aligned}\mathfrak{L}_y\mathfrak{L}_x[cf(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}(cf(x,y))dxdy \\ &= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}(cF(z_1,z_2))dz_1dz_2 \\ &= c\int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F(z_1,z_2)dz_1dz_2\end{aligned}$$

$$= \quad c\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$$

□

Proposition 5.3.3. *Let* $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ *be a separable hyperfunctions and* a *and* b *are two constants then*

$$\mathfrak{L}_y\mathfrak{L}_x[e^{ax+by}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u-a,v-b)$$

Proof.

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[e^{ax+by}f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}(e^{ax+by}f(x,y))dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}(e^{az_1+bz_2}F(z_1,z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}e^{bz_2}\int_0^\infty e^{-uz_1}e^{az_1}F(z_1,z_2)dz_1dz_2 \\
&= \int_0^\infty e^{-(v-b)z_2}\int_0^\infty e^{-(u-a)z_1}F(z_1,z_2)dz_1dz_2 \\
&= \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u-a,v-b)
\end{aligned}$$

□

Corollary 5.3.4. *Let* $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ *be a separable hyperfunctions and* a *and* b *are two constants then*

(a)

$$\mathfrak{L}_y\mathfrak{L}_x[e^{ax}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u-a,v)$$

(b)

$$\mathfrak{L}_y\mathfrak{L}_x[e^{by}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v-b)$$

Proposition 5.3.5. *Let $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunctions and a and b are two non zero real constants then*

$$\mathfrak{L}_y\mathfrak{L}_x[f(ax,by)](u,v) = \frac{1}{ab}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](\frac{u}{a},\frac{v}{b})$$

Proof.

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[f(ax,by)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}f(ax,by)dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F(az_1,bz_2)dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F_1(az_1)F_2(bz_2)dz_1dz_2 \\
&= \int_0^\infty e^{-uz_1}F_1(az_1)dz_1 \quad \int_0^\infty e^{-vz_2}F_2(bz_2)dz_2 \\
&= \int_0^\infty e^{-u\frac{\zeta_1}{a}}F_1(\zeta_1)\frac{1}{a}d\zeta_1 \quad \int_0^\infty e^{-v\frac{\zeta_2}{b}}F_2(\zeta_2)\frac{1}{b}d\zeta_2,
\end{aligned}$$

by putting $az_1 = \zeta_1$ and $bz_2 = \zeta_2$

$$= \frac{1}{ab}\int_0^\infty e^{-(\frac{v}{b})\zeta_2}\int_0^\infty e^{-(\frac{u}{a})\zeta_1}F_1(\zeta_1)F_2(\zeta_2)d\zeta_1d\zeta_2$$

$$= \frac{1}{ab}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](\frac{u}{a},\frac{v}{b})$$

□

Proposition 5.3.6. *Let $f(x,y) = [F(z_1,z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ be a separable hyperfunctions and a and b are two constants then*

$$\mathfrak{L}_y\mathfrak{L}_x[f(x+a,y+b)](u,v) = e^{au+bv}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)$$

Proof.

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[f(x+a,y+b)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}f(x+a,y+b)dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F(z_1+a,z_2+b)dz_1dz_2 \\
&= \int_0^\infty e^{-v(\zeta_2-b)}\int_0^\infty e^{-u(\zeta_1-a)}F(\zeta_1,\zeta_2)d\zeta_1d\zeta_2, \\
&\quad \text{By putting } z_1+a=\zeta_1 \text{ and } z_2+b=\zeta_2 \\
&= \int_0^\infty e^{bv}e^{-v\zeta_2}\int_0^\infty e^{au}e^{-u\zeta_1}F(\zeta_1,\zeta_2)d\zeta_1d\zeta_2 \\
&= e^{au+bv}\int_0^\infty e^{-v\zeta_2}\int_0^\infty e^{-u\zeta_1}F(\zeta_1,\zeta_2)d\zeta_1d\zeta_2 \\
&= e^{au+bv}\mathfrak{L}_y\mathfrak{L}_x[f(x,y)](u,v)
\end{aligned}$$

□

Proposition 5.3.7. *Let* $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ *be a separable hyperfunction. For positive integers* m *and* n*,*

$$\mathfrak{L}_y\mathfrak{L}_x[x^m y^n f(x,y)](u,v) = (-1)^{m+n}(\frac{d^m}{du^m}(\hat{f}_1(u)))(\frac{d^n}{dv^n}(\hat{f}_2(v))),$$

Where $\hat{f}_1(u) = \int_0^\infty e^{-uz_1}F_1(z_1)dz_1$ *and* $\hat{f}_2(v) = \int_0^\infty e^{-vz_2}F_2(z_2)dz_2$

Proof.

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[x^m y^n f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}x^m y^n f(x,y)dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}z_1^m z_2^n F_1(z_1)F_2(z_2)dz_1dz_2 \\
&= \int_0^\infty e^{-uz_1}z_1^m F_1(z_1)dz_1\int_0^\infty e^{-vz_2}z_2^n F_2(z_2)dz_2 \\
&= (-1)^m\frac{d^m}{du^m}(\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(-1)^n\frac{d^n}{dv^n}(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2) \\
&= (-1)^{m+n}(\frac{d^m}{du^m}(\hat{f}_1(u)))(\frac{d^n}{dv^n}(\hat{f}_2(v)))
\end{aligned}$$

□

Proposition 5.3.8. *Let* $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}^{exp}_{\mathcal{R}}(I_1 \times I_2)$ *be a separable hyperfunctions and* $\hat{f}_1(u) = \int_0^\infty e^{-uz_1}F_1(z_1)dz_1$*,* $\hat{f}_2(v) = \int_0^\infty e^{-vz_2}F_2(z_2)dz_2$ *Then*

(a)$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial x}f(x,y)](u,v) = u\hat{f}_1(u)\hat{f}_2(v)$

$(b)\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial y}f(x,y)](u,v) = v\hat{f}_1(u)\hat{f}_2(v)$

$(c)\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x^2}f(x,y)](u,v) = u^2\hat{f}_1(u)\hat{f}_2(v)$

$(d)\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y^2}f(x,y)](u,v) = v^2\hat{f}_1(u)\hat{f}_2(v)$

$(e)\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x\partial y}f(x,y)](u,v) = \mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y\partial x}f(x,y)](u,v) = uv\hat{f}_1(u)\hat{f}_2(v)$

Proof. (a)

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial x}f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\frac{\partial}{\partial x}(f(x,y))dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{\partial}{\partial z_1}(F(z_1,z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{d}{dz_1}(F_1(z_1)F_2(z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{d}{dz_1}(F_1(z_1))F_2(z_2)dz_1dz_2 \\
&= (\int_0^\infty e^{-uz_1}\frac{d}{dz_1}(F_1(z_1))dz_1)(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2) \\
&= u(\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2) \\
&= u\hat{f}_1(u)\hat{f}_2(v)
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial}{\partial y}f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\frac{\partial}{\partial y}(f(x,y))dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{\partial}{\partial z_2}(F(z_1,z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{d}{dz_2}(F_1(z_1)F_2(z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}F_1(z_1)\frac{d}{dz_2}(F_2(z_2))dz_1dz_2 \\
&= (\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2}\frac{d}{dz_2}(F_2(z_2))dz_2) \\
&= v(\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2) \\
&= v\hat{f}_1(u)\hat{f}_2(v)
\end{aligned}
$$

(c)

$$
\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x^2}f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\frac{\partial^2}{\partial x^2}(f(x,y))dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{\partial^2}{\partial z_1^2}(F(z_1,z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{d^2}{dz_1^2}(F_1(z_1)F_2(z_2))dz_1dz_2
\end{aligned}
$$

$$= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} \frac{d^2}{dz_1^2}(F_1(z_1))F_2(z_2)dz_1dz_2$$

$$= (\int_0^\infty e^{-uz_1} \frac{d^2}{dz_1^2}(F_1(z_1))dz_1)(\int_0^\infty e^{-vz_2} F_2(z_2)dz_2)$$

$$= u^2(\int_0^\infty e^{-uz_1} F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2} F_2(z_2)dz_2)$$

$$= u^2 \hat{f}_1(u)\hat{f}_2(v)$$

(d)

$$\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y^2} f(x,y)](u,v) = \int_0^\infty e^{-vy} \int_0^\infty e^{-ux} \frac{\partial^2}{\partial y^2}(f(x,y))dxdy$$

$$= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} \frac{\partial^2}{\partial z_2^2}(F(z_1,z_2))dz_1dz_2$$

$$= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} \frac{d^2}{dz_2^2}(F_1(z_1)F_2(z_2))dz_1dz_2$$

$$= \int_0^\infty e^{-vz_2} \int_0^\infty e^{-uz_1} F_1(z_1)\frac{d^2}{dz_2^2}(F_2(z_2))dz_1dz_2$$

$$= (\int_0^\infty e^{-uz_1} F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2} \frac{d^2}{dz_2^2}(F_2(z_2))dz_2)$$

$$= v^2(\int_0^\infty e^{-uz_1} F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2} F_2(z_2)dz_2)$$

$$= v^2 \hat{f}_1(u)\hat{f}_2(v)$$

(e)

$$\begin{aligned}
\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial x\partial y}f(x,y)](u,v) &= \int_0^\infty e^{-vy}\int_0^\infty e^{-ux}\frac{\partial^2}{\partial x\partial y}(f(x,y))dxdy \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{\partial^2}{\partial z_1\partial z_2}(F(z_1,z_2))dz_1dz_2 \\
&= \int_0^\infty e^{-vz_2}\int_0^\infty e^{-uz_1}\frac{\partial^2}{\partial z_1\partial z_2}(F_1(z_1)F_2(z_2))dz_1dz_2 \\
&= (\int_0^\infty e^{-uz_1}\frac{d}{dz_1}(F_1(z_1))dz_1)(\int_0^\infty e^{-vz_2}\frac{d}{dz_2}(F_2(z_2))dz_2) \\
&= uv(\int_0^\infty e^{-uz_1}F_1(z_1)dz_1)(\int_0^\infty e^{-vz_2}F_2(z_2)dz_2) \\
&= uv\hat{f}_1(u)\hat{f}_2(v)
\end{aligned}$$

Similarly, $\mathfrak{L}_y\mathfrak{L}_x[\frac{\partial^2}{\partial y\partial x}f(x,y)](u,v) = uv\hat{f}_1(u)\hat{f}_2(v)$ □

5.4. Inverse of Two Dimensional Laplace Transform of Hyperfunctions

If $f(x,y) = [F(z_1,z_2)] = [F_1(z_1)F_2(z_2)] \in \mathfrak{B}_{\mathcal{R}}^{exp}(I_1 \times I_2)$ be a separable hyperfunction with two dimensional Laplace transform $\hat{f}(u,v)$ the inverse transform is defined by

$$f(x,y) = \int_0^\infty e^{vy}\int_0^\infty e^{ux}\hat{f}(u,v)dudv$$

Chapter 6

Ordered Linear Space of Hyperfunctions, Norm Convergence and Completely Monotone Hyperfunctions

In this chapter some order theoretic properties of hyperfunctions are investigated. Order relation and norm convergence in the linear space of hyperfunctions are studied. The concept of completely monotone hyperfunctions is defined and some of its properties are proved.

6.1. Ordered Linear Space of Hyperfunctions

We establish an order relation in the linear space of hyperfunctions by defining a cone in it. Some properties of this cone are studied by introducing a topology to it.

Let the set of all real valued hyperfunctions be denoted by $\mathfrak{B}_{\mathbb{R}}(I)$. Then $\mathfrak{B}_{\mathbb{R}}(I)$ is a linear space over $\mathbb{R}$ and it is a subspace of $\mathfrak{B}(I)$.

Definition 6.1.1. Let $f(x), g(x) \in \mathfrak{B}_{\mathbb{R}}(I)$ where $f(x) = [F(z)]$, $g(x) = [G(z)]$. Define the order relation $' \leqslant '$ by $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in I$. In terms of defining functions $(F(x+i0) - F(x-i0)) \leqslant (G(x+i0) - G(x-i0)))$ for all $F(z) \in [F(z)]$ and $G(z) \in [G(z)]$.

Proposition 6.1.2. *The above defined relation* $' \leqslant '$ *is a partial order on* $\mathfrak{B}_{\mathbb{R}}(I)$.

Proof. Using the above definition of $' \leqslant '$, for $f(x) = [F(z)]$, $g(x) = [G(z)]$,
and $h(x) = [H(z)] \in \mathfrak{B}_{\mathbb{R}}(I)$

(1)$f(x) \leqslant f(x)$, the relation is reflexive

(2)$f(x) \leqslant g(x)$ and $g(x) \leqslant f(x)$ implies $f(x) = g(x)$. It is symmetric

(3)$f(x) \leqslant g(x)$ and $g(x) \leqslant h(x)$ implies $f(x) \leqslant h(x)$. It is transitive

Thus the relation $' \leqslant '$ is a partial order relation on $\mathfrak{B}_{\mathbb{R}}(I)$. □

Proposition 6.1.3. $\mathfrak{B}_{\mathbb{R}}(I)$ *is an ordered linear space*

Proof. Clearly $\mathfrak{B}_{\mathbb{R}}(I)$ is a linear space on $\mathbb{R}$ with respect to the addition and scalar multiplication defined on $\mathfrak{B}(I)$. Also by the previous proposition $' \leqslant '$ is a partial order on $\mathfrak{B}_{\mathbb{R}}(I)$. Hence it is an ordered linear space. □

Note: For the following results, consider I as a subset of $\mathbb{R}_+ = (0, \infty)$.

Let $\mathfrak{B}_B^M(I)$ denotes the linear subspace of $\mathfrak{B}_{\mathbb{R}}(I)$ of hyperfunctions of bounded exponential growth and has a complex measurable holomorphic function as defining function .

Let

$$\bar{N}(0,n) = \{z \in \mathcal{C} : |z| \leqslant n\},$$

i.e. $\bar{N}(0,n)$ is a closed complex neighbourhood of 0.

Definition 6.1.4. Let N be a complex neighbourhood of I. For $n = 1, 2, ...$ define

$$K_n = \bar{N}(0,n) \cap \{z : |z - w| \geqslant \frac{1}{n}, \forall w \in \mathbb{C} \backslash N\}$$

Then $\{K_n\}$ has the following properties

i) K_n is compact

ii)$K_n \subseteq K_{n+1}$

iii)If $K \subseteq N$ is compact then $K \subseteq K_n$ for sufficiently large n.

On each K_n and $f(x) = [F(Z)] \in \mathfrak{B}(I)$ define

$$\beta_{K_n,m}(f(x)) = \sup\{|\frac{d^m}{dz^m}F(z)| : \forall F(z) \in [F(z)], z \in K_n\}, m = 0, 1, 2, ...$$

Let $\mathfrak{B}^M_{B,K_n,m}(I)$ denotes the subspace of $\mathfrak{B}^M_B(I)$, consisting of all hyperfunctions with support contained in K_n. Then $\{\beta_{K_n,m}\}_{m=0}^{\infty}$ is a multinorm on $\mathfrak{B}^M_{B,K_n,m}(I)$. The defined set of multinorms generates a topology $\tau_{K_n,m}$ on $\mathfrak{B}^M_{B,K_n,m}(I)$.
$\mathfrak{B}^M_B(I)$ assigns the inductive limit topology τ when K_n varies over all compact sets $K_1, K_2, ...$

Proposition 6.1.5. *$\mathfrak{B}_B^M(I)$ is an ordered topological linear space.*

Proof. Since $\mathfrak{B}_B^M(I)$ is a subspace of the ordered linear space $\mathfrak{B}_{\mathbb{R}}(I)$, it is also an ordered linear space. By the above definition $\mathfrak{B}_B^M(I)$ assigns the inductive limit topology generated by multinorms. Hence $\mathfrak{B}_B^M(I)$ is an ordered topological linear space. □

Definition 6.1.6. The Cone, P of $\mathfrak{B}_B^M(I)$ is, when $\mathfrak{B}_B^M(I)$ restricted to the set of all non-negative hyperfunctions in $\mathfrak{B}_{\mathbb{R}}(I)$. The Positive Cone in $\mathfrak{B}_B^M(I)$ is $P + iP$ which is denoted as $\mathcal{P}$

Proposition 6.1.7. *The cone $\mathcal{P}$ in $\mathfrak{B}_B^M(I)$ has the following properties:*

i)$\mathcal{P} + \mathcal{P} \subset \mathcal{P}$

ii)$c\mathcal{P} \subset \mathcal{P}$ *for every real number* $c > 0$

iii)$\mathcal{P} \cap -\mathcal{P} = \{[0]\}$.

Proof. (i) Since the sum of two non-negative hyperfunction is again non-negative we have $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$

(ii) Let $f(x) \in \mathcal{P}$ and $c > 0$. Then the hyperfunction $cf(x)$ is clearly non negative. Hence $c\mathcal{P} \subset \mathcal{P}$ for every real number $c > 0$

(iii) All real functions belonging to the equivalence class of zero in the sense of hyperfunction is $\mathcal{P} \cap -\mathcal{P}$ □

Proposition 6.1.8. *$\mathcal{P}$ is a convex set in $\mathfrak{B}_B^M(I)$.*

Proof. Let $f(x) = [F(z)]$, $g(x) = [G(z)] \in \mathcal{P}$ and choose γ with $0 < \gamma < 1$. Then $\gamma f(x) + (1 - \gamma)g(x)$ is again a non negative hyperfunction in $\mathfrak{B}_B^M(I)$. □

Definition 6.1.9. For $f(x),\ g(x) \in \mathfrak{B}_B^M(I)$ with $f \leqslant g$, define the *order interval* between f and g by

$$[f,g] = \{h(x) \in \mathfrak{B}_B^M(I) : f(x) \leqslant h(x) \leqslant g(x)\}$$

Definition 6.1.10. A subset E of $\mathfrak{B}_B^M(I)$ is *order bounded* if there exists $f(x),\ g(x) \in \mathfrak{B}_B^M(I)$such that $E \subset [f,g]$

Proposition 6.1.11. *The cone $\mathcal{P}$ in $\mathfrak{B}_B^M(I)$ is generating.*

Proof. Since every complex measurable function $F = F^+ - F^-$,
we have $\mathcal{P} - \mathcal{P} = \mathfrak{B}_B^M(I)$ □

Definition 6.1.12. For $E \subset \mathfrak{B}_B^M(I)$ the *full hull*$[E]$ of E is defined as

$$[E] = \{h(x) \in \mathfrak{B}_B^M(I) : f(x) \leqslant h(x) \leqslant g(x), f(x), g(x) \in E\}$$

Proposition 6.1.13. *The cone $\mathcal{P}$ in $\mathfrak{B}_B^M(I)$ is normal*

Proof. The neighbourhood basis of 0 for τ consisting of real holomorphic hyperfunctions is a neighbourhood basis of 0 consisting of full sets. Hence the cone $\mathcal{P}$ is a normal cone. □

Proposition 6.1.14. *Every order bounded subset of $\mathfrak{B}_B^M(I)$ is τ bounded*

Proof. Proposition 1.4 in [43] states that if the cone K in an ordered topological vector space $E(\tau)$ is normal for τ, then every order bounded subset of E is $\tau-$ bounded. □

Proposition 6.1.15. *If $\mathcal{P}$ is a normal cone in $\mathfrak{B}_B^M(I)$ then $\mathcal{P} \cap \mathfrak{B}_{B,K_n,m}^M(I)$ is a normal cone in $\mathfrak{B}_{B,K_n,m}^M(I)$*

Proof. Proposition 1.8 in [43] is: if K is a normal cone in an ordered topological vector space $E(\tau)$ and M is a linear subspace of E then $K \cap M$ is a normal cone in M for the subspace topology. The result follows from this proposition. □

6.2. Norm Convergence in the Linear Space of Hyperfunctions

We define a norm to a subfamily of hyperfunctions having bounded exponential growth with compact support. We have established some properties of this convergence using the defining function of such hyperfunctions.

We consider a sub family of hyperfunctions having bounded exponential growth with compact support on $I \subset \mathbb{R}$. We denote it by $\mathfrak{B}_B^K(I)$

Definition 6.2.1. For $f(x) = [F(z)] \in \mathfrak{B}_B^K(I)$ the function $||.||_K$ is defined as

$$||f||_K = \sup_{K \subset I}\{|G(x+i0) - G(x-i0)| : G(z) \in [F(z)], x \in K, K \textit{ is compact subset of } I\}$$

Proposition 6.2.2. *$\mathfrak{B}_B^K(I)$ is a normed linear space.*

Proof. Let $f(x) = [F(z)],\ h(x) = [H(z)] \in \mathfrak{B}_B^K(I)$

For $f(x) = [F(z)] \in \mathfrak{B}_B^K(I)$, $||f||_K \geqslant 0$

$$\begin{aligned} ||f||_K = 0 \quad &\Leftrightarrow \quad \sup_{K \subset I}\{|G(x+i0) - G(x-i0)| : G(z) \in [F(z)], x \in K, K \textit{is compact subset of} I\} = 0 \\ &\Leftrightarrow \quad |G(x+i0) - G(x-i0)| = 0, \forall G(z) \in [F(z)] \\ &\Leftrightarrow \quad G(x+i0) - G(x-i0) = 0, \forall G(z) \in [F(z)] \end{aligned}$$

$$\Leftrightarrow \quad G(z) is\ a\ real\ analytic\ function, \forall G(z) \in [F(z)]$$

$$\Leftrightarrow \quad G(z) \in [0]$$

$$\Leftrightarrow \quad f(x) = [0]$$

Let c be a constant. Then

$||cf||_K$

$= \sup_{K \subset I}\{|cG(x+i0) - cG(x-i0)| : G(z) \in [F(z)], x \in K, K\ is\ compact\ subset\ of\ \ I\}$

$= |c| \sup_{K \subset I}\{|G(x+i0) - G(x-i0)| : G(z) \in [F(z)], x \in K, K\ is\ compact\ subset\ of\ I\}$

$= |c|.||f||_K$

Also for all $F(z) \in [F(z)]$ and $G(z) \in [G(z)]$,

$$|F(x+i0)-F(x-i0)+H(x+i0)-H(x-i0)| \leqslant |F(x+i0)-F(x-i0)|+|H(x+i0)-H(x-i0)|$$

Hence we have

$$||f+h||_K \leqslant ||f||_K + ||h||_K$$

Thus $||.||_K$ is a norm on $\mathfrak{B}_B^K(I)$. □

Definition 6.2.3. A sequence $f_n(x) = [F_n(x)]$ is a Cauchy sequence in $\mathfrak{B}_B^K(I)$ if $\forall\ \epsilon > 0$ there exists $n_0 \in \mathbb{N}$ satisfying the condition $||f_n - f_m||_K < \epsilon$ for $n, m \geqslant n_0$.

Proposition 6.2.4. *$\mathfrak{B}_B^K(I)$ is a Banach space*

Proof. Let sequence $(f_n(x))$ is a Cauchy sequence in $\mathfrak{B}_B^K(I)$ where $f_n(x) = [F_n(z)]$.

Let $\epsilon > 0$.

There is $n_0 \in \mathbb{N}$ satisfying $||f_n - f_m||_K < \epsilon$, for $n, m \geqslant n_0$.

Then

$$
\begin{aligned}
|F_n(x+i0) - F_n(x-i0) - F_m(x+i0) + F_m(x-i0)| &\leqslant ||f_n - f_m||_K \\
&< \epsilon, \forall n, m \geqslant n_0, \textit{ and for all} \\
&\quad F_n(z) \in [F_n(z)] and F_m(z) \in [F_m(z)]
\end{aligned}
$$

From this relation we have $(F_n(x+i0))$ and $(F_n(x-i0))$ are Cauchy sequences in $\mathbb{C}$. But $\mathbb{C}$ is complete. Hence we get $F_n(x+i0) \to F(x+i0)$ and $F_n(x-i0) \to F(x-i0)$ in $\mathbb{C}$.

Define $f(x) = [F(z)]$

Then $f_n(x) \to f(x)$ as $n \to \infty$ in the sense of hyperfunction.

Hence $\mathfrak{B}_B^K(I)$ is a Banach space. □

Proposition 6.2.5. *$\mathfrak{B}_B^K(I)$ is separable.*

Proof. Set of all hyperfunction in $\mathfrak{B}_B^K(I)$ taking rational points values will be a countable dense subset of $\mathfrak{B}_B^K(I)$ □

Proposition 6.2.6. *Let $f(x) = [F(z)]$, $g(x) = [G(z)] \in \mathfrak{B}_B^K(I)$ are non negative, real valued measurable hyperfunctions and if $f(x) \leqslant g(x)$ i.e. $F(z) \leqslant G(z)$ and it holds for every functions in the equivalence classes of $F(z)$ and $G(z)$) then*

$$
\int f(x)dx \leqslant \int g(x)dx
$$

Proof.

$$
\int f(x)dx = \int F(z)dz
$$

$$\begin{aligned} &\leqslant \int G(z)dz \\ &= \int g(x)dx \end{aligned}$$

□

Proposition 6.2.7. *Let* $f_k(x) = [F_k(z)], k = 1, 2, 3, ...$ *be a sequence of measurable hyperfunctions in* $\mathfrak{B}_B^K(I)$ *and* $f(x) = \lim f_k(x), f(x) = [F(z)]$*. Then*

$$\int ||f_k(x) - f(x)||_K dx \to 0 \;\; as \;\; k \to \infty$$

in the sense of hyperfunctions.

Proof. Let $f_k(x) = [F_k(z)], k = 1, 2, 3, ...$ be a sequence functions in $\mathfrak{B}_B^K(I)$ and $f(x) = \lim f_k(x),\ f(x) = [F(z)]$.

Then $||f_k(x) - f(x)||_K \to 0$ as $k \to \infty$

Applying the properties of integrals of measurable functions we get the result. □

6.3. Completely Monotone Hyperfunctions

Definition 6.3.1. A positive real valued hyperfunction $f(x) = [F(Z)]$ defined on $(0, \infty)$ is called a completely monotone hyperfunction if it satisfies

$$(-1)^n f^{(n)}(x) \geqslant 0, \forall x > 0, n = 0, 1, 2, ...$$

Proposition 6.3.2. *A positive real valued hyperfunction* $f(x) = [F(z)]$ *defined on* $(0, \infty)$ *is a completely monotone hyperfunction if there exists a positive valued hyper-*

function $g(x) = [G(z)]$ on $(0,\infty)$ with bounded exponential growth such that

$$f(s) = \mathcal{L}[g(x)](s), \forall s > 0$$

Proof. Suppose there exists a positive valued hyperfunction $g(x) = [G(z)]$ on $(0,\infty)$ with bounded exponential growth such that $f(s) = \mathcal{L}[g(x)](s), \forall s > 0$. Then

$$\begin{aligned}(-1)^n f^{(n)}(s) &= (-1)^n \frac{d^n}{ds^n}\mathcal{L}[g(x)](s) \\ &= \int_0^\infty x^n e^{-sx} g(x)dx \\ &\geqslant 0, \forall s > 0\end{aligned}$$

Hence $f(x)$ is a completely monotone hyperfunction. □

Proposition 6.3.3. *Let $f(x) = [F(x)]$ and $g(x) = [G(x)]$ be two completely monotone hyperfunctions. Then $f(x)g(x)$ is a completely monotone hyperfunction whenever the product is defined and $f(s) = \mathcal{L}[h(x)](s)$ and $g(x) = \mathcal{L}[j(x)](s)$, where $h(x) = [H(z)]$ and $j(x) = [J(z)]$ are two hyperfunctions, $s > 0$*

Proof. Product of $f(x)$ and $g(x)$ is defined when sing supp $f(x) \cap$ sing supp $g(x) = \phi$ If s is not an element in sing supp $f(x) \cap$ sing supp $g(x)$ then

$$\begin{aligned}f(s)g(s) &= \mathcal{L}[h(x)](s)\mathcal{L}[j(x)](s) \\ &= \mathcal{L}[h(x) * j(x)](s)\end{aligned}$$

Then by previous proposition the result follows. □

Proposition 6.3.4. *Let $f(x)$ be a completely monotone hyperfunction and $g(x)$ be a positive valued hyperfunction defined on $(0,\infty)$ such that $g\prime(x)$ is a completely mono-*

tone hyperfunction. Then $f \circ g$ is also a completely monotone hyperfunction.

Proof. We are going to prove the result using mathematical induction

Clearly $(f \circ g)(x) \geqslant 0$ for $x > 0$

$$(f \circ g)' = (f' \circ g)g' \leqslant 0, \ \ since \ \ (-1)f' \geqslant 0 \ \ and \ \ g' \geqslant 0$$

Hence the result true for $k = 1$

Suppose that the result is true for $k = n$

i.e. $(-1)^k (f \circ g)^k \geqslant 0$ for all $k = 0, 1, 2, ..., n$

Since $-f'$ and g' are completely monotone

$$\begin{aligned} (-1)^{n+1}(f \circ g)^{n+1} &= (-1)^n [((-f') \circ g)g']^{(n)} \\ &= (-1)^n \Sigma_0^n nc_k ((-f') \circ g)^{(k)} (g')^{(n-k)} \\ &= \Sigma_0^n nc_k [(-1)^k ((-f') \circ g)^{(k)}][(-1)^{n-k} (g')^{(n-k)}] \\ &\geqslant 0 \end{aligned}$$

Hence the result is true for $k = n + 1$ also. □

Chapter 7

Application

In this chapter we find solutions of partial differential equations involving hyperfunctions using the Weierstrass transforms defined in the previous chapter.

7.1. Application of Weierstrass Transform of Hyperfunction

Weierstrass transform can be used to find the solution of partial differential equation problems having hyperfunction solution and also for initial value problems having initial value a hyperfunction.

We can see from the following problem how Weierstrass transform can be used to find the solution of initial value problems having initial value a hyperfunction.

Examples 7.1.1. *Consider the equation* $(D_x^2 - D_t)\psi(x,t) = 0, -\infty < x < \infty$,
$t > 0$ *with initial condition* $\psi(x,0) = f(x) + c$ *where the initial value function* $f(x)$ *is a hyperfunction.*

Solution:

Let the Fourier transform with respect to x of ψ and f denoted by

$$\hat{\psi}(\zeta,t) = \mathfrak{F}[\psi(x,t)](\zeta),\ \hat{f}(\zeta) = \mathfrak{F}[f(x)](\zeta)$$

respectively.

Applying Fourier transform with respect to x to the given differential equation we have

$$(i\zeta)^2\hat{\psi} - D_t\hat{\psi} = 0$$

$$i.e.(D_t + \zeta^2)\hat{\psi} = 0$$

Then the general solution is

$$\hat{\psi}(\zeta,t) = \phi(\zeta)e^{-t\zeta^2}$$

Applying the initial condition $\hat{\psi}(\zeta,0) = \phi(\zeta) = \hat{f}(\zeta)$

Hence

$$\begin{aligned}
\hat{\psi}(\zeta,t) &= \hat{f}(\zeta)e^{-t\zeta^2} \\
&= \hat{f}(\zeta)\mathfrak{F}[\frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}](\zeta) \\
&= \mathfrak{F}[f(x)](\zeta).\mathfrak{F}[\frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}](\zeta) \\
&= \mathfrak{F}[f(x) * \frac{1}{\sqrt{4\pi t}}e^{\frac{-x^2}{4t}}](\zeta)
\end{aligned}$$

by convolution property.

Taking inverse Fourier transform on both sides

$$\begin{aligned}\psi(x,t) &= f(x) * \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} f(x)dx \\ &= \mathfrak{W}_{\mathfrak{t}}[f(x)](s)\end{aligned}$$

If

$$\begin{aligned}\psi(x,t) &= \mathfrak{W}_{\mathfrak{t}}[\delta(x-1)](s) \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(s-x)^2}{4t}} \delta(x-1)dx \\ &= \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} * \delta(x-1) \\ &= \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-1)^2}{4t}}\end{aligned}$$

Also for $x \neq 1, \psi(x,0) = 0 = \delta(x-1)$

Hence $f(x) = \delta(x-1)$ *is a hyperfunction satisfying* $\psi(x,0) = 0 = f(x)$. *Thus the solution is*

$$\psi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-(x-1)^2}{4t}}, -\infty < x < \infty, t > 0$$

Conclusion

The concept of hyperfunction is the contribution by the Japanese mathematician Mikio Sato. Hyperfunctions are the generalisation of generalized functions or distributions. The space of distributions forms a subspace of the linear space of hyperfunctions. The mathematical impossibility of explaining the concept of integration of certain generalized functions can be overcome through the introduction of hyperfunctions with the help of classical complex analysis.

Sato focused more on developing the theoretical concept of hyperfunctions. But Iso Imai discovered the computational power of hyperfunctions in solving partial differential equations. Inspired by Iso Imai's work, Urs Graf in his book 'Introduction to hyperfunctions and their integral transforms', extended different integral transforms like Laplace transform, Fourier transform, Mellin transform, Hilbert transforms and Hankel transforms to the linear space of hyperfunctions. He investigated on the practical approach than the theoretical concepts of hyperfunctions. The work in this thesis contains application of more different and combined integral transforms on hyperfunctions and their properties.

Chapter 1, consists of some preliminary definitions.

In Chapter 2, new integral transforms are applied to hyperfunctions. Integral transforms such as Weierstrass transform, Stieltjes transform and $\mathfrak{L}_2$ are extended to the subclasses of hyperfunctions. The inverse transforms are also obtained. Various operational properties of these transforms are also studied.

In Chapter 3, the combined transforms for hyperfunctions are investigated. The concepts of Fourier-Laplace transform, Laplace-Stieltjes transform and Fourier-Stieltjes transform are defined for hyperfunctions satisfying certain properties. Operational properties of these transforms are also obtained. The inverse transforms for Laplace-Stieltjes transform and Fourier-Stieltjes transform of hyperfunctions have been established.

In Chapter 4, Abelian - Tauberian type theorems are proved for Laplace transform of hyperfunctions, Stieltjes transform of hyperfunctions and Laplace- Stieltjes transform of hyperfunctions. Some supporting results are also proved for proving these theorems.

In Chapter 5, the concept of two dimensional Laplace transform of hyperfunctions is introduced. The idea of hyperfunctions in two variables are developed as in the case of hyperfunctions of one variable introduced by Urs Graf. Two dimensional Laplace transform is defined for hyperfunctions in two variables having a separable defining function. Some operational properties of this two dimensional transforms are obtained. Also the two dimensional inverse Laplace transform of hyperfunctions is defined. Using this two dimensional Laplace transform of hyperfunctions second order partial differential equations involving hyperfunctions can be solved without reducing it into first order equations.

In Chapter 6, the order relation in the linear space of real hyperfunctions has been studied. Also the concept of multinorm and cone are defined for the linear space of hyperfunctions. It is proved that every order bounded subset of the linear space of hyperfunctions having bounded exponential growth and compact support is topologically bounded with the topology generated by the set of multinorms. Some properties of these concepts are proved. The concept of completely monotone hyperfunctions is also defined.

In Chapter 7, as an application, an initial value problem involving hyperfunction is solved by applying Weierstrass transform for hyperfunction.

Future Study

In this present study only some basic concepts of these newly defined integral transforms for hyperfunctions have been established. Since hyperfunctions are infinitely differentiable we can establish many more results based on these integral transforms.

The following investigations are possible.

- Theoretical concepts of the above transforms can be developed through the Sheaf theoretic approach to hyperfunctions
- Support Kernel type theorems can be proved for these integral transforms with the help of Sheaf theory
- Application of numerical methods to the integral transform of hyperfunctions to solve partial differential equations and integral equations
- The existence of Abelian - Tauberian theorems for other transforms of hyperfunctions can be investigated
- Connection between hyperfunctions and Colombeau's generalized functions
- Embedding the space of Fourier distribution in the space of Fourier hyperfunction

- Mellin-Stieltjes transform of hyperfunctions
- Wavelet transform of hyperfunctions
- Existence of transforms in hyperfunctions in several variables
- Application of hyperfunctions in signal and image processing

Research Papers

1. Deepthi A.N., Mangalambal N.R, *A Note on Weierstrass Transform of Hyperfunctions*, International Journal of Innovative Technology and Exploring Engineering (IJITEE) , Volume-8, Issue-7, May 2019, pp 1247-1252 , ISSN: 2278-3075.

2. Deepthi A.N., Mangalambal N.R, *Norm Convergence in the Space of Hyperfunctions*, Asian Journal of Engineering and Applied Technology, The Research Publication, Vol. 8, No. 2, 2019, pp. 19-22, ISSN 2249-068X.

3. Deepthi A.N., Mangalambal N.R, *Stieltjes Transformation of Hyperfunctions*, International Journal of Research and Analytical Reviews (IJRAR), March 2019, Volume 6, Issue 1, pp 995-999(E-ISSN 23481269, P- ISSN 2349-5138).

4. Deepthi A.N., Mangalambal N.R, *Abelian - Tauberian Theorem for Laplace Transform of Hyperfunctions*, Global Journal of Pure and Applied Mathematics, Volume 15, Number 4 (2019), Research India Publications, pp 323–334, ISSN 0973-1768.

5. Deepthi A.N., Mangalambal N.R, *Ordered Linear Space of Hyperfunctions*, Malaya Journal of Matematik, Vol. S, No. 1, 485-489, 2019,ISSN (Online) : 2321 - 5666

6. Deepthi A.N., Mangalambal N.R, $\mathfrak{L}_2$ *Transform of Hyperfunctions*, Communicated

7. Deepthi A.N., Mangalambal N.R, *Fourier-Laplace Transform of Hyperfunctions*, Communicated

8. Deepthi A.N., Mangalambal N.R, *Laplace-Stieltjes Transform of Hyperfunctions*, Communicated

9. Deepthi A.N., Mangalambal N.R, *Abelian -Tauberian Theorem for Stieltjes Transform of Hyperfunctions*, Communicated

10. Deepthi A.N., Mangalambal N.R, *Abelian -Tauberian Theorem for Laplace-Stieltjes Transform of Hyperfunctions*, Communicated

11. Deepthi A.N., Mangalambal N.R, *Completely Monotone Hyperfunctions*, Communicated

12. Deepthi A.N., Mangalambal N.R, *Two Dimensional Laplace Transform of Hyperfunctions*, Communicated

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