STUDIES ON VARIATIONS OF DOMINATION IN GRAPHS

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By

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Contents

CHAPTER 1

Introduction

1.1 History

In the year 1736, the brilliant Swiss mathematician Leonhard Euler introduced the idea of graph made up of a set of objects called vertices and another set of objects called edges, which are made up of pairs of vertices. In order to traverse the seven bridges over the Pregel River, a well-known folklore problem known as the Königsberg Bridge Problem, he developed the concept of graphs[16]. The concept of graphs was effectively applied in many domains, after Euler. Recent years have seen an unparalleled rise in graph theory research. Even though graph theory was initially associated with recreational math problems, it is usually applicable to many areas of mathematics, including algebra, algebraic topology, number theory, algebraic geometry, numerical analysis, matrix theory, operations science, etc. Additionally, it promoted the development of other scientific fields, such as the physical, chemical, computer, and life sciences, as well as sociology, economics, and social sciences, as well as geography, genetics, architecture, electrical engineering and other fields. Numerous studies are being conducted in the area of graph theory, particularly in the area of domination.

1.2 Basic definitions

This section handles the basic terminology relevant to the work about graphs. We start by definition of graphs.

A graph G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices called its end vertices[81]. A graph having finite number of vertices and edges is called finite graph [18]. The number of vertices and number of edges of a finite graph G are called the order and size of G respectively [81]. When v_i is an end vertex of some edge e_j, v_i and e_j are said to be incident with each other[50]. When u and v are end vertices of an edge, they are adjacent and are neighbours[81].

An edge having the same vertex as both its end vertices is called loop[50]. More than one edge associated with a given pair of vertices are called parallel edges[50]. Two nonparallel edges are said to be adjacent if they are incident on a common vertex[50]. A graph that has neither loops nor parallel edges is called a simple graph[50]. A graph of size zero is called an empty graph.

The number of edges incident on a vertex v_i is called the degree, $d(v_i)$ of vertex v_i[50]. The maximum and minimum degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ respectively[18]. A vertex having no incident edge is called an isolated vertex[50]. A vertex of degree one is called a pendant vertex[50]. A vertex adjacent to a pendant vertex is called a support vertex[9]. An edge incident with a pendant vertex is called a pendant edge.

The distance between two vertices u and v of a graph G written $d(u, v)$, is the shortest length of a $u - v$ path in G. If G has no such path, then $d(u, v) = \infty$. The eccentricity $e(v)$ of a vertex v is $max\{d(u, v) : u \in V(G)\}$. Maximum of the eccentricities of the vertices of G is called the diameter of G. The radius of G is the minimum of the eccentricities of its vertices, $radius(G) = min\{e(u) : u \in V(G)\}$. To denote the diameter and radius of a graph G, the abbreviations $diam(G)$ and $rad(G)$ are used respectively. The center of a graph G, $c(G)$ is the subgraph induced by the vertices of minimum eccentricity[81].

The complement $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$[18].

A graph H is a subgraph of G written $H \subset G$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A spanning subgraph of G is a subgraph H with $V(H) = V(G)$. If V^1 is a nonempty subset of $V(G)$. The subgraph of G whose vertex set is V^1 and whose edge set is the set of those edges of G that have both ends in V^1 is called the subgraph induced by V^1 and is denoted by $< V^1 >$[18].

A walk is defined as a finite sequence of vertices and edges, beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. Vertices with which a walk begins and ends are called its terminal vertices. A walk which begins and ends at the same vertex is called a closed walk. A walk that is not closed is called an open walk. An open walk in which no vertex appears more than once is called a path[50]. A path with u and v as terminal vertices is called a $u - v$ path. A closed walk in which no vertex, except the initial and final vertex, appears more than once is called a cycle[50]. A graph with no cycle is called acyclic.

A graph G is said to be connected if there is at least one path between every pair of vertices in G. A disconnected graph is a graph which is not connected[50]. A connected acyclic graph is called a tree. The Components of a graph G are its maximal connected subgraphs[81].

A vertex cut of G is a subset V^1 of $V(G)$ such that $G - V^1$ is disconnected. k-vertex cut is a vertex cut of k-elements. The connectivity $\kappa(G)$ of G is the minimum k for which G has a k-vertex cut. A graph G is said to be k-connected if $\kappa(G) \geq k$.[18]

If a graph can be disconnected by the deletion of one vertex, that vertex is called cut vertex. A connected graph that has no cut vertices is called a block. A block of a graph is a subgraph that is a block and is maximal with respect to this property[18].

The path with n vertices is denoted by P_n[81]. The cycle with n vertices is denoted by C_n[81].

A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that any edge of G has one end vertex in X and the other end in Y; such (X, Y) is called a bipartition of the graph G. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y. If $|X| = m$ and $|Y| = n$ it is denoted by $K_{m,n}$[18].

A complete graph is a simple graph whose vertices are pairwise adjacent. The complete graph with n vertices is denoted by K_n[81].

The open neighborhood of v denoted by $N(v)$ is the set of vertices adjacent to v, $N(v) = \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of v denoted by $N[v]$ is the set $N(v) \cup \{v\}$[69]. For a set $S \subset V$ of vertices in a graph $G = (V, E)$ and $u \in S$, v is said to be a private neighbour of u if (with respect to S) $N[v] \cap S = \{u\}$[69].

Coloring all the vertices of a graph with colors so that no two adjacent vertices have the same color is called the proper coloring. The chromatic number of a graph G, written $\chi(G)$, is the minimum number of colors needed for proper coloring of graph G[50].

A subset M of $E(G)$ is called a matching in G if no two elements of M are adjacent in G. A matching M saturates a vertex v, and v is said to be M-saturated, if some edge of M is incident with v. If every vertex of G is M-saturated, then the matching M is perfect[18].

A property P of sets of vertices is said to be hereditary if whenever a set has property P, so does every subset $S^1 \subset S$. A property P of sets of vertices is said to be superhereditary if whenever a set has property P, so does every superset $S^1 \supset S$[69].

Investigating various graph parameters in various classes of graphs is a challenging problem. Various classes of graphs were constructed from standard graphs. Some of them given below.

From a simple graph G, Mycielski's construction produces a simple graph G^1 containing G. Beginning with G having vertex set $\{v_1, v_2, ..., v_n\}$, add vertices $U = \{u_1, u_2, ..., u_n\}$ and one more vertex w. Add edges to make u_i adjacent to all of neighbours of v_i in G, and finally let $N(w) = U$[81].

A subdivision of an edge uv is obtained by removing edge uv, adding a new vertex w, and adding edges uw and wv[69]. A wounded spider is the graph formed by subdividing at most $n-2$ of edges of a star $K_{1,n-1}$ for $n-1 \geq 0$[69].

Given a graph G, Trestled graph of index k denoted by $T_k(G)$, is the graph obtained from G by adding k copies K_2 for each edge uv of G and joining u and v to the respective end vertices of each K_2[69].

Let G and H be two graphs with disjoint vertex sets. Their union $G \cup H$ has $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Then their join is denoted $G + H$ and consists of $G \cup H$ and all edges joining $V(G)$ and $V(H)$[45]. The wheel W_n is defined to be the join of $K_1 + C_n$. The vertex corresponding to K_1 is known as apex and vertices corresponding to cycle are known as rim vertices while the edges corresponding to cycle are known as rim edges[72]. The helm H_n is the graph obtained from wheel W_n by attaching a pendant edge to each rim vertex[72].

The friendship graph F_n can be constructed by joining n copies of the cycle graph C_3 with a common vertex, which becomes a universal vertex for the graph.[66]

There are standard graph products, each with its own applications and theoretical interpretations.

The Cartesian product of G and H is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$. Two vertices (u_1, v_1) and (u_2, v_2) are adjacent precisely if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. Thus, $V(G\Box H) = \{(u,v) : u \in V(G)$ and $v \in V(H)\}$, $E(G\Box H) = \{(u_1,v_1)(u_2,v_2) : u_1 = u_2, v_1v_2 \in E(H)$ or $u_1u_2 \in E(G), v_1 = v_2\}$[62].

The strong product of G and H is the graph denoted as $G\boxtimes H$, and defined by[62]

$$\begin{aligned} V(G\boxtimes H) &= \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}, \\ E(G\boxtimes H) &= \{(u_1,v_1)(u_2,v_2) : u_1 = u_2, v_1v_2 \in E(H) \text{ or } u_1u_2 \in E(G), v_1 = v_2\}\cup \\ &\quad \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G) \text{ and } v_1v_2 \in E(H)\} \end{aligned}$$

The composition $G[H]$ is defined as [45],

$$\begin{aligned} V(G[H]) &= \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}, \\ E(G[H]) &= \{(u_1,v_1)(u_2,v_2) : u_1u_2 \in E(G) \text{ or } u_1 = u_2, v_1v_2 \in E(H)\} \end{aligned}.$$

Corona of graphs is defined by Frucht and Harary[45]. Later the concept of neighbourhood corona and edge corona of graphs were developed. Corona $G \circ H$, of two graphs G and H is obtained by taking one copy of G (which has n_1 vertices) and n_1 copies of H (which has n_2 vertices), and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H[45].

Let G and H be two graphs on n_1 and n_2 vertices, m_1 and m_2 edges respectively. Then the neighborhood corona, $G \star H$ is the graph obtained by taking n_1 copies of H and for each i, making all vertices in the i^{th} copy of H adjacent with the neighbors of v_i, $i = 1, 2, ..., n_1$[40].

Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, m_1 and m_2 edges, respectively. The edge corona $G \diamond H$ of G and H is defined as the graph obtained by taking one copy of G and m_1 copies of H, and then joining two end-vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H[80].

So many graph parameters like Vertex connectivity, matching number, chromatic number, and independence number are developed to meet real life applications. Concept of domination is one among them in which extensive research work is going on. In the book, domination in Graphs ; Advanced topics[70], surveys of recent developments are provided.

A set $S \subset V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of S or is adjacent to an element of S[69]. The cardinality of a minimum dominating set of graph G is called domination number and it is denoted by $\gamma(G)$[69].

The set of all minimal dominating sets of a graph G is denoted by $MDS(G)$. Upper domination number $\Gamma(G)$ is the maximum cardinality of a set in $MDS(G)$[69].

Cockayne introduced the concept of irredundance in [29].

For a subset $S \subset V(G)$ and $v \in S$, private neighbour of v in S, $pn[v, S]$ is defined as $N[v] - N[S - \{v\}]$. A set S is irredundant if for every $v \in S$, $pn[v, S] \neq \phi$. The minimum cardinality of a maximal irredundant set of a graph G is called the irredundance number and it is denoted by $ir(G)$. The maximum cardinality of an irredundant set is called the upper irredundance number and denoted by $IR(G)$[69].

An independent set in a graph is a set of pairwise nonadjacent vertices.[81].The independence number, $\beta_\circ(G)$ is the maximum cardinality of an independent set in G[69].

The minimum cardinality of an independent dominating set is called the independent domination number and it is denoted by $i(G)$[69].

A dominating set S in a graph G is said to be a perfect dominating set if for every vertex $u \in V(G) - S$, $|N[u] \cap S| = 1$. The cardinality of a minimum perfect dominating set is called perfect domination number and it is denoted by $\gamma_p(G)$[69].

A chain named domination chain which is obtained by connecting a chain of inequalities is given in [29].

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_o(G) \leq \Gamma(G) \leq IR(G)$$

This inequality chain is one of the most powerful focuses of research of domination.

So many bounds were found for domination number. Bound using the degree and order is a significant result among them. The upperbound was obtained by berge and lowerbound was obtained by Walikar, Acharya and Sampathkumar.[77, 12]

For any graph G of order n, $\lceil \frac{n}{1+\Delta(G)} \rceil \leq \gamma(G) \leq n - \Delta(G)$.

It is proved in [12] that every maximal independent set is a minimal dominating set. Since then comparing various domination parameters is a focal point in domination theory.

Nordaus and Gaddum established the following inequalities for chromatic number $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$ and $n\sqrt{n} \leq \chi(G)\chi(\overline{G}) \leq \frac{(n+1)^2}{n}$. Results related to sum (product) of a parameter of a graph G and the same parameter of its complement $\overline{G}$ is known as Nordaus-Gaddum type results[69].

The concept of P_3-convexity, which is more similar to domination, is also studied in this thesis.

A family C of subsets of a nonempty set X is called a convexity on X if

- $\phi \in C, X \in C$
- C is stable for intersections, and
- C is stable for nested unions

(X, C) is called a convexity space and members of C are called convex sets[53].

Here union of nested family of sets is nested union.

If (X, C) is a convexity space, then for a set $S \subset X$ convex hull of S, $H_G(S)$ is the smallest convex set containing S.

For a graph G, Given two vertices $v_1, v_2 \in V(G)$, the P_3- interval

$$\begin{aligned} I[\{v_1, v_2\}] &= \{v \in V : v \text{ adjacent to both } v_1 \text{ and } v_2\} \\ &\quad \cup\{v_1, v_2\}. \\ I[S] &= \cup_{v_1, v_2 \in S} I[v_1, v_2] \end{aligned}$$

S is a P_3- convex set, if $I[S] = S$.

A graph G together with P_3-convex sets in G form P_3-convexity C in G [36].

The P_3- convex hull can be formed from a sequence $I^p[S]$, where p is a non-negative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$, for every $p \geq 2$. Let $p \in N \cup \{0\}$ for which $I^p[S] = I^{p+1}[S]$, for all , then $I^p[S]$, is a convex set and $S \subset I^p[S]$. Hence $I^p[S]$ is the convex hull of S[36].

If $H_G(S) = V(G)$ then S is a P_3-hull set of G. The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-hull number of G[36]. In this thesis we are using hull number of G instead of P_3-hull number of G.

Convexity C on G is joint hull commutative provided that for each nonempty convex set S in C and for each vertex $p \in V(G)$, $H_G(S \cup \{p\}) = \cup_{u \in S} H_G(\{p, u\})$[74].

The caratheodory number is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_G(S)$, there is a set $F \subset S$ with $|F| \leq c$ and $u \in H_G(F)$[35]. In this thesis we are using $c(G)$ instead of c. A Radon partition of R is a partition of R into two disjoint sets R_1 and R_2 with $H_G(R_1) \cap H_G(R_2) \neq \phi$. The Radon number $r(G)$ of G is the minimum integer r such that every set of r vertices of G has a Radon partition [48].

The following significant results are given in [36].

- Let G and H be nontrivial graphs. If G is connected, then $h(G[H]) = 2$.
- $h(G \square H) \geq max\{h(G), h(H)\}$.
- Let G and H nontrivial connected graphs. Then, $h(G \boxtimes H) = 2$.

Caratheodory number and radon number in P_3-convexity were studied in [48, 35]

1.3 Background of the work

A well-researched graph parameter, domination is a classic subject in graph theory. According to the rules of chess, the queen is the player's most powerful piece because of its freedom or mobility. Since the queen is free to travel across any number of squares in any direction, the opposition player is constantly watchful about the moves in the directions dominated by the queen. The study of domination in graphs started by De Jaenisch in 1850 with a problem of finding the minimum number of queens that are needed to place on a Chess board such that each field not occupied by queen can be attacked by at least one[31]. It was proved in 1850's that, in an 8×8 chessboard five queens are required to completely dominate all of the squares[69]. Watkins provides a very thorough overview of the evolution and ongoing growth of this promising field of domination theory from the chessboard puzzles in [79]. The study of domination in graphs was further studied in the late 1950s and early 1960s. In his book [12]on graph theory, Berge introduced the concept coefficient of external stability, which is today known as the domination number. The phrases dominating set and domination number were first used by Oystein Ore [52] in his book on graph theory in 1962. In a survey published by Cockayne and Hedetniemi[28], the notation $\gamma(G)$ for the domination number of a graph G was first used.

Ore presented the first three theorems of dominating sets in his book "Theory of Graphs"[52].

- A dominating set S is a minimal dominating set if and only if, for each $u \in S$, one of the following condition holds
  - u is an isolate of S.
  - there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$
- Every connected graph G of order $n \geq 2$ has a dominating set S whose compliment $V - S$ is also a dominating set.
- If a graph G is a graph with no isolated vertices, then the compliment $V - S$ of every minimal dominating set S is also a dominating set.

Ore gave an upperbound for domination number $\gamma(G)$. Cockayne, Haynes and Hedetniemi characterized the graph forwhich $\gamma(G) = \lceil \frac{n}{2} \rceil$[27] . Later the bounds for $\gamma(G)$ in terms of degree and order are obtained[12, 67, 77]. Refining the bound, a domination chain $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_o(G) \leq \Gamma(G) \leq IR(G)$ was given in [69].

Numerous applications in real life situations have led to the introduction of numerous domination parameters. Total domination number, Roman domination number and fractional domination number are some of them. A beautiful survey on topic of domination is given in book Fundamentals of Domination in Graphs done by Teresa W. Haynes, Stephen T. Hedetniemi and Peter J. Slater[69]. If we can extend the domination chain using a new domination parameter that will be a a significant contribution.

One of the famous open problems in Mathematics is Vizing's conjecture. Vizing's conjecture proposed by V. G. Vizing in 1968 leads to the development of studies in domination number of cartesian product of graphs[76]. Bound for the domination number of grid graph is obtained by Jacobson and Kinch[46]. The domination number of cartesian product of specific graphs is an area in which developments are being made. Thus we are motivated to study domination number of cartesian product of some specific graphs.

In some reservation systems, reservations from particular special classes are required. We defined the Reserved Domination Number of Graph in light of the same motivation. Domination number of cartesian product of graphs is studied using reserved domination number.

We introduced a particular domination known as α-stable domination for the purpose of exploring the area of instability of domination in graphs.

In order to solve the problems like spreading a virus, disseminating ideas, and marketing strategies, we studied P_3-convexity which is more similar to domination in graphs. The concept of P_3-convexity was initially investigated, for directed graphs [34, 49]. Later P_3-convexity was studied in undirected graphs[24, 25, 26, 33, 11]. Radon number on P_3-convexity was well studied by Mitre C. Dourado, Dieter Rautenbach, Vinicius Fernandes dos Santos, Philipp M. Schaffer, Jayme L. Szwarcfiter and Alexandre Tomana in 2012[48]. P_3-convexity in different product of graphs was well studied by Erika M.M. Coelho, Hebert Coelho, Julliano R. Nascimento and Jayme L. Szwarcfiter in 2018[36], and caratheodory number in

P_3-convexity was well studied by Erika M.M. Coelho, Mitre C. Dourado, Dieter Rautenbach and Jayme L. Szwarcfiter in 2014[35]. Here we study some properties of P_3-convexity. Radon number, caratheodory number and hull number of product graphs is also studied.

1.4 Organization of the Thesis

This thesis is arranged in six chapters. The following provides a chapter-by-chapter summary of the thesis.

The first chapter is an introductory chapter. A brief introduction, thesis summary and preliminary concepts are discussed in this chapter. The terminologies and notations used in the next chapters are described in the section preliminaries.

In the second chapter, the lower bound for domination number of cartesian product of G and P_n is obtained. It is studied for the cartesian product of G and C_n. Upperbound for $\gamma(G\Box H)$, when $H = P_n$ or $H = C_n$ for which G has a minimum dominating set D such that $D = D_1 \cup D_2, D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, is obtained. These results are published in [59]

A- Reserved domination number of graph G, which would be useful in reservation systems is defined in the third chapter. A- Reserved domination number of various product graphs are studied. Generalized concept of A- Reserved domination number, $[A_1, A_2, ..., A_n]$-reserved domination number is defined and studied. The reserved domination number is used to study the domination number of cartesian product of graphs. Some of its results are published in [60]

α-d- stable domination number, α-a- stable domination number and α-stable domination, are introduced in fourth chapter with the aim of studying the stability of domination in graphs. Its properties are also studied. α-stable domination in product graphs are studied.

In fifth chapter, one section deals with general properties in P_3-convexity. P_3-convex invariants, P_3-hull number, radon number and caratheodory number of some classes of graphs are obtained. P_3-convexity in strong product, cartesian product and composition of graphs are studied. P_3-convexity in corona related graphs are also studied. Some of its results are published in [61].

Sixth chapter is a concluding chapter, consisting of summary and scope for further studies.

CHAPTER 2

Bounds of Domination Number of Cartesian Product of Graphs

2.1 Introduction

How a graph works on graph products is an important problem that is being discussed in detail. It is logical to assume that the value of the invariant on the product of two graphs G and H will relate to the value of G and H. This type of problems are well studied and simple for some invariants and products. The chromatic number of the Cartesian product of two graphs is maximum of their chromatic numbers is an illustration of this circumstance. There are still a number of invariants which has a conjectured behaviour, but not yet been determined. The domination number on a Cartesian product is an example of same. V. G. Vizing proposed the following conjecture in 1968 after posing it as a question in [76] $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.

In this chapter bounds for domination number of cartesian product of certain types of graphs and path are obtained. Bounds of domination number of cartesian product of some classes of graphs and cycle are also obtained. Examples that tighten the upper bound are obtained.

2.2 Bounds of domination number of cartesian product of a graph and path

Bounds of domination number of $G \Box P_n$, for which G satisfy certain conditions, is obtained in this section.

Theorem 2.2.1. *Let G be any graph and P_n be a path having n vertices. Then $\frac{n}{3}\gamma(G) < \gamma(G \Box P_n)$.*

Proof. Let P_n be the path with $V(P_n) = \{v_1, v_2, ..., v_n\}$.
Let D be a dominating set of $G \Box P_n$. Then, let $D_i = D \cap (V(G) \times \{v_i\})$.
Take $\pi_1(D_i) = \{u \in V(G) : (u, v_i) \in D_i\}$.
If $u \in V(G) - (\pi_1(D_1) \cup \pi_1(D_2))$ then $(u, v_1) \notin D$ and $(u, v_2) \notin D$. Since D is a dominating set (u, v_1) must be adjacent to a vertex $(w, v_1) \in D$. Then $w \in \pi_1(D_1)$. Hence u has a neighbour, $w \in \pi_1(D_1)$. Thus $\pi_1(D_1) \cup \pi_1(D_2)$ is a dominating set of G. Thus $\gamma(G) \leq |\pi_1(D_1) \cup \pi_1(D_2)|$.
Hence $\gamma(G) \leq |D_1 \cup D_2|$.

If $u \in V(G) - (\pi_1(D_{n-1}) \cup \pi_1(D_n))$ then $(u, v_{n-1}) \notin D$ and $(u, v_n) \notin D$. Since D is a dominating set (u, v_n) must be adjacent to a vertex $(w, v_n) \in D$. Then $w \in \pi_1(D_n)$. Hence u has a neighbour, $w \in \pi_1(D_n)$. Thus $\pi_1(D_{n-1}) \cup \pi_1(D_n)$ is a dominating set of G. Thus $\gamma(G) \leq |\pi_1(D_{n-1}) \cup \pi_1(D_n)|$.
Hence, $\gamma(G) \leq |D_{n-1} \cup D_n|$.

If $i \in 2, 3, ..., n-1$ and $u \in V(G) - (\pi_1(D_{i-1}) \cup \pi_1(D_i) \cup \pi_1(D_{i+1}))$, then $(u, v_{i-1}) \notin D$, $(u, v_i) \notin D$ and $(u, v_{i+1}) \notin D$. Then there is a vertex$(w, v_i) \in D$ such that (w, v_i) is adjacent to (u, v_i). Hence $w \in \pi_1(D_i)$ and w is adjacent to u in G. Thus $\pi_1(D_{i-1}) \cup \pi_1(D_i) \cup \pi_1(D_{i+1})$ is a dominating set of G.
Hence $\gamma(G) \leq |D_{i-1} \cup D_i \cup D_{i+1}|$.
Thus we can see that

$$\begin{aligned}
\gamma(G) &\leq |D_1 \cup D_2| \\
\gamma(G) &\leq |D_{n-1} \cup D_n| \\
\gamma(G) &\leq |D_{i-1} \cup D_i \cup D_{i+1}|, \forall i = 2, ..., n-1
\end{aligned}$$

Summing up all these inequalities we can see that

$n\gamma(G) \leq 2|D_1| + 2|D_n| + 3\sum_{i=2}^{n-1} D_i = 2|D| + |D_2| + |D_3| + ... + |D_{n-1}|$.

Hence, $n\gamma(G) + |D_1| + |D_n| \leq 3|D|$.

Then two cases arise.

Case 1 $|D_1| \neq 0$ or $|D_n| \neq 0$

$n\gamma(G) < 3|D|$

Case 2 $|D_1| = 0$

Then $D_2 = V(G)$ and since $\gamma(G) \leq \frac{|V(G)|}{2}$

$2\gamma(G) \leq |D_1 \cup D_2|$

Then $D_2 \cup D_3 = V(G)$ and since $\gamma(G) \leq \frac{|V(G)|}{2}$

$$\begin{aligned} 2\gamma(G) &\leq |D_1 \cup D_2 \cup D_3| \\ \gamma(G) &\leq |D_{n-1} \cup D_n| \\ \gamma(G) &\leq |D_{i-1} \cup D_i \cup D_{i+1}|, \forall i = 2, ..., n-1 \end{aligned}$$

Summing up all these inequalities we can see that

$n\gamma(G) + 3\gamma(G) \leq 3\sum_{i=1}^{n} D_i$

Hence, $|D| \geq \frac{n}{3}\gamma(G) + \gamma(G)$.

Thus $|D| > \frac{n}{3}\gamma(G)$.

Thus if D is a dominating set of $G\Box P_n$ then $|D| > \frac{n}{3}\gamma(G)$. Hence, $\gamma(G\Box P_n) > \frac{n}{3}\gamma(G)$
. □

Theorem 2.2.2. *Let G be any graph and P_n be a path having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, and if n is even then $\gamma(G\Box P_n) \leq \frac{n}{2}\gamma(G)$.*

Proof. Let D be a minimum dominating set of G such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2 and $V(P_n) = \{v_1, v_2, ..., v_n\}$, n is even.

Then $D^1 = (D_1 \times \{v_1\}) \cup (D_2 \times \{v_2\}) \cup (D_1 \times \{v_3\}) \cup (D_2 \times \{v_4\}) ..... \cup (D_2 \times \{v_n\})$

is a dominating set of $G\Box P_n$.

For, if $(u, v_i) \in G\Box P_n$ and i is odd. Then, there are three possibilities.

Case 1 If $u \in D_1$, then $(u, v_i) \in D^1$

Case 2 If $u \in D_2$, then (u, v_i) is adjacent to $(u, v_{i+1}) \in D^1$

Case 3 If $u \notin D_1 \cup D_2$, then u is adjacent to $w_1 \in D_1$. Therefore (u, v_i) is adjacent to $(w_1, v_i) \in D^1$ in $G\Box P_n$.

Thus D^1 can dominate (u, v_i).

If $(u, v_i) \in V(G\Box P_n)$ and i is even. Then, there are three possibilities.

Case 1 If $u \in D_2$, then $(u, v_i) \in D^1$

Case 2 If $u \in D_1$, then (u, v_i) is adjacent to $(u, v_{i-1}) \in D^1$

Case 3 If $u \notin D_1 \cup D_2$, then u is adjacent to $w_2 \in D_2$. Thus (u, v_i) is adjacent to $(w_2, v_i) \in D^1$ in $G\Box P_n$.

Thus D^1 can dominate (u, v_i).

Therefore D^1 is a dominating set of $G\Box P_n$.

Also

$$\begin{aligned}|D^1| &= |(D_1 \times \{v_1\}) \cup (D_2 \times \{v_2\}) \cup .... \cup (D_n \times \{v_n\})| \\ &= \underbrace{|D_1 \cup D_2| + ...|D_1 \cup D_2|}_{\frac{n}{2} times} \\ &= \frac{n}{2}\gamma(G)\end{aligned}$$

Hence, $\gamma(G\Box P_n) \leq \frac{n}{2}\gamma(G)$.

□

Corollary 2.2.3. *Let G be any graph and P_n be a path having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2, D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, and if n is even then* $\frac{n}{3}\gamma(G) < \gamma(G\Box P_n) \leq \frac{n}{2}\gamma(G)$

Remark 2.2.1. *There are graphs G which attains sharp upper bound in 2.2.2. For the graph G in 2.1, $\gamma(G\Box P_n) = \frac{n}{2}\gamma(G)$.*

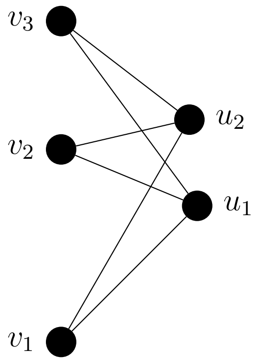

Figure 2.1

The sequential join $G_1 + G_2 + ... + G_n$ is obtained by joining all vertices of G_i to all the vertices of G_{i+1} for $i = 1, 2, ..., (n-1)$[53].

Definition 2.2.4. *Define Wave graph,*
$W(n,m) = K_1 + \overline{K_m} + K_1 + K_1 + \underbrace{K_1 + \overline{K_{m-1}} + K_1 + \overline{K_{m-1}} + ...}_{(n-2) times} + K_1 + K_1 + K_1 + \overline{K_m} + K_1$ *for* $n \geq 3$, $m \geq 2$. *Then* $G = W(4,3)$ *is the graph described in the figure* 2.2

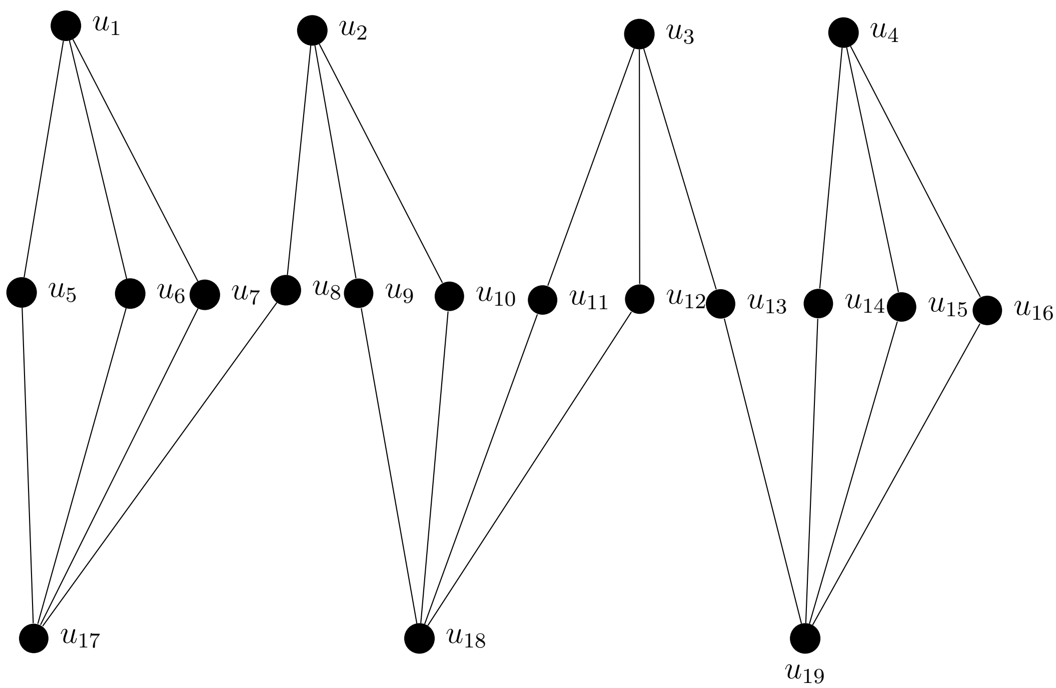

Figure 2.2

Remark 2.2.2. *Strict inequality may occur in upper bound of* 2.2.2. *For the graph*

$G = W(4,3)$, $\gamma(G\Box P_6) < 3\gamma(G)$.

Proof. Here $D = \{u_1, u_2, u_3, u_4, u_{17}, u_{18}, u_{19}\}$ is a minimum dominating set of G. If $D_1 = \{u_1, u_2, u_3, u_4\}$, $D_2 = \{u_{17}, u_{18}, u_{19}\}$ then $D = D_1 \cup D_2$ is a minimum dominating set, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2. Then $D^1 = (D_2 \times \{v_1\}) \cup (D_1 \times \{v_2\}) \cup (D_2 \times \{v_4\}) \cup (D_1 \times \{v_5\}) \cup (D_2 \times \{v_6\})$ is a dominating set of $G\Box P_6$.

For if $(u, v) \in G\Box P_6$, then consider the possiblity $v = v_1$. Then if $u \in D_2$ then $(u, v) \in D^1$. If $u \in V - D$ then there is a $w \in D_2$ such that w is adjacent to u in G. Then $(w, v_1) \in D^1$is adjacent to (u, v) in $G\Box P_6$. If $v \in D_1$ then, $(u, v_2) \in D_1 \times \{v_2\} \subset D^1$. And (u, v) is adjacent to $(u, v_2) \in D^1$. Hence D^1 can dominate all the vertices in $G\Box\{v_1\}$. Similarly D^1 can dominate $G\Box\{v_i\}, \forall i = 1, 2, ...6$. Thus D^1 is a dominating set of $G\Box P_6$.
$|D^1| = 20$
But $\gamma(G).n = 3 \times 7 = 21$

Hence, $\gamma(G\Box P_6) < |D| \leq 21 = \gamma(G)n$. □

Corollary 2.2.5. *Let G be any graph and P_n be a path having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2, D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, then $\frac{n}{3}\gamma(G) < \gamma(G\Box P_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.*

2.3 Domination number of cartesian product of a graph and cycle

Bounds of domination number of $G\Box C_n$, for which G satisfy certain conditions, is obtained in this section.

Theorem 2.3.1. *Let G be a graph having a minimum dominating set D which can be partitioned into two nonempty sets D_1 and D_2 with the property every vertex not in D is adjacent with atleast one vertex in D_1 and atleast one vertex in D_2. Then $\gamma(G\Box C_4) \leq 2\gamma(G)$.*

Proof. Let G be a graph having a minimum dominating set D which can be partitioned into two nonempty sets D_1 and D_2 with the property every vertex not in D is adjacent with atleast one vertex in D_1 and atleast one vertex in D_2. Let the vertices of C_4 be v_1, v_2, v_3, v_4 with v_i adjacent to v_{i+1} (addition is with respect to addition modulo 4).
If $D^1 = (D_1 \times \{v_1\}) \cup (D_2 \times \{v_2\}) \cup (D_1 \times \{v_3\}) \cup (D_2 \times \{v_4\})$, then we claim that D^1 is a dominating set of $G \Box C_4$. For, if $(u, v) \in V(G \Box C_4)$, then we distinguish into four cases.

Case 1 If $v = v_1$,

If $u \in D_1$ then $(u, v) \in D^1$
If $u \in D_2$ then $(u, v_2) \in D^1$ and adjacent to (u, v).
If $u \notin D$ then there is some u^1 in D_1 which is adjacent to u. Then $(u^1, v_1) \in D^1$ is adjacent to (u, v).

Similarly we can prove in all other cases that (u, v) is adjacent to atleast one vertex in D^1.

Thus D^1 is a dominating set of $G \Box C_4$ with $|D^1| = 2\gamma(G)$. Hence $\gamma(G \Box C_4) \leq 2\gamma(G)$. □

This theorem can be generalized as follows.

Theorem 2.3.2. *Let G be any graph and C_n be a cycle having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, then $\gamma(G \Box C_n) \leq \lceil \frac{n}{2} \rceil \gamma(G)$.*

Proof. Let D be a minimum dominating set of G such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2 and $V(C_n) = \{v_1, v_2, ..., v_n\}$, n is even.
Then $D^1 = (D_1 \times \{v_1\}) \cup (D_2 \times \{v_2\}) \cup (D_1 \times \{v_3\}) \cup (D_2 \times \{v_4\}) .... \cup (D_2 \times \{v_n\})$ is a dominating set of $G \Box C_n$.
For, if $(u, v_i) \in V(G \Box C_n)$ and i is odd. Then ,there are three possibilities.

Case 1 If $u \in D_1$, then $(u, v_i) \in D^1$

Case 2 If $u \in D_2$, then (u, v_i) is adjacent to $(u, v_{i+1}) \in D^1$

Case 3 If $u \notin D_1 \cup D_2$, then u is adjacent to $w_1 \in D_1$. Therefore (u, v_i) is adjacent to $(w_1, v_i) \in D^1$ in $G \Box C_n$.

Thus D^1 can dominate (u, v_i) .

If $(u, v_i) \in V(G \Box C_n)$ and i is even. Then, there are three possibilities.

Case 1 If $u \in D_2$, then $(u, v_i) \in D^1$

Case 2 If $u \in D_1$,then (u, v_i) is adjacent to $(u, v_{i-1}) \in D^1$

Case 3 If $u \notin D_1 \cup D_2$, then u is adjacent to $w_2 \in D_2$. Thus (u, v_i) is adjacent to $(w_2, v_i) \in D^1$ in $G \Box C_n$.

Thus D^1 can dominate (u, v_i) .
Hence D^1 is a dominating set of $G \Box C_n$.

$$\begin{aligned} |D^1| &= |(D_1 \times \{v_1\}) \cup (D_2 \times \{v_2\}) \cup .... \cup (D_n \times \{v_n\})| \\ &= \underbrace{|D_1 \cup D_2| + ...|D_1 \cup D_2|}_{\frac{n}{2} times} \\ &= \tfrac{n}{2}\gamma(G) \end{aligned}$$

Hence, $\gamma(G \Box C_n) \leq \frac{n}{2}\gamma(G)$.
Similarly, if n is odd $\gamma(G \Box C_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.

□

Theorem 2.3.3. *Let G be any graph and C_n be a cycle having n vertices, then* $\frac{n}{3}\gamma(G) \leq \gamma(G \Box C_n)$.

Proof. Let C_n be the cycle with $V(C_n) = \{v_1, v_2, ..., v_n\}$.
Let D be a dominating set of $G \Box C_n$. Then, let $D_i = D \cap (V(G) \times \{v_i\})$.
Take $\pi_1(D_i) = \{u \in V(G) : (u, v_i) \in D_i\}$.

If $i \in 2, 3, ..., n-1$ and $u \in V(G) - (\pi_1(D_{i-1}) \cup \pi_1(D_i) \cup \pi_1(D_{i+1}))$, then $(u, v_{i-1}) \notin D$, $(u, v_i) \notin D$ and $(u, v_{i+1}) \notin D$. Then there is a vertex$(w, v_i) \in D$ such that (w, v_i) is adjacent to (u, v_i). Hence $w \in \pi_1(D_i)$ and w is adjacent to u in G. Thus $\pi_1(D_{i-1}) \cup \pi_1(D_i) \cup \pi_1(D_{i+1})$ is a dominating set of G.

Hence, $\gamma(G) \leq |D_{i-1} \cup D_i \cup D_{i+1}|$, $\forall i = 2, ..., n-1$.
Similarly, $\gamma(G) \leq |D_{n-1} \cup D_n \cup D_1|$.
$\gamma(G) \leq |D_n \cup D_1 \cup D_2|$.

Summing up all these inequalities we can see that
$n\gamma(G) \leq 3\sum_{i=1}^{n} |D_i|$

Hence, $n\gamma(G) \leq 3|D|$ for a dominating set D of $G\Box C_n$.
Thus, $\gamma(G\Box C_n) \geq \frac{n}{3}\gamma(G)$. □

Corollary 2.3.4. *Let G be any graph and C_n be a cycle having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, then $\frac{n}{3}\gamma(G) \leq \gamma(G\Box C_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.*

Remark 2.3.1. *Converse of theorem 2.3.2 is not true. There exists graphs G with $\gamma(G\Box C_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$ but G has no minimum dominating set D with the property, D can be partitioned such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2. In 2.3 G has no minimum dominating set D with the property, D can be partitioned as such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2 but $\gamma(G\Box C_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.*

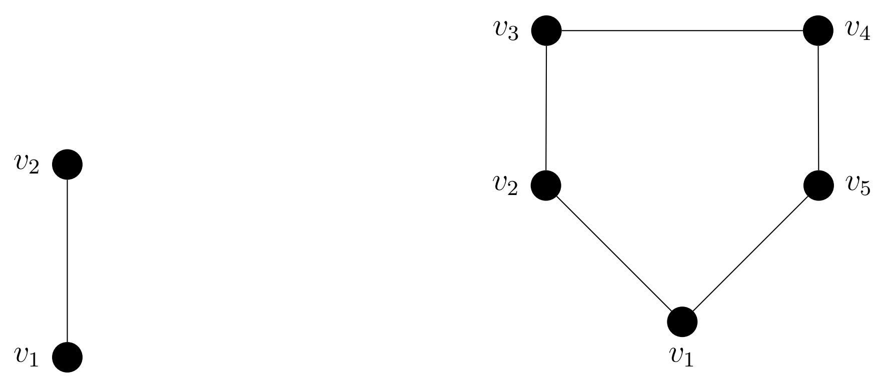

Figure 2.3

Remark 2.3.2. *In [69] we can see that if D is a minimum dominating set of a graph G, then at least one vertex in $V - D$ is dominated by no more than two vertices* *in D. Hence, it is not possible to partition, a minimum dominating set D into more than three disjoint sets each vertex not in D has atleast one neighbour in each sets.*

2.4 Conclusion

In this chapter the lower bound for domination number of cartesian product of G and P_n is obtained. It is studied for the cartesian product of G and C_n. Upperbound for $\gamma(G\Box H)$, when $H = P_n$ or $H = C_n$ for which G has a minimum dominating set D such that $D = D_1 \cup D_2, D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, is obtained.

CHAPTER 3

Reserved Domination in Graphs

3.1 Introduction

The variety of applications to 'coverage' or 'location' problems in mathematics and in the real world causes for the majority of the sudden increase in the number of domination parameters. Recently many new domination parameters have been developed.

In certain reservation systems allocations are compulsorily made from some special classes. Motivated from the same, Reserved domination number of graph G is defined in this chapter. This chapter is divided into seven sections. A-reserved domination number is introduced in the second section of this chapter. In the third Section, A-reserved domination number in various product graphs are studied. A-reserved domination number in corona of graphs are studied in the fourth section. In the fifth section, $[A_1, A_2, ..., A_n]$-reserved domination number is defined and studied. In the sixth section, the reserved domination number is used to study the domination number of cartesian product of graphs. Seventh section is a concluding section

3.2 A-Reserved domination number

In this section, A-reserved domination is introduced and characterization of minimal A-reserved dominating set is obtained. The special cases for which the A-reserved domination domination number exceeds domination number is obtained. Calculating the precise value of A- reserved domination number is a hard problem for specific classes of graphs. Here A- reserved domination number for some classes of graphs are obtained.

3.2.1 A-reserved domination

Definition 3.2.1. *Let $A \subset V(G)$, $A \neq \phi$, a dominating set D of a graph G is an A- reserved dominating set, if $D \cap A \neq \phi$. The A-reserved domination number of G, $r\gamma_A(G)$ is the cardinality of a minimum A-reserved dominating set.*

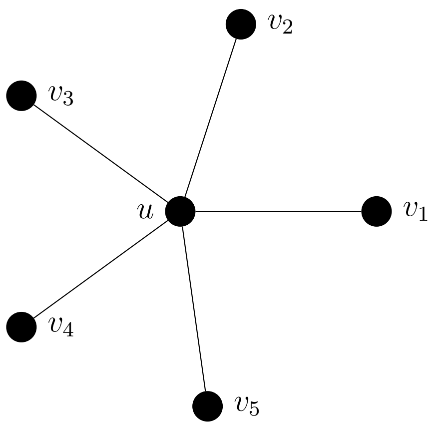

Figure 3.1

For illustration , consider the graph G in 3.1

If $A = \{v_1, v_2, v_3\}$, then $\{u, v_1\}$ is a minimum A- reserved dominating set .

Hence, $r\gamma_A(G) = 2 = \gamma(G) + 1$.

If $B = \{u, v_1, v_2, v_3\}$, then $\{u\}$ is a minimum B- reserved dominating set . Hence, $r\gamma_B(G) = 1 = \gamma(G)$.

Remark 3.2.1. *There are graphs with $r\gamma_A(G) = \gamma(G)$ for any $A \subset V(G)$. For any symmetric graph $r\gamma_A(G) = \gamma(G)$ for every $A \subset V(G)$. For Petersen graph G and for every $A \subset V(G)$, $r\gamma_A(G) = 3 = \gamma(G)$.*

Proof. Let $A \subset V(G)$. Choose any $v \in A$, and choose two vertices u and w so that $D = \{u, v, w\}$ is an independent set. Then D is a minimum A-reserved dominating set. Hence $r\gamma_A(G) = 3 = \gamma(G)$. □

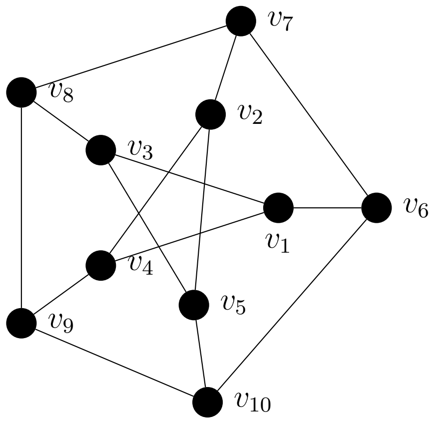

Figure 3.2

Theorem 3.2.2. *For any Graph G and $A \subset V(G)$,*

$$\gamma(G) \leq r\gamma_A(G) \leq \gamma(G) + 1$$

Proof. Every A-reserved dominating set is a dominating set of G. Thus $\gamma(G) \leq r\gamma_A(G)$. A minimum dominating set together with an element in A form an A-reserved dominating set. Hence G has an A-reserved dominating set with cardinality $\gamma(G) + 1$. Thus $r\gamma_A(G) \leq \gamma(G) + 1$.

□

Theorem 3.2.3. *An A-reserved dominating set D is a minimal A-reserved dominating set if and only if for each vertex v in D one of the following conditions holds*

1. *v is an isolate of D.*

2. *v has a private neighbour u in $V - D$.*

3. *$D \cap A = \{v\}$.*

Proof. If an A-reserved dominating set D is minimal, then D is an A-reserved dominating set and for each vertex v in D, $D - \{v\}$ is not an A-reserved dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or $(D - \{v\}) \cap A = \phi$.

Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u in $V - D$.

If $(D - \{v\}) \cap A = \phi$, since D is an A-reserved dominating set, $D \cap A \neq \phi$. Hence $D \cap A = \{v\}$.

Conversely, suppose that D is an A-reserved dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal A-reserved dominating set. If D is not a minimal A-reserved dominating set, then there exists a vertex $v \in D$ such that $D - \{v\}$ is an A-reserved dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. And v has no private neighbour in $V - D$ and condition 2 does not hold. $D - \{v\}$ is an A-reserved dominating set implies $(D - \{v\}) \cap A \neq \phi$. Hence condition 3 does not hold. Hence D is a minimal A-reserved dominating set.

□

Theorem 3.2.4. *If $A \subset B$, then $r\gamma_A(G) \geq r\gamma_B(G)$.*

Proof. If $A \subset B$, then every A-reserved dominating set is a B-reserved dominating set. Thus $r\gamma_A(G) \geq r\gamma_B(G)$.

□

Remark 3.2.2. $r\gamma_A(G) = 1$ *if and only if A contains a universal vertex of G.*

Remark 3.2.3. $r\gamma_A(G) = n$ *if and only if $G = \overline{K_n}$.*

Proof. If G has atleast one edge uv, choose any $w \in A$. If $\{w\} \cap \{u, v\} = \phi$, then $D = (V(G) - \{u\})$ is an A-reserved dominating set. If $\{w\} \cap \{u, v\} = \{v\}$, then

$D = (V(G) - \{u\})$ is an A-reserved dominating set. Hence $r\gamma_A(G) < n$. Thus $r\gamma_A(G) = n$ if and only if $G = \overline{K_n}$. □

Definition 3.2.5. *[73] $core(G)$ is defined as the set of all vertices belonging to all γ-set of G. $core(G) = \cap\{S : S \in \Omega(G)\}$, where $\Omega(G)$ is the family of all γ-sets of G.*

Definition 3.2.6. *[73] $anticore(G)$ is defined as the set of all vertices not belonging to any γ-set of G. $anticore(G) = V(G) - \cup\{S : S \in \Omega(G)\}$, where $\Omega(G)$ is the family of all γ-sets of G..*

Definition 3.2.7. *[73] Given $G = (V, E)$, $G_v + u$ for $u \notin V$ and $v \in V$, is defined as $G_v + u = (V^1, E^1)$, where $V^1 = V \cup \{u\}$ and $E^1 = E \cup \{uv\}$.*

Theorem 3.2.8 ([73]). *$v \in anticore(G)$ if and only if $\gamma(G_v + u) = \gamma(G) + 1$.*

Remark 3.2.4. *For a graph G and $A \subset V(G)$,*

- *$r\gamma_A(G) = \gamma(G) + 1$ if and only if $A \subset anticore(G)$.*
- *$r\gamma_A(G) = \gamma(G)$ if and only if $A \cap \overline{(anticore(G))} \neq phi$.*

Remark 3.2.5. *For graph G, $r\gamma_A(G) = \gamma(G)$ for every $A \subset V(G)$ if and only if every vertex of G can be a part of a minimum dominating set. Thus $r\gamma_A(G) = \gamma(G)$ for every $A \subset V(G)$ if and only if $anticore(G) = \phi$.*

Theorem 3.2.9. *For a graph G and $A \subset V(G)$,*

- *$r\gamma_A(G) = \gamma(G) + 1$ if and only if $\gamma(G_v + u) = \gamma(G) + 1$ for every $v \in A$.*
- *$r\gamma_A(G) = \gamma(G)$ if and only if $\exists v \in A$ satisfying $\gamma(G_v + u) \neq \gamma(G) + 1$.*

Proof. From theorem 3.2.4, $r\gamma_A(G) = \gamma(G) + 1$ if and only if $A \subset anticore(G)$. From theorem 3.2.8, $v \in anticore(G)$ if and only if $\gamma(G_v + u) = \gamma(G) + 1$. Hence $r\gamma_A(G) = \gamma(G) + 1$ if and only if $\gamma(G_v + u) = \gamma(G) + 1$ for every $v \in A$.

From theorem 3.2.4, $r\gamma_A(G) = \gamma(G)$ if and only if $A \cap \overline{(anticore(G))} \neq \phi$. Hence $r\gamma_A(G) = \gamma(G)$ if and only if $\exists v \in A \cap \overline{(anticore(G))}$. Thus $r\gamma_A(G) = \gamma(G)$ if and only if there exists $v \in A$ satisfying $\gamma(G_v + u) \neq \gamma(G) + 1$.

□

Theorem 3.2.10. *For cycle C_n and any $A \subset V(C_n)$, $r\gamma_A(C_n) = \lceil \frac{n}{3} \rceil = \gamma(G)$.*

Proof. If $A \subset V(C_n)$ and $u \in A$, then $D = \{v \in V(C_n) : d(u,v) = 3k, k \in N\}$ is an A-reserved dominating set with $|D| = \lceil \frac{n}{3} \rceil = \gamma(G)$. Hence, $r\gamma_A(C_n) = \lceil \frac{n}{3} \rceil = \gamma(G)$.

□

Theorem 3.2.11. *For complete graph K_n and any $A \subset V(K_n)$, $r\gamma_A(K_n) = 1$.*

Proof. If $A \subset V(K_n)$, then choose any $u \in A$. Then $D = \{u\}$ is an A-reserved dominating set with $|D| = 1$. Hence $r\gamma_A(K_n) = 1$.

□

Theorem 3.2.12. *For complete bipartite graph $K_{m,n}$ with $2 \leq m, n$ and any $A \subset V(K_{m,n})$, $r\gamma_A(K_{m,n}) = 2$.*

Proof. Let X, Y be the bipartition of $K_{m,n}$. Let $A \subset V(K_{m,n})$, $u \in A$, if $u \in X$ choose any vertex v from the other partition Y. Then $D = \{u, v\}$ is an A-reserved dominating set with $|D| = 2$. Hence $r\gamma_A(K_{m,n}) = 2$.

□

Theorem 3.2.13. *If $n \equiv 1(mod3)$ and $A \subset V(P_n)$, then $r\gamma_A(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$.*

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$, $A \subset V(P_n)$ and choose any $u \in A$.

Case 1 $u = v_{3l+1}, l \in N, 0 \leq l \leq \lfloor \frac{n}{3} \rfloor$.
Then let $D = \{v \in V(P_n) : d(u,v) = 3k, k \in N\} \cup \{u\}$. Then D is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Case 2 $u = v_{3l}, l \in N$, $0 \leq l \leq \lfloor \frac{n}{3} \rfloor$.
Then let $D^1 = \{v \in V(P_n) : d(u,v) = 3k, k \in N\}$. Then $|D^1| = \lceil \frac{n}{3} \rceil - 2$. Then $D = \{v \in V(P_n) : d(u,v) = 3k, k \in N\} \cup \{u, u_1\}$ is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Case 3 $u = v_{3l+2}, l \in N$, $0 \leq l \leq \lfloor \frac{n}{3} \rfloor$.
Then let $D^1 = \{v \in V(P_n) : d(u,v) = 3k, k \in N\}$. Then $|D| = \lceil \frac{n}{3} \rceil - 2$. Then $D = \{v \in V(P_n) : d(u,v) = 3k, k \in N\} \cup \{u, u_n\}$ is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Hence $r\gamma_A(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$. □

Theorem 3.2.14. *If* $n \equiv 0(mod3)$ *and* $A \subset V(P_n)$, *then*

$$r\gamma_A(P_n) = \begin{cases} \frac{n}{3} & if A \cap \{v_i : i \equiv 2(mod3)\} \neq \phi \\ \frac{n}{3}+1 & Otherwise \end{cases}$$

Proof. Let $A \subset V(P_n)$

Case 1 $A \cap \{v_i : i \equiv 2(mod3)\} \neq \phi$

Let $u \in A \cap \{v_i : i \equiv 2(mod3)\}$, then $D = \{v_i \in V(P_n) : i \equiv 2(mod3)\}$ is minimum dominating set of P_n and $u \in D \cap A$. Then D is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Case 2 $A \cap \{v_i : i \equiv 2(mod3)\} = \phi$

$D = \{v_i \in V(P_n) : i \equiv 2(mod3)\}$ is the unique minimum dominating set of P_n and $D \cap A = \phi$. Hence $r\gamma_A(P_n) = \lceil \frac{n}{3} \rceil + 1$.

□

Theorem 3.2.15. *If* $n \equiv 2(mod3)$ *and* $A \subset V(P_n)$, *then*

$$r\gamma_A(P_n) = \begin{cases} \lceil\frac{n}{3}\rceil & if A \cap \{v_i : i \equiv 1,2(mod3)\} \neq \phi \\ \lceil\frac{n}{3}\rceil+1 & Otherwise \end{cases}$$

Proof. Let $A \subset V(P_n)$

Case 1 $A \cap \{v_i : i \equiv 1,2(mod3)\} \neq \phi$

Let $u \in A \cap \{v_i : i \equiv 1(mod3)\}$, then $D = \{v_i \in V(P_n) : i \equiv 1(mod3)\}$ is a minimum dominating set of P_n and $u \in D \cap A$. Then D is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Let $u \in A \cap \{v_i : i \equiv 2(mod3)\}$, then $D = \{v_i \in V(P_n) : i \equiv 2(mod3)\}$ is minimum dominating set of P_n and $u \in D \cap A$. Then D is an A-reserved dominating set and $|D| = \lceil \frac{n}{3} \rceil = r\gamma_A(P_n)$.

Case 2 $A \cap \{v_i : i \equiv 1,2(mod3)\} = \phi$

$D = \{v_i \in V(P_n) : i \equiv 2(mod3)\}$ and $D^1 = \{v_i \in V(P_n) : i \equiv 1(mod3)\}$ are the minimum dominating sets of P_n and $D \cap A = \phi$, $D^1 \cap A = \phi$. Hence $r\gamma_A(P_n) = \lceil \frac{n}{3} \rceil + 1$.

□

Remark 3.2.6. $r\gamma_{A\cup B}(G) = min\ \{r\gamma_A(G), r\gamma_B(G)\}$.

Proof. Every A- reserved dominating set is an $A \cup B$- reserved dominating set too. Hence, $r\gamma_{A\cup B}(G) \leq r\gamma_A(G)$.

Similarly, $r\gamma_{A\cup B} \leq r\gamma_B(G)$.

If D is a minimum $A\cup B$- reserved dominating set. Then it is an A-reserved dominating set or a B-reserved dominating set. Hence $r\gamma_{A\cup B}(G) \geq \min\ \{r\gamma_A(G), r\gamma_B(G)\}$. Thus, $r\gamma_{A\cup B}(G) = \min\ \{r\gamma_A(G), r\gamma_B(G)\}$.

□

3.3 A-Reserved domination in graph products

A-reserved domination in cartesian products is studied in the first section. A-reserved domination in strong products is studied in the second section.

3.3.1 A-Reserved domination in cartesian products

Theorem 3.3.1. *Let G and H be two graphs of order n_1 and n_2, then for any $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_{A\times B}(G\Box H) \leq min\{n_1 r\gamma_B(H), n_2 r\gamma_A(G)\}$.*

Proof. Let S_H be a minimum B-reserved dominating set of H. Let us see that $S = V(G) \times S_H$ is an $A \times B$-reserved dominating set of $G\Box H$. If $(u, v) \in (V(G) \times V(H)) - S$. Then (u, v) is adjacent to atleast one vertex in S. And since $B \cap S_H \neq \phi$ and $V(G) \supset A$, $(A \times B) \cap (V(G) \times S_H) \neq \phi$. Hence S is an $A \times B$-reserved dominating set of $G\Box H$.

Similarly, if S_G is a minimum A-reserved dominating set of G, then $S = S_G \times V(H)$ is an $A \times B$-reserved dominating set of $G\Box H$.

Thus,

$$r\gamma_{A\times B}(G\Box H) \leq min\{n_1 r\gamma_B(H), n_2 r\gamma_A(G)\}.$$

□

Remark 3.3.1. *The bound in theorem 3.3.1 is tight. For example for the graphs G and H given in figure 3.3 with $A = \{v_1\}, B = \{u_1\}$, $r\gamma_{A\times B}(G\Box H) = 2 = min\{n_1 r\gamma_B(H), n_2 r\gamma_A(G)\}$.*

And the strict inequality in theorem 3.3.1 attains for the graphs G and H given in figure 3.3 with $A = \{v_3\}, B = \{u_1\}$, $r\gamma_{A\times B}(G\Box H) = 3 < min\{n_1 r\gamma_B(H), n_2 r\gamma_A(G)\}$.

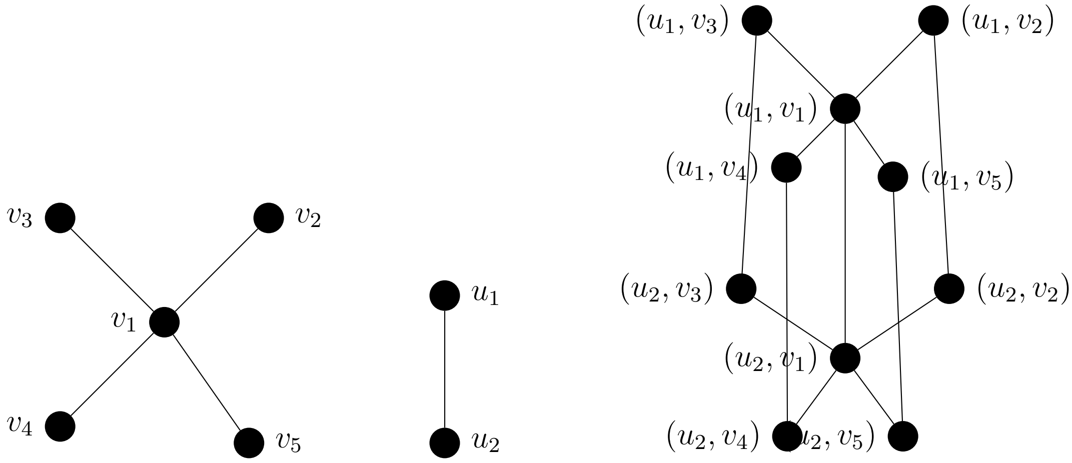

Figure 3.3

A graph G satisfies Vizing's conjecture, if the inequality $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ is true for arbitrary graph H. [19]

Theorem 3.3.2. *If a graph G satisfy the inequality $r\gamma_{A\times B}(G\Box H) \geq r\gamma_A(G)r\gamma_B(H)$, for every graph H and for $A\times B = \{(u, v)\}$ where (u, v) is an element in a minimum dominating set of $G\Box H$. Then G satisfies Vizing's inequality.*

Proof. If a graph G satisfy the inequality $r\gamma_{A\times B}(G\Box H) \geq r\gamma_A(G)r\gamma_B(H)$, for every graph H and for $A \times B = \{(u, v)\}$ where (u, v) is an element in a minimum dominating set of $G\Box H$. Then,

$$\begin{aligned} \gamma(G\Box H) &= r\gamma_{A\times B}(G\Box H) \\ &\geq r\gamma_A(G)r\gamma_B(H) \ . \\ &\geq \gamma(G)\gamma(H). \end{aligned}$$

Thus, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$ for every graph H and G satisfies Vizing's conjecture.

□

We can see graphs G and H and $A \subset V(G)$, $B \subset V(H)$ satisfy the inequality $r\gamma_{A\times B}(G\Box H) < r\gamma_A(G)r\gamma_B(H)$. For example in figure 3.3 if $A = \{v_1\}, B = \{u_1\}$, then $r\gamma_{A\times B}(G\Box H) < r\gamma_A(G)r\gamma_B(H)$.

There are graphs G and H so that $r\gamma_{A\times B}(G\Box H) = r\gamma_A(G)r\gamma_B(H)$, for every $A \subset V(G)$ and $B \subset V(H)$.
For example, if $G = C_4$ and $H = K_2$, then for every $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_A(G) = 2$, $r\gamma_B(H) = 1$ and $r\gamma_{A\times B}(G\Box H) = 2 = r\gamma_A(G)r\gamma_B(H)$.

We can see graph G , H , $A \subset V(G)$ and $B \subset V(H)$ for which $r\gamma_{A\times B}(G\Box H) > r\gamma_A(G)r\gamma_B(H)$.
For example, if $G = K_{1,4}$ and $H = K_2$ as given in figure 3.3, then for $A = \{v_3\}$ and for any $B \subset V(H)$, $r\gamma_A(G) = 2$, $r\gamma_B(H) = 1$ and $r\gamma_{A\times B}(G\Box H) = 3 > r\gamma_A(G)r\gamma_B(H)$.

Theorem 3.3.3. *Let G be a graph that satisfies $r\gamma_{A\times B}(G\Box H) \geq r\gamma_A(G)r\gamma_B(H)$, for every $A \subset V(G)$ and for every graph H and $B \subset V(H)$, and let G^1 be a spanning subgraph of G such that $r\gamma_A(G) = r\gamma_A(G^1)$. Then G^1 also satisfies $r\gamma_{A\times B}(G^1\Box H) \geq r\gamma_A(G^1)r\gamma_B(H)$, for every $A \subset V(G)$ and for every graph H and $B \subset V(H)$.*

Proof. Let $A \subset V(G)$ and $B \subset V(H)$, then

$$\begin{aligned} r\gamma_{A\times B}(G^1\Box H) &\geq r\gamma_{A\times B}(G\Box H) \\ &\geq r\gamma_A(G)r\gamma_B(H) \\ &\geq r\gamma_A(G^1)r\gamma_B(H) \end{aligned}$$

□

Theorem 3.3.4. *If n is an odd integer and $x_1, x_2, ..., x_n$ be the vertices in the first copy of P_n in $P_n\Box P_2$ and $y_1, y_2, ..., y_n$ be the vertices in the second copy of P_n in $P_n\Box P_2$, then for any $A \subset V(P_n\Box P_2)$,*

$$r\gamma_A(P_n\Box P_2) = \begin{cases} \lceil \frac{n+1}{2} \rceil & if A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} \neq \phi \\ \lceil \frac{n+2}{2} \rceil & Otherwise \end{cases}$$

.

Proof. Let $A \subset V(P_n\Box P_2)$

Case 1 If $A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} = \phi$

Choose any $x_{2k} \in D \cap A$. Then $D = \{x_i : i \equiv 1(mod4)\} \cup \{y_j : j \equiv$

$3(mod4)\} \cup \{x_{2k}\}$ is an A-reserved dominating set with $|D| = \lceil\frac{n+2}{2}\rceil$. Thus $r\gamma_A(P_n \Box P_2) \leq \lceil\frac{n+2}{2}\rceil$.

Let D be a minimum A-reserved dominating set. Then $u \in D \cap A$ and $u = x_{2k}$ or $y_{2k}, 2 \leq k \leq \frac{n+1}{2}$.

If $x_{2k} \in D \cap A$, then removing $x_{2k} \cup \{x_i, y_i, i \geq 2k\}$ from D and if necessary, replacing y_{2k-1} by y_{2k-2} the resulting set D^1 would dominate the induced subgraph $< \{x_1, y_1, x_2, y_2 ....., x_{2k-2}, y_{2k-2}\} >$. Since $\gamma(P_{2k-2} \Box P_2) = \lceil\frac{2k-2+1}{2}\rceil = k$, $|D^1| \geq k$.

And removing $x_{2k} \cup \{x_i, y_i, i \leq 2k\}$ from D and if necessary, replacing y_{2k+1} by y_{2k+2} the resulting set D^{11} would dominate the induced subgraph $< \{x_{2k+2}, y_{2k+2}, ..., x_n, y_n\} >$. Since $\gamma(P_{n-(2k+1)} \Box P_2) = \lceil\frac{n-2k-1+1}{2}\rceil = \lceil\frac{n-2k}{2}\rceil$, $|D^{11}| \geq \lceil\frac{n-2k}{2}\rceil$.

Hence,

$$\begin{aligned} |D| &\geq |D^1| + |D^{11}| + 1 \\ &\geq k + \lceil\tfrac{n-2k}{2}\rceil + 1 \\ &= \tfrac{n+1}{2} + 1 \\ &\geq \lceil\tfrac{n+2}{2}\rceil \end{aligned}$$

Thus $r\gamma_A(P_n \Box P_2) \geq \lceil\frac{n+2}{2}\rceil$.

Hence $r\gamma_A(P_n \Box P_2) = \lceil\frac{n+2}{2}\rceil$.

Case 2 If $A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} \neq \phi$

Choose any $u \in A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\}$.

If $u = x_{2k+1}, 3 \leq k \leq \frac{n-1}{2}$, then let $D = \{x_i : 1 \leq i \leq 2k, d(x_i, x_{2k+1}) \equiv 0(mod4)\} \cup \{y_i : 1 \leq i \leq n, d(y_{2k-1}, y_i) \equiv 0(mod4)\} \cup \{x_{2k+1}, y_{2k-1}\}$.

If $u = x_1$, then let $D = \{x_i : i \equiv 1(mod4)\} \cup \{y_j : j \equiv 3(mod4)\}$.

If $u = y_{2k+1}, 3 \leq k \leq \frac{n-1}{2}$, then let $D = \{y_i : 1 \leq i \leq 2k, d(y_i, y_{2k+1}) \equiv 0(mod4)\} \cup \{y_i : 1 \leq i \leq n, d(x_{2k-1}, x_i) \equiv 0(mod4)\} \cup \{y_{2k+1}, x_{2k-1}\}$.

If $u = y_1$, then let $D = \{y_i : i \equiv 1(mod4)\} \cup \{x_j : j \equiv 3(mod4)\}$.

Then D is an A-reserved dominating set with $|D| = \lceil\frac{n+1}{2}\rceil$. Thus $r\gamma_A(P_n \Box P_2) \leq \lceil\frac{n+1}{2}\rceil$. Since $\gamma(P_n \Box P_2) = \lceil\frac{n+1}{2}\rceil$, $r\gamma_A(P_n \Box P_2) \geq \lceil\frac{n+1}{2}\rceil$.

Hence $r\gamma_A(P_n \Box P_2) = \lceil\frac{n+1}{2}\rceil$.

□

Theorem 3.3.5. *If n is an even integer , then for any $A \subset V(P_n \Box P_2)$,*

$$r\gamma_A(P_n \Box P_2) = \lceil \frac{n+1}{2} \rceil$$

Proof. Let $A \subset V(P_n \Box P_2)$

Case 1 If $A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} \neq \phi$

Choose any $u \in A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\}$.

If $u = x_{2k+1}, 3 \leq k \leq \frac{n}{2}$, then let $D = \{x_i : 1 \leq i \leq n, d(x_i, x_{2k+1}) \equiv 0(mod4)\} \cup \{y_i : 1 \leq i \leq n, d(y_{2k-1}, y_i) \equiv 0(mod4)\} \cup \{x_{2k+1}, y_{2k-1}, x_n\}$.

If $u = x_1$, then let $D = \{x_i : i \equiv 1(mod4)\} \cup \{y_j : j \equiv 3(mod4)\} \cup \{x_n\}$.

If $u = y_{2k+1}, 3 \leq k \leq \frac{n}{2}$, then let $D = \{y_i : 1 \leq i \leq 2k, d(y_i, y_{2k+1}) \equiv (0mod4)\} \cup \{y_i : 1 \leq i \leq n, d(x_{2k-1}, x_i) \equiv (0mod4)\} \cup \{y_{2k+1}, x_{2k-1}, x_n\}$.

If $u = y_1$, then let $D = \{y_i : i \equiv 1(mod4)\} \cup \{x_j : j \equiv 3(mod4)\} \cup \{x_n\}$.

Then D is an A-reserved dominating set with $|D| = \lceil \frac{n+1}{2} \rceil$. Thus $r\gamma_A(P_n \Box P_2) \leq \lceil \frac{n+1}{2} \rceil$. Since $\gamma(P_n \Box P_2) = \lceil \frac{n+1}{2} \rceil$, $r\gamma_A(P_n \Box P_2) \geq \lceil \frac{n+1}{2} \rceil$. Hence $r\gamma_A(P_n \Box P_2) = \lceil \frac{n+1}{2} \rceil$.

Case 2 If $A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} = \phi$

Choose any $u \in D \cap A$.

If $u = x_{2k}, 2 \leq k \leq \frac{n}{2}$, then let $D = \{x_i : 1 \leq i \leq n, d(x_i, x_{2k}) \equiv 0(mod4)\} \cup \{y_i : 1 \leq i \leq n, d(y_{2k-2}, y_i) \equiv 0(mod4)\} \cup \{x_{2k}, y_{2k-2}, x_1\}$.

If $u = x_2$, then let $D = \{x_i : 1 \leq i \leq n, i \equiv (2mod4)\} \cup \{y_i : 1 \leq i \leq n, i \equiv 0(mod4)\} \cup \{x_1\}$.

If $u = y_{2k}, 2 \leq k \leq \frac{n}{2}$, then let $D = \{y_i : 1 \leq i \leq n, d(y_i, y_{2k}) \equiv 0(mod4)\} \cup \{x_i : 1 \leq n, d(x_{2k-2}, x_i) \equiv 0(mod4)\} \cup \{xy_{2k}, x_{2k-2}, x_1\}$.

If $u = y_2$, then let $D = \{y_i : 1 \leq i \leq n, i \equiv 2(mod4)\} \cup \{x_i : 1 \leq i \leq n, i \equiv 0(mod4)\} \cup \{x_1\}$.

Then D is an A-reserved dominating set with $|D| = \lceil \frac{n+1}{2} \rceil$. Thus $r\gamma_A(P_n \Box P_2) \leq \lceil \frac{n+1}{2} \rceil$. Since $\gamma(P_n \Box P_2) = \lceil \frac{n+1}{2} \rceil$, $r\gamma_A(P_n \Box P_2) \geq \lceil \frac{n+1}{2} \rceil$. Hence $r\gamma_A(P_n \Box P_2) = \lceil \frac{n+1}{2} \rceil$.

□

Theorem 3.3.6. *If* $A \subset V(P_n \Box P_3)$,

$$n - \lfloor \frac{n-1}{4} \rfloor \leq r\gamma_A(P_n \Box P_3) \leq n + 1 - \lfloor \frac{n-1}{4} \rfloor$$

.

Proof. From [46], $\gamma(P_n \Box P_3) = n - \lfloor \frac{n-1}{4} \rfloor$. Using 3.2.2, $n - \lfloor \frac{n-1}{4} \rfloor \leq r\gamma_A(P_n \Box P_3) \leq n + 1 - \lfloor \frac{n-1}{4} \rfloor$.

□

Theorem 3.3.7. *If* $A \subset V(P_n \Box P_4)$,

$$\begin{array}{ll} n + 1 \leq r\gamma_A(P_n \Box P_4) \leq n + 2 & \textit{if n=1,2,3,4,5,6 or 9} \\ n \leq r\gamma_A(P_n \Box P_4) \leq n + 1 & \textit{Otherwise} \end{array}$$

.

Proof. From [46],

$$\begin{array}{ll} \gamma(P_n \Box P_4) = n + 1 & \text{if n=1,2,3,4,5,6 or 9} \\ \gamma(P_n \Box P_4) = n & \text{Otherwise} \end{array}$$

Using 3.2.2,

$$\begin{array}{ll} n + 1 \leq r\gamma_A(P_n \Box P_4) \leq n + 2 & \text{if n=1,2,3,4,5,6 or 9} \\ n \leq r\gamma_A(P_n \Box P_3) \leq n + 1 & \text{Otherwise} \end{array}$$

□

Theorem 3.3.8. *For any* $A \subset V(K_m)$ *and* $B \subset V(K_n), m \leq n$, $r\gamma_{A\times B}(K_m \Box K_n) = m$.

Proof. Let $u_1, u_2, ..., u_m$ be the vertices of K_m and $v_1, v_2, ..., v_n$ be the vertices of K_n. Choose any $(u_i, v_j) \in A \times B$, then $D = \{(u_1, v_j), (u_2, v_j), (u_3, v_j), ..., (u_i, v_j), ...(u_m, v_j)\}$ is an $(A \times B)$-reserved dominating set with $|D| = m$. Hence $r\gamma_{A\times B}(K_m \Box K_n) \leq m$. Since $\gamma(K_m \Box K_n) = m$, $r\gamma_{A\times B}(K_m \Box K_n) \geq m$. Thus $r\gamma_{A\times B}(K_m \Box K_n) = m$.

□

Theorem 3.3.9. *For any* $A \subset V(P_m)$ *and* $B \subset V(K_n), m, n > 2 \in N$, $r\gamma_{A\times B}(P_m \Box K_n) = m$.

Proof. Let $u_1, u_2, ..., u_m$ be the vertices of P_m and $v_1, v_2, ..., v_n$ be the vertices of K_n. Choose any $(u_i, v_j) \in A \times B$, then $D = \{(u_1, v_j), (u_2, v_j), (u_3, v_j), ..., (u_i, v_j), ...(u_m, v_j)\}$ is an $(A \times B)$-reserved dominating set with $|D| = m$. Hence $r\gamma_{A\times B}(P_m \square K_n) \leq m$. Since $\gamma(P_m \square K_n) = m$, $r\gamma_{A\times B}(P_m \square K_n) \geq m$. Thus $r\gamma_{A\times B}(P_m \square K_n) = m$.

□

Theorem 3.3.10. *For any $A \subset V(C_m)$ and $B \subset V(K_n), m, n > 2 \in N$, $r\gamma_{A\times B}(C_m \square K_n) = m$.*

3.3.2 Reserved domination in strong products

Theorem 3.3.11. *For any two nontrivial graphs G and H and for every $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_{A\times B}(G \boxtimes H) \leq r\gamma_A(G) r\gamma_B(H)$.*

Proof. Let D_1 be the minimum A-reserved dominating set of G and D_2 be the minimum B-reserved dominating set of H.
Then there are vertices u, v such that $u \in D_1 \cap A$ and $v \in D_2 \cap B$. Hence $(u, v) \in (D_1 \times D_2) \cap (A \times B)$

Let $(u_1, v_1) \in V(G \boxtimes H)$ so that $(u_1, v_1) \notin D_1 \times D_2$, then

Case 1 $u_1 \in D_1$ and $v_1 \notin D_2$,
then v_1 is dominated by $v_2 \in D_2$. Hence (u_1, v_1) is dominated by $(u_1, v_2) \in D_1 \times D_2$.

Case 2 $u_1 \notin D_1$ and $v_1 \in D_2$,
then u_1 is dominated by $u_2 \in D_1$. Hence (u_1, v_1) is dominated by $(u_2, v_1) \in D_1 \times D_2$.

Case 3 $u_1 \notin D_1$ and $v_1 \notin D_2$,
then u_1 is dominated by $u_2 \in D_1$ and v_1 is dominated by $v_2 \in D_2$. Hence (u_1, v_1) is dominated by $(u_2, v_2) \in D_1 \times D_2$.

Thus $D_1 \times D_2$ is an $(A \times B)$-reserved dominating set and $r\gamma_{A\times B}(G \boxtimes H) \leq r\gamma_A(G) r\gamma_B(H)$.

□

We can see graphs for which $r\gamma_{A\times B}(G\boxtimes H) = r\gamma_A(G)r\gamma_B(H), \forall A \subset V(G), \forall B \subset V(H)$. If $G = C_4$ and $H = K_2$, then for every $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_A(G) = 2$, $r\gamma_B(G) = 1$ and $r\gamma_{A\times B}(G \boxtimes H) = 2 = r\gamma_A(G)r\gamma_B(H)$.

We can see graphs G and H and $A \subset V(G), B \subset V(H)$ for which $r\gamma_{A\times B}(G \boxtimes H) < r\gamma_A(G)r\gamma_B(H), \forall A \subset V(G), \forall B \subset V(H)$. If $G = P_3$ and $H = P_3$, $A = \{v_1\}$ and $B = \{u_1\}$ the pendant vertices of G and H respectively, then $r\gamma_A(G) = 2$, $r\gamma_B(G) = 2$ and $r\gamma_{A\times B}(G \boxtimes H) = 3 < r\gamma_A(G)r\gamma_B(G)$.

3.4 Reserved domination in the corona of graphs

Theorem 3.4.1. *Let G be a graph order n and H be a graph order m and H_i be the copy of H corresponding to $v_i \in G$, then for $A \subset V(G \circ H)$,*

$$r\gamma_A(G \circ H) = \begin{cases} n & \text{if } A \cap (V(G) \cup \{v : v \text{ is a universal vertex of } H_i\}) \neq \phi \\ n+1 & \text{Otherwise} \end{cases}$$

.

Proof. Let $A \subset V(G \circ H)$

Case 1 If $A \cap (V(G) \cup \{v :$ v is a universal vertex of $H_i\}) = \phi$,

Let D be an A-reserved dominating set of $G \circ H$. Since D is a dominating set of $G \circ H$, $|D \cap (V(H_i) \cup \{v_i\})| \geq 1$, $\forall i = 1, 2, ..., n$.

And if $|D \cap (V(H_i) \cup \{v_i\})| = 1$, $\forall i = 1, 2, ..., n$, then $w_i \in D \cap (V(H_i) \cup \{v_i\})$ dominates $V(H_i)$ and there exist a vertex $w_k \in A \cap (H_k \cup \{v_k\})$ for some $k \in \{1, 2, ...n\}$ and w_k dominates H_k. Then, either $w_k = v_k$ or w_k is a universal vertex of H_k, which is not possible. Hence $|D \cap (V(H_j) \cup \{v_j\})| > 1$ for some $j \in \{1, 2, ..., n\}$. Thus $|D| \geq n + 1$ and $r\gamma_A(G \circ H) \geq n + 1$.
Choose any $w \in A$, then $\{v_1, v_2, ..., v_n\} \cup \{w\}$ is an A-reserved dominating set of GoH. Hence $r\gamma_A(GoH) \leq n + 1$.
Thus $r\gamma_A(GoH) = n + 1$.

Case 2 If $A \cap (V(G) \cup \{v :$ v is a universal vertex of $H_i\}) \neq \phi$,

Let $w \in A \cap (V(G) \cup \{v :$ v is a universal vertex of $H_i\}$ and $w \in V(H_k) \cup \{v_k\}$, $k \in \{1, 2, ...n\}$. Let D be an A-reserved dominating set of GoH. Since D is a dominating set of $G \circ H$, $|D \cap (V(H_i) \cup \{v_i\})| \geq 1$, $\forall i = 1, 2, ..., n$. Hence $|D| \geq n$ and $r\gamma_A(G \circ H) \geq n$

Hence $\{w\} \cup (\{v_1, v_2, ..., v_n\} - \{v_k\})$ is an A- reserved dominating set of $G \circ H$. Hence $r\gamma_A(G \circ H) \leq n$

Thus $r\gamma_A(GoH) = n$

□

3.5 $[A_1, A_2, ..., A_k]$-reserved domination

$[A_1, A_2, ..., A_k]$-reserved domination as a generalized concept of A-reserved nomination number is defined in this section and studied it.

Definition 3.5.1. *Let $A_1, A_2, ..., A_k \subset V(G)$. If $D \subset V(G)$ is a dominating set with the property $D \cap A_i \neq \phi, \forall i = 1, 2, ..., k$, then D is said to be an $[A_1, A_2, ...A_k]$-reserved dominating set of G. Cardinality of a minimum $[A_1, A_2, ...A_k]$- reserved dominating set is called the $[A_1, A_2, ...A_k]$-reserved domination number denoted by $r\gamma_{[A_1, A_2, ..., A_k]}(G)$.*

For example , consider the graph fig.3.4 .

If $A_1 = \{v_1, v_2\}, A_2 = \{v_7\}$, Then $\{v_1, v_2, v_5, v_7\}$is a minimum $[A_1, A_2]$- reserved dominating set.

Hence $r\gamma_{[A_1, A_2]}(G) = 4$.

If $B_1 = \{v_1, v_2, v_6\}, B_2 = \{v_3, v_4, v_5\}$, Then $\{v_5, v_6\}$ is a minimum $[B_1, B_2]$- reserved dominating set.

Here $r\gamma_{[B_1, B_2]}(G) = 2$.

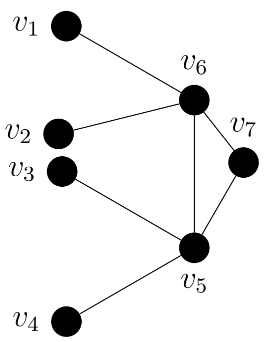

Figure 3.4

Remark 3.5.1. *For any* G *and any* $A_1, A_2, ...A_k \subset V(G)$, $\gamma(G) \leq r\gamma_{[A_1,A_2,...,A_k]}(G) \leq \gamma(G) + k$.

Theorem 3.5.2. *If* $A_1, A_2, ...A_k \subset V(G)$ *then an* $[A_1, A_2, ..., A_k]$*-reserved dominating set* D *is a minimal* $[A_1, A_2, ..., A_k]$*-reserved dominating set if and only if for each vertex* v *in* D *one of the following conditions holds*

1. v *is an isolate of* D.
2. v *has a private neighbour* u *in* $V - D$.
3. *There exists* $l \in \{1, 2, ...k\}$ *such that* $D \cap A_l = \{v\}$.

Proof. If an $[A_1, A_2, ..., A_k]$-reserved dominating set D is minimal, then D is an $[A_1, A_2, ..., A_k]$-reserved dominating set and for each vertex v in D, $D - \{v\}$ is not an $[A_1, A_2, ..., A_k]$-reserved dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or there exists $l \in \{1, 2, ...k\}$ such that $(D - \{v\}) \cap A_l = \phi$.

Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u in $V - D$.

If there exists $l \in \{1, 2, ...k\}$ such that $(D - \{v\}) \cap A_l = \phi$, but since D is an $[A_1, A_2, ..., A_k]$-reserved dominating set, $D \cap A_i \neq \phi, \forall i \in \{1, 2, ...k\}$. Hence $D \cap A_l = \{v\}$. Thus there exists $l \in \{1, 2, ...k\}$ such that $D \cap A_l = \{v\}$.

Conversely, suppose that D is an $[A_1, A_2, ..., A_k]$-reserved dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal $[A_1, A_2, ..., A_k]$-reserved dominating set. If D is not a minimal $[A_1, A_2, ..., A_k]$-reserved dominating set, then there exists a vertex $v \in D$ such that $D - \{v\}$ is an $[A_1, A_2, ..., A_k]$-reserved dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. Then v has no private neighbour in $V - D$ and condition 2 does not hold. $D - \{v\}$ is an $[A_1, A_2, ..., A_k]$-reserved dominating set implies $(D - \{v\}) \cap A_i \neq \phi, \forall i \in \{1, 2...., n\}$. Hence condition 3 does not hold. Hence D is a minimal $[A_1, A_2, ..., A_k]$-reserved dominating set.

□

3.6 Domination number of cartesian product of graphs using reserved domination number

In this section, domination number of cartesian product of graphs are studied using reserved domination number.

Theorem 3.6.1. *Let $G = K_{1,n}$ with u as center and $V(G) = \{u, u_1, u_2, ...u_n\}$ and let H be any graph. Let $D \subset V(G \square H)$ such that D dominates $\{u\} \square H$. Then, $|D| \geq 2\gamma(H) - \gamma(< F >)$, where $F = \{v \in H : (u_i, v)$ is not adjacent to D for some $i\} \cup \{v \in H : (u_i, v) \notin D, \forall i\}$.*

Proof. Define , $\pi : (G \square H) \to H$ as $\pi(u, v) = v, \forall (u, v) \in (G \square H)$

Then, $\pi(S) = \{v \in H : (u, v) \in S\}$.

$D_0 = (\{u\} \times H) \cap D$

$D_1 = D \cap (\{u_1\} \times (H - \pi(N(D_0))))$

$D_2 = D \cap (\{u_2\} \times (H - \pi(N(D_0 \cup D_1))))$

$D_3 = D \cap (\{u_2\} \times (H - \pi(N(D_0 \cup D_1 \cup D_2))))$

⋮

⋮

$D_n = D \cap (\{u_n\} \square (H - \pi(N(D_0 \cup D_1 \cup D_2 .... \cup D_{n-1}))))$

Then $\pi(D_0) \cup \pi(D_1) \cup \pi(D_2) .... \pi(Dn)$ dominates H

Therefore,

$$\sum_{i=0}^{n} |D_i| \geq \gamma(H) \tag{3.6.1}$$

Now, let $D^1 = D - (D_0 \cup D_1 \cup .... \cup D_n)$ and $K = \{v \in H : v$ is not dominated by $\pi(D^1)\}$

Then, $K \subset \{v \in H : (u_i, v)$ is not dominated by D for some $u_i\} \cup \{v \in H : (u_i, v) \notin D, \forall i\}$

For, Let $v \in K$ and if $v \notin \{v \in H : (u_i, v)$ is not dominated by D for some $i\}$

ie, (u_i, v) is dominated by $D, \forall i$

Then, (u_i, v) is dominated by $\{D_0 \cup D_1 \cup ....D_n\}$

If $(u_i, v) \in D_i$ .Then for l different from $i, (u_l, v)$ is not dominated by $D_0 \cup D_1 \cup ......D_n$ and hence not dominated by D.

Hence $(u_i, v) \notin D_i, \forall i$

Therefore , $v \in N(\pi(D_i))$ for $i = 0, 1, ..., n$

Since (u_i, v) is not dominated by $D^1, \forall i = 1, 2, ..., n$

We have , $(u_i, v) \notin D^1$ for $i = 1, 2, ..., n$

Hence , $(u_i, v) \notin D$ for $i = 1, 2, ..., n$

Hence, $K \subset \{v \in H : (u_i, v)$ is not dominated by D for some $u_i\} \cup \{v \in H : (u_i, v) \notin D, \forall i\}$.

Put $F = \{v \in H : (u_i, v)$ is not dominated by D for some $u_i\} \cup \{v \in H : (u_i, v) \notin D, \forall i\}$

Then , $|D^1| + \gamma(< K >) \geq \gamma(H)$

And so ,

$$|D^1| + \gamma(< F >) \geq \gamma(H) \tag{3.6.2}$$

From equation 1 and 2 ,

$\sum_{i=0}^{n} |D_i| + |D^1| + \gamma(< F >) \geq \gamma(H) + \gamma(H)$

Therefore , $|D| \geq 2\gamma(H) - \gamma(< F >)$

Hence , the result.

□

The following result generalizes theorem 3.6.1.

Theorem 3.6.2. *Let G be any graph with $\gamma(G) = 1$ and $N \subset V(G)$ such that*

$r\gamma_N(G) > 1$.*And let* H *be any graph. Let* $D \subset G\Box H$ *with the property that* D *dominates* $(G-N)\Box H$.*Then,* $|D| \geq 2\gamma(H) - \gamma(< F >)$, *where* $F = \{v \in H : (u,v)$ *is not dominated by* D *for some* $u \in N\} \cup \{v \in H : (u,v) \notin D, \forall u \in N\}$.

Theorem 3.6.3. *Let* $G = K_{1,N}$ *with* u *as center and* $V(G) = \{u\} \cup N$ *where* $N = N_1 \cup N_2 \cup ....N_k$, $|N_i| \geq 2, N_i = \{u_{i1}, u_{i2}, ..., u_{ir}\}$ *for* $i = 1, 2, ...k$ *and* H *be any graph. If* $D \subset V(G\Box H)$ *so that* D *dominates* $\{u\}\Box H$, *then* $|D| \geq (k+1)\gamma(H) - \sum_{i=1}^{k} \gamma(< F_i >)$, *where* $F_i = \{v \in H : (u_{ij}, v)$ *is not dominated by* D *for some* $j\} \cup \{v \in H : (u_{ij}, v) \notin D \cap N_i, \forall j\}$.

Proof. We are using induction on k. As by above theorem , we can see that the result is true for $k = 1$.

Assume the result is true for $k - 1$,

Now, let $N_k = \{u_{k1}, u_{k2}, ..., u_{kr}\}$

$D_k = N_k \cap D$

$B = \{v \in H : (u, v)$ is only dominated by $D_k\}$

And let,

$D_{k1} = D_k \cap \{u_{k1}\} \times B$

$D_{k2} = D_k \cap (\{u_{k2}\} \times (B - \pi(N(D_{k1}))$

$\vdots$

$\vdots$

$\vdots$

$D_{kr} = D_k \cap (\{u_{k2}\} \times (B - \pi(N(D_{k1})) \cup \pi(N(D_{k2})).... \cup \pi(N(D_{k(r-1))}))$

And $D_k^1 = D_{k1} \cup D_{k2} \cup ....D_{kr}$

Then ,

$$|D_k^1| \geq \gamma(< B >) \tag{3.6.3}$$

Also

Let , $C_k = \{v \in H : v$ is not dominated by $\pi((D_k) - (D_k^1))\}$

Then

$C_k \subset F_k$ where

$F_k = \{v \in H : (u_{kj}, v)$ is not dominated by D for some $j\}$

$\cup\{v \in H : (u_{kj}, v) \notin D \cap N_k, \forall j\}$

Therefore,

$$|((D_k) - (D_k^1))| \geq \gamma(H) - \gamma(< F_k >) \tag{3.6.4}$$

From equation 3 and 4

$$|D_k| \geq \gamma(< B >) + \gamma(H) - \gamma(< F_k >) \tag{3.6.5}$$

Now, $(D - D_k)$ together with $\gamma(< B >)$ elements in $\{u\}\Box H$ dominates $\{u\}\Box H$

Therefore by induction hypothesis ,

$$|D - D_k| + \gamma(< B >) \geq (k)\gamma(H) - \sum_{i=1}^{k-1} \gamma(< F_i >) \tag{3.6.6}$$

where $F_i = \{v \in H : (u_{ij}, v)$ is not dominated by D for some $j\} \cup \{v \in H : (u_{ij}, v) \notin D \cap N_i, \forall j\}$

Therefore , from equation 5 and 6

$$|D| \geq (k + 1)\gamma(H) - \sum_{i=1}^{k} \gamma(< F_i >) \tag{3.6.7}$$

Hence , the result.

□

3.7 Conclusion

The concept of A-reserved domination number in graphs is introduced in this chapter. Necessary and sufficient condition for an A-reserved dominating set to be a minimal A-reserved dominating set is obtained. A-reserved domination number in some classes of graphs are obtained. These parameters in certain product graphs are studied. Generalized concept of A-reserved domination number is studied. In the sixth section domination number of cartesian product of graphs is studied using reserved domination number.

CHAPTER 4

Stability of Domination in Graphs

4.1 Introduction

Motivated by numerous applications various types of dominating sets such as Roman domination, Total domination, fractional domination were introduced and studied in graph theory. For the purpose of studying stability of domination in graphs, one such domination called α-stable domination is introduced in this chapter. This chapter is divided into seven sections. In the second Section of this Chapter, the concept of α-d- stable domination number of a graph G is introduced. The concept of α-a- stable domination number of a graph G is introduced in the third section. In section four, α- stable domination number of a graph G is introduced. In the fifth section α-stable domination in product graphs are studied. α-stable domination in corona of graphs is studied in sixth section. Seventh section is a concluding section.

In this chapter the elements of a dominating set are called donors and the other vertices are called acceptors. A subset D of vertices in a social network graph with the condition that each member in D dominates almost equally many members in $V - D$, or that each member in $V - D$ is dominated by almost equally many members in D, or both, plays a key role. This concept of equitable domination in graphs was defined and studied by A Anitha, S Arumugam and Mustapha Chellali[2].

In social network problems related to marketing, banking and others the instability affects the system when adjacent acceptors are dominated by unequal number of donors or adjacent donors dominates unequal number of acceptors. This situation become worse when the instability is large. Motivated from this idea the concept of α-stable of domination number is being introduced.

4.2 α-d-stable domination

In this section, α-d-stable domination is introduced and characterization of a minimal α-d-stable dominating set is obtained. And relation between α-d-stable domination number and independent domination number is obtained. Nordhaus-Gaddum type result is obtained. For any non negative integer β, graph G with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)...... > \gamma_d^\beta(G)$ is constructed.

Definition 4.2.1. *Let D be a dominating set. For a vertex u in D let $\psi_D(u) = |N(u) \cap (V - D)|$. The donor instability or d-instability of an edge e connecting two donor vertices u and v, $d_{inst}^D(e)$= $|\psi_D(u) - \psi_D(v)|$. Let $D \subset V$, the d-instability of D, is the sum of d-instabilities of all edges connecting vertices in D, $\psi_d(D) = \sum_{e \in <D>} d_{inst}^D(e)$.*

Definition 4.2.2. *Let D be a dominating set. Given a non negative integer α, D is an α-d-stable dominating set, if $d_{inst}^D(e) \leq \alpha$ for any edge e connecting two donor vertices. Cardinality of a minimum α-d-stable dominating set is the α-d-stable domination number and denoted by $\gamma_d^\alpha(G)$.*

Definition 4.2.3. *A dominating set D is d-stable if $\psi_d(D) = 0$. Cardinality of a minimum d-stable dominating set is the d-stable domination number and denoted by $\gamma_d^0(G)$.*

Remark 4.2.1. *If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma_d^\alpha(G) \leq \gamma_d^\beta(G)$.*

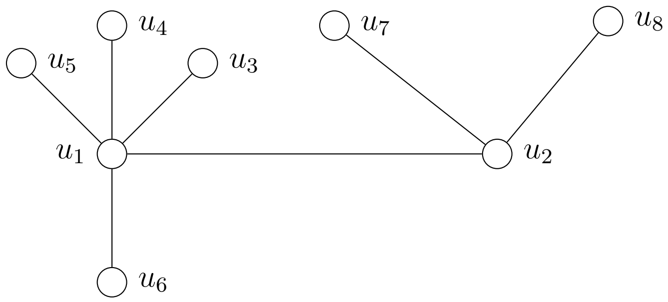

Figure 4.1

Example 4.2.1. *In Figure 4.1, $D = \{u_1, u_2\}$ is the minimum dominating set. $\psi_D(u_1) = 4$ and $\psi_D(u_2) = 2$. And $d^D_{inst}(u_1u_2) = 2$. Hence D is a minimum 2-d-stable dominating set. And for $\alpha \geq 2$, $\gamma^{\alpha}_d(G) = 2$.*
If $S = \{u_1, u_7, u_8\}$, $\psi_S(u_1) = 3$, $\psi_S(u_7) = 0$ and $\psi_S(u_8) = 0$. And S is an independent set. Hence $\gamma^1_d(G) = \gamma^0_d(G) = 3$.

Remark 4.2.2. *Property of being α-d-stable dominating set is neither superhereditary nor hereditary.*

Theorem 4.2.4. *An α-d- stable dominating set D is a minimal α-d- stable dominating set if and only if for each vertex v in D one of the following conditions holds*

1. *v is an isolate of D.*
2. *v has a private neighbour u in $V - D$.*
3. *There exist two adjacent vertices u_1 and u_2 different from v in D, u_1 adjacent to v, u_2 not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$.*

Proof. If an α-d- stable dominating set D is minimal, then D is an α-d- stable dominating set and for each vertex v in D, $D - \{v\}$ is not an α-d- stable dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or there exist two adjacent vertices u_1 and u_2 different from v in D with $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$ but $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$.

Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u

in $V - D$.

If $|\psi_D(u_1) - \psi_D(u_2)| \leq \alpha$ and $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| > \alpha$, let $\alpha = 0$, then $\psi_D(u_1) = \psi_D(u_2)$ and $|\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2)| = \alpha + 1$. Assume $\psi_{D-\{v\}}(u_1) > \psi_{D-\{v\}}(u_2)$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$. If $\alpha > 0$, then assume $\psi_D(u_1) > \psi_D(u_2)$. Then $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) = \alpha + 1$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$.

Conversely, suppose that D is an α-d-stable dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal α-d-stable dominating set. If D is not a minimal α-d-stable dominating set, then there exists a vertex $v \in D$ such that $D - \{v\}$ is an α-d-stable dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. And v has no private neighbour in $V - D$ and condition 2 does not hold. $D - \{v\}$ is an α-d-stable dominating set implies for any two adjacent vertices u_1 and u_2 in $D - \{v\}$, $\psi_{D-\{v\}}(u_1) - \psi_{D-\{v\}}(u_2) \leq \alpha$. Hence condition 3 does not hold. Hence D is a minimal α-d-stable dominating set.

□

Remark 4.2.3. *For non negative integer α, $\gamma_d^\alpha(G) = 1 \iff \gamma(G) = 1$.*

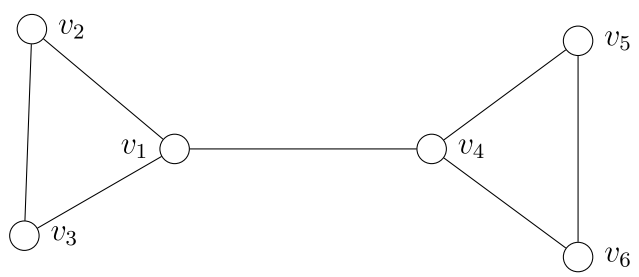

Figure 4.2

Theorem 4.2.5. *For a graph G and non negative integer α, $\beta_o(G) \geq \gamma_d^\alpha(G)$.*

Proof. Let S be a maximum independent set. Then, every vertex in $V - S$ is adjacent with atleast one vertex in S. Thus S is a dominating set. No two vertices in S are adjacent. It follows that S is an α-d-stable dominating set. Hence, $\beta_o(G) \geq \gamma_d^\alpha(G)$. And this bound is sharp. In figure 4.2, $\gamma_d^\alpha(G) = 2 = \beta_o(G)$.

□

Theorem 4.2.6. *For a graph G and non negative integer α, $i(G) \geq \gamma_d^\alpha(G)$.*

Proof. Let S be a minimum independent dominating set. No two vertices in S are adjacent. It follows that S is an α-d-stable dominating set. Hence, $i(G) \geq \gamma_d^\alpha(G)$. And this bound is sharp. In figure 4.2, $\gamma_d^\alpha(G) = 2 = i(G)$.

□

Proposition 4.2.7. *The domination chain $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_o(G) \leq \Gamma(G) \leq IR(G)$ can be extended as $ir(G) \leq \gamma(G) \leq \gamma_d^\alpha(G) \leq i(G) \leq \beta_o(G) \leq \Gamma(G) \leq IR(G)$.*

Theorem 4.2.8. *For any graph G of order $n \geq 2$ and non negative integer α, $3 \leq \gamma_d^\alpha(G) + \gamma_d^\alpha(\overline{G}) \leq n + 1$.*

Proof. Let α be any non negative integer and G be any graph order $n \geq 2$, since $\gamma_d^\alpha(G) \leq i(G)$ and $i(G) \leq n - \Delta(G)$

$$\begin{aligned} \gamma_d^\alpha(G) + \gamma_d^\alpha(\overline{G}) &\leq i(G) + i(\overline{G}) \\ &\leq (n - \Delta(G)) + (n - \Delta(\overline{G})) \\ &\leq n + 1 \end{aligned}.$$

Let G be a graph order $n \geq 2$. If $\gamma_d^\alpha(G) = 1$, then there is a vertex u of degree $n - 1$ in G. Hence u will be an isolated vertex in $\overline{G}$. Hence $\gamma_d^\alpha(\overline{G}) \geq 2$.. In a similar way if $\gamma_d^\alpha(\overline{G}) = 1$ then $\gamma_d^\alpha(G) \geq 2$. Hence $\gamma_d^\alpha(G) + \gamma_d^\alpha(\overline{G}) \geq 3$. And the lower bound is obviously obtained if $\gamma_d^\alpha(G) > 1$.

□

The bounds given in the above theorem are sharp. For $G = K_n$, $\gamma_d^\alpha(G) = 1$ and $\gamma_d^\alpha(\overline{G}) = n$ and $\gamma_d^\alpha(G) + \gamma_d^\alpha(\overline{G}) = n + 1$. For $K_{1,n-1}$, $\gamma_d^\alpha(G) = 1$ and $\gamma_d^\alpha(\overline{G}) = 2$.

Theorem 4.2.9. *For a connected triangle free graph G with $|V(G)| \geq 2$ and any non negative integer α, $\gamma_d^\alpha(\overline{G}) = 2$.*

Proof. Since G is a connected graph there is an edge uv in G. If G is isomorphic to K_2 then $\overline{G}$ is an empty graph with two vertices. Hence $\gamma_d^\alpha(\overline{G}) = 2$. If $|V(G)| > 2$, then each vertex of G is not adjacent to at least one of u or v. Hence u and v are non adjacent vertices in $\overline{G}$ and every other vertices are adjacent with either u or v or both. Hence $\{u, v\}$ is an α-d-stable dominating set of $\overline{G}$. Since G has no isolated vertex $\gamma_d^\alpha(\overline{G}) \neq 1$. Hence $\gamma_d^\alpha(\overline{G}) = 2$.

□

Theorem 4.2.10. *If D is an α-d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) = d(u) + k + \alpha$, $k \in Z^+$, then D contains at least k elements from $(N[v] - N[u])$.*

Proof. If D is an α-d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) = d(u) + k + \alpha$, $k \in Z^+$, then $|\psi_D(v) - \psi_D(u)| \leq \alpha$. Hence $|N[v] \cap (V - D)| \leq |N[u] \cap (V - D)| + \alpha$. Thus, $d(v) - d(u) \leq |(N[v] - N[u]) \cap D| + \alpha$.

Hence D contains at least k elements from $(N[v] - N[u])$.

□

Corollary 4.2.11. *If D is a d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) > d(u)$, then D contains at least $d(v) - d(u)$ elements from $(N[v] - N[u])$.*

Corollary 4.2.12. *If u is a pendant vertex adjacent to v, D is a d-stable dominating set and $u, v \in D$, then $N[v] \subset D$.*

Theorem 4.2.13. *For any non negative integer β, there exist graph G with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G) ...... > \gamma_d^\beta(G)$*

Proof. Take $k = \beta + 1$

Construct G as follows,

Step 1:- Let H be the complete graph with vertex set $\{a_1, a_2, ..., a_k\}$.

Step 2:- Let $A_i = \{a_{i,1}, a_{i,2}, ..., a_{i,i+1}\}$ for $i = 1, 2, ..., k$.

Form G by joining each vertices in A_i with a_i in H for $i = 1, 2..., k$.

Let D be a d-stable dominating set, $B_i = \{a_i, a_{i,1}, a_{i,2}, ..., a_{i,i+1}\}$ and $C_i = B_i \cap D$.

If $a_r, a_s \in D$ with $r < s$ then by corollary 1.11, D contains $s - r$ elements from A_s and hence by corollary 1.12, $N[a_s] \subset D$. Thus $B_s \subset D$. Thus $C_i = B_i$ for all $i = 1, 2, ..., k$.

Hence $D = V(G)$ or $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{s-1} \cup A_{s+1} \cup ..... \cup A_{k-1} \cup A_k \cup \{a_s\}$. Hence we take $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{s-1} \cup A_{s+1} \cup ..... \cup A_{k-1} \cup A_k \cup \{a_s\}$.

Thus $|C_i| = i + 1$ for $i \neq s$ and $|C_s| = 1$. Then,

$$\begin{aligned} |D| &= 2 + 3 + ... + (s) + 1 + (s+2) + ... + (k+1) \\ &= \tfrac{(k+1)(k+2)}{2} - (s+1). \end{aligned}$$

Hence $|D|$ is minimum when $\frac{(k+2)(k+1)}{2} - (s+1)$ is minimum. That is when $s = k$. And $D = A_1 \cup A_2 \cup A_3 \cup ...... \cup A_{k-1} \cup \{a_k\}$ will form a d- stable dominating set with $|D| = \frac{(k+2)(k+1)}{2} - (k+1)$.

Hence $\gamma_d^0(G) = \frac{(k+2)(k+1)}{2} - (k+1) = \frac{(k)(k+1)}{2}$.

Similarly,

$$\begin{aligned} \gamma_d^0(G) &= \tfrac{(k)(k+1)}{2} \\ \gamma_d^1(G) &= \tfrac{k(k-1)}{2} + 1 \\ &\vdots \\ \gamma_d^\beta(G) &= \tfrac{(k-\beta+1)(k-\beta)}{2} + \beta. \end{aligned}$$

Figure 4.3 illustrates the graph with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)$.

□

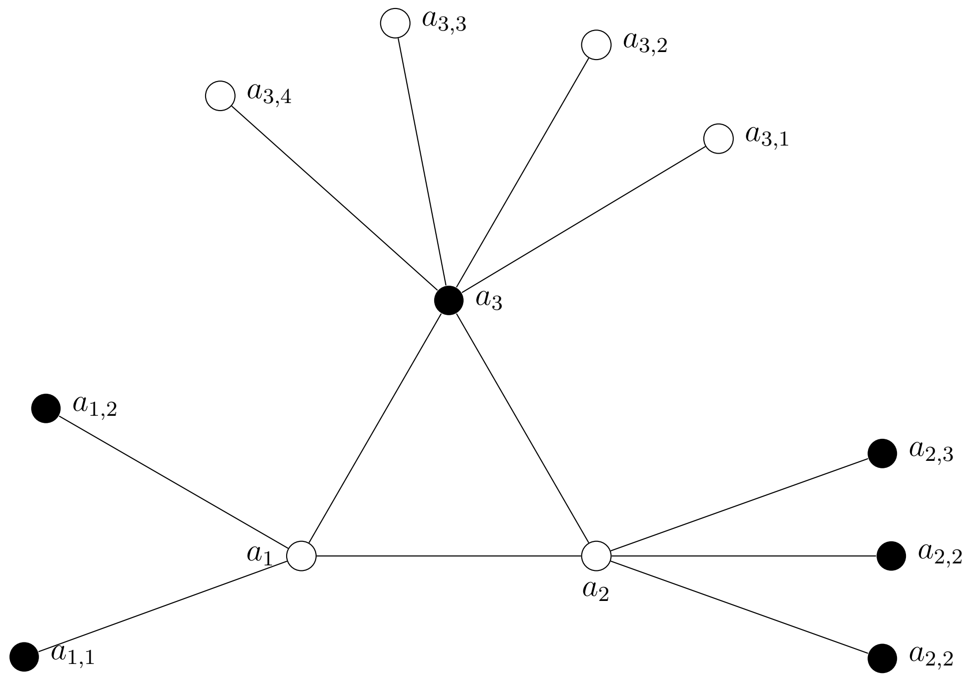

Figure 4.3

4.3 α-a-stable domination

In this section, α-a-stable domination is introduced and characterization of minimal α-a-stable dominating set is obtained. And relation between α-a-stable domination number and perfect domination number is obtained. For any non negative integer β, graph G with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)...... > \gamma_a^\beta(G)$ is obtained.

Definition 4.3.1. *Let D be a dominating set. For a vertex u not in D, let $\phi_D(u)$ $= |N(u) \cap D|$. The Acceptor Instability or a-instability of an edge e connecting two acceptor vertices u and v is, $a_{inst}^D(e) = |\phi_D(u) - \phi_D(v)|$. The a-instability of D, $\phi_a(D)$ is the sum of a-instabilities of all edges connecting vertices in $V - D$, $\phi_a(D) = \sum_{e \in <V-D>} a_{inst}^D(e)$.*

Definition 4.3.2. *Let D be a dominating set. Given a non negative integer α, D is an α-a-stable dominating set, if $a_{inst}^D(e) \leq \alpha$ for any edge e connecting two acceptor vertices. Cardinality of a minimum α-a-stable dominating set is α-a-stable domination number and denoted by $\gamma_a^\alpha(G)$.*

Definition 4.3.3. *The dominating set D is a-stable if $\phi_a(D) = 0$. Minimum* *cardinality of an a-stable dominating set is a-stable domination number and denoted by* $\gamma_a^0(G)$.

Remark 4.3.1. *If* $\alpha \geq \beta$, *then* $\gamma(G) \leq \gamma_a^\alpha(G) \leq \gamma_a^\beta(G)$

Example 4.3.1. *In Figure 4.1,* $D = \{u_1, u_2\}$ *is the minimum dominating set.* $\phi_D(u_3) = \phi_D(u_4) = \phi_D(u_5) = \phi_D(u_6) = \phi_D(u_7) = \phi_D(u_8) = 1$. *Hence* D *is a minimum a-stable dominating set and* $\gamma_a^\alpha(G) = 2$ *for all non negative integer* α.

Remark 4.3.2. *Property of being* α*-a-stable dominating set is neither superhereditary nor hereditary.*

Theorem 4.3.4. *An* α*-a- stable dominating set* D *is a minimal* α*-a- stable dominating set if and only if for each vertex* v *in* D *one of the following conditions holds*

1. v *is an isolate of* D.
2. v *has a private neighbour* u *in* $V - D$.
3. *There exist two adjacent vertices* u_1 *and* u_2 *in V-D,* u_1 *adjacent to* v, u_2 *not adjacent to* v *and* $\phi_D(u_2) = \phi_D(u_1) + \alpha$.

Proof. If an α-a- stable dominating set D is minimal then D is an α-a- stable dominating set and for each vertex v in D, $D - \{v\}$ is not an α-a- stable dominating set. This means that some vertex u in $(V - D) \cup \{v\}$ is not dominated by $D - \{v\}$ or there exist two adjacent vertices u_1 and u_2 in $V - D$ with $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$ but $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$.

Now if some vertex u in $(V - D) \cup \{v\}$ is not dominated by any vertex in $D - \{v\}$, either $u = v$, means v is an isolate of D or $u \in V - D$. If u is not dominated by $D - \{v\}$, then u is adjacent only to vertex v in D. ie, v has a private neighbour u in $V - D$.

If $|\phi_D(u_1) - \phi_D(u_2)| \leq \alpha$ and $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| > \alpha$, let $\alpha = 0$, then $\phi_D(u_1) = \phi_D(u_2)$ and $|\phi_{D-\{v\}}(u_1) - \phi_{D-\{v\}}(u_2)| = \alpha + 1$. Assume $\phi_{D-\{v\}}(u_2) > \phi_{D-\{v\}}(u_1)$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$. If $\alpha > 0$, then assume $\phi_D(u_2) > \phi_D(u_1)$. Then $\phi_D(u_2) - \phi_D(u_1) = \alpha$

and $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) = \alpha+1$. Then u_1 is adjacent to v but u_2 is not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$.

Conversely, suppose that D is an α-a-stable dominating set and for each vertex $v \in D$, one of the three statements holds. We show that D is a minimal α-a-stable dominating set. If D is not a minimal α-a-stable dominating set,then there exists a vertex $v \in D$ such that $D - \{v\}$ is an α-a-stable dominating set. Then each vertex u in $(V - D) \cup \{v\}$ is adjacent with atleast one vertex in $D - \{v\}$. Then v is not an isolate of D and condition 1 does not hold. And v has no private neighbour in $V - D$ and condition 2 does not hold. If $D - \{v\}$ is an α-a-stable dominating set then for any adjacent vertices u_1 and u_2 in $(V - D) \cup \{v\}$, $\phi_{D-\{v\}}(u_2) - \phi_{D-\{v\}}(u_1) \leq \alpha$. Hence condition 3 does not hold. Hence D is a minimal α-a-stable dominating set.

□

Remark 4.3.3. *For non negative integer α, $\gamma_a^\alpha(G) = 1 \iff \gamma(G) = 1$*

Theorem 4.3.5. *For $\alpha \geq 1$, $\gamma_a^\alpha(G) = 2 \iff \gamma(G) = 2$*

Proof. If $\gamma(G) = 2$, then for a minimum dominating set D, $|D| = 2$

$$|D| = 2 \Rightarrow \phi_D(v) = 1 \quad or \quad \phi_D(v) = 2 \quad \forall v \in V - D$$

$$\Rightarrow |\phi_D(v_1) - \phi_D(v_2)| \leq 1, \qquad \forall v_1, v_2 \in V - D$$

$$\Rightarrow \gamma_a^\alpha(G) = 2$$

Conversely, if $\gamma_a^\alpha(G) = 2$, then $\gamma(G) \neq 1$. If D is a minimum α-a- stable dominating set , then $|D| = 2$, and D is a dominating set. Thus, $\gamma(G) = 2$

□

Theorem 4.3.6. *For any graph G and non negative integer α, $\gamma_a^\alpha(G) \leq \gamma_p(G)$. And this bound is sharp.*

Proof. If D is a perfect dominating set, then every vertex in $V - D$ is adjacent with exactly one vertex in D. And hence $\phi_D(v) = 1$, for all $v \in (V - D)$. And so D

is an a-stable dominating set . Thus every perfect dominating set is an a-stable dominating set . Hence, $\gamma_a^\alpha(G) \leq \gamma_p(G)$.

For $G = P_{3n}$, $\gamma_p(G) = n = \gamma_a^\alpha(G)$. So we can see that the bound is sharp.

□

Theorem 4.3.7. *For any positive integer β, there exist graph G with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G) ...... > \gamma_a^\beta(G)$.*

Proof. Let $k = \beta + 1$

Construct G as follows

Step 1:Let H be the complete graph with vertex set $\{a_1, a_2, ..., a_k\}$

Step 2: For each i take i copies of P_3 with vertex set $\{b_{i,j}, b'_{i,j}, b_{i,j}"\}$ for $j = 1, 2, ..., i$ and join $b'_{i,j}$ with a_i for each $j = 1, 2, ..., i$.

Let $A_i^j = \{b_{ij}, b'_{ij}, b_{ij}"\}$ for $j = 1, 2, ...i$ and $A_i = \{a_i\} \cup \cup_{j=1}^{i} \{b_{ij}, b'_{ij}, b_{ij}"\}$ for all $i \in \{1, 2, ..., k\}$.

Let D be an a-stable dominating set. If $a_i \in D$ then $|A_i \cap D| \geq i + 1$.

Let r be the smallest integer such that $a_r \notin D$. Then $|A_r \cap D| \geq r$.

If $s > r$ and $a_s \notin D$, since $\gamma^a_{inst}(a_r, a_s) = 0$, $b'_{sj} \in D$ for atmost r values of j. And if there exist j for which $b'_{sj} \notin D$ then $b_{sj}, b_{sj}" \in D$.

$$\Longrightarrow \begin{array}{rcl} |A_s \cap D| & \geq & r + 2(s - r) \\ & = & 2s - r \geq s + 1 \end{array}$$

$$\begin{array}{rcl} Hence, |D| & \geq & |A_1 \cap D| + |A_2 \cap D| + |A_3 \cap D| + ... + |A_{r-1} \cap D| + \\ & & |A_r \cap D| + |A_{r+1} \cap D| + ... + |A_k \cap D| \\ & \geq & 2 + 3 + ... + (r - 1 + 1) + r + (r + 2) + ... + (k + 1) \ . \\ & = & 1 + 2 + ... + k + k - 1 \\ & = & \frac{k(k+1)}{2} + (k - 1) \end{array}$$

Thus, $\gamma_d^0(G) \geq \frac{k(k+1)}{2} + (k - 1)$.

And $D' = \{a_1, ..., a_{k-1}\} \cup \cup_{i=1}^{k} \{b'_{i1}, b'_{i2}, ...b'_{ii}\}$ is an a-stable dominating set with $|D'| = \frac{k(k+1)}{2} + (k - 1)$.

Hence, $\gamma_a^0(G) \leq \frac{k(k+1)}{2} + (k - 1)$

Thus, $\gamma_a^0(G) = \frac{k(k+1)}{2} + (k - 1)$

Similarly,

$$\begin{aligned}
\gamma_a^1(G) &= \tfrac{k(k+1)}{2} + (k-2)\\
\gamma_a^2(G) &= \tfrac{k(k+1)}{2} + (k-3)\\
\gamma_a^3(G) &= \tfrac{k(k+1)}{2} + (k-4)\\
&\vdots\\
\gamma_a^{\beta-1}(G) &= \tfrac{k(k+1)}{2} + 1\\
\gamma_a^{\beta}(G) &= \tfrac{k(k+1)}{2}
\end{aligned}$$

Figure 4.4 illustrates the graph with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)$.

□

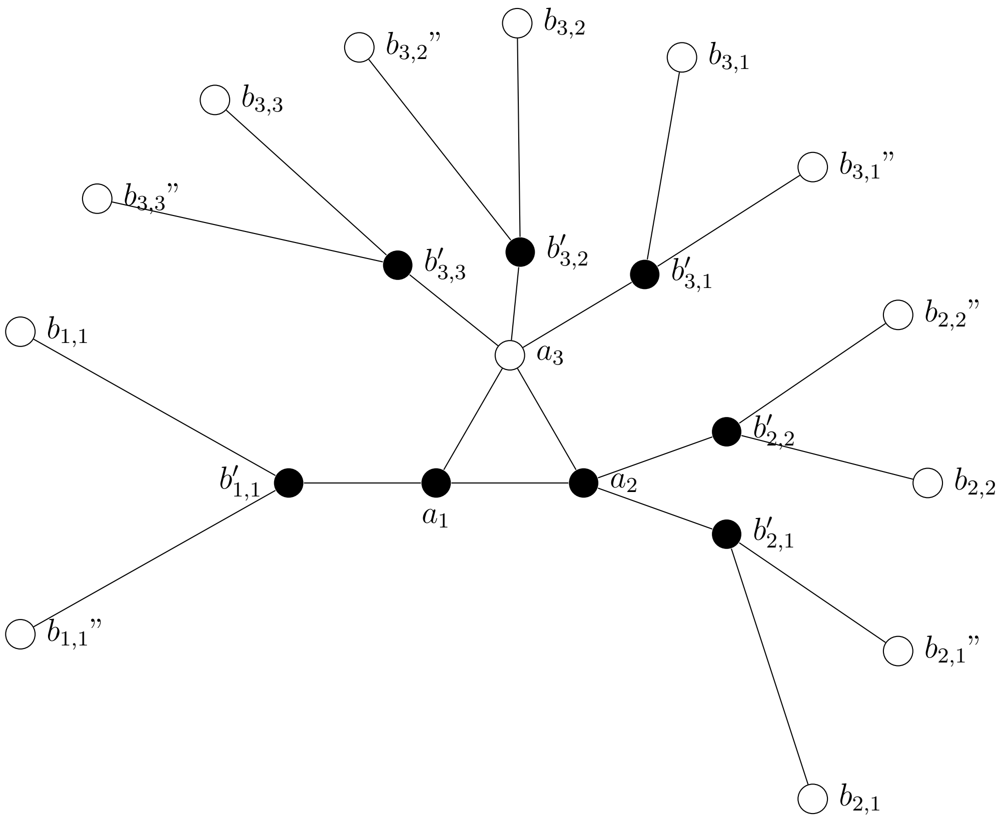

Figure 4.4

4.4 α-stable domination

In this section, the concept of α -stable domination and stable dominating index is introduced and bound for α -stable domination number is obtained. α-stable domination number for standard graphs are obtained.

Definition 4.4.1. *A dominating set D is stable, if $\psi_d(D) = 0$ and $\phi_a(D) = 0$. Minimum cardinality of a stable dominating set is called stable domination number and denoted by $\gamma^0(G)$.*

Definition 4.4.2. *If a dominating set D is an α-d-stable dominating set and α-a -stable dominating set, then D is called an α-stable dominating set and cardinality of a minimum α-stable dominating set is defined as α-stable domination number and denoted by $\gamma^\alpha(G)$*

Remark 4.4.1. *If a minimum α -a-stable dominating set is an α-d-stable dominating set, then $\gamma^\alpha(G) = \gamma_a^\alpha(G)$. And if a minimum α-d- stable dominating set is an α-a-stable dominating set, then $\gamma^\alpha(G) = \gamma_d^\alpha(G)$.*

Remark 4.4.2. *If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma^\alpha(G) \leq \gamma^\beta(G)$.*

Definition 4.4.3. *Minimum α so that $\gamma^\alpha(G) = \gamma(G)$ is called stable dominating index and denoted by $I_{sd}(G)$.*

Example 4.4.1. *In figure 4.1, the minimum d-stable dominating set $\{u_1, u_7, u_8\}$ is an a-stable dominating set. Hence $\{u_1, u_7, u_8\}$ is a minimum stable dominating set and $\gamma^0(G) = 3$.*
A minimum 1-d-stable dominating set $\{u_1, u_7, u_8\}$ form a 1-a- stable dominating set and hence $\{u_1, u_7, u_8\}$ is a minimum 1-stable dominating set and $\gamma^1(G) = 3$.
And minimum dominating set $\{u_1, u_2\}$ is a 2-a- stable dominating set and a 2-d-stable dominating set. $\{u_1, u_2\}$ form a minimum 2-stable dominating set. Hence, $\gamma^2(G) = 2$.
And $\forall \alpha \geq 2$, $\gamma^\alpha(G) = 2 = \gamma(G)$. Hence, $I_{sd}(G) = 2$.

Compliment of a minimum α-stable dominating set need not be an α-stable dominating set. In graph figure 4.1, $\{u_1, u_2, u_3\}$ is a minimum 1-stable dominating set but its compliment is not a 1-stable dominating set.

Remark 4.4.3. *Property of being α-stable dominating set is neither superhereditary nor hereditary.*

Theorem 4.4.4. *For any graph G and for any non-negative integer α, $\gamma(G) = 1 \iff \gamma^\alpha(G) = 1$.*

Proof. If $\gamma(G) = 1$, then the single vertex set $\{v\}$ which dominates all vertices of G, is an α-d-stable dominating set and an α-a-stable dominating set. Then $\gamma^\alpha(G) = 1$. Also any α-stable dominating set is a dominating set. So, if $\gamma^\alpha(G) = 1$ then $\gamma(G) = 1$.

□

Lemma 4.4.5. *For any graph G and for any non negative integer α, $\gamma^\alpha(G) = n \iff G = \overline{K_n}$.*

Proof. If $G \neq \overline{K_n}$, there is atleast one vertex v with $d(v) \geq 1$. Then $V - \{v\}$ is an α-stable dominating set. This means that $\gamma^\alpha(G) \leq n - 1$. Hence if $\gamma^\alpha(G) = n$, then $G = \overline{K_n}$. If $G = \overline{K_n}$, then $\gamma^\alpha(G) = n$ trivially.

□

Theorem 4.4.6. *For a graph G with $\delta(G) \geq 1$, $\gamma^\alpha(G) \leq n - 1$.*

Proof. From lemma 4.4.5 it is clear that $\gamma^\alpha(G) \leq n - 1$.

□

Theorem 4.4.7. *For every graph G of order n and maximum degree Δ and for any non negative integer α, $\gamma^\alpha(G) \geq \frac{n}{\Delta+1}$.*

Proof. Since $\gamma^\alpha(G) \geq \gamma(G)$ and $\gamma(G) \geq \frac{n}{\Delta+1}$, $\gamma^\alpha(G) \geq \frac{n}{\Delta+1}$.

□

Theorem 4.4.8. *For any non negative integer α, $\gamma^\alpha(G) = \gamma(G)$ for the following Graphs*

- *Complete graph K_n*
- *Path P_n*
- *Cycle C_n*
- *Wheel graph W_n*

- *Helm graph* H_n

Proof. In these graphs minimum dominating set D, form an α-stable dominating set. Hence α-stable domination number is same as its domination number.

□

Remark 4.4.4. *If* G *is the corona* $C_p \circ K_1$*, then for any non negative integer* α*,* $i(G) = \gamma_d^\alpha(G) = \gamma_a^\alpha(G) = \gamma^\alpha(G) = \gamma(G) = p$*.*

Proof. Clearly the pendant vertices will form an α-stable dominating set and independent dominating se. Hence $i(G) = \gamma_d^\alpha(G) = \gamma_a^\alpha(G) = \gamma^\alpha(G) = \gamma(G) = p$.

□

Theorem 4.4.9. *For complete bipartite graph* $G = K_{m,n}, m \leq n$ *and non-negative integer* α

$$\gamma_a^\alpha(G) = 2$$

$$\gamma_d^\alpha(G) = \begin{cases} min\{(n-m+2-\alpha), m\} & if n-m+2-\alpha \geq 2 \\ m & otherwise \end{cases}$$

$$\gamma^\alpha(G) = \begin{cases} 2 & if n-m \leq \alpha \\ m & otherwise \end{cases}$$

Proof. Let $X = \{v_1, v_2, ..., v_m\}$ and $Y = \{u_1, u_2, ..., u_n\}$ be the bipartition of $V(G)$. Then $\{v_1, u_1\}$ is a minimum α-a- stable dominating set of G. Hence $\gamma_a^\alpha(G) = 2$.

Let D be a minimum α-d-stable dominating set.

Case 1

Let D intersects both X and Y and $|D \cap X| = l$ and $|D \cap Y| = k$. Then $d_{inst}^D(e) = |(n-k)-(m-l)|$. Thus $d_{inst}^D(e) \leq \alpha$ only if $|(n-k)-(l-m)| \leq \alpha$. Then $(n-k)-(l-m) \leq \alpha$ or $(n-k)-(l-m) \geq -\alpha$. Since D is minimum α-d-stable dominating set $(n-k)-(m-l) \leq \alpha$. Thus $(m-l)-(n-k) \leq \alpha$. Hence $k \geq n-m+l-\alpha$. Thus k and l are minimum when $l = 1$ and $k = n-m+1-\alpha$. Thus $|D| \geq n-m+2-\alpha$. And $D = \{u_1, v_1, v_2....., v_{n-m+1-\alpha}\}$ is an α-d-stable dominating set of $|D| = n-m+2-\alpha$. Hence $\gamma_d^\alpha(G) = n-m+2-\alpha$.

Case 2

Let D intersects with X only. Then $|D| = m$.

Case 3

Let D intersects with Y only. Then $|D| = n$.

Thus $\gamma_d^{\alpha}(G) = \begin{cases} min\{(n-m+2-\alpha), m\} & if n-m+2-\alpha \geq 2 \\ m & otherwise \end{cases}$

Let D be a minimum α-stable dominating set. Case 1

Let $|D \cap X| = l$ and $|D \cap Y| = k$. Since $d_{inst}^{D}(e) \leq \alpha$ and $a_{inst}^{D}(e) \leq \alpha$, $|l-k| \leq \alpha$ and $|(n-k)-(m-l) \leq \alpha$. Hence if $n-m > \alpha$, $k=0$ and $l=m$ and if $n-m \leq \alpha$, $k=1$ and $l=1$. Thus, $\gamma^{\alpha}(G) = \begin{cases} 2 & if n-m \leq \alpha \\ m & otherwise \end{cases}$

□

4.5 α-stable domination in product graphs

4.5.1 α-stable domination in cartesian product of graphs

Theorem 4.5.1. *Let G and H be two graphs of order n_1 and n_2 , then for any non-negative integer α ,*

- $\gamma_a^{\alpha}(G \Box H) \leq min\{n_1\gamma_a^{\alpha}(H), n_2\gamma_a^{\alpha}(G)\}$
- $\gamma_d^{\alpha}(G \Box H) \leq min\{n_1\gamma_d^{\alpha}(H), n_2\gamma_d^{\alpha}(G)\}$
- $\gamma^{\alpha}(G \Box H) \leq min\{n_1\gamma^{\alpha}(H), n_2\gamma^{\alpha}(G)\}$.

Proof. Let S_H be a minimum α-a-stable dominating set of H. Let us see that $S = V(G) \times S_H$ is an α-a-stable dominating set of $G \Box H$. If $(u,v) \in (V(G) \times V(H)) - S$. Then (u,v) is adjacent to atleast one vertex in S. And if $(u_1,v_1) \in (V(G) \times V(H)) - S$ and $(u_2,v_2) \in (V(G) \times V(H)) - S$ and (u_1,v_1) adjacent to (u_2,v_2). Then,

$$\begin{aligned} \phi_S(u_1,v_1) &= |\{(u,v) \in S : (u_1,v_1) \ \ adjacent \ \ to \ \ (u,v)\}| \\ &= |\{(u,v) \in S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\} \cup \\ & \quad \{(u,v) \in S : u_1 \ \ adjacent \ \ to \ \ u \ \ and \ \ v_1 = v\}| \end{aligned}$$

Since $\{(u,v) \in S : u_1 \ \ adjacent \ \ to \ \ u \ \ and \ \ v_1 = v\} = \phi$

$$\begin{aligned} \phi_S(u_1,v_1) &= |\{(u,v) \in S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\}| \\ &= \phi_{S_H}(v_1). \end{aligned}$$

Similarly $\phi_S(u_2, v_2) = \phi_{S_H}(v_2)$. And

$$\begin{aligned} |\phi_S(u_1,v_1) - \phi_S(u_2,v_2)| &= |\phi_{S_H}(v_1) - \phi_{S_H}(v_2) \\ &\leq \alpha. \end{aligned}$$

Hence S is an α-a-stable dominating set of $G \square H$.

Similarly, if S_G is a minimum α-a-stable dominating set of G, then $S_G \times V(H)$ is an α-a-stable dominating set of $G \square H$.

Thus,

$\gamma_a^\alpha(G \square H) \leq min\{n_1\gamma_a^\alpha(H), n_2\gamma_a^\alpha(G)\}$.

Let S_H be a minimum α-d-stable dominating set of H. Let $S = V(G) \times S_H$. If $(u, v) \in (V(G) \times V(H)) - S$, then (u, v) is adjacent to atleast one vertex in S. And if $(u_1, v_1) \in S$ and $(u_2, v_2) \in S$ and (u_1, v_1) adjacent to (u_2, v_2). Then,

$$\begin{aligned} \psi_S(u_1,v_1) &= |\{(u,v) \in (V(G) \times V(H)) - S : (u_1,v_1) \ \ adjacent \ \ to \ \ (u,v)\}| \\ &= |\{(u,v) \in (V(G) \times V(H)) - S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\} \cup \\ & \quad \{(u,v) \in (V(G) \times V(H)) - S : u_1 \ \ adjacent \ \ to \ \ u \ \ and \ \ v_1 = v\}| \end{aligned}$$

Since, $\{(u,v) \in (V(G) \times V(H)) - S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\} = \phi$

$$\begin{aligned} \psi_S(u_1,v_1) &= |\{(u,v) \in (V(G) \times V(H)) - S : u_1 = u \ \ and \ \ v_1 \ \ adjacent \ \ to \ \ v\}| \\ &= \psi_{S_H}(v_1). \end{aligned}$$

Similarly $\psi_S(u_2, v_2) = \psi_{S_H}(v_2)$.

Thus

$$\begin{aligned} |\psi_S(u_1,v_1) - \psi_S(u_2,v_2)| &= |\psi_{S_H}(v_1) - \psi_{S_H}(v_2) \\ &\leq \alpha \end{aligned}$$

Thus S is an α-d-stable dominating set of $G \square H$.

Similarly, if S_G is a minimum α-d-stable dominating set of G, $S_G \times V(H)$ is an α-d-stable dominating set of $G \square H$. Thus,

$\gamma_d^\alpha(G \square H) \leq min\{n_1\gamma_d^\alpha(H), n_2\gamma_d^\alpha(G)\}$.

Hence, $\gamma^\alpha(G \square H) \leq min\{n_1\gamma^\alpha(H), n_2\gamma^\alpha(G)\}$.

□

Remark 4.5.1. *The bound in theorem 4.5.1 is attained if $G = K_n$ and $H = K_2$; because $\gamma^\alpha(K_n \square K_2) = 2 = min\{2\gamma^\alpha(K_n), n\gamma^\alpha(K_2)\}$.*

Theorem 4.5.2. *For any graph G of order m and any non negative integer α, $\gamma^{\alpha}(C_n \Box G) \geq \frac{mn}{\Delta(G)+3}$.*

Proof. Since $\Delta(C_n \Box G) = \Delta(G) + 2$ and $|V(C_n \Box G)| = mn$, by theorem 4.4.7 $\gamma^{\alpha}(C_n \Box G) \geq \frac{mn}{\Delta(G)+3}$.

□

Theorem 4.5.3. *If $m, n \geq 2$ for any non negative integer α, $\gamma^{\alpha}(C_m \Box C_n) \geq \frac{mn}{5}$.*

Proof. By theorem 4.5.2,

$$\begin{aligned} \gamma^{\alpha}(C_m \Box C_n) &\geq \frac{mn}{\Delta(C_m)+3} \\ &= \frac{mn}{5} \end{aligned}$$

□

Theorem 4.5.4. *If $m, n \geq 2$ for any non negative integer α, $\gamma^{\alpha}(C_m \Box C_n) \geq \gamma^{\alpha}(C_m)\gamma^{\alpha}(C_n)$.*

Proof. By theorem 4.5.3,

$$\begin{aligned} \gamma^{\alpha}(C_m \Box C_n) &\geq \frac{mn}{5} \\ &\geq \lceil \frac{m}{3} \rceil \lceil \frac{n}{3} \rceil \\ &= \gamma^{\alpha}(C_m)\gamma^{\alpha}(C_n) \end{aligned}$$

Hence, $\gamma^{\alpha}(C_m \Box C_n) \geq \gamma^{\alpha}(C_m)\gamma^{\alpha}(C_n)$.

□

Theorem 4.5.5. *For non-negative integer α, $\gamma^{\alpha}(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$.*

Proof. Let $x_1, x_2, ..., x_n$ be the vertices in the first copy of P_n and $y_1, y_2, ..., y_n$ be the vertices in the second copy of P_n.

Case 1

If n is odd, D consists of those vertices x_i, y_j where $i \equiv 1(mod4)$ and $j \equiv 3(mod4)$ will form an α-stable dominating set with $|D| = |\{x_i : i \equiv 1(mod4)\}| + |\{y_j : j \equiv 3(mod4)\}| = \lceil \frac{n+1}{2} \rceil$. Hence $\gamma^{\alpha}(P_2 \Box P_n) \leq \lceil \frac{n+1}{2} \rceil$. Since $\gamma^{\alpha}(P_2 \Box P_n) \geq \gamma(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$, $\gamma^{\alpha}(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$.

Case 2

If n is even and $n \equiv 0(mod4)$ then $D = \{x_i : i \equiv 1(mod4)\} \cup \{x_{n-2}\} \cup \{y_j : j \equiv 3(mod4) \& j \leq n-5\} \cup \{y_n\}$ is an α-stable dominating set with $|D| = |\{x_i : i \equiv 1(mod4)\}| + |\{x_{n-2}\}| + |\{y_j : j \equiv 3(mod4) \& j \leq n-5\}| + |\{y_n\}| = \frac{n}{4} + 1 + \frac{n}{4} -$

$1 + 1 = \lceil \frac{n+1}{2} \rceil$. Hence $\gamma^\alpha(P_2 \Box P_n) \leq \lceil \frac{n+1}{2} \rceil$. Since $\gamma^\alpha(P_2 \Box P_n) \geq \gamma(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$, $\gamma^\alpha(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$.

Case 3

If n is even and $n \equiv 2(mod4)$ then $D = \{x_i : i \equiv 1(mod4) \& i \leq n-5\} \cup \{x_n\} \cup \{y_j : j \equiv 3(mod4)\} \cup \{y_{n-2}\}$ is an α-stable dominating set with $|D| = |\{x_i : i \equiv 1(mod4) \& i \leq n-5\}| + |\{x_n\}| + |\{y_j : j \equiv 3(mod4)\}| + |\{y_{n-2}\}| = \lfloor \frac{n}{4} \rfloor - 1 + 1 + \lfloor \frac{n}{4} \rfloor + 1 = \lceil \frac{n+1}{2} \rceil$. Hence $\gamma^\alpha(P_2 \Box P_n) \leq \lceil \frac{n+1}{2} \rceil$. Since $\gamma^\alpha(P_2 \Box P_n) \geq \gamma(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$, $\gamma^\alpha(P_2 \Box P_n) = \lceil \frac{n+1}{2} \rceil$. □

4.6 α-stable domination in corona of graphs

Theorem 4.6.1. *For any two graphs G and H and non negative integer α, α-stable domination number of its corona, $\gamma^\alpha(GoH) = |V(G)|$.*

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices of G and $\{u_1, u_2, ...u_m\}$ be the vertices of H. H_i be the i^{th} copy H in GoH. To make sure that each vertex of H_i is dominated, we need atleast one vertex of H_i or v_i. Thus the dominating set contains atleast n vertices. Let $D = \{v_1, v_2, ..., v_n\}$. Then each vertex of $V - D$ is adjacent with exactly one vertex of D and each vertex of D dominates exactly m vertices in $V - D$. And so $\psi_D(v) = m$, $\forall v \in D$ and $\phi_D(v) = 1$, $\forall v \in V - D$. Therefore, $|\phi_D(v_1) - \phi_D(v_2)| = 0, \forall v_1, v_2 \in V - D$ and $|\psi_D(v_1) - \psi_D(v_2)| = 0, \forall v_1, v_2 \in D$. And so D is a minimum α-stable dominating set. Thus $\gamma^\alpha(GoH) = n = |V(G)|$.

□

4.7 Conclusion

Any real life situation in social network such as banking and marketing, can be modelled by Graphs. In this chapter, to study the stability of domination in Graphs the concept of α-d-stable domination number, α-a-stable domination number and α-stable domination number are introduced. Necessary and sufficient condition for an α-d-stable dominating set to be a minimal α-d-stable dominating set is obtained.

Necessary and sufficient condition for an α-a-stable dominating set to be a minimal α-a-stable dominating set is obtained. The relation between α-d-stable domination number and independent domination number is discussed. And these parameters in certain product graphs are studied.

Since the parameters introduced here have large scale of applications, this area has great scope for further studies.

CHAPTER 5

P_3- Convexity in graphs

5.1 Introduction

Any network can be modelled by graphs. If a set of vertices S initially possessing a property spreads the property to the vertices having two neighbours S, then finding the minimum number of vertices required to spread the property to all vertices in the graph is an important problem in the field of Graph theory. Sharing an idea or spreading a virus or the strategy in some sort of marketing are some examples of this. We can approach the problem through P_3-convexity. The P_3-convexity was first studied for directed graphs [34, 49]. Later P_3-convexity considered for undirected graphs [24, 25, 26, 33, 11]. As the adjacency is the main property discussed in P_3-convexity, the concept of P_3- convexity resembles domination in graphs. This motivated us to study P_3- convexity in graphs. In this chapter the concept P_3-convexity C, exclusively P_3-convex invariants P_3-hull number, radon number and caratheodory number is studied.

This chapter is divided into six sections. In which first one is an introductory section. Second section deals with general properties in P_3-convexity. Third section contains P_3-convex invariants, P_3-hull number, radon number and caratheodory number of some classes of graphs. And fourth section deals with P_3-convexity on strong product, cartesian product and composition of graphs. P_3-convexity in corona related graphs are studied in fifth section. And sixth section is a concluding section.

For a graph G, Given a set $S \subset V(G)$, the P_3- interval $I[S] = S \cup \{v : |N(v) \cap S| \geq 2\}$. S is a P_3- convex set, if $I[S] = S$. A graph G together with P_3-convex sets in G form P_3-convexity C in G[36].

The P_3- convex hull $H_C(S)$ of S in G is the smallest P_3- convex set containing S. [36] P_3-convexity C is uniquely determined by the P_3-convexity in a graph G. Hence in this chapter we are using $H_G(S)$ instead of $H_C(S)$.

The P_3- convex hull can be formed from a sequence $I^p[S]$, where p is a nonnegative integer, $I^0[S] = S$, $I^1[S] = I[S]$, and $I^p[S] = I[I^{p-1}[S]]$, for every $p \geq 2$. When, for some $p \in N$ we have if $I^q[S] = I^p[S]$, for all $q \geq p$, then $I^p[S]$, is a convex set.

If $H_G(S) = V(G)$ then S is a P_3-hull set of G. The cardinality $h(G)$ of a minimum P_3-hull set in G is called the P_3-hull number of G.[36] In this chapter we are using hull number of G instead of P_3-hull number of G.

The caratheodory number is the smallest integer c such that for every set S of vertices of G and every vertex u in $H_G(S)$, there is a set $F \subset S$ with $|F| \leq c$ and $u \in H_G(F)$[35]. In this chapter we are using $c(G)$ instead of c.

A Radon partition of R is a partition of R into two disjoint sets R_1 and R_2 with $H_G(R_1) \cap H_G(R_2) \neq \phi$. The Radon number $r(G)$ of G is the minimum integer r such that every set of r vertices of G has a Radon partition. [48]

An outerplanar graph is a planar graph that allows an embedding in the plane such that all vertices are on the outer face. A maximal outerplanar graph is an outerplanar graph with a maximum number of edges. In the plane embedding the boundary of the outer face, provided it has at least three vertices, is then a hamiltonian cycle. All other edges form a triangulation of this outer cycle.[44]

Convexity C on G is joint hull commutative provided that for each nonempty convex set S in C and for each vertex $p \in V(G)$, $H_G(S \cup \{p\}) = \cup_{u \in S} H_G(\{p, u\})$[74].

A wounded spider is the graph formed by subdividing at most $n-2$ edges of star $K_{1,n-1}$.

5.2 General properties in P_3-convexity

This section deals with characterization of graphs with certain hull number. Characterization of tree using P_3-convexity is given in this section. Joint hull commutative property in P_3-convexity is discussed.

Theorem 5.2.1. *$h(G) = n$ if and only if each vertex has degree less than or equal to 1.*

Proof. Let G be a graph with each vertex has degree less than or equal to 1. If S is a proper subset of $V(G)$, then $H_G(S) = S$. Thus minimum P_3- hull set is $V(G)$ itself. Hence $h(G) = n$.

If $h(G) = n$, then no vertex is adjacent to more than two vertex. Each vertex is adjacent to atmost on vertex. Hence degree of each vertex is less than or equal to 1.

□

Theorem 5.2.2. *Let G be a star with atleast 3 vertices, then $h(G) = n - 1$.*

Proof. If G is a star $K_{1,n-1}$, then all the pendant vertices must be there in minimum P_3- hull set. And set of all pendant vertices will form a P_3- hull set. Hence $h(K_{1,n-1}) = n - 1$.

□

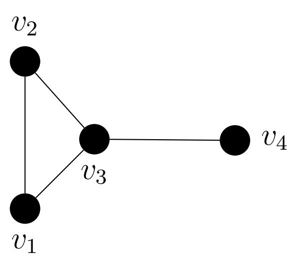

Figure 5.1

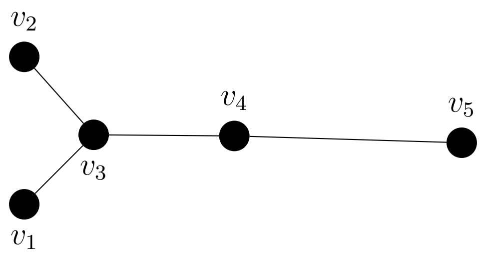

Figure 5.2

Theorem 5.2.3. *Let G be a graph with $|V(G)| \geq 4$ and $h(G) = n - 1$, then the graphs isomorphic to graphs given in figure 5.1, $P_m, m \geq 5$ and 5.2 are forbidden subgraphs.*

Proof. If G has a subgraph isomorphic to graphs given in figure 5.1 or $P_m, m \geq 5$ or figure 5.2,

Case 1 G has a subgraph isomorphic to figure 5.1.
If $S = V(G) - \{v_2, v_3\}$, then $H_G(S) = V(G)$. Hence $h(G) \leq n - 2$.

Case 2 G has a subgraph isomorphic to $P_m, m \geq 5$.
Let $v_1, v_2, ..., v_m$ the vertices of P_m. If $S = (V(G) - \{v_i : i \equiv 0(mod2)\}) \cup \{v_m\}$, then $H_G(S) = V(G)$. Hence $h(G) \leq n - 2$.

Case 3 G has a subgraph isomorphic to figure 5.2.
If $S = V(G) - \{v_3, v_4\}$, then $H_G(S) = V(G)$. Hence $h(G) \leq n - 2$.

Thus if $h(G) = n - 1$, then then graphs given in figure 5.1, $P_m, m \geq 5$ and figure 5.2 are forbidden subgraphs. □

Corollary 5.2.4. *Let G be a graph with order n and $h(G) = n - 1$*

- *If* $n = 3$*, then either* G *is a triangle or a star.*
- *If* $n = 4$*, then either* G *is a path or a star.*
- *If* $n \geq 5$*, then* G *is a star.*

Proof. Let G be a graph with order n and $h(G) = n - 1$.

If n=3 Then it is clear that either G is a triangle or a star.

If n=4 Graph which has no subgraph as stated in theorem 5.2.3 is either a path or a star.

If $n \geq 5$ Graph which has no subgraph as stated in theore 5.2.3 is a star.

□

Theorem 5.2.5. *Let* G *be a 2-connected graph with a universal vertex. Then* $h(G) = 2$*.*

Proof. Let v be a universal vertex of G and u be any vertex in $V(G)$ which has eccentricity e in $G - u$.
Then, $I[\{u, v\}] = N[u]$
$I^2[\{u, v\}] = N^2[u]$
$I^3[\{u, v\}] = N^3[u]$
$\vdots$
$I^e[\{u, v\}] = V(G)$.
Hence $H_G(\{u, v\}) = V(G)$ and $h(G) = 2$ □

The condition given in 5.2.5 is not necessary. For the graph in 5.3 $h(G) = 2$ but G has no universal vertex.

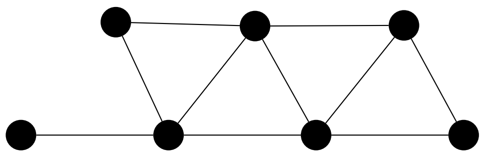

Figure 5.3

There are 1-connected graphs with universal vertex, but $h(G) > 2$. Figure 5.4 is a 1- connected graph having universal vertex. But $h(G) = 3$.

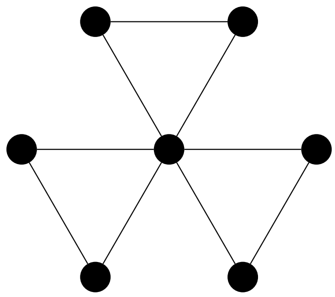

Figure 5.4

Remark 5.2.1. *The minimum size of a graph G for which $order(G) = n$ and $h(G) = 2$ is $2(n-2)$.*

Proof. Let G be a graph of order n. If $h(G) = 2$, then there exists two vertices u and v so that $H_G(\{u, v\}) = V(G)$. Let $S = \{u, v\}$ and d be the minimum integer so that $I^d[\{u, v\}] = V(G)$. Then every vertex in $I^1[S] - S$ is incident with at least two vertices in S. And every vertex in $I^k[S] - I^{k-1}S$ is incident with at least two vertices in $I^{k-1}S$ $\forall k \in \{1, 2, ..., d\}$. Hence there should be at least $2(n-2)$ edges. Figure 5.3 illustrates a graph having minimum size of graph with order 7 and $h(G) = 2$. Here $order(G) = 7$ and $size(G) = 2(7-2)$

□

Theorem 5.2.6. *For $2 \leq a \leq n-1$, there exist a connected graph G with $|V(G)| = n$ and $h(G) = a$.*

Proof. When $a = n-1$, if $G = K_{1,n-1}$ then $h(G) = n-1$ and $|V(G)| = n$.

For $a \leq n-2$,
Let $K_{1,n-1}$ is the star with centre v and pendant vertices $v_1, v_2, \ldots\ldots, v_{n-1}$. And let G be the graph obtained from the star $K_{1,n-1}$ by joining vertices x_{i-1}, x_i for $1 \leq i \leq n-a$. Then $x_{n-1}, x_{n-2}, \ldots\ldots, x_{n-a}$ will be a minimum P_3- hull set of G and hence $h(G) = a$ and $|V(G)| = n$.

□

Remark 5.2.2. *Being a P_3-convex set is neither hereditary nor superhereditary property.*

Theorem 5.2.7. *If G is disconnected with atleast two components G_1 and G_2, $|V(G_1)| \geq 2$, $|V(G_2)| \geq 2$. Then $h(\overline{G}) = 2$.*

Proof. $K_{r,s}$, $r, s \geq 2$ is a spanning subgraph of $\overline{G}$. Also $h(K_{r,s}) = 2$, $r, s \geq 2$ and hence $h(K_{r,s}) \geq h(\overline{G})$. Hull number of a graph is always greater than or equal to 2. Thus $h(\overline{G}) = 2$.

□

Theorem 5.2.8. *Let G be a tree with n vertices. Then there exist a sequence of sets $V(G) = V_n \supset V_{n-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i| = i$.*

Proof. Let G be a tree. Let $V(G) = V_n$ and v_1 be a pendant vertex of the tree G. If $V_{n-1} = V_n - \{v_1\}$. Then clearly V_{n-1} is convex in G. Let v_2 be a pendant vertex of $G - v_1$ and $V_{n-2} = V_{n-1} - \{v_2\}$. Then clearly V_{n-2} is convex in G and so on . The sets V_i, thus formed have the property $V(G) = V_n \supset V_{n-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i| = i$.

□

For the graph G in figure 5.5 there exist a sequence of sets $V(G) = V_n \supset V_{n-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i| = i$. But G is not a tree. Hence converse may not be true in theorem 5.2.8.

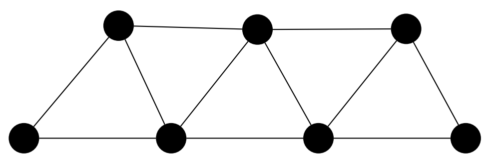

Figure 5.5

Theorem 5.2.9. *Let G be a connected graph. Then G is a tree if and only if for each connected subgraph H of G, $V(H)$ is a convex set of G.*

Proof. Let G be a connected graph. Suppose that for each connected subgraph H of G, $V(H)$ is a convex set of G. If G has a cycle C then $C - v$ is a connected subgraph of G. But $V(C - v)$ not convex. Thus G contains no cycle. Thus G is a tree.

Let G be a tree and H be a connected subgraph of G. If $V(H)$ is not a convex set there exists a vertex $u \in V(G) - V(H)$ which is adjacent to two vertices $v_1, v_2 \in V(H)$. Then there exists two v_1v_2 paths in G. Since G is a tree it is not possible. Hence for each connected subgraph H of G, $V(H)$ is a convex set of G.

□

Theorem 5.2.10. *If G is a graph having no cycle with length ≤ 4 and $|V(G)| \geq 4$, then $h(G) \geq 3$.*

Proof. Let G be a graph, having no cycle with length ≤ 4 and $|V(G) \geq 4$. If $S = \{u_1, u_2\} \subset V(G)$, then we distinguish into three cases.

Case 1 If $d(u, v) = 1$,
Since G is triangle free, $I^1[S] = S$. Hence $H_G(S) = S$.

Case 2 If $d(u, v) = 2$,
Then if u, w, v be the minimum uv path in G. Then $I^1[S] = \{u, v, w\}$ and $I^2[S] = \{u, v, w\}$. Hence $H_G(S) = \{u, v, w\}$.

Case 3 If $d(u, v) \geq 3$,
Then $I^1[S] = S$. Hence $H_G(S) = S$.

Thus $H_G(S) = I^1[S]$ and $|I^1[S]| \leq 3$. Hence $H_G(S) \neq V(G)$ and S cannot be a hull set. Thus, $h(G) \geq 3$.

□

Theorem 5.2.11. *If G is a graph with the property $I^1[S] = H_G(S)$, $\forall S \subset V(G)$, then G has joint hull commutative propery.*

Proof. Let G be a graph with the property $I^1[S] = H_G(S)$, $\forall S \subset V(G)$.
For any convex set C and $p \in V(G)$, if $w \in H_G(C \cup \{p\}) - (C \cup \{p\})$, then $w \in I^1(C \cup \{p\})$. Then there exists $v_1 \in C$ such that w is adjacent to v_1 and p.

Then $\forall w \in H_G(C \cup \{p\}) - (C \cup \{p\})$, there is some $v_1 \in C$ such that $w \in H_G(\{v_1, p\})$.

Hence, $H_G(C \cup \{p\}) = \cup\{H_G(\{c,p\}) : c \in C\}$. Thus, G has joint hull commutative property.

□

Following theorem shows that converse may not be true.

Theorem 5.2.12. *If G is a maximal outer planar graph with $diam(G) > 2$, then G has joint hull commutative property. But there are $S \subset V(G)$ such that $I^1[S] \neq H_G(S)$.*

Proof. In maximal outer planar graph G, convex sets are singleton sets or $V(G)$ or vertex set S having property distance between any two vertices is at least 3.
Let $p \in V(G)$,
If S is either singleton or $V(G)$ then trivially it satisfy the property $H_G(S \cup \{p\}) = \cup\{H_G(\{v,p\}) : v \in S\}$.
If S is a vertex set in which any two distinct distance are at distance at least 3,
If there is a vertex $v \in S$, such that $d(v,p) = 2$, then $H_G(\{v,p\}) = V(G)$. And so, $H_G(S \cup \{p\}) = \cup\{H_G(\{v,p\}) : v \in S\}$.
If there is no vertex $v \in S$, such that $d(v,p) = 2$, then $H_G(\{v,p\}) = \{v,p\}$ for all $v \in S$. And so $H_G(S \cup \{p\}) = S \cup \{p\} = \cup\{H_G(\{v,p\}) : v \in S\}$. Thus maximal outer planar graphs have joint hull commutative P_3- convex property.

Also, if $|V(G)| \geq 5$, then for any adjacent vertices u and w in $V(G)$, $H_G(\{u,v\}) = V(G)$ and $I^1[\{u,v\}] \neq H_G(S)$.

□

Theorem 5.2.13. *Let H be a connected subgraph of a graph G. Then if $V(H)$ is the only nontrivial convex set of H, then H is a block. In particular, if $V(G)$ is the only nontrivial convex set of G, then G is a block.*

Proof. Let H be a connected subgraph of a graph G and $V(H)$ is the only nontrivial convex set of H. If w is a cut vertex of H and H_1 and H_2 are the components of $H - w$. Let $v \in H_1$, then convex hull of $\{v,w\} \subset V(H_1)$ which is a contradiction. Thus, H is a block.

□

Theorem 5.2.14. *Every block in a graph G is a P_3- convex set.*

Proof. Let B be a block which is not P_3-convex set. Then, there exist a vertex $u \in \overline{B}$ such that u has two neighbours v_1 and v_2 in B. $< B \cup \{u\} >$ is a non separable subgraph containing B which is a contradiction to B is a block. Thus B is a P_3-convex set.

□

Theorem 5.2.15. *Let G^1 be the graph generated by Mycielski's construction of a graph G. Then $h(G^1) \leq h(G) + 1$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$, $n \geq 2$ and $u_1, u_2, u_3, ..., u_n, v$ be the vertices added for Mycielski's construction. If S is a minimum P_3- hull set of a graph G. Then if $S^1 = S \cup \{v\}$. Then $V(G) \subset H_G(S^1)$. Each vertex in $\{u_1, u_2, ..., u_n\}$ has atleast two neighbours in $V(G)$. Hence $\{u_1, u_2, ..., u_n\} \subset H_G(S^1)$. And v has is adjacent to every vertices in $\{u_1, u_2, ..., u_n\}$. Thus $H_G(S) = V(G^1)$. Hence, S^1 is a P_3-convex hull set of G^1. And $h(G^1) \leq h(G) + 1$.

If $G = P_3$, then $h(G) = 2$. And if G^1 is the graph generated by Mycielski's construction of a graph G, then $h(G^1) = 3$. Thus, this bound is sharp.

If $G = K_n$, then $h(G) = 2$. And if G^1 is the graph generated by Mycielski's construction of a graph G, then $h(G^1) = 2$. Thus, strict inequality may occur.

□

Theorem 5.2.16. *Let G be a graph of order $n \geq 3$ and size m, $T_k(G)$ the trestled graph of G with $k \geq 2$, then $h(T_k(G)) = km$.*

Proof. Let $u_{il}^p u_{jl}^p$ be the p^{th} copy of K_2 added to the edge e_l with end vertices $v_i v_j$. Let $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_{h(G)}\}$ be a minimum P_3 -hull set. Choose an edge e_1 incident with v_1. Inductively choose e_i incident with v_i which is different from $e_1, e_2, \ldots\ldots..e_{i-1}$ for $1 \leq i \leq h(G)$. Rename the remaining edges as $e_{h(G)+1}, e_{h(G)+2}, ..., e_m$. Take $S = \{u_{il}^p : 1 \leq i, l \leq h(G); 1 \leq p \leq k\}$. Choose v_{i_l}

a vertex incident with e_l for $h(G)+1 \leq l \leq m$. Then take $T = \{u^p_{i_l l} : 1 \leq p \leq k; h(G)+1 \leq l \leq m\}$. Then $H_{T_k(G)}(S \cup T) = V(T_k(G))$. Thus $|SUT| = km$ and these km vertices of $S \cup T$ will form a convex hull set of $T_k(G)$. Hence $h(T_k(G)) \leq km$. We need atleast one vertex from each copy of K_2 to form the convex hull set. Thus $h(T_k(G)) \geq km$. Thus $h(T_k(G)) = km$.

□

5.3 P_3-Convexity in some classes of graphs

P_3-convex invariants- hull number, radon number and caratheodory number of some classes of graphs are obtained in this section.

Theorem 5.3.1. *For a path P_n with $n \geq 3$, $c(P_n) = 2$, $r(P_n) = \lfloor \frac{n}{2} \rfloor + 2$ and $h(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. Let $U \subseteq V(G)$, consider $v \in H(U)$.
If $|U| \leq 2$ or $v \in U$, there is nothing to prove.
Otherwise, if $u \in I^p[U] - I^{p-1}[U]$ for some $p \geq 1$, then there exists two vertices v_1 and v_2 in $I^{p-1}[U]$ which are adjacent to v. But $d(v_1) \leq 2$ and $d(v_2) \leq 2$. Hence $p = 1$. Thus, if $F = \{v_1, v_2\}$, $v \in H_G(F)$ and $F \subset U$. Hence $c(P_n) = 2$.

Let $v_1, v_2, ..., v_n$ be the vertices of P_n.
Then if $R = \{v_n\} \cup \{v_i : i \equiv 1(mod2), 1 \leq i \leq n\}$ has no radon partition in P_n.
Hence $r(P_n) \geq \lfloor \frac{n}{2} \rfloor + 2$.
And if $|R| \geq \lfloor \frac{n}{2} \rfloor + 2$, then there exists $i \in \{1, 2, ..., (n-2)\}$ so that $v_i, v_{i+1}, v_{i+2} \in R$.
Then $R_1 = \{v_{i+1}\}$ and $R_2 = R - \{v_{i+1}\}$ is a radon partition of R. Hence $r(P_n) \leq \lfloor \frac{n}{2} \rfloor + 2$.
Thus $r(P_n) = \lfloor \frac{n}{2} \rfloor + 2$.

If $S = \{v_n\} \cup \{v_i : i \equiv 1(mod2), 1 \leq i \leq n\}$, then $H_G(S) = V(P_n)$. Hence $h(P_n) \leq \lfloor \frac{n}{2} \rfloor + 1$.
If $|S| < \lfloor \frac{n}{2} \rfloor + 1$, then there exist $i \in \{1, 2, ..., n-1\}$ so that $v_i, v_{i+1} \notin S$. Hence there exist $i \in \{1, 2, ..., n-1\}$ so that $v_i, v_{i+1} \notin H_G(S)$. Thus $h(P_n) \geq \lfloor \frac{n}{2} \rfloor + 1$.

Hence $h(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.

□

Theorem 5.3.2. *For a complete graph K_n with $n \geq 3$, $c(K_n) = 2$, $r(K_n) = 3$ and $h(K_n) = 2$.*

Proof. Let $U \subset V(G)$, consider $v \in H_G(U)$. If $F = \{v_1, v_2\} \subset U$, then $H_G(F) = V(G)$. Thus, $v \in h(F)$. Hence $c(K_n) = 2$.

P_3-Convex hull of every set having atleast two elements contain all the other vertices. Thus every three element set can be partitioned into two sets for which the convex hull of one set contains the vertex from the other set. Hence $r(K_n) = 3$.

Choose any vertices $u, v \in V(K_n)$ and let $S = \{u, v\}$ then $H_G(S) = V(K_n)$. Hence $h(K_n) = 2$.

□

Theorem 5.3.3. *For a cycle C_n, $c(C_n) = 2$, $r(C_n) = \lceil \frac{n}{2} \rceil + 1$ and $h(C_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $U \subset V(G)$, consider $v \in H_G(U)$. If $v \in H_G(U) - U$, since the degree of each vertex is 2, $v \in I^1[U]$. Thus there exists two vertices v_1 and v_2 in U which are adjacent to v. Thus, if $F = \{v_1, v_2\} \subset U$, $v \in H_G(F)$. Hence $c(C_n) = 2$.

Let $v_1, v_2, ..., v_n$ be the vertices of C_n.
Then choose $R = \{v_1, v_3, v_5, ...v_n\}$, if n is odd and choose $R = \{v_1, v_3, v_5, ...v_{n-1}\}$, if n is even. Then R has no radon partition in C_n. Thus $r(C_n) \geq \lceil \frac{n}{2} \rceil + 1$.
And if $|R| \geq \lceil \frac{n}{2} \rceil + 1$, then there exists $i \in \{1, 2, ..., (n-2)\}$ so that $v_i, v_{i+1}, v_{i+2} \in R$ or $v_{n-1}, v_n, v_1 \in R$ or $v_n, v_1, v_2 \in R$. Then, if $v_i, v_{i+1}, v_{i+2} \in R$, $R_1 = \{v_{i+1}\}$ and $R_2 = R - \{v_{i+1}\}$ is a radon partition of R. If $v_{n-1}, v_n, v_1 \in R$ then $R_1 = \{v_n\}$ and $R_2 = R - \{v_n\}$ is a radon partition of R. If $v_n, v_1, v_2 \in R$ then $R_1 = \{v_1\}$ and $R_2 = R - \{v_1\}$ is a radon partition of R. Hence $r(C_n) \leq \lceil \frac{n}{2} \rceil + 1$.
Thus $r(C_n) = \lceil \frac{n}{2} \rceil + 1$.

Choose $S = \{v_1, v_3, v_5, ...v_n\}$, if n is odd and choose $S = \{v_1, v_3, v_5, ...v_{n-1}\}$, if n is even. Then $H_G(S) = V(C_n)$. Hence $h(C_n) \leq \lceil \frac{n}{2} \rceil$.
If $|S| < \lceil \frac{n}{2} \rceil$, then $v_1, v_n \notin S$ or there exist $i \in \{1, 2, ..., n-1\}$ so that $v_i, v_{i+1} \notin S$. Hence $v_1, v_n \notin H_G(S)$ or there exist $i \in \{1, 2, ..., n-1\}$ so that $v_i, v_{i+1} \notin H_G(S)$.
Thus $h(C_n) \geq \lceil \frac{n}{2} \rceil$.
Thus $h(C_n) = \lceil \frac{n}{2} \rceil$. □

Theorem 5.3.4. *For a star $K_{1,n-1}$, $n \geq 4$, $c(K_{1,n-1}) = 2$, $r(K_{1,n-1}) = 4$, and $h(K_{1,n-1}) = n$.*

Proof. Let $U \subset V(G)$, consider $v \in h(U)$. If $v \in H_G(U) - U$, then since the only one vertex having degree greater than 2 is the center, $v \in I^1[S]$ and v is the center. Thus there exists two vertices u_1 and u_2 in U which are adjacent to v. Thus, if $F = \{u_1, u_2\}, v \in H_G(F)$. Hence $c(K_{1,n-1}) = 2$.

Every four element set can be partitioned into two sets having two elements for which the centre as the common element of convex hull. And for the set containing three vertices having degree one has no radon partition. Thus $r(K_{1,n-1}) = 4$.

Let $v_1, v_2, ..., v_{n-1}$ be the pendant vertices of $K_{1,n-1}$. Then every P_3-hull set contains $v_1, v_2, ..., v_{n-1}$. Hence $h(K_{1,n-1}) \geq n - 1$.
And $\{v_1, v_2, ..., v_{n-1}\}$ is a P_3-hull set. Hence $h(K_{1,n-1}) \leq n - 1$.
Thus $h(K_{1,n-1}) = n - 1$

□

Theorem 5.3.5. *For a complete bipartite graph $K_{m,n}$, $m, n \geq 2$, $r(K_{m,n}) = 3$, $c(K_{m,n}) = 2$, $h(K_{m,n}) = 2$.*

Proof. Every three element set contains atleast two vertices which are independent. And the convex hull of these two vertices contain all the other vertices. Thus every three element set can be partitioned into two sets for which the convex hull of one set contains the vertex from the other set. Hence $r(K_{m,n}) = 3$.

Let $U \subset V(K_{m,n})$ and $v \in H_G(U)$. If $v \in U$, then there is a subset $F \subset U$ with $|F| \leq 2$ and $v \in H_G(F)$.

If $v \in H_G(U) - U$, then there exist two vertices in $v_1, v_2 \in U$ so that $I^1[\{v_1, v_2\}] \neq \{v_1, v_2\}$. Then $I^2[\{v_1, v_2\}] = V(K_{m,n})$. Hence there is a subset $F = \{v_1, v_2\} \subset U$ with $|F| \leq 2$ and $v \in H_G(F)$.
Thus $c(K_{m,n}) = 2$.

Let v_1, v_2 be two vertices from one partite set of $K_{m,n}$, then $H_G(\{v_1, v_2\}) = V(K_{m,n})$. Hence $h(K_{m,n}) = 2$.

□

Theorem 5.3.6. *For a wounded spider G,*

$$h(G) = \begin{cases} \Delta(G) & \textit{if centre is incident with more than one pendant vertex} \\ \Delta(G)+1 & \textit{otherwise} \end{cases}.$$

Proof. If $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_t\}$ be the pendant vertices of the graph. Then $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_t\}$ must be contained in a minimum P_3- hull set. If atleast two vertices from this set is adjacent to the vertex with degree $\Delta(G)$. Then this set is a minimum P_3 -hull set and $h(G) = \Delta$. Otherwise, $\{v_1, v_2, \ldots\ldots\ldots\ldots, v_t\} \cup \{v\}$ where v is the vertex with degree Δ, will form a minimum P_3- hull set. Thus,

$$h(G) = \begin{cases} \Delta & \text{if centre is incident with more than one pendant vertex} \\ \Delta+1 & \textit{otherwise} \end{cases}$$

□

5.4 P_3-Convexity in product graphs

A detailed study in the P_3-convex invariants- hull number, radon number and caratheodory number of product graphs-strong product of graphs, cartesian product of graphs and composition of graphs- is done in this section.

5.4.1 P_3-Convexity in strong product of graphs

Hull number in strong product of graphs were studied in [36] and theorem 5.4.1 is obtained in [36]. Here we deal with caratheodory number and radon number of strong product of graphs.

If $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(H) = \{v_1, v_2, ..., v_n\}$. Here we refer the set of vertices $\{(u_i, v_1), (u_i, v_2), ..., (u_i, v_n)\}$ as line L_i and set of vertices $\{(u_1, v_j), (u_2, v_j), ..., (u_m, v_j)\}$ as column C_j.[36]

Theorem 5.4.1. *[36] Let G and H be nontrivial connected graphs. Then,* $h(G \boxtimes H) = 2$.

Lemma 5.4.2. *[36] Let G and H be nontrivial connected graphs,* $S \subset V(G \square H)$*. Let* $L_i^1 \subset L_i$*, for some* $i \in \{1, 2, ..., m\}$ *and* $C_j^1 \subset C_j$*, for some* $j \in \{1, 2, ..., n\}$*, such that* L_i^1 *and* C_j^1 *induce connected graphs and* $L_i^1 \cap C_j^1 \neq \phi$*. Let* $R = \{(u_k, v_l) \in V(G \square H) : (u_k, v_j) \in C_j^1 and (u_i, v_l) \in L_i^1\}$*. If* $L_i^1 \cup C_j^1 \subset I^p[S]$ *for some integer* $p > 0$*, then* $R \subset H_{G \square H}(S)$*.*

We are using lemma 5.4.2 from [36] to obtain 5.4.3.

Lemma 5.4.3. *Let G and H be nontrivial connected graphs,* $S \subset V(G \boxtimes H)$*. Let* $L_i^1 \subset L_i$*, for some* $i \in \{1, ..., m\}$ *and* $C_j^1 \subset C_j$*, for some* $j \in \{1, ..., n\}$*, such that* L_i^1 *and* C_j^1 *induce connected graphs and* $L_i^1 \cap C_j^1 \neq \phi$*. Let* $R = \{(u_k, v_l) \in V(G \boxtimes H) : (u_k, v_j) \in C_j^1 and (u_i, v_l) \in L_i^1\}$*. If* $L_i^1 \cup C_j^1 \subset I^p[S]$ *for some integer* $p > 0$*, then* $R \subset H_{G \boxtimes H}(S)$*.*

Proof. $G \square H$ is a spanning subgraph of $G \boxtimes H$. Hence for any $U \subset V(G) \times V(H)$, $H_{G \square H}(U) \subset H_{G \boxtimes H}(U)$.
Here, L_i^1 and C_j^1 induce connected subgraphs of L_i and C_j respectively and $L_i^1 \cap C_j^1 \neq \phi$. And $L_i^1 \cup C_j^1 \subset I^p[S]$ for some integer $p > 0$. Then by [36],
$R = \{(u_k, v_l) \in V(G \square H) : (u_k, v_j) \in C_j^1 \text{and} (u_i, v_l) \in L_i^1\} \subset H_{G \square H}(S)$.
Since $H_{G \square H}(S) \subset H_{G \boxtimes H}(S)$, $R \subset H_{G \boxtimes H}(S)$.

□

Lemma 5.4.4. *Let G and H be non trivial connected graphs. Then, for* $S \subset V(G \boxtimes H)$ *either* $H_{G \boxtimes H}(S) = V(G \boxtimes H)$ *or* $H_{G \boxtimes H}(S) = S$*.*

Proof. Let $S \subset V(G \boxtimes H)$ and $H_{G \boxtimes H}(S) \neq S$. Then there exists a vertex $(a, b) \in I^1[S]$ and $(a, b) \notin S$. Thus there exist two vertices (a_1, b_1) and (a_2, b_2) adjacent to (a, b) in $G \boxtimes H$.

Case 1 a_1 adjacent to a and b_1 adjacent to b.

Rename a_1, b_1, a and b as u_1, v_1, u_2, v_2 respectively. And remaining vertices of G as u_3, u_4,....,u_m and remaining vertices of H as v_3, v_4,....,v_n respectively.

Then $\{(u_1, v_1), (u_2, v_2), (u_1, v_2), (u_2, v_1)\} \subset I^2[S]$.

There is a path from $u_1(u_2)$ to u_i, $\forall i \in \{3, 4, ..., m\}$. Thus each vertices $(u_i, v_1), (u_i, v_2) \in H_{G\boxtimes H}(S)$, $\forall i \in \{3, 4, ..., m\}$. And There is a path from $v_1(v_2)$ to v_j, $\forall j \in \{3, 4, ..., n\}$. Thus each vertices $(u_1, v_j), (u_2, v_j) \in H_{G\boxtimes H}(S)$, $\forall j \in \{3, 4, ..., n\}$. Hence Ł$_1$ and $C_1 \subset H_{G\boxtimes H}(S)$.

Hence by lemma 5.4.3, $H_{G\boxtimes H}(S) = V(G \boxtimes H)$.

Case 2 a_1 adjacent to a and $b_1 = b$.

Rename a_1, a and b_1 as u_1, u_2 and v_1 respectively. Choose any vertex adjacent to v_1 and rename it as v_2. And rename remaining vertices of G as u_3, u_4,....,u_m and remaining vertices of H as v_3, v_4,....,v_n respectively.

Then $\{(u_1, v_1), (u_2, v_1)\} \subset I^2[S]$ and $\{(u_1, v_1), (u_2, v_2), (u_1, v_2), (u_2, v_1)\} \subset I^3[S]$.

There is a path from $u_1(u_2)$ to u_i, $\forall i \in \{3, 4, ..., m\}$. Thus each vertices $(u_i, v_1), (u_i, v_2) \in H_{G\boxtimes H}(S)$, $\forall i \in \{3, 4, ..., m\}$. And there is a path from $v_1(v_2)$ to v_j, $\forall j \in \{3, 4, ..., n\}$. Thus each vertices $(u_1, v_j), (u_2, v_j) \in H_{G\boxtimes H}(S)$, $\forall j \in \{3, 4, ..., n\}$. Hence Ł$_1$ and $C_1 \subset H_{G\boxtimes H}(S)$.

Hence by lemma 5.4.3 $H_{G\boxtimes H}(S) = V(G \boxtimes H)$.

Thus, if $S \subset V(G \boxtimes H)$ and $H_{G\boxtimes H}(S) \neq S$, then $H_{G\boxtimes H}(S) = V(G \boxtimes H)$. Hence, either $H_{G\boxtimes H}(S) = V(G \boxtimes H)$ or $H_{G\boxtimes H}(S) = S$.

□

Theorem 5.4.5. *Let G and H be nontrivial connected graphs, then $c(G \boxtimes H) = 2$.*

Proof. Let S be a subset of $V(G \boxtimes H)$ and $(u,v) \in H_{G\boxtimes H}(S)$.

Case 1 $(u,v) \in S$, then there is $F \subset S$ with $|F| = 1$ and $(u,v) \in H_{G\boxtimes H}(F)$.

Case 2 $(u,v) \notin S$ and $(u,v) \in H_{G\boxtimes H}(S)$. Choose any $(a,b) \in I^1[S]$, but $(a,b) \notin S$. Then there exist vertices $(a_1,b_1),(a_2,b_2) \in S$ which are adjacent to (a,b). Take $F = \{(a_1,b_1),(a_2,b_2)\} \subset S$. Then $H[F] \neq F$. Hence by lemma5.4.4 $H_{G\boxtimes H}(F) = V(G \boxtimes H)$. Hence $(u,v) \in H_{G\boxtimes H}(F)$, $F \subset S$ and $|F| = 2$.

Thus $c(G \boxtimes H) \leq 2$. Since G and H are nontrivial connected graphs, $c(G \boxtimes H) \geq 2$.

Thus $c(G \boxtimes H) = 2$.

□

Theorem 5.4.6. *Let G and H be nontrivial connected graphs, then $r(G \boxtimes H) \geq max\{\lceil \frac{diam(G)+4}{3} \rceil, \lceil \frac{diam(H)+4}{3} \rceil\}$.*

Proof. Let $diam(G) = k$ and $diam(H) = l$, and $v_1, v_2, ..., v_{l+1}$ be a maximum path in H. If $u \in V(G)$, then if $R = \{(u,v_1),(u,v_4),(u,v_7),...,(u,v_{\lceil \frac{l+1}{3} \rceil})\}$, then, each vertex in $V(G\boxtimes H)$ is adjacent to atmost one vertex in R. Hence, for any partition R_1, R_2 of R, $H_{G\boxtimes H}(R_1) = R_1$ and $H_{G\boxtimes H}(R_2) = R_2$. And thus $H_{G\boxtimes H}(R_1) \cap H_{G\boxtimes H}(R_2) = \phi$. Thus R with $|R| = \lceil \frac{l+1}{3} \rceil$ has no radon partition in $G\boxtimes H$. Hence $r(G\boxtimes H) \geq \lceil \frac{l+1}{3} \rceil + 1$. Similarly $r(G \boxtimes H) \geq \lceil \frac{k+1}{3} \rceil + 1$. Thus $r(G \boxtimes H) \geq \max\{\lceil \frac{k+1}{3} \rceil + 1, \lceil \frac{l+1}{3} \rceil + 1\}$. Hence $r(G \boxtimes H) \geq \max\{\lceil \frac{diam(G)+4}{3} \rceil, \lceil \frac{diam(H)+4}{3} \rceil\}$.

□

This bound is sharp since $r(G \boxtimes H) = \lceil \frac{n+4}{3} \rceil = \max\{\lceil \frac{diam(G)+4}{3} \rceil, \lceil \frac{diam(H)+4}{3} \rceil\}$, when $G = P_n$ and $H = K_m$.

For $G = P_4$ and $H = P_4$, $r(G\boxtimes H) = 4 > \max\{\lceil \frac{diam(G)+4}{3} \rceil, \lceil \frac{diam(H)+4}{3} \rceil\}$. Hence strict inequality may occur in the above result.

5.4.2 P_3-Convexity in cartesian product of graphs

P_3 hull number of cartesian product of graphs were well studied in [36]. In this section bounds for caratheodory number and radon number of cartesian product of graphs are obtained.

Theorem 5.4.7. *Let G and H be connected graphs different from complete graph with caratheodory number $c(G)$ and $c(H)$ respectively. Then $c(G\Box H) \geq max\{c(G)+1, c(H)+1\}$.*

Proof. Let $|V(G)| = m$, $|V(H)| = n$ and $V(H) = \{u_1, u_2, ..., u_m\}$. Choose $S = \{v_1, v_2, ..., v_{c(G)}\} \subset V(G)$ and $v_{c(G)+1} \in H_G(S)$, but S has no subset F such that $|F| \leq c(G)-1$ and $v_{c(G)+1} \in H_G(F)$. Then, let u_i, u_j be vertices in H with $d(u_i, u_j) = 2$ and $u_p \in N[u_i, u_j]$. Take $S^1 = \{(u_i, v_1), (u_i, v_2), (u_i, v_3), ..., (u_i, v_{c(G)}), (u_j, v_{c(G)})\}$. Then $(u_p, v_{c(G)+1}) \in H_{G\Box H}(S^1)$ and S^1 has no subset F such that $|F| \leq c(G)$ and $(u_p, v_{c(G)+1}) \in H_{G\Box H}(F)$. Hence $c(G\Box H) \geq c(G)+1$. Similarly, we can show that $c(G\Box H) \geq c(H)+1$. Hence $c(G\Box H) \geq \max\{c(G)+1, c(H)+1\}$.

□

The bound is sharp, since $c(G\Box H) = 3 = max\{c(G)+1, c(H)+1\}$, when $G = P_3$ and $H = P_3$.

For $G = P_8$ and $H = P_3$, $c(G\Box H) = 8 > \max\{c(G)+1, c(H)+1\}$. Hence strict inequality may occur in the above result.

Remark 5.4.1. *If $G = K_m$ and $H = K_n$ then $c(G\Box H) = 2 = c(G) = c(H)$.*

Proof. For any $S \subset V(K_m\Box K_n)$ with $|S| \geq 2$, $H_{K_m\Box K_n}(S) = V(K_m\Box K_n)$. Hence $c(K_m\Box K_n) = 2$.

□

Remark 5.4.2. *Let G and H be connected graphs and G is different from complete graph. If caratheodory number of G is $c(G)$, then $c(G\Box H) \geq c(G)+1$.*

Theorem 5.4.8. *Let G and H be connected graphs with radon number $r(G)$ and $r(H)$ respectively. Then* $r(G\Box H) \geq max\{\lfloor\frac{diam(H)+1}{3}\rfloor(r(G)-1)+1, \lfloor\frac{diam(G)+1}{3}\rfloor(r(H)-1)+1\}$.

Proof. Let $diam(H) = l$ and $v_1, v_2, ..., v_{l+1}$ be a maximum path of H. Choose $R = \{u_1, u_2, ..., u_{r(G)-1}\} \subset V(G)$ which has no radon partition in G. Then $R^1 = \{u_1, u_2, ..., u_{r(G)-1}\} \times \{v_1, v_4, v_7, ..., v_{\lfloor\frac{l+1}{3}\rfloor}\}$ has no radon partition in $G\Box H$.

If R^1 has a partition R_1^1, R_2^1 and $H_{G\Box H}(R_1^1) \cap H_{G\Box H}(R_2^1) \neq \phi$. Let $R_1^1 \cap (R \times \{v_i\}) = R_1^{1i}$, $R_2^1 \cap (R \times \{v_i\}) = R_2^{1i}$ and $R^1 \cap (R \times \{v_i\}) = R^{1i}$. Then $H_{G\Box H}(R^1) = \cup_{i=1}^{\lfloor\frac{l+1}{3}\rfloor} H_{G\Box H}(R^{1i})$, $H_{G\Box H}(R_1^1) = \cup_{i=1}^{\lfloor\frac{l+1}{3}\rfloor} H_{G\Box H}(R_1^{1i})$ and $H_{G\Box H}(R_2^1) = \cup_{i=1}^{\lfloor\frac{l+1}{3}\rfloor} H_{G\Box H}(R_2^{1i})$. Hence $H_{G\Box H}(R_1^{1i}) \cap H_{G\Box H}(R_2^{1i}) \neq \phi$ for some $i \in \{1, 3, ..., \lfloor\frac{l+1}{3}\rfloor\}$. If $R_1 = \{u : (u, v) \in R_1^{1i}\}$ and $R_2 = \{u : (u, v) \in R_2^{1i}\}$, then R_1, R_2 is a radon partition of R in G, which is a contradiction. Hence R^1 has no radon partition in $G\Box H$.

Hence $r(G\Box H) \geq \lfloor\frac{diam(H)+1}{3}\rfloor(r(G)-1)+1$.

In a similar way, $r(G\Box H) \geq \lfloor\frac{diam(G)+1}{3}\rfloor(r(H)-1)+1$.

Thus $r(G\Box H) \geq \max\{\lfloor\frac{diam(H)+1}{3}\rfloor(r(G)-1)+1, \lfloor\frac{diam(G)+1}{3}\rfloor(r(H)-1)+1\}$.

□

For $G = P_{11}$ and $H = P_4$,
$r(G\Box H) > max\{\lfloor\frac{diam(H)+1}{3}\rfloor(r(G)-1)+1, \lfloor\frac{diam(G)+1}{3}\rfloor(r(H)-1)+1\}$. Hence strict inequality may occur in the above result.

5.4.3 P_3-Convexity in composition of graphs

The composition of graphs G and H can be defined as follows: every vertex $v_i \in V(G)$, for every $1 \leq i \leq |V(G)|$, is replaced by a copy of H, denoted by G_i, and if there exists an edge $v_iv_j \in E(G)$, every vertex of G_i is adjacent to every vertex of G_j, for all $i, j \in \{1, ..., |V(G)|\}$, $i \neq j$. For $i \neq j$, $i, j \in \{1, ..., |V(G)|\}$ if there exists a path from v_i to v_j in G, then a subgraph G_j of $G[H]$, is said to be reachable from a subgraph G_i.[36]

The P_3-convex hull number of composition of graphs is obtained in [36] and it is obtained that $h(G[H]) = 2$, when G and H are nontrivial graphs and G is connected. In this section caratheodory number and radon number of composition of graphs is studied.

Lemma 5.4.9. *Let G and H be non trivial connected graphs. Then, for $S \subset V(G[H])$ either $H_{G[H]}(S) = V(G[H])$ or $H_{G[H]}(S) = S$.*

Proof. Let $V(G) = \{u_1, u_2, ..., u_m\}$ and $V(H) = \{v_1, v_2, ..., v_n\}$. Let $U \subset V(G[H])$ and $H_{G[H]}(U) \neq U$. Then choose any vertex $v \in I^1[U]$. Then there exists two vertices $x, y \in U$ which are adjacent to v.

If $x, y \in G_i$ for $i \in \{1, ..., |V(G)|\}$, then for some $j \in \{1, ..., |V(G)|\}$, $v_iv_j \in E(G)$ and x and y is adjacent every vertices of G_j. Hence $V(G_j) \subset I^1[\{x, y\}]$. And $V(G_i) \subset I^2[\{x, y\}]$. Since G is connected, every subgraph G_k is reachable from G_j, for $k \neq j$, $k \in \{1, 2, ..., |V(G)|\}$. Hence $H_{G[H]}(\{x, y\}) = V(G[H])$.

If $x \in G_i$ and $y \in G_j$, $i \neq j$.
Then if $v \in G_i$, $V(G_j) \subset I^2[\{x, y\}]$. Since for $k \neq j$, $k \in \{1, 2, ..., |V(G)|\}$, every vertex of G_k is reachable from G_j, $H_{G[H]}(\{x, y\}) = V(G[H])$.
If $v \in G_p$, $p \neq i, j$, then $G_p \subset I^2[\{x, y\}]$. And G_k is reachable from G_p, $\forall k \neq p$, $k \in \{1, 2, ..., |V(G)|\}$. Hence $H_{G[H]}(\{x, y\}) = V(G[H])$.

Hence $H_{G[H]}(\{x, y\}) = V(G[H])$.

Hence if $S \subset V(G[H])$ then either $H_{G[H]}(S) = V(G[H])$ or $H_{G[H]}(S) = S$.

□

Theorem 5.4.10. *Let G and H be nontrivial connected graphs. Then $c(G[H]) = 2$.*

Proof. Let S be a subset of $V(G[H])$ and $(u, v) \in H_{G[H]}(S)$.

Case 1 $(u,v) \in S$, then there is $F \subset S$ with $|F| = 1$ and $(u,v) \in H_{G \boxtimes H}(F)$.

Case 2 $(u,v) \notin S$ and $(u,v) \in H_{G[H]}(S)$. Choose any $(a,b) \in I^1[S]$, but $(a,b) \notin S$. Then there exist vertices $(a_1,b_1),(a_2,b_2) \in S$ which are adjacent to (a,b). Take $F = \{(a_1,b_1),(a_2,b_2)\} \subset S$. Then $H_{G[H]}[F] \neq F$. Hence by lemma5.4.9 $H_{G[H]}(F) = V(G[H])$. Hence $(u,v) \in H_{G[H]}(F)$, $F \subset S$ and $|F| = 2$.

Thus $c(G[H]) \leq 2$. Since G and H are nontrivial connected graphs, $c(G[H]) \geq 2$.

Thus $c(G[H]) = 2$. □

Theorem 5.4.11. *Let G and H be nontrivial connected graphs. Then $r(G[H]) = max\{3, \lfloor \frac{diam(G)+1}{3} \rfloor + 1\}$.*

Proof. Let $V(G) = \{u_1, u_2, ..., u_m\}$, $\lfloor \frac{diam(G)+1}{3} \rfloor > 2$ and $v_1, v_2, ..., v_{l+1}$ be a maximum path in G.

If $R = \{(u_1,v_1),(u_4,v_1),....,(u_{\lfloor \frac{l+1}{3} \rfloor},v_1)\}$. Then no two vertices in R has common neighbour in $V(G[H])$. Hence $H_{G[H]}(R) = R$ and R has no radon partition in $G[H]$. Hence $r(G[H]) \geq \lfloor \frac{diam(G)+1}{3} \rfloor + 1$.

Let $R \subset V(G[H])$ with $|R| \geq \lfloor \frac{diam(G)+1}{3} \rfloor + 1$.
If $|G_i \cap R| \geq 2$ for some $i \in \{1, 2, ...m\}$ and $x, y \in G_i \cap R$. Then by 5.4.9, $H_{G[H]}(\{x,y\}) = V(G[H])$. Hence $R_1 = \{x,y\}$ and $R_2 = R - \{x,y\}$ is a radon partition of R.
If $|G_i \cap R| \leq 1$ for all $i \in \{1, 2, ...m\}$, since $|R| \geq \lfloor \frac{diam(G)+1}{3} \rfloor + 1$, there exists two vertices $x \in G_i$ and $y \in G_j$, with x, y has a common neighbour $w \in G_k$ for some $k \neq i, j$, $k \in \{1, 2, ...m\}$. Thus by 5.4.9 $H_{G[H]}(\{x,y\}) = V(G[H])$. Then $R_1 = \{x,y\}$ and $R_2 = R - \{x,y\}$ is a radon partition of R.

Hence $r(G[H]) \leq \lfloor \frac{diam(G)+1}{3} \rfloor + 1$.

Thus $r(G[H]) = \lfloor \frac{diam(G)+1}{3} \rfloor + 1$.

If $\lfloor \frac{diam(G)+1}{3} \rfloor \leq 2$, then Let $R \subset V(G[H])$ with $|R| \geq 3$.
Then, if $|G_i \cap R| \geq 2$ for some $i \in \{1, 2, ...m\}$ and $x, y \in G_i \cap R$. $H_{G[H]}(\{x,y\}) =$

$V(G[H])$. Hence $R_1 = \{x, y\}$ and $R_2 = R - \{x, y\}$ is a radon partition of R.
If $|G_i \cap R| \leq 1$ for all $i \in \{1, 2, ...m\}$, since $|R| \geq \lfloor \frac{diam(G)}{4} \rfloor + 1$, there exists two vertices $x \in G_i$ and $y \in G_j$, with x, y has a common neighbour $w \in G_k$ for some $k \neq i, j$, $k \in \{1, 2, ...m\}$. Thus $H_{G[H]}(\{x, y\}) = V(G[H])$. Then $R_1 = \{x, y\}$ and $R_2 = R - \{x, y\}$ is a radon partition of R. Hence $r(G[H]) = 3$.

Thus $r(G[H]) = \max\{3, \lfloor \frac{diam(H)+1}{3} \rfloor + 1\}$. □

5.5 P_3-Convexity in corona related graphs

Study in the P_3-convex invariants- hull number and radon number of corona related graphs is done in this section.

5.5.1 P_3-Convexity in corona of graphs

Detailed study on hull number and radon number of corona of graphs is done in this section.

Theorem 5.5.1. *Let G be a graph with radon number r, and $|V(G)| = n$, then $2r - 1 \leq r(G \circ H) \leq r + n$.*

Proof. Let $V(G) = \{v_1, v_2, ..., v_n\}$, $V(H) = \{u_1, u_2, ..., u_n\}$ and copy of H corresponding to v_i be H_i.

Let R^1 be any subset of $V(G \circ H)$ with $|R^1| \geq r + n$.

Case 1 If $|R^1 \cap V(G)| \geq r$. Then $R^1 \cap V(G)$ has a radon partition in G, say R^1_1 and R^1_2. Then $R^1_1 \cup (R^1 - V(G))$ and R^1_2 is a radon partition of $G \odot H$.

Case 2 If $|R^1 \cap V(G)| < r$. Then $|R^1 \cap \cup_{i=1}^{n} V(H_i)| \geq n + 1$. Hence there is some $k \in \{1, 2, ..., n\}$ such that $|V(H_k) \cap R^1| \geq 2$.

If $|(V(H_k) \cup \{v_k\}) \cap R^1| \geq 3$, say $\{w_1, w_2, w_3\} \subset (R^1 \cap V(H_k))$. Then $\{w_1, w_2\}$, $\{w_3\}$is a radon partition of $\{w_1, w_2, w_3\}$ and $R^1 - \{w_3\}$ and $\{w_3\}$ is a radon partition of R^1.

Otherwise $|(V(H_i) \cup \{v_i\}) \cap R^1| \leq 2, \forall i \in \{1, 2, ..., n\}$. Then since $|R^1| \geq r + n$, there are atleast r number of H_i where $i \in \{1, 2, ..., n\}$ with $|(V(H_i) \cup \{v_i\}) \cap R^1| = 2$, say for $i = 1, 2, ..., r$. Since, $|R^1 \cap V(G)| = t < r$, there are atleast $r - t$ number of H_i where $i \in \{1, 2, ...r\}$ with $|V(H_i) \cap R^1| = 2$, say for $i = t + 1, t + 2, ..., r$ and t number of H_i with $|(V(H_i) \cup \{v_i\}) \cap R^1| = 2$, say for $i = 1, 2, ..., t$. Let $T_i = ((V(H_i) \cup \{v_i\}) \cap R^1)$, $\forall i = 1, 2, ..., t$ and $T_i = (V(H_i) \cap R^1)$, $\forall i = t + 1, t + 2, ...r$. Then $\{v_1, v_2, ..., v_r\}$ has a radon partition say R_1 and R_2. Let $A = \{i : i \in R_1\}$ and $B = \{i : i \in R_2\}$. Then $R_1^1 = \cup_{i \in A} T_i$ and $R_2^1 = \cup_{i \in B} T_i$ is a radon partition of R^1.

Hence, $r(G \circ H) \leq r + n$.

Choose $R = \{v_1, v_2, ..., v_{r-1}\}$ so that R has no Radon partition in G. Then take u_i^1, u_i^2 any two vertices in $V(K_i)$. Then $R^1 = \cup_{i=1}^{r-1}\{u_i^1, u_i^2\}$ has no Radon partition in $G \circ H$. Thus $2r - 1 \leq r(G \circ H)$.

□

Strict inequality may occur in the above result.

For $G =$Friendship graph F_3 and $H = K_m$, $r(G \circ H) = 9 < 4 + 7 = r(G) + n$, $G =$Friendship graph F_3 and $H = K_n$, is an example for $r(G \circ K) < r + n$.

And for $G = C_8$ and $H = K_m$, $r(G \circ K_n) = 13 > 2 \times 5 - 1 = 2r(G) - 1$. Hence $G = C_8$ and $H = K_m$ is an example for $2r - 1 < r(G \circ H)$.

And upper bound may attain. Since for $G = C_n$ and $H = K_n$, $r(G \circ K_m) = \lceil \frac{n}{2} \rceil + 1 + n = r(G) + n$, $G = C_n$ and $H = K_m$ is an example.

Lower bound may also attain. Since for $G = K_n$ and $H = K_n$, $r(G \circ K_n) = 5 = 2 \times 3 - 1 = 2r(G) - 1$, $G = K_n$ and $H = K_n$ is an example.

Theorem 5.5.2. *For any two graphs G and H, $h(G \circ H) = |V(G)| + 1$.*

Proof. Let G and H be graphs with $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(K) = \{u_1, u_2, ..., u_m\}$ and H_i be the copy of H corresponding to v_i, $\forall i = 1, 2, ..., n$.

Let $S^1 = \{v_1\} \cup \{u_1^1, u_2^1, ..., u_n^1\}$ where u_i^1 is any vertex from H_i, $\forall i = 1, 2, ..., n$.

Let $l = eccentricity(v_1)$ in G and $p = diameter(H)$.Then,

$$\begin{aligned} I[S^1] &\supset N[v_1] \cup \{u_1^1, u_2^1, ..., u_m^1\} \\ I^2[S^1] &\supset N^2[v_1] \cup \{u_1^1, u_2^1, ..., u_m^1\} \\ I^l[S^1] &\supset V(G) \cup \{u_1^1, u_2^1, ..., u_m^1\} \\ I^{l+p}[S^1] &= V(G \circ H) \end{aligned}$$

Hence $h(G \circ H) \leq |V(G)| + 1$.

Conversely, let $S^1 \subset V(G \circ H)$, with $|S^1| < |V(G)| + 1$. Then, $|S^1| \leq |V(G)|$ and hence $|S^1 \cap (\cup_{i=1}^n V(H_i))| < |V(G)|$ or $S^1 \cap V(G) = \phi$.

If $|S^1 \cap (\cup_{i=1}^n V(H_i))| < |V(G)|$, there is some $k \in \{1, 2, ...n\}$ with $V(H_k) \cap S^1 = \phi$. Then $H_{G \circ H}[S^1] \not\supseteq V(H_k)$

If $S^1 \cap V(G) = \phi$. Then,

Case 1 $|V(H_i) \cap S^1| = 1$, $\forall i = 1, 2, ..., n$.

Then $H_{G \circ H}[S^1] = S^1 \neq V(G \circ H)$

Case 2 $|V(H_k) \cap S^1| = 0$ for some $k \in \{1, 2, ..., n\}$.
Then $H_{G \circ H}[S^1] \not\supseteq V(H_k)$
Hence $H_{G \circ H}[S^1] \neq V(G \circ H)$.

Thus, $h(G \circ H) \geq |V(G)| + 1$.

Hence it follows that $h(G \circ H) = |V(G)| + 1$.

□

5.5.2 P_3-Convexity in neighbourhood corona of graphs

In this section bounds for hull number and radon number of neighbourhood corona of graphs are obtained.

Theorem 5.5.3. *Let G and H be nontrivial graphs with hull number $h(G)$ and $h(H)$ respectively and $\delta(G) \geq 2$. Then $h(G \star H) \leq h(G)$.*

Proof. Let $S = \{v_1, v_2, ..., v_{h(G)}\}$ be a minimum P_3-convex hull set and $I^d[S] \supset V(G)$ for some d. Since G has no pendant vertex every vertex in $G \star H$ is adjacent with atleast two vertices in G. Hence $I^{d+1}[S] = V(G \star H)$. Hence $h(G \star H) \leq h(G)$.

□

The bound given in 5.5.3 is sharp, since the bound is attained for $G = C_8$ and $H = P_5$. Here $h(G \star H) = 4 = h(G)$.
Following remark shows that the difference in the inequality 5.5.3 may be large.

Remark 5.5.1. *For any integer $a \geq 2$ there exists graph G and H with $h(G \star H) = h(G) + a$.*

Proof. Let $n = a - 2$, G be the friendship graph F_n and H be any graph with $|V(H)| \geq 2$. Then $h(G \star H) = 2$ and $h(G) = n$.

□

Following remark shows that theorem 5.5.3 may not hold for graph G with pendant vertices.

Remark 5.5.2. *For the graph $G = P_5$ and $H = K_3$, $h(G \star H) = 4$ while $h(G) = 3$.*

Theorem 5.5.4. *Let G be any graph and $|V(H)| \geq 2$, then*

$$r(G \star H) \geq 2 \times \lfloor \frac{diam(G) + 1}{3} \rfloor + 1$$

.

Proof. Let $v_1, v_2, ..., v_{l+1}$ be a maximum path in G and remaining vertices of $V(G)$ be $\{v_{l+2}, v_{l+3}, ..., v_n\}$.Let H_i be the copy of H corresponding to v_i, $\forall i \in \{1, 2, ..., n\}$

in $G \star H$. Choose two vertices $v_{i,1}, v_{i,2}$ from H_i, $\forall i \in A = \{1, 4, 7, ..., \lfloor \frac{diam(G)+1}{3} \rfloor\}$. Then $R = \cup_{i \in A}\{v_{i,1}, v_{i,2}\}$ has no radon partition in $G \star H$. Hence $r(G \star H) \geq 2 \times \lfloor \frac{diam(G)+1}{3} \rfloor + 1$.

□

5.5.3 P_3-Convexity in edge corona of graphs

In this section bounds for hull number and radon number of edge corona of graphs are obtained.

Theorem 5.5.5. *Let G and H be nontrivial graphs with hull number $h(G)$ and $h(H)$ respectively. Then $h(G \diamond H) \leq h(G)$.*

Proof. Let $S = \{v_1, v_2, ..., v_h\}$ be a minimum P_3-convex hull set of G and $I^d[S] \supset V(G)$ for some d. Then every vertex in $G \diamond H$ is adjacent with atleast two vertices in G. Hence $I^{d+1}[S] = V(G \diamond H)$. Hence $h(G \diamond H) \leq h(G)$.

□

Theorem 5.5.6. *Let G be a graph having perfect matching and $|V(H)| \geq 2$, then $r(G \diamond H) \geq |V(G)| + 1$.*

Proof. Let $e_1, e_2, ..., e_q$ be a perfect matching in G and $e_{q+1}, e_{q+2}, ..., e_r$ be the remaining edges of G. Let H_i be the copy of H in $G \diamond H$ corresponding to e_i , $\forall i \in \{1, 2, ..., r\}$. Then $|\{e_1, e_2, ..., e_q\}| = \frac{|V(G)|}{2}$. Choose two vertices $v_{i,1}, v_{i,2}$ from H_i, $\forall i \in \{1, 2, ..., q\}$. Let $R = \cup_{i=1}^{q}\{v_{i,1}, v_{i,2}\}$. Then $|R| = |V(G)|$ and R has no radon partition in $G \diamond H$. Hence $r(G \diamond H) \geq |V(G)| + 1$.

□

Since $r(G \diamond H) = 7 = |V(G)| + 1$, for $G = C_6$ and for any graph H, the bound given in theorem 5.5.6 is sharp.

5.6 Conclusion

The concept of P_3- convexity is studied here. The motivation behind this is the problem like sharing an idea or marketing, having the property:- a set of vertices S initially possessing a property spreads the property to the vertices having two neighbours in S. These problems like sharing or marketing can be extended in a way that set of vertices having the property a set of vertices S initially possessing a property spreads the property to the vertices having k neighbours in S. Thus there is a large scale of application and scope in this area.

This chapter includes general properties in P_3-convexity, P_3-convex invariants, hull number, radon number and caratheodory number of some classes of graphs, strong product, cartesian product, composition of graphs. and corona related graphs.

CHAPTER 6

Conclusion and Results

6.1 Conclusion

This thesis is an attempt to introduce various types of domination parameters and study their properties. And domination number of cartesian product of graphs is studied using one of this domination parameter.

First chapter is an introductory chapter in which background information and preliminary information about the work are provided.

In the second chapter a study of domination number of cartesian product of graphs is done. Main results from this chapter are listed below. These results are published in [59]

1. Let G be any graph and P_n be a path having n vertices. Then $\frac{n}{3}\gamma(G) < \gamma(G \Box P_n)$.
2. Let G be any graph and P_n be a path having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2, D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, then $\gamma(G \Box P_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.

3. Let G be a graph having a minimum dominating set D which can be partitioned into two nonempty sets D_1 and D_2 with the property every vertex not in D is adjacent with atleast one vertex in D_1 and atleast one vertex in D_2. Then $\gamma(G\Box C_4) \leq 2\gamma(G)$.

4. Let G be any graph and C_n be a cycle having n vertices, then $\frac{n}{3}\gamma(G) \leq \gamma(G\Box C_n)$.

5. Let G be any graph and C_n be a cycle having n vertices. If G has a minimum dominating set D such that $D = D_1 \cup D_2$, $D_1 \cap D_2 = \phi$ and every vertex not in D has a neighbour in D_1 and a neighbour in D_2, then $\gamma(G\Box C_n) \leq \left\lceil \frac{n}{2} \right\rceil \gamma(G)$.

In the third chapter Reserved domination number is introduced and some of its properties are studied. Result on domination number of cartesian product of graphs is obtained using reserved domination number. Some of the results from third chapter is published in [60]

Let $A \subset V(G)$, $A \neq \phi$, a dominating set D of a graph G is an A- reserved dominating set, if $D \cap A \neq \phi$. The A-reserved domination number of G, $r\gamma_A(G)$ is the cardinality of minimum A-reserved dominating set.

Let $A_1, A_2, ..., A_k \subset V(G)$, $A_i \neq \phi, \forall i = 1, 2, ..., k$. If $D \subset V(G)$ is a dominating set with the property $D \cap A_i \neq \phi, \forall i = 1, 2, ..., k$, then D is said to be an $[A_1, A_2, ...A_k]$- reserved dominating set of G. Cardinality of minimum $[A_1, A_2, ...A_k]$- reserved dominating set is called the $[A_1, A_2, ...A_k]$-reserved domination number denoted by $r\gamma_{[A_1,A_2,...,A_k]}(G)$.

6. For any Graph G and $A \subset V(G)$, $\gamma(G) \leq r\gamma_A(G) \leq \gamma(G) + 1$

7. An A-reserved dominating set D is a minimal A-reserved dominating set if and only if for each vertex v in D one of the following conditions holds

   (a) v is an isolate of D.

   (b) v has a private neighbour u in $V - D$.

   (c) $D \cap A = \{v\}$.

8. If $A \subset B$, then $r\gamma_A(G) \geq r\gamma_B(G)$.

9. $r\gamma_A(G) = 1$ if and only if A contains a universal vertex of G.

10. $r\gamma_A(G) = n$ if and only if $G = \overline{K_n}$.

11. For a graph G and $A \subset V(G)$,

    - $r\gamma_A(G) = \gamma(G) + 1$ if and only if $A \subset anticore(G)$.
    - $r\gamma_A(G) = \gamma(G)$ if and only if $A \cap \overline{(anticore(G))} \neq phi$.

12. For a graph G and $A \subset V(G)$,

    - $r\gamma_A(G) = \gamma(G) + 1$ if and only if $\gamma(G_v + u) = \gamma(G) + 1$ for every $v \in A$.
    - $r\gamma_A(G) = \gamma(G)$ if and only if $\exists v \in A$ satisfying $\gamma(G_v + u) \neq \gamma(G) + 1$.

13. For cycle C_n and any $A \subset V(C_n)$, $r\gamma_A(C_n) = \lceil \frac{n}{3} \rceil = \gamma(G)$.

14. For complete graph K_n and any $A \subset V(K_n)$, $r\gamma_A(K_n) = 1$.

15. For complete bipartite graph $K_{m,n}$ with $2 \leq m, n$ and any $A \subset V(K_{m,n})$, $r\gamma_A(K_{m,n}) = 2$.

16. If $n \equiv 1(mod3)$ and $A \subset V(P_n)$, then $r\gamma_A(P_n) = \lceil \frac{n}{3} \rceil = \gamma(P_n)$.

17. If $n \equiv 0(mod3)$ and $A \subset V(P_n)$, then
$$r\gamma_A(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if A \cap \{v_i : i \equiv 2(mod3)\} \neq \phi \\ \lceil \frac{n}{3} \rceil + 1 & Otherwise \end{cases}$$

18. If $n \equiv 2(mod3)$ and $A \subset V(P_n)$, then
$$r\gamma_A(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil & if A \cap \{v_i : i \equiv 1, 2(mod3)\} \neq \phi \\ \lceil \frac{n}{3} \rceil + 1 & Otherwise \end{cases}$$

19. $r\gamma_{A \cup B}(G) = \min \{r\gamma_A(G), r\gamma_B(G)\}$.

20. Let G and H be two graphs of order n_1 and n_2, then for any $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_{A \times B}(G \Box H) \leq min\{n_1 r\gamma_B(H), n_2 r\gamma_A(G)\}$.

21. If a graph G satisfy the inequality $r\gamma_{A \times B}(G \Box H) \geq r\gamma_A(G) r\gamma_B(H)$, for every graph H and for $A \times B = \{(u, v)\}$ where (u, v) is an element in a minimum dominating set of $G \Box H$. Then G satisfies Vizing's inequality.

22. Let G be a graph that satisfies $r\gamma_{A\times B}(G\Box H) \geq r\gamma_A(G)r\gamma_B(H)$, for every $A \subset V(G)$ and for every graph H and $B \subset V(H)$, and let G^1 be a spanning subgraph of G such that $r\gamma_A(G) = r\gamma_A(G^1)$. Then G^1 also satisfies $r\gamma_{A\times B}(G^1\Box H) \geq r\gamma_A(G^1)r\gamma_B(H)$, for every $A \subset V(G^1)$ and for every graph H and $B \subset V(H)$.

23. If n is an odd integer and $x_1, x_2, ..., x_n$ be the vertices in the first copy of P_n in $P_n\Box P_2$ and $y_1, y_2, ..., y_n$ be the vertices in the second copy of P_n in $P_n\Box P_2$, then for any $A \subset V(P_n\Box P_2)$,

$$r\gamma_A(P_n\Box P_2) = \begin{cases} \lceil\frac{n+1}{2}\rceil & if A \cap \{x_i, y_i : i = 2k+1, 1 \leq i \leq n\} \neq \phi \\ \lceil\frac{n+2}{2}\rceil & Otherwise \end{cases}$$

.

24. If n is an even integer , then for any $A \subset V(P_n\Box P_2)$,

$$r\gamma_A(P_n\Box P_2) = \lceil\frac{n+1}{2}\rceil$$

25. If $A \subset V(P_n\Box P_3)$,

$$n - \lfloor\frac{n-1}{4}\rfloor \leq r\gamma_A(P_n\Box P_3) \leq n + 1 - \lfloor\frac{n-1}{4}\rfloor$$

.

26. If $A \subset V(P_n\Box P_4)$,

$$n + 1 \leq r\gamma_A(P_n\Box P_4) \leq n + 2 \quad \text{if n=1,2,3,4,5,6 or 9}$$
$$n \leq r\gamma_A(P_n\Box P_4) \leq n + 1 \quad \text{Otherwise}$$

.

27. For any $A \subset V(K_m)$ and $B \subset V(K_n), m \leq n$ $r\gamma_{A\times B}(K_m\Box K_n) = m$.

28. For any $A \subset V(P_m)$ and $B \subset V(K_n), m, n > 2 \in N$, $r\gamma_{A\times B}(P_m\Box K_n) = m$.

29. For any $A \subset V(C_m)$ and $B \subset V(K_n), m, n > 2 \in N$, $r\gamma_{A\times B}(C_m\Box K_n) = m$.

30. For any two nontrivial graphs G and H and for every $A \subset V(G)$ and $B \subset V(H)$, $r\gamma_{A\times B}(G \boxtimes H) \leq r\gamma_A(G)r\gamma_B(H)$.

31. Let G be a graph order n and H be a graph order m and H_i be the copy of H corresponding to $v_i \in G$, then for $A \subset V(GoH)$,

$$r\gamma_A(GoH) = \begin{cases} n & \text{if } A \cap (V(G) \cup \{v : \text{v is a universal vertex of } H_i\}) \neq \phi \\ n+1 & Otherwise \end{cases}$$

.

32. If $A_1, A_2, ...A_k \subset V(G)$ then an $[A_1, A_2, ..., A_k]$-reserved dominating set D is a minimal $[A_1, A_2, ..., A_k]$-reserved dominating set if and only if for each vertex v in D one of the following conditions holds
    (a) v is an isolate of D.
    (b) v has a private neighbour u in $V - D$.
    (c) $\exists l \in \{1, 2, ...k\}$ such that $D \cap A_l = \{v\}$.

33. Let $G = K_{1,n}$ with u as center and $V(G) = \{u, u_1, u_2, ...u_n\}$ and let H be any graph . Let $D \subset V(G\Box H)$ such that D dominates $\{u\}\Box H$. Then , $|D| \geq 2\gamma(H) - \gamma(< F >)$, where $F = \{v \in H : (u_i, v)$ is not adjacent to D for some $i\} \cup \{v \in H : (u_i, v) \notin D, \forall i\}$.

34. Let G be any graph with $\gamma(G) = 1$ and $N \subset V(G)$ such that $r\gamma_N(G) > 1$.And let H be any graph. Let $D \subset G\Box H$ with the property that D dominates $(G - N)\Box H$ .Then, $|D| \geq 2\gamma(H) - \gamma(< F >)$, where $F = \{v \in H : (u, v)$ is not dominated by D for some $u \in N\} \cup \{v \in H : (u, v) \notin D, \forall u \in N\}$.

35. Let $G = K_{1,N}$ with u as center and $V(G) = \{u\} \cup N$ where $N = N_1 \cup N_2 \cup ....N_k$, $|N_i| \geq 2, N_i = \{u_{i1}, u_{i2}, ..., u_{ir}\}$ for $i = 1, 2, ...k$ and H be any graph. If $D \subset V(G\Box H)$ so that D dominates $\{u\}\Box H$, then $|D| \geq (k + 1)\gamma(H) - \sum_{i=1}^{k} \gamma(< F_i >)$, where $F_i = \{v \in H : (u_{ij}, v)$ is not dominated by D for some $j\} \cup \{v \in H : (u_{ij}, v) \notin D \cap N_i, \forall j\}$.

    α-stable domination number is introduced and some of its properties are studied in the fourth chapter. Main results from the chapter are listed below.

    Let D be a dominating set. For a vertex u in D let $\psi_D(u) = |N(u) \cap (V - D)|$. The donor instability or d-instability of an edge e connecting two donor vertices u and v, $d^D_{inst}(e)= |\psi_D(u) - \psi_D(v)|$. Let $D \subset V$, the d-instability of D, is the sum of d-instabilities of all edges connecting vertices in D, $\psi_d(D) = \sum_{e\in<D>} d^D_{inst}(e)$. Let D be a dominating set. Given a non negative integer α, D is an α-d-stable dominating set, if $d^D_{inst}(e) \leq \alpha$ for any edge e connecting two donor vertices. Cardinality of a minimum α-d-stable dominating set is the α-d-stable domination number and denoted by $\gamma^\alpha_d(G)$. A dominating set D is d-stable if $\psi_d(D) = 0$. Cardinality of a minimum d-stable dominating set is d-stable domination number and denoted by $\gamma^0_d(G)$.

For a vertex u not in D, let $\phi_D(u) = |N(u) \cap D|$. The Acceptor Instability or a-instability of an edge e connecting two acceptor vertices u and v is, $a^D_{inst}(e)$ $= |\phi_D(u) - \phi_D(v)|$. The a-instability of D, $\phi_a(D)$ is the sum of a-instabilities of all edges connecting vertices in $V - D$, $\phi_a(D) = \sum_{e\in<V-D>} a^D_{inst}(e)$. Let D be a dominating set. Given a non negative integer α, D is an α-a-stable dominating set, if $a^D_{inst}(e) \leq \alpha$ for any edge e connecting two acceptor vertices. Cardinality of a minimum α-a-stable dominating set is α-a-stable domination number and denoted by $\gamma^\alpha_a(G)$. The dominating set D is a-stable if $\phi_a(D) = 0$. Minimum cardinality of an a-stable dominating set is a-stable domination number and denoted by $\gamma^0_a(G)$.

A dominating set D is stable, if $\psi_d(D) = 0$ and $\phi_a(D) = 0$. Minimum cardinality of a stable dominating set is called stable domination number and denoted by $\gamma^0(G)$. If a dominating set D is an α-d-stable dominating set and α-a -stable dominating set, then D is called an α-stable dominating set and cardinality of a minimum α-stable dominating set is defined as α-stable domination number and denoted by $\gamma^\alpha(G)$.

36. If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma^\alpha_d(G) \leq \gamma^\beta_d(G)$.
37. Property of being α-d-stable dominating set is neither superhereditary nor hereditary.
38. An α-d- stable dominating set D is a minimal α-d- stable dominating set if and only if for each vertex v in D one of the following conditions holds

(a) v is an isolate of D.

(b) v has a private neighbour u in $V - D$.

(c) There exist two adjacent vertices u_1 and u_2 different from v in D, u_1 adjacent to v, u_2 not adjacent to v and $\psi_D(u_1) = \psi_D(u_2) + \alpha$.

39. For non negative integer α, $\gamma_d^\alpha(G) = 1 \iff \gamma(G) = 1$.

40. For a graph G and non negative integer α, $\beta_o(G) \geq \gamma_d^\alpha(G)$.

41. For a graph G and non negative integer α, $i(G) \geq \gamma_d^\alpha(G)$.

42. $ir(G) \leq \gamma(G) \leq \gamma_d^\alpha(G) \leq i(G) \leq \beta_o(G) \leq \Gamma(G) \leq IR(G)$.

43. For any graph G of order $n \geq 2$ and non negative integer α, $3 \leq \gamma_d^\alpha(G) + \gamma_d^\alpha(\overline{G}) \leq n + 1$.

44. For a connected triangle free graph G with $|V(G)| \geq 2$ and any non negative integer α, $\gamma_d^\alpha(\overline{G}) = 2$

45. If D is an α-d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) = d(u) + k + \alpha$, $k \in Z^+$, then D contains at least k elements from $(N[v] - N[u])$.

46. If D is a d-stable dominating set of a graph G and u and v are adjacent vertices in D with $d(v) > d(u)$, then D contains at least $d(v) - d(u)$ elements from $(N[v] - N[u])$.

47. If u is a pendant vertex adjacent to v, D is a d-stable dominating set and $u, v \in D$, then $N[v] \subset D$.

48. For any non negative integer β, there exist graph G with $\gamma_d^0(G) > \gamma_d^1(G) > \gamma_d^2(G)...... > \gamma_d^\beta(G)$

49. If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma_a^\alpha(G) \leq \gamma_a^\beta(G)$

50. Property of being α-a-stable dominating set is neither superhereditary nor hereditary.

51. An α-a- stable dominating set D is a minimal α-a- stable dominating set if and only if for each vertex v in D one of the following conditions holds

(a) v is an isolate of D.

(b) v has a private neighbour u in $V - D$.

(c) There exist two adjacent vertices u_1 and u_2 in V-D, u_1 adjacent to v, u_2 not adjacent to v and $\phi_D(u_2) = \phi_D(u_1) + \alpha$.

52. For non negative integer α, $\gamma_a^\alpha(G) = 1 \iff \gamma(G) = 1$

53. For $\alpha \geq 1$, $\gamma_a^\alpha(G) = 2 \iff \gamma(G) = 2$

54. For any graph G and non negative integer α, $\gamma_a^\alpha(G) \leq \gamma_p(G)$. And this bound is sharp.

55. For any positive integer β, there exist graph G with $\gamma_a^0(G) > \gamma_a^1(G) > \gamma_a^2(G)...... > \gamma_a^\beta(G)$.

56. If $\alpha \geq \beta$, then $\gamma(G) \leq \gamma^\alpha(G) \leq \gamma^\beta(G)$.

57. Property of being α-stable dominating set is neither superhereditary nor hereditary.

58. For any graph G and for any non-negative integer α, $\gamma(G) = 1 \iff \gamma^\alpha(G) = 1$.

59. For any graph G and for any non negative integer α, $\gamma^\alpha(G) = n \iff G = \overline{K_n}$.

60. For a graph G with $\delta(G) \geq 1$, $\gamma^\alpha(G) \leq n - 1$.

61. For every graph G of order n and maximum degree Δ and for any non negative integer α, $\gamma^\alpha(G) \geq \frac{n}{\Delta+1}$.

62. For any non negative integer α, $\gamma^\alpha(G) = \gamma(G)$ for the following Graphs

- Complete graph K_n
- Path P_n
- Cycle C_n
- Wheel graph W_n
- Helm graph H_n

63. If G is the corona $C_p \circ K_1$, then for any non negative integer α, $i(G) = \gamma_d^\alpha(G) = \gamma_a^\alpha(G) = \gamma^\alpha(G) = \gamma(G) = p$.

64. For complete bipartite graph $G = K_{m,n}, m \leq n$ and non-negative integer α

    $\gamma_a^\alpha(G) = 2$

    $$\gamma_d^\alpha(G) = \begin{cases} min\{(n-m+2-\alpha), m\} & if n-m+2-\alpha \geq 2 \\ m & otherwise \end{cases}$$

    $$\gamma^\alpha(G) = \begin{cases} 2 & if n-m \leq \alpha \\ m & otherwise \end{cases}$$

65. Let G and H be two graphs of order n_1 and n_2 , then for any non-negative integer α ,

    - $\gamma_a^\alpha(G\Box H) \leq min\{n_1\gamma_a^\alpha(H), n_2\gamma_a^\alpha(G)\}$
    - $\gamma_d^\alpha(G\Box H) \leq min\{n_1\gamma_d^\alpha(H), n_2\gamma_d^\alpha(G)\}$
    - $\gamma^\alpha(G\Box H) \leq min\{n_1\gamma^\alpha(H), n_2\gamma^\alpha(G)\}$.

    This bound is sharp.

66. For any graph G of order m and any non negative integer α, $\gamma^\alpha(C_n\Box G) \geq \frac{mn}{\Delta(G)+3}$.

67. If $m, n \geq 2$ for any non negative integer α, $\gamma^\alpha(C_m\Box C_n) \geq \frac{mn}{5}$.

68. If $m, n \geq 2$ for any non negative integer α, $\gamma^\alpha(C_m\Box C_n) \geq \gamma^\alpha(C_m)\gamma^\alpha(C_n)$.

69. For non-negative integer α, $\gamma^\alpha(P_2\Box P_n) = \lceil\frac{n+1}{2}\rceil$.

70. For any two graphs G and H and non negative integer α, α-stable domination number of its corona, $\gamma^\alpha(GoH) = |V(G)|$.

    Chapter five deals with the concept of P_3-convexity. Radon number, caratheodory number and hull number of some classes of graphs are obtained. P_3-convexity in product graphs and corona related graphs are well studied. A part of this chapter is published in [61]

71. $h(G) = n$ if and only if each vertex has degree less than or equal to 1.

72. Let G be a star with atleast 3 vertices, then $h(G) = n - 1$.

73. Let G be a graph with order n and $h(G) = n - 1$

    - If $n = 3$, then either G is a triangle or a star.
    - If $n = 4$, then either G is a path or a star.
    - If $n \geq 5$, then G is a star.

74. Let G be a 2-connected graph with a universal vertex. Then $h(G) = 2$.

75. The minimum size of a graph G for which $order(G) = n$ and $h(G) = 2$ is $2(n - 2)$.

76. For $2 \leq a \leq n - 1$, there exist a connected graph G with $|V(G)| = n$ and $h(G) = a$.

77. Being a P_3-convex set is neither hereditary nor superhereditary property.

78. If G is disconnected with atleast two components G_1 and G_2, $|V(G_1)| \geq 2$, $|V(G_2)| \geq 2$. Then $h(\overline{G}) = 2$.

79. Let G be a tree with n vertices. Then there exist a sequence of sets $V(G) = V_n \supset V_{n-1} \supset \ldots\ldots\ldots \supset V_1$ where for each i, V_i is convex and $|V_i| = i$.

80. Let G be a connected graph. Then G is a tree if and only if for each connected subgraph H of G, $V(H)$ is a convex set of G.

81. If G is a graph having no cycle with length ≤ 4 and $|V(G)| \geq 4$, then $h(G) \geq 3$.

82. If G is a graph with the property $I^1[S] = H_G(S)$, $\forall S \subset V(G)$, then G has joint hull commutative propery.

83. If G is a maximal outer planar graph with $diam(G) > 2$, then G has joint hull commutative property. But there are $S \subset V(G)$ such that $I^1[S] \neq H_G(S)$.

84. Let H be a connected subgraph of a graph G. Then if $V(H)$ is the only nontrivial convex set of H, then H is a block. In particular, if $V(G)$ is the only nontrivial convex set of G, then G is a block.

85. Every block in a graph G is a P_3- convex set.

86. Let G^1 be the graph generated by Mycielski's construction of a graph G. Then $h(G^1) \leq h(G) + 1$.

87. Let G be a graph of order $n \geq 3$ and size m, $T_k(G)$ the trestled graph of G with $k \geq 2$, then $h(T_k(G)) = km$.

88. For a path P_n with $n \geq 3$, $c(P_n) = 2$, $r(P_n) = \lfloor \frac{n}{2} \rfloor + 2$ and $h(P_n) = \lfloor \frac{n}{2} \rfloor + 1$.

89. For a complete graph K_n with $n \geq 3$, $c(K_n) = 2$, $r(K_n) = 3$ and $h(K_n) = 2$.

90. For a cycle C_n, $c(C_n) = 2$, $r(C_n) = \lceil \frac{n}{2} \rceil + 1$ and $h(C_n) = \lceil \frac{n}{2} \rceil$.

91. For a star $K_{1,n-1}$, $n \geq 4$, $c(K_{1,n-1}) = 2$, $r(K_{1,n-1}) = 4$, and $h(K_{1,n-1}) = n$.

92. For a complete bipartite graph $K_{m,n}$, $m, n \geq 2$, $r(K_{m,n}) = 3$, $c(K_{m,n}) = 2$, $h(K_{m,n}) = 2$.

93. For a wounded spider G,
$$h(G) = \begin{cases} \Delta(G) & \text{if centre is incident with more than one pendant vertex} \\ \Delta(G) + 1 & \textit{otherwise} \end{cases}.$$

94. Let G and H be nontrivial connected graphs. Then, $h(G \boxtimes H) = 2$.

95. Let G and H be non trivial connected graphs. Then, for $S \subset V(G \boxtimes H)$ either $H_{G \boxtimes H}(S) = V(G \boxtimes H)$ or $H_{G \boxtimes H}(S) = S$.

96. Let G and H be nontrivial connected graphs, then $c(G \boxtimes H) = 2$.

97. Let G and H be nontrivial connected graphs, then $r(G \boxtimes H) \geq \max\{\lceil \frac{diam(G)+4}{3} \rceil, \lceil \frac{diam(H)+4}{3} \rceil\}$.

98. Let G and H be connected graphs different from complete graph with caratheodory number $c(G)$ and $c(H)$ respectively. Then $c(G \square H) \geq \max\{c(G)+1, c(H)+1\}$.

99. If $G = K_m$ and $H = K_n$ then $c(G \square H) = 2 = c(G) = c(H)$.

100. Let G and H be connected graphs and G is different from complete graph. If caratheodory number of G is $c(G)$, then $c(G \square H) \geq c(G) + 1$.

101. Let G and H be connected graphs with radon number $r(G)$ and $r(H)$ respectively. Then $r(G \square H) \geq \max\{\lfloor \frac{diam(H)+1}{3} \rfloor (r(G) - 1) + 1, \lfloor \frac{diam(G)+1}{3} \rfloor (r(H) - 1) + 1\}$.

102. Let G and H be non trivial connected graphs. Then, for $S \subset V(G[H])$ either $H_{G[H]}(S) = V(G[H])$ or $H_{G[H]}(S) = S$.

103. Let G and H be nontrivial connected graphs. Then $c(G[H]) = 2$.

104. Let G and H be nontrivial connected graphs. Then $r(G[H]) = \max\{3, \lfloor \frac{diam(G)+1}{3} \rfloor + 1\}$.

105. Let G be a graph with radon number r, and $|V(G)| = n$, then $2r - 1 \leq r(G \circ H) \leq r + n$.

106. For any two graphs G and H, $h(G \circ H) = |V(G)| + 1$.

107. Let G and H be nontrivial graphs with hull number $h(G)$ and $h(H)$ respectively and $\delta(G) \geq 2$. Then $h(G \star H) \leq h(G)$.

108. Let G be any graph and $|V(H)| \geq 2$, then

$$r(G \star H) \geq 2 \times \lfloor \frac{diam(G) + 1}{3} \rfloor + 1$$

.

109. Let G and H be nontrivial graphs with hull number $h(G)$ and $h(H)$ respectively. Then $h(G \diamond H) \leq h(G)$.

110. Let G be a graph having perfect matching and $|V(H)| \geq 2$, then $r(G \diamond H) \geq |V(G)| + 1$.

6.2 Proposals for further study

We are aware that the research done for this work is not complete. Even so, it continues to be a vibrant and accessible area for research. Some of the problems on our thoughts that are still open are listed here.

1. Characterize the graphs G and $A \subset V(G)$ for which $r\gamma_A(G) = \gamma(G)$ in terms of degree.

2. Find the bounds for $r\gamma_{[A_1, A_2, \ldots, A_k]}(G)$ in terms of order and degree.

3. Study the change of reserved domination number of a graph with the atomic variations of graph, such as vertex removal, edge removal.

4. To study the stability of domination in real life problems it is more beneficial to define α-stable domination for weighted graph and study it in detail.

5. Characterize the graphs with $\gamma^0(G) = \gamma(G)$.

6. Characterize the graphs for which the compliment of minimum α-stable dominating set is an α-stable dominating set.

7. Develop an algorithm to find the α-stable domination number of a graph.

8. Study the change of hull number of a graph with the atomic variations of graph, such as vertex removal, edge removal.

Publications in Journals and Presentations

Publications:

1. Reeja Kuriakose, Parvathy K. S., "On the P Convexity of Graphs", *International Journal of Recent Technology and Engineering*, vol.*7,Issue 6S2*, pp. *870-873*, 2019.

2. Reeja Kuriakose, Parvathy K. S. "Bounds of domination number of Cartesian product of certain types of Graphs", *Bulletin of Kerala Mathematics Association*, vol. 15, 2, pp. *173-178*, 2017.

3. Reeja Kuriakose, Parvathy K. S., "Reserved Domination number of Graphs", *Contemporary studies in Discrete Mathematics*, Vol. *1*, pp. *67-70*, 2017.

Presentations:

1. Reeja Kuriakose, *On P_3 convex hull number of graphs*, 2nd International Conference on Pure and applied Mathematics, SCSVMV. Enathur, Kanchipuram, Tamilnadu held on 17-19 December, 2018.

2. Reeja Kuriakose, *Absorption number of Graphs*, International Conference on "Discrete Mathematics and its Applications to Network Science", Birla Institute of Technology and Science- K K Birla, Goa Campus, held on 07-10 July, 2018.

3. Reeja Kuriakose, *Conditional domination and domination extension on graphs*, National seminar on discrete mathematics and applications, Centre for studies in Discrete Mathematics Thrissur, Kerala held on 19-21 July, 2017.

4. Reeja Kuriakose, *Domination degree and powerful domination*, Two day national seminar on "Cryptography , Number Theory and Algebra", Mercy College, Palakkad, Kerala held on 22-23 July, 2015

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