CONTRIBUTIONS TO THE THEORY OF PROBABILITY DISTRIBUTIONS

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By

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CERTIFICATE

This is to certify that the work reported in this thesis entitled **CONTRIBUTIONS TO THE THEORY OF PROBABILITY DISTRIBUTIONS** submitted to the University of Calicut for the award of degree of Doctor of Philosophy in Statistics is a bona fide research work carried out by Smt. Mariamma Antony under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

Dr. N. Raju
(Supervising Teacher)

DECLARATION

I hereby declare that the matter embodied in this thesis is the result of investigations carried out by me in the Department of Statistics, University of Calicut, under the supervision and guidance of Dr.N. Raju, Reader, Department of Statistics, University of Calicut. This thesis contains no material which has been accepted for award of any degree or diploma in any University or Institute and to the best of my knowledge and belief, it contains no material previously published by any other person, except where the due references are made in the text of the thesis.

Calicut University Campus,

April 25, 2008 Mariamma Antony

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MARIAMMA ANTONY

NOTATIONS AND ABBREVIATIONS

		Page in which the Notation first appears
Laplace	$La(m, s)$	**14**
Linnik	$L(a, l)$	**16**
Autoregressive Moving Average	*ARMA*	**27**
Autoregressive	*AR*	**27**
Moving Average	*MA*	**28**
Autoregressive Integrated Moving Average	*ARIMA*	**28**
Geometric Linnik	$GL(a, l)$	**32**
Generalized Linnik	$GeL(a, l, p)$	**35**
Type I Generalized Geometric Linnik	$GeGL_1(a, l, p)$	**35**
Type II Generalized Geometric Linnik	$GeGL_2(a, l, t)$	**47**
Generalized Asymmetric Laplace	$GeALa(l, m, t)$	**57**
Generalized Asymmetric Linnik	$GeAL(a, l, m, t)$	**58**
Geometric Asymmetric Linnik	$GAL(a, l, m)$	**60**

CHAPTER I

INTRODUCTION

1.1 INTRODUCTION

Hald (1998) gave an account of the history of Mathematical Statistics from 1752 to 1930. Klebanov et al. (2006) discussed the role of sums of a random number of random variables in the study of limit theorems.

The work of Bernoulli and de Moivre in the first half of the 17 th century is considered to be the beginning of the theory of limiting distributions for sums of independent and identically distributed random variables. This fundamental area of Probability Theory attracted many researchers, including Poisson, Gauss and Laplace (see Klebanov et al. (2006)). The next important period in the development of the theory during the 19th century, is connected with the names of Chebyshev, Markov and Lyapunov, who developed effective methods for proving limit theorems for sums of independent but arbitrarily distributed random variables. The modern period in Probability Theory begins with the

Kolmogorov’s axiomatization. In the 1930`s the classical questions about necessary and sufficient conditions for the convergence of normalized sums of independent random variable to degenerate and normal laws were answered by Levy, Bernstein , Feller and Khintchine. The original ideas of Levy and Kolmogorov gave rise to a new line of research on limiting distributions of scaled sums of independent and identically distributed random variables without finite variance. A new class of distributions referred to as stable laws, emerged as the only possible limits of such sequences.

The most widely known and used theorem in various areas of science is the Central Limit Theorem, which gives necessary and sufficient conditions for the convergence of sums of independent and identically distributed random variables to the normal law. Consequently, many scientists believe that if the number of summands is large, their sum can always be approximated by a normal distribution. This, however may not be the case. If the summands have infinite variance, then the sum may converge only to a stable non-Gaussian law. Moreover, even if the variables are independent and normally distributed, the sum of their random number may not be distributed according to the normal law, as is illustrated by the following example (see Kruglov and Korolev (1990)).

EXAMPLE 1.1.1

Let X_k, $k = 1, 2, \ldots$ be independent and identically distributed random variables with the standard normal distribution. Consider the sum of X_k's up to a random moment n_n, where the distribution of n_n is uniform on the set of integers $\{1, 2, \ldots, n\}$. Then, the characteristic function of the normalized random sum

$$S_n = \frac{(X_1 + X_2 + \ldots + X_{n_n})}{\sqrt{n}} \quad \text{is equal to}$$

$$f_n(t) = \frac{1}{n}\sum_{i=1}^{n} e^{-it^2/2n}.$$

Thus, when $n \to \infty$, the limiting characteristic function of the sum, S_n becomes $f(t) = \left(\frac{2}{t^2}\right)\left(1 - e^{-t^2/2}\right)$, which is not the characteristic function of a normal law.

Therefore, we must study the sum of a random number of random variables. Apart from its interesting theoretical properties, the random summation scheme appears naturally in various fields, such as Physics, Biology, Economics, Reliability and Queuing Theory. The following examples, which have many generalizations, illustrate how random summation can arise in practice:

(i) **Marketing** : When ordering supplies to a store, the owner would like to know the total amount, T, of an article A sold during given period of time. If X_k is the (random) amount of A sold to the k^{th} customer, and N is the number of customers buying A during the time period considered, then the total amount of A sold can be written as

$$T = X_1 + X_2 + ... + X_N.$$

(ii) **Insurance Mathematics / Risk Theory** : In risk theory, one is interested in the distribution of aggregate claims generated by a portfolio of insurance policies (collective risk model). If the individual claims are denoted by X_k's (assumed to be independent and identically distributed) and the random variable N denotes the number of claims in a given time period, then the aggregate (total) claim S is given by

$$S = X_1 + X_2 + ... + X_N.$$

(iii) **Reliability Theory**: Many systems studied in reliability theory can be described by the following scheme and its various generalizations (see, e.g., Gertsbakh (1984) and Pillai and Sandhya (1996)) . A system consists of two operating units. The working time (until failure) of each unit has the same distribution F (x). At time 0, the first unit starts working while the second is on stand by. When the operating unit fails, its place is taken by the stand by unit, and the first unit goes to repair. If during that time the second unit fails, the system fails. An important issue in this setting is the nature of the distribution of the time T until failure of the system. Let X denote the length of the cycle, where the cycle starts with putting first unit into operation and ends with completion of its repair (before the second unit fails). Let Y be the length of the incomplete cycle (terminated by the system failure). Clearly $T = X_1 + X_2 + ... + X_{N-1} + Y$, where N – 1 is the (random) number of cycles prior to the failure, so that

$$P(N = i) = p(1 - p)^{i-1},\ i \geq 1$$

and p is the probability that the second unit fails before the first completes its repair. Typically, p is very small, reflecting the fact that the operational (average) time is larger than the repair (renewal) time. One is usually interested in the asymptotic distribution of T when $p \to 0$.

Another sequence of examples of practical appearance of random sums is connected with the problem of parameter estimation under sampling with a random sample size. Recall that many common estimators (including maximum likelihood estimators , M–estimators and others) are well–approximated by sums of independent random variables (see, Ibragimov and Khasmmskii (1979)). It is expected that for estimation problems with random sample size, such approximation will be provided by sums with random number of terms.

EXAMPLE 1.1.2

Suppose that a statistician observes data transmitted through a device. Assume that the device may fail with probability p in time of transmission of any observation. In such situation, the statistician will have a sample of random volume N, where N has a geometric distribution with parameter p and is independent of the values of the transmitted observations.

- One may come across samples of random volume in reliability theory, for instance when testing operational safety (Gnedenko (1989)). If the test is connected with the life time, then we observe a random number of failures in a given time interval.
- When we want to find the distribution of the velocities of cars on a highway, and observe the velocities of cars at a given point of the highway during a given time interval, we obtain a sample of random volume.

The number of cases where sums occur is enormous, and it is clear that random summation scheme plays an important role in many applied probability problems. In addition, random sums appear in various branches of Mathematics, including Mathematical Statistics and the Theory of Stochastic Processes (see, e.g., Gnedenko and Korolev (1996) , Kalashnikov (1997) and Rahimov (1995)). Klebanov et al. (1984) discussed the properties of distribution of geometric sums of random variables.

In Chapter II, some properties of geometric Linnik distribution are studied. Type I generalized Linnik distribution is also studied in this Section. Estimation of parameters of geometric Linnik distribution is done is Section 3. Autoregressive process with geometric Linnik marginal distribution is studied in Section 4. As generalizations of geometric Linnik distributions, type I and type II generalized geometric Linnik distributions are introduced and studied.

Asymmetric generalizations of geometric Linnik distribution are studied in Chapter III. A representation of the asymmetric Linnik distribution is obtained. Type I generalized geometric asymmetric Linnik distribution is introduced. It is shown that this distribution arises as the limit distribution of the geometric sums of generalized asymmetric Linnik random variables. The stability property of type I generalized geometric asymmetric Linnik distribution is examined. Autoregressive models with type I generalized geometric asymmetric Linnik marginals are developed. Various forms of geometric asymmetric Laplace distributions are also introduced in this Chapter.

Tailed distributions are found to be useful in the study of life testing experiments and clinical trials. Tailed forms of type I and type II generalized geometric Linnik distribution and their asymmetric forms are studied in Chapter IV. Using Marshall – Olkin scheme, Marshall – Olkin forms of type I and type II generalized geometric Linnik distributions are introduced and studied. A representation of tailed type I generalized geometric Linnik distribution is obtained. Also a first order autoregressive model with tailed type I generalized geometric Linnik distribution is introduced. It is shown that the process is not time reversible. The model is extended to higher order cases. The tailed type II generalized geometric Linnik distribution is also introduced and studied in this Chapter. As a generalization of tailed type I and type II generalized geometric Linnik distributions, tailed type I and type II generalized geometric asymmetric Linnik distributions are introduced and studied in this Chapter. Marshall – Olkin scheme is applied to geometric Linnik characteristic function and its generalizations, and the distributions so generated are examined.

In Chapter V, geometric marginal asymmetric Laplace and asymmetric Linnik distribution is introduced and studied. Time series models with geometric marginal asymmetric Laplace and asymmetric Linnik distributions are introduced. Also in this Chapter we study the properties of geometric marginal asymmetric Linnik - asymmetric Linnik distribution. A bivariate time series model with this marginal distribution is developed and studied. Geometric bivariate semi-a - Laplace distribution is also introduced and studied in this Chapter. The summary and conclusion of the Thesis is presented in Chapter VI.

1.2 LAPLACE DISTRIBUTION

The double exponential distribution was introduced by Laplace (1774) (see Kotz et al. (2001)) as the distribution form for which the likelihood function is maximized by setting the location parameter equal to the median of the observed values of an odd number of independent and identically distributed random variables. This result appeared in Laplace's fundamental paper on symmetric distributions for describing errors of measurement and is known as the first law of Laplace (see Kotz et al. (2001)). A random variable X on R is said to have Laplace distribution if its probability density function is

$$f(x) = \frac{1}{2s} e^{-\frac{|x-m|}{s}}, \; s > 0, -\infty < m < \infty. \qquad (1.2.1)$$

Another mode of genesis of this distribution is as the distribution of the difference of two independent and identically distributed exponential random variables.

The Laplace distribution, being heavier tailed than the normal, has been used quite commonly as an alternative to the normal distribution, in robustness studies. Kotz et al. (2001) discussed the applications of Laplace distributions in Engineering Sciences, Financial Data Modeling, Inventory Management and Quality Control, Astronomy, Biological and Environmental Sciences. Detection of a known constant signal that is distorted by the presence of a random noise was discussed on Communications Theory on various occasions. For the detection of noise in presence of Laplace noise, see Marks et al. (1978) and Dadi and Marks (1987) . A standard problem in communication theory is encoding and decoding of analog signals. The distribution of such signals depends on their nature. Among the most important one's are the speech signals. It has been found that the Laplace distribution accurately models speech signals. Laplace distribution has potential applications in modeling the fracturing of materials under applied forces. Another area where Laplace distribution can find most interesting and successive application is modeling of financial data. This is due to the fact that the traditional models based on Gaussian distribution are very often not supported by real life data because of long tails and asymmetry present in these data. Since Laplace distributions can account for leptokurtic and skewed data, they are natural candidates to replace Gaussian distribution and processes.

In the last several decades various skewed Laplace distributions have appeared in the literature. McGill (1962) considered the distribution with probability density function

$$f(x)=\begin{cases}\dfrac{1}{2s_1}e^{-\frac{|x-m|}{s_1}}, & x\le m\\[2ex] \dfrac{1}{2s_2}e^{-\frac{|x-m|}{s_2}}, & x>m\end{cases} \qquad (1.2.2)$$

while Holla and Bhattacharya (1968) studied the distribution with probability density function

$$f(x)=\begin{cases}\dfrac{p}{s}e^{-\frac{|x-m|}{s}}, & x\le m\\[2ex] \dfrac{(1-p)}{s}e^{-\frac{|x-m|}{s}}, & x>m\end{cases}, \quad 0<p<1.$$

Lingappaiah (1988) derived some properties of (1.2.2), terming the distribution as two–piece double exponential. Poiraud-Casanova and Thomas-Agnan (2000) considered a skewed Laplace distribution with probability density function

$$f(x) = p(1-p)\begin{cases} e^{-(1-p)|x-m|} & for \quad x < m \\ e^{-p|x-m|} & for \quad x \geq m \end{cases}$$

where $m \in (-\infty, \infty)$ and $0 < p < 1$. To show the equivalence of certain quantile estimators using the method of Azzalini (1985) , Balakrishnan and Ambagaspitiya (1994) (see Kotz et al. (2001)) studied a skewed Lapalce distribution with density

$$f(x) = \begin{cases} \frac{1}{2}e^{-(1+l)x}, & -\infty < x \leq 0 \\ e^{-x} - \frac{1}{2}e^{-(1+l)x}, & 0 < x < \infty \end{cases} .$$

Using the method of Fernandez and Steel (1988), Kozubowki and Podgorski (2000) introduced an asymmetric Lapalce distribution with density

$$f(x) = \frac{1}{s}\frac{k}{1+k^2}\begin{cases} e^{-\frac{k}{s}(x-m)} & for \quad x \geq m \\ e^{-\frac{1}{s\,k}(x-m)} & for \quad x < m \end{cases}$$

where $\mu = \frac{\sigma}{\sqrt{2}}\left(\frac{1}{k} - k\right)$ They have named this distribution as asymmetric Laplace distribution and studied various properties of this distribution. Kozubowski and Podgorski (2000) suggested asymmetric Laplace models for modeling interest rates, arguing that the asymmetric Laplace model is capable of capturing the peakedness, fat-tailedness, skewness and high kurtosis observed in the data. Kozubowski and Podgorski (2001) presented an application of asymmetric Laplace distribution in modeling foreign currency exchange rates. They fitted asymmetric Laplace laws to a bivariate data sets on two currency commodities: the German Deutschmark Vs. the U.S. Dollar and the Japanese Yen Vs. the U.S. Dollar. The asymmetric Laplace laws are proved to be useful for modeling stock market returns and modeling price changes of commodities. Rachev and Sen Gupta (1993) proposed Laplace – Weibull mixtures for modeling price changes.

For the applications of Laplace distribution in different fields see Kotz et al. (2001) and Johnson et al. (1995). Kanji (1985) and Jones and Mc Lachlan (1990) have discussed the Laplace normal mixture distribution with density function

$$f(x) = \frac{p}{2s_1} e^{\frac{-|x-m|}{s_1}} + \frac{(1-p)}{\sqrt{2p}\, s_2} e^{-\frac{(x-m)^2}{2s_2{}^2}}, \tag{1.2.3}$$

$$-\infty < x, m < \infty, 0 < p < 1, s_1, s_2 > 0$$

and applied the distribution to fit wind shear data. Maximum likelihood estimation of parameters of this distribution has been discussed by Scallan (1992). Generalized normal Laplace distribution was introduced and studied in

Reed (2004, 2005) and Reed and Jorgensen (2004) . The normal – Laplace distribution is defined by the characteristic function $e^{(imt-\frac{t^2 s_1^2}{2})}\frac{s_2^2}{s_2^2+t^2}$. The corresponding random variable X_1 can be expressed as $X_1 \stackrel{d}{=} Y+Z$ where Y and Z are independent with $Y \stackrel{d}{=} N(m, s_1^2)$ and $Z \stackrel{d}{=} La(0, s_2)$ where $La(m, s)$ denotes the Laplace distribution defined in (1.2.1). The normal Laplace distribution discussed in Reed and Jorgenson (2004) has the characteristic function

$$f_{X_2}(t) = \left[e^{imt-\frac{s_1^2 t^2}{2}} \right] \left[\frac{db}{(d\text{-it})(b+it)} \right]. \tag{1.2.4}$$

X_2 can be expressed as $X_2 \stackrel{d}{=} Y + E_1 - E_2$ where E_1 and E_2 are independent exponential random variables with parameters δ and β respectively.

The generalized normal Laplace distribution is defined by the characteristic function

$$f(t) = \left\{ \left[e^{imt-\frac{t^2 s_1^2}{2}} \right] \left[\frac{s_2^2}{(s_2^2+t^2)} \right] \right\}^p .$$

The corresponding random variable X_3 can be expressed as

$$X_3 = Y + La_1 + La_2 + \ldots + La_p$$

where $Y \overset{d}{=} \left(pm, ps_1^2 \right)$ and $La_1, La_2, \ldots, La_p$ are independent and identically distributed Laplace random variables with $La_i \overset{d}{=} La(0, s_2), i = 1, 2, \ldots, p.$ As in the case of (1.2.4), we can write the characteristic function of generalized normal Laplace distribution as

$$f_{X_4}(t) = \left[e^{imt - \frac{t^2 s_1^2}{2}} \left[\frac{db}{(d - it)(b + it)} \right] \right]^p ,$$

$$d, b, p > 0 \text{ and } -\infty < m < \infty.$$

The corresponding random variable $X_4 = Y + G_1 - G_2$, where Y, G_1 and G_2 are independent random variables with $Y \overset{d}{=} N\left(pm, ps_1{}^2 \right)$ and G_1 and G_2 have gamma distribution with scale parameters δ and β respectively having common shape parameter p.

1.3 LINNIK DISTRIBUTION

Linnik (1963) proved that the function

$$f(t) = \frac{1}{1 + l\,|t|^a}, \; 0 < a \le 2, \; l > 0 \tag{1.3.1}$$

is the characteristic function of a probability distribution. The distribution corresponding to the characteristic function (1.3.1) is called Linnik (or a - Laplace) distribution. A random variable X with characteristic function f in (1.3.1) is denoted by $X \underset{=}{d} L(a, l)$. Note that the $L(a, l)$ distributions are symmetric and for $a = 2$, it becomes the classical symmetric Laplace distribution. Devroye (1986) presented a simple algorithm for generating pseuodo random observations from this distribution. Kotz and Ostrovskii (1996) have obtained a mixture representation of Linnik distribution. Let X_a and X_b be two random variables possessing $L(a, l)$ and $L(b, l)$ distributions respectively, $0 < a < b \le 2$ and Y_{ab} a non-negative random variable (independent of X_b) with density function $f(x) = \left(\frac{b}{p}\sin\frac{pa}{b}\right)\frac{x^{a-1}}{1 + x^{2a} + 2x^a Cos\frac{pa}{b}}$, $0 < x < \infty$. Kotz and Ostrovstkii (1996) have shown that $X_a \underset{=}{d} X_b Y_{ab}$.

Linnik laws are special cases of strictly geometric stable distributions introduced in Klebanov et al. (1984). A random variable Y (and its probability distribution) is called strictly geometric stable if for every $p \in (0,1)$, there is an

$a_p > 0$ such that $a_p \sum_{i=1}^{N_p} Y_i \underline{\underline{d}}\ Y$ where N_p is a geometric random variable with mean $\frac{1}{p}$, Y_i's are independent and identically distributed copies of Y, independent of N_p. Note that strictly geometric stable law is a special case of geometric stable laws. The characteristic function of strictly geometric stable laws can be written as

$$f(t) = \frac{1}{1 + l\,|t|^{a}\, e^{-(ipad\,sgn(t)/2)}},\ t \in R$$

where δ is such that $|d| \leq \min\left(1, \frac{2}{a} - 1\right)$.

The following properties of Linnik distributions are immediate. For proofs of these results, refer Kotz et al. (2001).

(1) Let $X, X_1, X_2, \ldots$ be symmetric, independent and identically distributed random variables and let N_p be a geometric random variable with mean $\frac{1}{p}$, independent of the X_i's. Then the following statements are equivalent:

(a) X is stable with respect to geometric summation,

$$a_p \sum_{i=1}^{N_p} (X_i + b_p)\ \underline{\underline{d}}\ X \quad \text{for all} \quad p \in (0,1) \qquad (1.3.2)$$

where $a_p > 0$ and $b_p \in R$.

(b) X has a symmetric Linnik distribution . Moreover, the constants a_p and b_p are necessarily of the form : $a_p = p^{1/a}, b_p = 0$.

(2) The distribution function and density of the Linnik $L(a,1)$ distribution with $0 < a < 2$ admit the following representations for $x > 0$:

$$F_{a,1}(x) = 1 - \frac{\sin\frac{pa}{2}}{p} \int_0^{\infty} \frac{n^{a-1} e^{-nx}}{1 + n^{2a} + 2n^{a}.\cos\frac{pa}{2}} dv \qquad (1.3.3)$$

and

$$p_{a,1}(x) = \frac{\sin\frac{pa}{2}}{p} \int_0^{\infty} \frac{n^{a} e^{-n|x|}}{1 + n^{2a} + 2n^{a}\cos\frac{pa}{2}} dv. \qquad (1.3.4.)$$

For $x < 0$, use $F_{a,1}(x) = 1 - F_{a,1}(-x)$ and $p_{a,1}(x) = p_{a,1}(-x)$.

(3) The density p_a of Linnik $L(a,1)$ distribution has the following representation for $x > 0$:

$$\forall n > 0, \quad p_a(\pm x) = \frac{1}{p} \sum_{k=1}^{n} C_k x^{-ka-1} + R_n(x) \qquad (1.3.5)$$

where $C_k = (-1)^{k+1} \Gamma(ka+1) \sin(kpa/2)$,

$$|R_n(x)| \le a \frac{\Gamma(a(n+1)+1)}{p\left|\sin\left(\frac{pa}{2}\right)\right|} x^{-a(n+1)-1}.$$

(4) Let $X \underset{=}{d} L(a, s)$ with $0 < a \leq 2$. Then for every $0 < p < a$, we have

$$e(p) = E|X|^p = \frac{p(1-p)s^p p}{a\,\Gamma(2-p)\sin\frac{p\,p}{a}\cos\frac{p\,p}{2}}. \qquad (1.3.6)$$

(5) All symmetric Linnik distributions are in Class L, that is, for all $c \in (0,1)$ the Linnik characteristic function f given by (1.3.1) can be written as

$$f(t) = f(ct)y_c(t) \qquad (1.3.7)$$

where y_c is a characteristic function.

(6) The characteristic function (1.3.1) of the Linnik distribution $L(a, l)$ admits the representation

$$f(t) = \exp\left(\int_R \left(e^{itm} - 1\right) d\Lambda(u)\right) \qquad (1.3.8)$$

where

$$\frac{d\Lambda(u)}{du} = \frac{a}{2|u|} E\exp\left(-\left|\frac{u}{s\,X}\right|^a\right) = \frac{1}{s}\int_0^\infty g_a\left(\frac{u}{l\,w^{1/a}}\right)\frac{e^{-w}dw}{w^{1+1/a}}$$

and X has the stable distribution $f(t) = e^{-|t|^a}$ and g is the density of X.

Kotz, et al. (2001) has discussed the multivariate forms of Laplace distributions. Anderson (1992) gave an example of bivariate Gumbel type Laplace model with density function

$$f(x_1, x_2) = \frac{1}{4}\left\{\left(1+q\,|x_1|\right)\left(1+q\,|x_2|-q\right)\right\}e^{-\left(1+|x_1|+|x_2|+q\,|x_1||x_2|\right)},$$

$$(x_1, x_2) \in R^2.$$

1.4 INFINITE DIVISIBILITY

A distribution function F is infinitely divisible if for every positive integer n, there exists a distribution function F_n such that

$$F = F_n^{n*}.$$

Equivalently, a characteristic function ϕ is said to be infiniety divisible if for every positive integer n, there exists a characteristic function ϕ_n such that $f(t) = (f_n(t))^n$.

It is known that the class of infinitely divisible distributions coincides with the class of limit distributions of the row sums of certain triangular arrays. Note that the class of all infinite divisible distributions coincides with the class of all continuous convolution semi groups. For the properties of infinite divisibility of distributions, see Laha and Rohatgi (1979).

Klebanov et al. (1984) introduced the concept of geometric infinite divisibility of random variables. A random variable Y is said to be geometrically infinitely divisible, if Y can be expressed as

$$Y \underset{=}{d} \sum_{j=1}^{Np} X_p^{(j)} \tag{1.4.1}$$

for every $p \in (0,1)$ where $P(Np = k) = pq^{k-1}, k = 1, 2, \ldots$ and N_p and $X_p^{(j)}$ $(j = 1, 2, \ldots)$ are independent and q=1-p.

Pillai (1990b) proved that every geometrically infinitely divisible distribution is infinitely divisible. For the applications of geometric infinite divisibility in time series modeling, see Pillai and Jose (1995).

1.5 SELF- DECOMPOSABLE DISTRIBUTIONS

Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables, and let $\{b_n\}$ be a sequence of positive real numbers such that the following condition holds:

$$\lim_{n \to \infty} \max_{1 \leq k \leq n} P\left\{|X_k| \geq b_n e\right\} = 0 \tag{1.5.1}$$

for every $e > 0$. By writing $X_{nk} = \frac{X_k}{b_n}$, $1 \leq k \leq n$, $n \geq 1$, we see that the sequence $\{X_{nk}\}$ of row-independent random variables satisfies the uniformly asymptotic negligible condition. Set $S_n = \sum_{k=1}^{n} X_k$ for $n \geq 1$.

Let L be the class of distributions which are the weak limits of the distribution of the sums $b_n^{-1}S_n - a_n$, $n \geq 1$ where a_n and $b_n > 0$ are suitably chosen constants.

Now it is easy to obtain the following results related to self-decomposable distributions (see Laha and Rohatgi (1979)).

(i) Let $\{X_n\}$ be a sequence of independent random variables, and $\{b_n\}$ a sequence of positive constants such that (1.5.1) is satisfied. Suppose that the sequence $\{b_n^{-1}S_n - a_n\}$ converges in law to a non-degenerate random variable for some sequence of constants $\{a_n\}$. Then $b_n \to \infty$ and $\frac{b_{n+1}}{b_n} \to 1$ as $n \to \infty$.

(ii) A distribution function F with characteristic function ϕ is said to be self decomposable if and only if, for every $0 < c < 1$, there exists a characteristic function ϕ_c such that $f(t) = f(ct)\, f_c(t)$ for $t \in R$.

Based on the above result, we can see that a random variable X is self decomposable if for every $c, 0 < c < 1$, $X \underset{=}{d}\, c X' + X_c$, where X and X' are identically distributed and X' and X_c are independent.

It is known that any non-degenerate self decomposable distribution is absolutely continuous. Also every self decomposable distribution is infinitely divisible. For the application of self decomposable distributions in time series modeling, see Bondesson (1981). Pillai (1990a) proved the self-decomposability of Mittag-Leffler distributions. Jayakumar and Pillai (1992) discussed the self-decomposability of Linnik distributions.

1.6 STABLE DISTRIBUTIONS

A random variable X on R is said to have stable distribution if for any two positive constants a and b,

$$a\,X_1 + b\,X_2 \overset{d}{=} cX + d \tag{1.6.1}$$

where X_1 and X_2 are independent and identically distributed as X, $c > 0, d \in R$. Let f denotes the characteristic function of X, X_1 and X_2. Then (1.6.1) in terms of characteristic functions is $f(at)\,f(bt) = e^{idt} f(ct)$.

When $d = 0$, the distribution is said to be strictly stable. For the properties of stable distributions, see Laha and Rohatgi (1979).

The characteristic function of a stable distribution can be expressed as

$$f(t) = \begin{cases} e^{imt--l\,|t|^{a}\,(1-d\,sgn(t)\,\tan\frac{pa}{2})} & if \quad a \neq 1 \\ e^{imt--l\,|t|(1-d\frac{2}{p}sgn(t)\,\ln|t|)} & if \quad a = 1 \end{cases}.$$

α is called the exponent of the distribution, $0 < a \le 2$, μ is the location parameter, $-\infty < m < \infty$, l is the scale parameter, $\lambda > 0$; δ is the symmetry parameter, $-1 \le \delta \le 1$ and

$$Sgn(t) = \begin{pmatrix} 1 & if & t>0 \\ 0 & if & t=0. \\ -1 & if & t<0 \end{pmatrix}$$

Two important members of the class of stable distributions are normal and Cauchy. The Laplace distribution is self-decomposable but not stable.

The following properties of stable distributions are well known:

The class of stable distributions is a proper subclass of the class of infinitely divisible distributions. Let L_1 be the family of distributions, which appear as limit distributions of $\frac{S_n}{b_n} - a_n$, where $S_n = \sum_{1=1}^{n} X_i$, as $b_n \to \infty$ with $n \to \infty$.. Then F is in class L_1 if and only if F is stable.

THEOREM 1.6.1

In order that a distribution function F with characteristic function ϕ is in class L_1, it is necessary and sufficient that F be infinitely divisible and $\ln f$ has either one of the following representations:

$$\ln f(t) = imt + c_1 \int_{-\infty}^{0} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{dx}{|x|^{1+b}} + c_2 \int_{0}^{\infty} \left(e^{itx} - 1 - \frac{1+x}{1+x^2} \right) \frac{dx}{|x|^{1+b}}$$

with $c_1 \geq 0, c_2 \geq 0, c_1 + c_2 > 0$ and $0 < b < 2$

or

$$\ln f(t) = imt - \frac{s^2 t^2}{2}.$$

As a generalization of stable distributions, Pillai (1971) studied semi stable distributions. A random variable X on R with characteristic function $\phi(t)$ is said to be semi stable if $f(t) = (f(bt))^a, a > 1, 0 < |b| < 1.$

1.7 GEOMETRIC STABLE DISTRIBUTIONS

Let $X_1, X_2, \ldots$ be independent, identically distributed random variables. Define

$$S_{N_p} = X_1 + X_2 + \ldots + X_{N_p}$$

where N_p is geometric with $P(N_p = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$.

Then the week limit of

$$a(p) \sum_{i=1}^{N_p} (X_i + b(p)) \qquad (1.7.1)$$

when $p \to 0, \quad a(p) > 0, \quad -\infty < b(p) < \infty$ is called geometric stable distributions. The sums of the form (1.7.1) usually appear in many applied problems in different areas such as Insurance Mathematics/ Risk Theory, Queuing Theory, etc. For various properties of geometric stable distributions, see Kozubowski and Rachev (1999).

1.8 MODELS OF TIME SERIES

A time series is a series of observations made sequentially in time. Main objective of time series analysis is to reveal the probability law that governs the observed time series. A simple way of describing a time series to the give one dimensional distribution function and the mean value function if it exists. To describe the association between two values on the same series at different times we use the auto correlation function

$$r_k = \frac{E\left[(X_t - m_t)\,(X_{t+k} - m_{t+k})\right]}{\sqrt{Var\left(X_t\right)Var\left(X_{t+k}\right)}} \qquad (1.8.1)$$

where $m_{t+i} = E\left(X_{t+i}\right), i = 0,1,2,\ldots$.

A time series can be viewed as a realization of a stochastic process. A class of time series we encounter in practical situations is the stationary series. If for any set of time t_1, t_2, …, t_n and at times t_1 +h, t_2+h, …, t_n +h, the joint distribution of $\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right)$ and $\left(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h}\right)$ are the same for all $h > 0$ and for every n, then the process $\{X_t\}$is said to be strictly stationary. A time series $\{X_t\}$ is said to be weakly stationary if (i) $E(X_t{}^2) < \infty$, (ii) $E(X_t) = m$

for all t and (iii) $Cov(X_t, X_{t+s})$ depends only on the length of the interval s. Note that $\{X_t\}$ is weakly stationary if it is strictly stationary with finite second moments.

The most popular class of linear time series models consists of autoregressive moving average (ARMA) models, purely autoregressive (AR) and purely moving average (MA) models. For modeling non-stationary data, autoregressive integrated moving average (ARIMA) models are used.

An autoregressive model of order $p \geq 1$, abbreviated as AR(p), is defined as

$$X_n = r_1 X_{n-1} + r_2 X_{n-2} + ... + r_p X_{n-p} + e_n \tag{1.8.2}$$

where $\{e_n\}$ is a sequence of independent and identically distributed random variables and $r_1, r_2, ...$ are constants. Note that (1.8.2) represents the current value of the process X_n through its immediate p past values $X_{n-1}, X_{n-2}, ..., X_{n-p}$ and a random shock e_n. The simplest form of an autoregressive model is AR (1) and is given by

$$X_n = r X_{n-1} + e_n. \tag{1.8.3}$$

Let $f_{X_n}(t)$ denotes the characteristic function of X_n and $f_{e_n}(t)$, that of e_n. Then (1.8.3) in terms of characteristic function becomes

$$f_{e_n}(t) = \frac{f_{X_n}(t)}{f_{X_{n-1}}(rt)}. \tag{1.8.4}$$

That is, the innovation process exists if and only if (1.8.4) is a characteristic function for every $r, |r| \le 1$. In the case of stationary process $\{X_n\}$, $f_{e_n}(t)$ exists for every r, $0 < r \le 1$ if and only if the distribution of X_n is in class L or self decomposable. The autocorrelation function of the process (1.8.3) is $r(k) = r^k$. Even though Gaussian models have dominated in the development of time series modeling, autoregressive processes with non Gaussian marginal distributions are a fast growing area of investigation due to the applications of the same (see, Jayakumar and Pillai (1993) and Jayakumar and Kuttykrishnan (2007)).

A moving average model of order $q \ge 1$, denoted by MA(q), is given by

$$X_n = d_1 e_{n-1} + d_2 e_{n-2} + ... + d_q e_{n-q} + e_n \tag{1.8.5}$$

where $\{e_n\}$ is a sequence of independent and identically distributed random variables and $d_1, d_2, ..., d_q$ are constants. Here the current value of X_n is linearly dependent on the q previous values of e_n's. For q=1, (1.8.5) reduces to the MA(1) model given by

$$X_n = d_1 e_{n-1} + e_n. \tag{1.8.6}$$

Combining AR and MA models, the general linear time series model, namely autoregressive moving average model, denoted by ARMA (p,q) has the form

$$X_n - r_1 X_{n-1} - r_2 X_{n-2} - \ldots - r_p X_{n-p} = e_n + d_1 e_{n-1} + d_2 e_{n-2} + \ldots + d_q e_{n-q} \tag{1.8.7}$$

where $r_1, r_2, \ldots, r_p$; $d_1, d_2, \ldots, d_q$ are constants and $\{e_n\}$ is a sequence of independent and identically distributed random variables.

As noted in Section 2, Laplace distribution is a natural and some times superior alternative to the Gaussian distribution. Andel (1983) developed an AR(1) model with Laplace marginal. The corresponding process $\{X_n\}$ is of the form

$$X_0 = e_1$$

and for $n = 1, 2, \ldots$,

$$X_n = \begin{cases} r\ X_{n-1} & w.p. \quad r^2 \\ r\ X_{n-1} + e_n & w.p. \quad 1 - r^2 \end{cases}$$

where $\{e_n\}$ is a sequence of independent and identically distributed Laplace random variable with characteristic function

$$f_{e_n}(t) = \frac{1}{1 + s^2 t^2}. \tag{1.8.8}$$

Dewald and Lewis (1985) developed and studied a second order Laplace autoregressive time series model. For the development of autoregressive time series models with Laplace marginals, see Gibson (1986) and Damsleth and El-Shaarawi (1989).

CHAPTER II

GEOMETRIC LINNIK AND GENERALIZED GEOMETRIC LINNIK DISTRIBUTION

2.1 INTRODUCTION

Pillai (1985) introduced a generalization of the Linnik distribution with characteristic function (1.3.1), namely semi α-Laplace distribution. A random variable X on R has semi α-Laplace distribution if its characteristic function $\phi(t)$ is of the form

$$\phi(t) = \frac{1}{1+|t|^{\alpha}\delta(t)} \tag{2.1.1}$$

where $\delta(t)$ satisfies the functional equation

$$d(t) = d\left(p^{1/a}t\right), 0<p<1, 0<a\leq 2. \tag{2.1.2}$$

George & Pillai (1987) derived expression for the density function of α-Laplace random variables in terms of Meijer's G-function and obtained a multivariate generalization of α-Laplace distribution.

This Chapter is based on Mariamma Antony and Raju (2005)

Pakes (1998) introduced generalized Linnik law with characteristic function

$$\phi(t) = \frac{1}{\left(1+|t|^{\alpha}\right)^{\nu}}, \quad \nu > 0, \quad 0 < \alpha \le 2. \qquad (2.1.3)$$

This distribution is known as Pakes generalized Linnik distribution. When $\nu = 1$, it reduces to α-Laplace distribution where as when $\alpha = 2$, it reduces to the generalized Laplacian distribution of Mathai (1993a) .

DEFINITION 2.1.1

A random variable X on R is said to have geometric Linnik distribution and write $X \underline{\underline{d}} GL(\alpha, \lambda)$ if its characteristic function $\phi(t)$ is

$$f(t) = \frac{1}{1+\ln(1+l\,|t|^{a})}, t \in R, 0 < a \le 2, \; l > 0. \qquad (2.1.4)$$

In Section 2, we study some properties of geometric Linnik distribution. Type I generalized Linnik distribution is studied in this Section. Estimation of parameters of geometric Linnik distribution is done is Section 3. Autoregressive process with geometric Linnik marginal distribution is studied in Section 4. In Section 5, another generalization of geometric Linnik distribution is introduced and the properties of this type II generalized geometric Linnik distribution are studied. In this case, representation of type II generalized geometric Linnik random variable is obtained, limit theorems concerning this generalized Linnik distribution are proved along with estimation of parameters of this distributions using empirical characteristic function. Note that some of the results that we present in this Chapter on geometric Linnik and type I generalized geometric Linnik distribution are available in Leksmi and Jose (2004,2006) and for the sake of completeness, we include the same in our discussion.

2.2 SOME PROPERTIES OF GEOMETRIC LINNIK DISTRIBUTION

THEOREM 2.2.1

$GL(\alpha,\lambda)$ distribution is infinitely divisible .

PROOF

For $GL(\alpha,\lambda)$ distribution, $f(t) = \dfrac{1}{1+\ln(1+l\,|t|^{a})}$,

$$e^{1-\frac{1}{\phi(t)}} = \frac{1}{1+\lambda|t|^{\alpha}}. \tag{2.2.1}$$

But $\dfrac{1}{1+\lambda|t|^{\alpha}}$ is the characteristic function an infinitely divisible distribution.

Thus the distribution with characteristic function $\dfrac{1}{1+\ln(1+l\,|t|^{a})}$, $l>0, 0<a\le 2$

is the characteristic function of a geometrically infinitely divisible distribution.

By Pillai & Sandhya (1990), every geometrically infinitely divisible distribution is infinitely divisible. This completes the proof.

Pillai (1990b) introduced geometric exponential distribution while studying the geometric infinite divisibility of harmonic mixtures of random variables.

A random variable X on $[0,\infty)$ is said to have geometric exponential distribution if it has the Laplace transform

$$f(d) = E\left(e^{-dX}\right) = \frac{1}{1+1n(1+d)}, \quad d > 0. \qquad (2.2.2)$$

A representation of geometric Linnik random variables in terms of geometric exponential and stable random variables is presented below.

THEROM 2.2.2

Let X and Y be independent random variables such that X has geometric exponential distribution with Laplace transform $\frac{1}{1+1\mathrm{n}(1+\delta)}$ and Y is stable with characteristic function $e^{-\lambda|t|^{\alpha}}, \quad 0 < \alpha \leq 2$. Then $X^{1/a}Y \underline{\underline{d}} GL(a, l)$.

PROOF

$$f_{X^{1/a}Y}(t) = E\left(e^{itX^{1/a}Y}\right)$$

$$= \int_0^\infty E\left[e^{itX^{1/a}Y} \Big/ X = x \right] dF(x)$$

$$= \int_0^\infty f_Y(tx^{1/a})\, dF(x)$$

$$= \frac{1}{1 + 1n\left(1 + l\,|t|^a\right)}.$$

DEFINITION 2.2.1

A random variable X on R has the generalized Linnik distribution and write $X \underline{\underline{d}}\, GeL(a, l, p)$ if it has the characteristic function

$$f(t) = \frac{1}{(1 + l\,|t|^a)^p}, \quad p > 0,\ l > 0,\ 0 < a \le 2. \tag{2.2.3}$$

Erdogan and Ostrovskii (1997, 1998) considered a generalization of $GeL\,(a, l, p)$ distribution and studied its properties. They discussed the analytic and asymptotic properties of this distribution and obtained some integral and series representation of its probability density.

DEFINITION 2.2.2

A random variable X on R is said to have type I generalized geometric Linnik distribution and write $X \underline{\underline{d}}\, GeGL_1(a, l, p)$ if it has the characteristic function

$$f(t) = \frac{1}{1 + p\ln(1 + l\,|t|^{a})}, \; 0 < a \le 2, \; p > 0, \; l > 0. \qquad (2.2.4)$$

THEOREM 2.2.3.

Let $X_1, X_2, \ldots$ be independent and identically distributed GL(α,λ) random variables and N_p be geometric with mean $\frac{1}{p}$. Define $Y = X_1 + X_2 + \ldots + X_{N_p}$ where N_p is independent of X_i's. Then $Y \underline{\underline{d}}\; GeGL_1(a, l, 1/p)$.

Proof follows easily.

THEOREM 2.2.4

Let $X_1, X_2, \ldots$ be independent and identically distributed $GeL(a, l, 1/n)$ random variables. Let N be geometric with mean n independent of X_i's. Then $X_1 + X_2 + \ldots + X_N$ converge in distribution to Z where $Z \underline{\underline{d}}\; GL(a, l)$.

PROOF

Consider

$$\frac{1}{(1 + l\,|t|^{a})^{1/n}} = \frac{1}{1 + \left[(1 + l\,|t|^{a})^{1/n} - 1\right]}.$$

Since Linnik distribution is infinitely divisible, $\frac{1}{(1+\lambda|t|^{\alpha})^{1/n}}$ is the characteristic function of a probability distribution.

Let $$\phi_n(t) = \frac{1}{1+n\left[(1+\lambda|t|^{\alpha})^{1/n}-1\right]}.$$

Then

$$f(t) = \lim_{n\to\infty} f_n(t)$$

$$= \frac{1}{1+\lim_{n\to\infty} n\left[(1+l\,|t|^{a})^{1/n}-1\right]}$$

$$= \frac{1}{1+ln(1+l\,|t|^{a})}.$$

THEOREM 2.2.5

The function $f(t) = \frac{1}{1+ln(1+l\,|t|^{a})}$ on R is a characteristic function if and only if $a \in (0,2]$.

PROOF

Suppose for some $a > 0$, the function $f(t)$ is a characteristic function. Then we have to prove that $a \in (0,2]$. The case $\alpha < 0$ is impossible due to the requirement that $\lim_{t\to 0} f(t) = 1$ for the characteristic function ϕ. Note that for each positive integer n, the function $f_n(t) = \left[\frac{1}{1+\frac{1}{n} ln(1+ l\ |t|^{a})}\right]^n$ is also a characteristic function. Let F_n denote the distribution function with characteristic function f_n. Then F_n converges weakly to a Linnik characteristic function $f(t) = \frac{1}{1+ l\ |t|^{a}}$.

This implies that $a \in (0,2]$.

For fixed $a \in (0,2]$, the function $\frac{1}{1+\lambda|t|^{\alpha}}$ is the characteristic function of Linnik distribution. By Theorem 2.2.4, $\frac{1}{1+ln(1+ l\ |t|^{a})}$ is a characteristic function.

THEOREM 2.2.6

For each $a \in (0,2]$, the distribution $F_{a,1}$ with characteristic function $f(t) = \frac{1}{1+ln(1+ l\ |t|^{a})}$ is absolutely continuous.

PROOF follows from Theorem 2.2.2 and hence is omitted.

THEOREM 2.2.7

$\mathrm{GL}(\alpha,\lambda)$ is normally attracted to stable law.

PROOF

Let $S_n = X_1 + X_2 + ... + X_n$ where $X_i's$ are independent and identically distributed $\mathrm{GL}(\alpha,\lambda)$ random variables. The characteristic function of $n^{-1/a} S_n$ is

$$f_{n^{-1/a} S_n}(t) = \left[\frac{1}{1 + ln(1 + \frac{l\,|t|^a}{n})} \right]^n$$

$$= \left[\frac{1}{1 + \frac{l\,|t|^a}{n} + o(\frac{1}{n^2})} \right]^n$$

$$\to e^{-\lambda |t|^\alpha}, \text{ as } \quad n \to \infty .$$

THEOREM 2.2.8

$X_{a,l}(s) \underline{\underline{d}} Y_{a,l}\, Z_s^{1/a}$ where $Y_{a,l}$ is symmetric stable with characteristic function $e^{-l\,|t|^a}$ and Z_s is geometric gamma with Laplace transform $\left(\frac{1}{1 + \ln(1 + d)} \right)^s$.

PROOF

$$f_X(t) = E\left[e^{itYZ^{1/a}} \right]$$

$$= \int_0^\infty e^{-l\,z|t|^a}\, dF(z) \quad = \left[\frac{1}{1 + ln(1 + l\,|t|^a)} \right]^s .$$

This completes the proof.

2.3 ESTIMATION OF PARAMETERS OF GEOMETRIC LINNIK DISTRIBUTION

Press (1972) used empirical characteristic function to estimate the parameters of a stable law. Jacques et al. (1999) used characteristic function technique to estimate the parameters of geometric stable law (see also, Kozubowski (1999)). Here we estimate the parameters of geometric Linnik distribution using empirical characteristic function.

Consider the geometric Linnik distribution with characteristic function

$$f(t) = \frac{1}{1 + ln(1 + l\,|t|^{a})}, \quad l > 0,\ 0 < a \le 2.$$

The function $\hat{f}_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j}$ is called the sample (empirical) characteristic function. We have $E\left[\hat{f}_n(t)\right] = f(t)$ and by the strong law of large numbers, $\hat{f}_n(t) \xrightarrow{a.s} f(t)$.

Take

$$d(t) = e^{(\frac{1}{f(t)} - 1)} - 1 = l\,|t|^{a}.$$

Then

$$d(t_i) = l\,|t_i|^{a}, \quad i = 1, 2.$$

Taking logarithms on both sides, we get

$$\ln d(t_1) = \ln l + a \ln|t_1|,$$

$$\ln d(t_2) = \ln l + a \ln|t_2|.$$

Hence,

$$a\left[\ln|t_1| - \ln|t_2|\right] = \ln d(t_1) - \ln d(t_2).$$

That is,

$$a = \frac{\ln \frac{d(t_1)}{d(t_2)}}{\ln \frac{|t_1|}{|t_2|}}.$$

For $a \neq 1$,

$$\ln d(t_1) \ln|t_2| = \ln|t_2| \ln l + a \ln|t_1| \ln|t_2|$$

and

$$\ln d(t_2) \ln|t_1| = \ln|t_1| \ln l + a \ln|t_2| \ln|t_1|.$$

Hence,

$$\ln l \{\ln|t_1| - \ln|t_2|\} = \ln d(t_2) \ln|t_1| - \ln d(t_1) \ln|t_2|$$

Therefore,

$$l = \exp\left\{\frac{\ln d(t_2) \ln(t_1) - \ln d(t_1) \ln(t_2)}{\ln|t_1| - \ln|t_2|}\right\}.$$

That is,

$$\hat{a} = \frac{\ln \dfrac{\hat{d}_n(t_1)}{\hat{d}_n(t_2)}}{\ln \dfrac{|t_1|}{|t_2|}}$$

and for $\alpha \neq 1$,

$$\hat{l} = \exp\left\{ \frac{\ln \hat{d}_n(t_2) \ln(t_1) - \ln \hat{d}_n(t_1) \ln(t_2)}{\ln|t_1| - \ln|t_2|} \right\}$$

where $\hat{d}_n(t) = \exp\left\{ \dfrac{1}{f_n(t)} - 1 \right\} - 1$ is the sample counterpart of $d(t)$.

2.4 AN AUTOREGRESSIVE PROCESS WITH GEOMETRIC LINNIK MARGINALS

Here we develop an autoregressive process with $\mathrm{GL}(\alpha, \lambda)$ marginal distribution.

The analysis of time series in the classical set up is based on the assumption that the observed series is a realization from a Gaussian sequence. However, there are many situations where the naturally occurring data show a tendency to follow heavy tailed distributions that can not be modeled by a Gaussian distribution. The usual technique of transferring data to use a Gaussian model also fails in certain situations (see, Lawrance (1991)). Hence a number of non-Gaussian autoregressive models have been introduced by various researchers (see, Jayakumar and Pillai (1993, 2002), Pillai and Jayakumar (1994), Lawrance and Lewis (1985)).

The study of non-Gaussian autoregressive models began with the pioneering work of Gaver and Lewis (1980). They have considered an AR(1) model with exponential(m) marginal distribution. The model is given by

$$X_0 = e_1$$

and for $n = 1, 2, \ldots$

$$X_n = r\,X_{n-1} + \begin{cases} 0 & w.p. \quad r \\ e_n & w.p. \quad (1-r) \end{cases} \qquad (2.4.1)$$

and $w.p.$ stands for with probability, $0 \le \rho \le 1$ and $\{e_n\}$ is a sequence of independent and identically distributed exponential random variables.

Another exponential AR(1) process is obtained by interchanging $\mathrm{X}_{\mathrm{n}-1}$ and e_n and this can have no effect on the marginal distribution of $X_n{}'s$. Proceeding this way with r replaced by $1-d$, we have for $n = 0, 1, 2, \ldots$

$$X_n = (1-d)e_n + \begin{cases} X_{n-1} & w.p. \quad d \\ 0 & w.p. \quad (1-d) \end{cases}. \qquad (2.4.2)$$

The exponential AR(1) model in (2.4.2), called TEAR(1), is Markovian and has the d^r correlation structure of the exponential AR(1) model. For the properties of TEAR (1) model, see Lawrance and Lewis (1981).

THEOREM 2.4.1

Let $\{X_{n,} n \geq 1\}$ be defined as

$$X_n = \begin{cases} e_n & w.p. \quad p \\ X_{n-1} + e_n & w.p. \quad (1-p) \end{cases} \tag{2.4.3}$$

where $\{e_n\}$ is a sequence of independent and identically distributed random variables. A necessary and sufficient condition that $\{X_n\}$ is strictly stationary Markov process with $GL(\alpha, \lambda)$ marginals is that $\{e_n\}$ are distributed as $GeGL_1(a, l, r)$.

PROOF

Taking characteristic functions on both sides of (2.4.3), we get

$$f_{X_n}(t) = pf_{e_n}(t) + (1-p) f_{X_{n-1}}(t) f_{e_n}(t).$$

If $\{X_n\}$ is stationary, then

$$f_X(t) = pf_e(t) + (1-p) f_X(t) f_e(t).$$

That is,

$$f_e(t) = \frac{f_X(t)}{p + (1-p) f_X(t)}.$$

If $f_X(t) = \dfrac{1}{1 + \ln(1 + l\,|t|^a)}$, then

$$f_e(t) = \frac{1}{1 + p\ln(1 + l\,|t|^a)}.$$

Conversely, if $\{e_n\}$ are independent and identically distributed as $GeGL_1(a, l, p)$, then

$$f_{X_1}(t) = p\frac{1}{1 + p\ln(1 + l\,|t|^a)} + (1-p)\frac{1}{1 + ln(1 + l\,|t|^a)} \quad \frac{1}{1 + p\ln(1 + l\,|t|^a)}$$

$$= \frac{1}{1 + p\ln(1 + l\,|t|^a)}\left[\frac{p + p\,ln(1 + l\,|t|^a) + 1 - p}{1 + \ln(1 + l\,|t|^a)}\right]$$

$$= \frac{1}{1 + \ln(1 + l\,|t|^a)}.$$

If $X_{n-1} \underset{=}{d}\, GL(a, l)$, then we get $X_n \underset{=}{d}\, GL(a, l)$.

Hence the process $\{X_n\}$ is strictly stationary. This completes the proof.

Consider the k^{th} order autoregressive process

$$X_n = \begin{cases} e_n & w.p. \quad p \\ X_{n-1} + e_n & w.p. \quad p_1 \\ X_{n-2} + e_n & w.p. \quad p_2 \\ \quad . & \\ \quad . & \\ \quad . & \\ X_{n-k} + e_n & w.p. \quad p_k \end{cases} \tag{2.4.4}$$

where $p + p_1 + p_2 + ... + p_k = 1, \;\; 0 < p_i < 1, i = 1, 2, ..., k$ and $\{e_n\}$ is a sequence of independent and identically distributed random variables independent of $X_{n-1}, X_{n-2}, ...$

Taking characteristic functions on both sides of (2.4.4), we get

$$f_{X_n}(t) = p f_{e_n}(t) + p_1 f_{X_{n-1}}(t) f_{e_n}(t) + p_2 f_{X_{n-2}}(t) f_{e_n}(t) + ... + p_k f_{X_{n-k}}(t) f_{e_n}(t).$$

That is,

$$f_{e_n}(t) = \frac{f_X(t)}{p + (1-p) f_X(t)}.$$

Following similar lines in Theorem 2.4.1, we get the following result.

THEOREM 2.4.2

A necessary and sufficient condition that the model (2.4.4) defines AR(k) process with $\mathrm{GL}(\alpha, \lambda)$ distribution is that $\{e_n\}$ is distributed as $GeGL_1(a, l, p)$.

2.5 GENERALIZED GEOMETRIC LINNIK DISTRIBUTION

Here we introduce type II generalized geometric Linnik distribution and study its properties.

DEFINITION 2.5.1

A random variable X on R has type II generalized geometric Linnik distribution and writes $X \underline{\underline{d}}\, Ge\,GL_2(a, l, t)$, if it has the characteristic function

$$f(t)=\left[\frac{1}{1+ln(1+l\,|t|^{a})}\right]^{t},\ t\in R, 0<a\le 2,\ l, t>0.$$

Note that when $t=1$, type II generalized geometric Linnik distribution reduces to geometric Linnik distribution.

THEOREM 2.5.1

$GeGL_2(a, l, t)$ distributions are infinitely divisible.

PROOF

Follows from Theorem 2.2.1

DEFINITION 2.5.2

A random variable X on $(0,\infty)$ has geometric gamma distribution if it has Laplace transform

$$f_1(d)=\left[\frac{1}{1+ln(1+d)}\right]^{t},\quad d>0, t>0.$$

For properties of geometric gamma distribution, see Jose and Lekshmi (1999).

A representation of type II generalized geometric Linnik random variable in terms of geometric gamma and stable random variable is presented below.

THEOREM 2.5.2

Let X and Y be independent random variables such that X has geometric gamma distribution with Laplace transform $\left[\frac{1}{1+ln(1+d)}\right]^{t}$ and Y be stable with characteristic function $e^{-l|t|^{a}}$, $0<a\le 2$, $l>0$. Then $X^{1/a}Y \underset{=}{d} GeGL_2(a, l, t)$.

PROOF

$$f_{X^{1/a}Y}(t) = E\left(e^{itX^{1/a}Y}\right)$$

$$= \int_0^\infty E\left(e^{itX^{1/a}Y} / X = x\right) dF(x)$$

$$= \int_0^\infty f_Y(tx^{1/a})\, dF(x)$$

$$= \int_0^\infty e^{-l|t|^a x} dF(x)$$

$$= \left[\frac{1}{1+\ln(1+l\,|t|^a)}\right]^t .$$

Now we shall consider a limiting property of the type II generalized Linnik distribution.

THEOREM 2.5.3

Let $X_1, X_2, ...$ be independent and identically distributed random variables with characteristic function $\left(\frac{1}{1+l\,|t|^a}\right)^{1/n}$ and N be a negative binomial random variable with probability generating function $\left(\frac{pz}{1-qz}\right)^n, n > 0,$ $p = \frac{1}{n}$, $q = 1 - p$. Then $X_1 + X_2 + ... + X_N$ converges in distribution to Z where $Z \overset{d}{=} GeGL_2$.

PROOF

Let $\qquad S_N = X_1 + X_2 + ... + X_N$.

$$f_{S_N}(t) = E\left(e^{it(X_1+X_2+...+X_N)}\right)$$

$$= \sum_{n=1}^{\infty} [f(t)]^n P(N = n).$$

Therefore,

$$f_{S_N}(t) = \left[\frac{pf(t)}{1-qf(t)}\right]^n, \quad \text{where } f(t) = \left(\frac{1}{1+l\,|t|^a}\right)^{1/n}.$$

$$=\left[\frac{p}{\left(1+l\,|t|^{a}\right)^{1/n}-q}\right]^{n}$$

$$=\left[\frac{1/n}{\left(1+l\,|t|^{a}\right)^{1/n}-\left(1-\frac{1}{n}\right)}\right]^{n}$$

$$=\left[\frac{1}{n\left(1+l\,|t|^{a}\right)^{1/n}-(n-1)}\right]^{n}$$

$$=\left[\frac{1}{1+n\left[\left(1+l\,|t|^{a}\right)^{1/n}-1\right]}\right]^{n}.$$

Let $\quad f_n(t)=f_{S_N}(t)$

$$\lim_{n\to\infty} f_n(t)=\left[\frac{1}{1+\lim\limits_{n\to\infty}\left\{n\left[1+l\,|t|^{a}\right]^{\frac{1}{n}}-1\right\}}\right]^{n}$$

$$=\left[\frac{1}{1+\ln(1+l\,|t|^{a})}\right]^{n}.$$

THEOREM 2.5.4

The function $f(t)=\left[\frac{1}{1+\ln(1+l\,|t|^{a})}\right]^{n}$ on R is a characteristic function if and only if $0<\alpha\le 2$ and $n>0$.

THEOREM 2.5.5

For each $\alpha\in(0,2]$ and $n>0$, the distribution $F_{a,n}(x)$ with characteristic function $f(t)=\left[\frac{1}{1+\ln(1+l\,|t|^{a})}\right]^{n}$ is absolutely continuous.

Proofs of Theorems 2.5.4 and 2.5.5 follow in the same lines as in Theorems 2.2.5 and 2.5.2

THEOREM 2.5.6

$GeGL_2(a, l, n)$ is normally attracted to stable law.

PROOF

The characteristic function of $n^{-1/a}\left(X_1+X_2+\ldots+X_n\right)$ is

$$f_{n^{-1/a}S_n}(t)=f_{\frac{X_1+X_2+\ldots+X_n}{n^{1/a}}}(t)$$

$$=f_{X_1+X_2+\ldots+X_n}\left(t/n^{1/a}\right)$$

$$= \left[f_{X_i}\left(t / n^{1/a} \right) \right]^n$$

$$= \left[\frac{1}{1 + \ln\left(1 + \frac{l\,|t|^a}{n} \right)} \right]^{n\,n}$$

$$= \left[\frac{1}{1 + \frac{l\,|t|^a}{n} + o(\frac{1}{n^2})} \right]^{n\,n}$$

$$\to e^{-l\,n\,|t|^a} \quad \text{as} \quad n \to \infty.$$

This completes the proof.

THEOREM 2.5.7

The generalized geometric Linnik stochastic process admits the representation

$$X_{a\,,\,l\,,n}{}^{(s)} \underline{\underline{d}}\, Y_{a\,,\,l}\ Z_{s,n}^{1/a}$$

where $Y_{a\,,\,l}$ is symmetric stable with characteristic function $\mathrm{e}^{-\lambda|t|^{\alpha}}$ and $Z_{s,n}$ is geometric gamma process with Laplace transform $\left[\frac{1}{1+\ln(1+d)} \right]^{n\,s}$.

PROOF

$$f_{Y_{a,l}\ Z_{s,n}^{1/a}}(t) = E\left(e^{itYZ^{1/a}}\right)$$

$$= \int_0^\infty E\left(e^{itYZ^{1/a}} \Big/ Z = z\right) dF(z)$$

$$= \int_0^\infty f_Y(tz^{1/a})\, dF(z)$$

$$= \int_0^\infty e^{-l\,z|t|^a}\, d\ F(z)$$

$$= \left[\frac{1}{1+\ln(1+l\ |t|^a)}\right]^{ns}.$$

Therefore, $\quad X_{a,l,n}{}^{(s)} \underset{=}{d}\ Y_{a,l}\ Z_{s,n}^{1/a}$.

Following the method of empirical characteristic function used in the case of GL distribution, we can estimate the $GeGL_2$ distribution parameters.

Consider the $GeGL_2(a, l, n)$ characteristic function

$$f(t) = \left[\frac{1}{1+\ln(1+l\ |t|^a)}\right]^{n}.$$

We have, the empirical characteristic function is

$$\hat{f}_n(t) = \frac{1}{n}\sum_{j=1}^{n} e^{itX_j}.$$

$$\ln f(t) = -n \ \ln\left[1+\ln(1+l\,|t|^{a})\right].$$

Proceeding as in Section 3, we get

$$\hat{a} = \frac{\ln \dfrac{\hat{d}_n(t_1)}{\hat{d}_n(t_2)}}{\ln \dfrac{|t_1|}{|t_2|}} \quad \text{and for } a \neq 1,$$

$$\hat{l} = \frac{e\text{xp}\left(\ln \hat{d}_n(t_2) \ln|t_1| - \ln \hat{d}_n(t_1) \ln|t_2|\right)}{\ln \dfrac{|t_1|}{|t_2|}},$$

where

$$\hat{d}_n(t) = \exp\left(\left[\hat{f}_n(t)\right]^{-1/n} - 1\right) - 1 \quad \text{and}$$

$$\hat{n} = \frac{-\ln \hat{f}_n(t)}{\ln\left[1+\ln(1+\hat{l}\,|t|^{\hat{a}})\right]}.$$

CHAPTER III

SOME ASYMMETRIC GENERALIZATIONS

3.1 INTRODUCTION

Due to the applications of Laplace distribution and the asymmetric nature of the data sets, several asymmetric generalizations of the Laplace distribution are introduced in the literature by different authors (see, Yu and Zhang (2005) and Kozubowski and Podgorski (2000)).

Let X and Y be two independent gamma random variables with parameters (α_1, β_1) and (α_2, β_2) where $a_i, b_i > 0$, i = 1, 2. Then the characteristic function of $Z = X - Y$ is

$$\phi_Z(t) \quad = \quad (1 + i\beta_1 t)^{-\alpha_1}(1 - i\beta_2 t)^{-\alpha_2}.$$

Mathai (1993a) introduced the generalized Laplace distribution with characteristic function

$$\phi_Z(t) = \frac{1}{\left(1+\lambda^2 t^2\right)^{\tau}}, \ t, l > 0. \tag{3.1.1}$$

When τ is a positive integer, the probability density function corresponding to (3.1.1) is

$$f_Z(x) = \frac{1}{(2l)^t} |x|^{t-1} e^{\frac{|x|}{l}} \sum_{r=0}^{t-1} \frac{(t)_r}{r!(t-r-1)!} \left(2\frac{|x|}{l}\right)^g$$

where

$$(t)_r = t(t+1)...(t+r-1), \quad (t)_0 = 1, \quad t \neq 0, \quad -\infty < x < \infty.$$

When $\tau = 1$, the above distribution reduces to Laplace distribution and is a member of class of self decomposable distributions. The generalized Laplace density has applications in various fields such as the production of the chemical melatonin in human body, growth decay mechanism like formation of sand dunes in nature, input- output situations in economic contexts, industrial production etc.(see, Mathai 1993a, 1993b, 1994, 2000).

Note that most of the real life contexts may not be symmetric in nature and we introduce and study the asymmetric form of the generalized Laplace distribution considered above. This asymmetric form is defined by the characteristic function

$$f_z(t) \quad = \quad \left(\frac{1}{1+ l\, t^2 - imt}\right)^t \qquad (3.1.2)$$

and the corresponding random variable Z is denoted as $\mathrm{GeALa}(\lambda, \mu.\tau)$.

In Section2, a generalization of the asymmetric distribution in (3.1.2) is considered and a representation of this distribution is obtained. Type I generalized geometric asymmetric Linnik distribution is introduced in Section 3 and some properties of this distribution are studied. Autoregressive models of type I generalized geometric asymmetric Linnik marginal distribution is introduced and studied in Section 4. In Section 5, various skewed versions of Laplace distributions are discussed and geometric versions of these distributions are introduced and their extensions to the Linnik case are discussed.

3.2 GENERALIZED ASYMMETRIC LINNIK DISTRIBUTION

In this Section, we obtain a representation for an asymmetric version of the generalized Linnik distribution in (2.2.3). Consider the distribution with characteristic function

$$f(t) \quad = \quad \left(\frac{1}{1+ l\, |t|^a - imt}\right)^t, -\infty < m < \infty,\ l, t \geq 0, 0 < a \leq 2. \qquad (3.2.1)$$

We shall refer this distribution as the generalized asymmetric Linnik distribution and denote it by $\text{GeAL}(\alpha,\lambda,\mu,\tau)$. When α=2, τ=1, it reduces to asymmetric Laplace distribution of Kozubowski and Podgorski (2000).

THEOREM 3.2.1

A $\text{GeAL}(\alpha,\lambda,\mu,\tau)$ random variable X with characteristic function (3.2.1) admits the representation $X \underset{=}{d} \mu W + (\lambda W)^{1/\alpha} Z$ where Z is symmetric stable with characteristic function $\psi(t) = e^{-\lambda|t|^{\alpha}}$ and W is a Gamma random variable with probability density function $g(w) = \frac{1}{\Gamma(\tau)} w^{\tau-1} e^{-w}, w > 0, \tau > 0$ independent of Z.

PROOF

Conditioning on W we obtain the characteristic function $\phi(t)$ of $\mu W + (\lambda W)^{1/\alpha} Z$ as

$$f(t) = E\left(E\left(e^{it\left(mW + (lW)^{1/a} Z\right)} \Big/ W\right)\right)$$

$$= \int_0^\infty E\left(e^{it\left(mw + (lw)^{1/a} Z\right)}\right) g(w)dw$$

$$=\left[\frac{1}{1+l\,|t|^{a}-imt}\right]^{t}.$$

Hence the Theorem.

3.3 GENERALIZED GEOMETRIC ASYMMETRIC LINNIK DISTRIBUTION

Since the distribution with characteristic function (3.2.1) is infinitely divisible, using the result of Klebanov et al. (1984), we can define a geometrically infinitely divisible distribution with characteristic function $\psi(\mathrm{t})$ as $f(t)=\exp\left\{1-\frac{1}{y\,(t)}\right\}$ where $f(t)$ is the characteristic function of an infinitely divisible distribution. The characteristic function (3.2.1) can be written as

$$\left(\frac{1}{1+l\,|t|^{a}-imt}\right)^{t}=\exp\left\{1-\frac{1}{\left[1+t\,\ln(1+l\,|t|^{a}-imt)\right]^{-1}}\right\}$$

Hence $\psi(\mathrm{t})=\frac{1}{1+\tau\ln(1+\lambda|\mathrm{t}|^{\alpha}-\mathrm{i}\mu\mathrm{t})}$ is a characteristic function of a geometrically infinitely divisible distribution.

The distribution with characteristic function

$$y\,(t) = \frac{1}{1+t\,\ln(1+l\,|t|^{a}-imt)}, -\infty < m < \infty,\; l\,,t\, \geq 0,\, 0 < a\, \leq 2 \qquad (3.3.1)$$

is called Type I generalized geometric asymmetric Linnik distribution with parameters $\mu, \sigma, \alpha, \tau$.

If X is a random variable with characteristic function (3.3.1), we represent it as $X \underline{\underline{d}}\, GeGAL_1(a\,,l\,,m,t\,)$. It may be noted that when $t\, = 1$ in (3.3.1) the corresponding distribution is the geometric version of asymmetric Linnik distribution and we call it as geometric asymmetric Linnik distribution and is denoted by $X \overset{d}{=} GAL(a\,,l\,,m)$.

Now we consider the asymmetric behavior of the GeGAL_1 distribution.

THEOREM 3.3.1

The $GeGAL_1(a\,,l\,,m,t\,)$ distribution is the limit distribution of the geometric sums of $GeAL\left(a\,,l\,,m,\frac{t}{n}\right)$ random variables.

PROOF

Let $\phi(t)$ be the characteristic function of a $GeAL\left(a\,,l\,,m,\frac{t}{n}\right)$random variable. Then

$$f(t) \quad = \quad \left(\frac{1}{1+ l\,|t|^{a} - imt}\right)^{\frac{t}{n}}.$$

Define

$$\Theta(t) = \frac{1}{f(t)} - 1 = \left(1+ l\,|t|^{a} - imt\right)^{\frac{t}{n}} - 1.$$

Hence using Lemma 3.2 of Pillai (1990b),

$$f_n(t) = \frac{1}{1+ p\Theta(t)}$$

where p > 1, is the characteristic function of geometric sum of random variables.

By choosing p = n, we have

$$\phi_n(t) = \left[1 + n\left[\left(1 + \lambda|t|^{\alpha} - i\mu t\right)^{\tau/n} - 1\right]\right]^{-1}$$

So $\phi_n(t)$ is the characteristic function of geometric sum of $GeAL\left(a, l, m, \frac{t}{n}\right)$.

Consider

$$\lim_{n\to\infty} f_n(t) = \frac{1}{1+ \lim\limits_{n\to\infty} n\left[\left(1+ l\,|t|^{a} - imt\right)^{t/n} - 1\right]}$$

$$= \frac{1}{1+\tau \ln(1+\lambda|t|^{\alpha} - i\mu t)}$$

which is the characteristic function of $GeGAL_1(a, l, m, t)$ random variables. Hence the theorem

Now we prove stability property of $GeGAL_1(a, l, m, t)$ random variables with respect to geometric summation.

THEOREM 3.3.2

Let $\{X_n\}$ be a sequence of independent and identically distributed random variables and let N_p be geometric random variable with mean $\frac{1}{p}$. Further, assume that N_p is independent of X_i's. If $U_{N_p} = \sum_{i=1}^{N_p} X_i$, then the random variables U_{N_p} and X_i are identically distributed if X_i follows $GeGAL_1(a, l, m, t)$.

PROOF

Let $\phi(t)$ and $\Theta(t)$ be the characteristic functions of X_i and U_{N_p} respectively. Then

$$\Theta(t) = \frac{pf(t)}{1-(1-p)f(t)}. \qquad (3.3.2)$$

Suppose $X_i \underline{\underline{d}}\ GeGAL_1(a, l, m, t)$.

Then, by (3.3.2) we have

$$\Theta(t) = \frac{p}{p + t\ \ln\left(1 + l\ |t|^a - imt\right)}$$

$$= \frac{1}{1 + \frac{t}{p}\ln\left(1 + l\ |t|^a - imt\right)}.$$

Hence the theorem.

3.4 AUTOREGRESSIVE MODELS WITH GENERALIZED GEOMETRIC ASYMMETRIC LINNIK MARGINAL DISTRIBUTION

Here we develop a time series model with $GeGAL_1(a, l, m, t)$ marginal distribution on the basis of geometric infinite divisibility property of the distribution.

THEOREM 3.4.1

Let $\{X_n, n \geq 1\}$ be defined as

$$X_n = \begin{cases} e_n & w.p. \quad q \\ X_{n-1} + e_n & w.p. \quad 1-q \end{cases} \tag{3.4.1}$$

where $0 < \theta \leq 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables. A necessary and sufficient condition that $\{X_n\}$ is a stationary process with $GeGAL_1(a, l, m, t)$ marginal is that $\{\varepsilon_n\}$ is distributed as $GeGAL_1(a, l, m, qt)$.

PROOF

Let $f_{X_n}(t)$ be the characteristic function of $\{X_n\}$. Then from (3.4.1), we get

$$f_{X_n}(t) = q\, f_{e_n}(t) + (1-q) f_{X_{n-1}}(t) f_{e_n}(t) \tag{3.4.2}$$

Assuming stationarity, we have

$$f_X(t) = q\, f_e(t) + (1-q) f_X(t) f_e(t).$$

Hence

$$f_e(t) = \frac{f_X(t)}{q + (1-q) f_X(t)}. \tag{3.4.3}$$

Suppose $X_n \underset{=}{d}\, GeGAL_1(a, l, m, t)$.

Then

$$f_X(t) \quad = \quad \frac{1}{1 + t\, \ln\left(1 + l\, |t|^a - imt\right)}.$$

Substituting this in (3.4.3) and simplifying we get,

$$f_e(t) \quad = \quad \frac{1}{1 + qt\, \ln\left(1 + l\, |t|^a - imt\right)}.$$

Hence $e_n \underset{=}{d}\, GeGAL_1(a\,,\,l\,,m,qt\,)$.

Conversely, assume that $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(a\,,\,l\,,m,qt\,)$ random variables and $X_0 \underset{=}{d}\, GeGAL_1(a\,,\,l\,,m,t\,)$. Then from (3.4.2), for n = 1 we have $\phi_{X_1}(t) \;=\; \dfrac{1}{1+\tau\ln\left(1+\lambda|t|^{\alpha}-i\mu t\right)}$.

If $X_{n-1} \underset{=}{d}\, GeGAL_1(a\,,\,l\,,m,t\,)$, then we get $X_n \underset{=}{d}\, GeGAL_1(a\,,\,l\,,m,t\,)$..

Thus using inductive argument, $\{X_n\}$ is a stationary process with $GeGAL_1(a\,,\,l\,,m,t\,)$ marginal distribution. Hence the Theorem.

We call the process defined by (3.4.1) with $X_0 \underset{=}{d}\, GeGAL_1(a\,,\,l\,,m,t\,)$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(a\,,\,l\,,m,qt\,)$ random variables as the first order autoregressive process with $GeGAL_1(a\,,\,l\,,m,t\,)$ marginal distribution .

From the Definition of the model (3.4.1), it is easily verified that

$$f_{X_n}(t) = q f_{e_n}(t)\frac{1-(1-q)^n f_{e_n}^n(t)}{1-(1-q)f_{e_n}(t)} + (1-q)^n f_{X_0}(t) f_{e_n}^n(t).$$

When $n \to \infty$, $f_{X_n}(t) \;=\; q f_{e_n}(t)\dfrac{1}{1-(1-q)f_{e_n}(t)}$.

Let X_0 is distributed arbitrary and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(a\,,\,l\,,m,qt\,)$ random variables. Then as $n \to \infty$

$$f_{X_n}(t) \quad = \quad \frac{1}{1+t\,\ln\left(1+\,l\,|t|^{a}\,-imt\right)}.$$

Hence if X_0 is distributed arbitrary, then the autoregressive process is asymptotically stationary with $GeGAL_1(a\,,\,l\,,m,t\,)$ marginal distribution.

Now from the joint characteristic function of $\left(X_n\,,X_{n+1}\right)$ of the process, it can be easily verified that the first order autoregressive process (3.4.1) with $GeGAL_1(a\,,\,l\,,m,t\,)$ marginal distribution is not time reversible.

An autoregressive model of k^{th} order with $GeGAL_1(a\,,\,l\,,m,t\,)$ as marginal distribution can be defined as

$$X_n = \begin{cases} e_n & w.p. \quad p_0 \\ X_{n-1}+e_n & w.p. \quad p_1 \\ X_{n-2}+e_n & w.p. \quad p_2 \\ \quad . & \\ \quad . & \\ \quad . & \\ X_{n-k}+e_n & w.p. \quad p_k \end{cases}$$

where $\sum_{i=0}^{k} p_i = 1$, $0 < p_i < 1$, $i = 1..k$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $GeGAL_1(a, l, m, t)$ random variables.

3.5 GEOMETRIC ASYMMETRIC LAPLACE DISTRIBUTIONS-VARIOUS FORMS

Kozubowski and Podgorski (2007a, b) considered different forms of asymmetric/skew Laplace distribution. The type I form is discussed in Section 1, has the characteristic function

$$f(t) = \frac{1}{1 + l t^2 - imt}, m \in R, l > 0.$$

The Type II asymmetric Laplace distribution is obtained by Azzalini's method (see, Azzalini (1985)) and is given by the characteristic function

$$y(t) = \frac{e^{iqt}\left[l t + (1+d)^2 i \right]}{(l t + i)\left[l^2 t^2 + (1+d)^2 \right]}, t, q \in R, l > 0, d \geq 0. \tag{3.5.1}$$

The corresponding density is

$$f(x) = \frac{1}{l} \begin{cases} e^{-\left|\frac{x-q}{l}\right|} - \frac{1}{2} e^{-(1+d)\left|\frac{x-q}{l}\right|}, & x \geq q \\ \frac{1}{2} e^{-(1+d)\left|\frac{x-q}{l}\right|}, & x < q \end{cases}.$$

The geometric asymmetric Laplace distribution based on the Type II asymmetric Laplace distribution is defined by the characteristic function $\phi(t) = \frac{1}{1-\ln(\psi(t))}$ where $\psi(t)$ is given by (3.5.1). Note that (3.5.1) can be extended to the asymmetric Linnik case by considering

$$g(x) = 2f(x)F(g\,x),\ x \in R, g \in R$$

where $f(x)$ is the probability density function of Linnik distribution and $F(x)$ is the corresponding distribution function. The geometric version of this will lead to the geometric asymmetric Linnik law corresponding to the Azzalini's method.

Type III asymmetric Laplace distribution is the distribution of X_g where

$$X_g = \frac{1}{\sqrt{1+g^2}} X + \frac{g}{\sqrt{1+g^2}} |Y|, g \in R \tag{3.5.2}$$

where X and Y are independent and identically distributed standard Laplace random variables. Denoting $c = \frac{\gamma}{\sqrt{1+\gamma^2}} \in (-1,1)$ and introducing the location parameter $\theta \in R$ and scale parameter $\lambda > 0$, density of X_γ can be written as

$$f(x)=\frac{1}{l}\begin{cases}\dfrac{1}{2\left(\sqrt{1-c^2}-c\right)}e^{-\frac{1}{\sqrt{1-c^2}}\left|\frac{x-q}{l}\right|}-\dfrac{c}{1-2c^2}e^{-\frac{1}{c}\left|\frac{x-q}{l}\right|}, & x\geq q\\ \dfrac{1}{2\left(\sqrt{1-c^2}+c\right)}e^{-\frac{1}{\sqrt{1-c^2}}\left|\frac{x-q}{l}\right|}, & x<q\end{cases}.$$

The corresponding characteristic function is

$$y\ (t)=\frac{e^{iqt}}{\left[1+\left(1-c^2\right)l^2t^2\right]\left[1-i\,l\,ct\right]},q\,,t\in R,c\in(-1,1).$$

The geometric asymmetric Laplace distribution is defined by the characteristic function $\frac{1}{1-\ln(\psi(\mathrm{t}))}$.

For the interrelations between type I , Type II and Type III asymmetric Laplace laws, see Kozubowski and Podgorski (2007a,b).

CHAPTER IV

TAILED DISTRIBUTIONS

4.1 INTRODUCTION

Tailed distributions have found applications in various fields and were studied by many authors (see, Littlejohn (1994), Muraleedharan (1999), Muraleedharan and Kale (2002) and Hutton (1990)). We encounter tailed distributions in life testing experiments where an item fails instantaneously. In clinical trials, some times a medicine has no response initially with a certain probability and on a later stage there may be response, the length of the response is described by certain probability distribution.

DEFINITION 4.1.1

Let the random variable X has distribution function F(x) and characteristic function $f_X(t)$. A tailed random variable U with tail probability θ associated with X is defined by the characteristic function

$$f_U(t) = q + (1-q) f_X(t) \qquad (4.1.1)$$

This Chapter is based on Mariamma Antony and Raju (2008a)

In Section 2, we introduce tailed distributions associated with type I generalized geometric Linnik distribution and study its properties. Tailed Type II generalized geometric Linnik distribution is also discussed in this Section. Tailed type I generalized geometric asymmetric Linnik distribution is studied in Section 3. Marshal and Olkin (1997) considered a method of introducing new parameters in to a distribution F. For the applications of Marshall Olkin scheme, see Marshall and Olkin (1997) and Jayakumar and Mathew (2006). Jayakumar and Kuttykrishnan (2007) redefined Marshall -Olkin schemes and used them to describe reparametrized forms of a distribution in terms of characteristic function. In Section 4, we derive the Marshall-Olkin form of geometric Linnik, Type I generalized geometric Linnik, Type II generalized geometric Linnik, geometric semi-alpha Laplace distributions and introduce the tailed distributions generated by the Marshall-Olkin forms.

4.2 TAILED GENERALIZED GEOMETRIC LINNIK DISTRIBUTIONS

Now we introduce tailed Type I generalized geometric Linnik distribution and obtain a representation of the same.

In (4.1.1), when $\phi_X(t) = \dfrac{1}{1+\tau \ln\left(1+\lambda|t|^{\alpha}\right)}$, then

$$f_U(t) = \frac{1+tq \ln\left(1+ l\, |t|^{a}\right)}{1+t \ln\left(1+ l\, |t|^{a}\right)}. \tag{4.2.1}$$

The random variable U with characteristic function (4.2.1) is called tailed Type I generalized geometric Linnik and denoted by $TGeGL_1(a, l, t, q)$

THEOREM 4.2.1

Let X and Y be independent random variables such that X has tailed generalized geometric exponential distribution with Laplace transform $q+(1-q)\dfrac{1}{1+t \ln(1+d)}$ and Y is stable with characteristic function $e^{-\lambda|t|^{\alpha}}$, $0 < a \le 2,\ l, d, t > 0,\ 0 < q < 1.$ Then $Z = X^{1/\alpha} Y$ has $TGeGL_1(a, l, t, q)$ distribution.

PROOF

$$\phi_Z(t) = E\left[e^{itX^{1/\alpha} Y} \right]$$

$$= \int_0^{\infty} f_Y\left(tx^{1/a}\right) dF(x)$$

$$= \int_0^\infty e^{-l|t|^a x} dF(x)$$

$$= q + (1-q)\frac{1}{1+t\ \ln\left(1+l\ |t|^a\right)}$$

$$= \frac{1+tq\ \ln\left(1+l\ |t|^a\right)}{1+t\ \ln\left(1+l\ |t|^a\right)}.$$

This completes the proof.

Now we develop a first order autoregressive model with TGeGL_1 distribution as marginal.

Consider the model

$$X_n = \begin{cases} e_n & w.p. \quad p \\ X_{n-1} + e_n & w.p. \quad 1-p \end{cases} \tag{4.2.2}$$

$$= \mathrm{I}_n \mathrm{X}_{n-1} + \varepsilon_n$$

where $\{\varepsilon_n\}$ and $\{\mathrm{I}_n\}$ are two sequences of independent and identically distributed random variables with I_n, X_{n-1} and ε_n mutually independent and

$$P(I_n = 0) = p = 1 - P(I_n = 1).$$

We have the model (4.2.2) in terms of characteristic functions is

$$f_{e_n}(t) = \frac{f_{X_n}(t)}{p + (1-p) f_{X_{n-1}}(t)}.$$

In the stationary case,

$$f_{e_n}(t) = \frac{f_X(t)}{p + (1-p) f_X(t)}.$$

If X has characteristic function (4.2.1), then

$$f_{e_n}(t) = \frac{1 + tq \ln\left(1 + l\,|t|^a\right)}{1 + t\,c \ln\left(1 + l\,|t|^a\right)}$$

where $c = p + (1-p)q$.

That is,

$$f_{e_n}(t) = \frac{tq}{c} + \left(1 - \frac{tq}{c}\right) \frac{1}{1 + t\,c \ln\left(1 + l\,|t|^a\right)}.$$

Hence, if the model (4.2.2) is stationary with $TGeGL_1(a, l, t, q)$ marginal distribution, then the distribution of the innovation sequence $\{\varepsilon_n\}$ is $TGeGL_1\left(a, l, t\,c, \frac{tq}{c}\right)$.

If $X_0 \stackrel{d}{=} TGeGL_1(a, l, t, q)$ and $\{\varepsilon_n\}$ are independent and identically distributed as $TGeGL_1\left(a, l, t\, c, \frac{tq}{c}\right)$, then the characteristic function of X_1 is

$$f_{X_1}(t) = \left[p + (1-p) f_{X_0}(t)\right] f_{e_1}(t)$$

$$= \left[\mathrm{p} + (1-\mathrm{p}) \frac{1+\tau\theta \ln\left(1+\lambda|\mathrm{t}|^{\alpha}\right)}{1+\tau \ln\left(1+\lambda|\mathrm{t}|^{\alpha}\right)}\right]\left[\frac{1+\tau\theta \ln\left(1+\lambda|\mathrm{t}|^{\alpha}\right)}{1+\tau\mathrm{c} \ln\left(1+\lambda|\mathrm{t}|^{\alpha}\right)}\right]$$

$$= \frac{1+tq \ln\left(1+l\,|t|^{a}\right)}{1+t \ln\left(1+l\,|t|^{a}\right)}.$$

That is, $X_1 \underline{\underline{d}} X_0$.

If $X_{n-1} \underline{\underline{d}} X_0$, we can prove that $X_n \underline{\underline{d}} X_0$ and hence the process $\{\mathrm{X}_\mathrm{n}\}$ is stationary.

Based on this, we now define stationary first order autoregressive tailed Type I generalized geometric Linnik process as follows:

Let

$$X_0 \underline{\underline{d}} TGeGL_1(a, l, t, q)$$

and for $n = 1, 2, \ldots$

$$X_n = \begin{cases} e_n & w.p. \quad p \\ X_{n-1} + e_n & w.p. \quad 1-p \end{cases}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $TGeGL_1\left(a\,,\,l\,,t\,c,\frac{tq}{c}\right)$ random variables where $c = p + (1-p)q$.

It can be shown that the process $\{X_n\}$ is not time reversible. For this, consider the characteristic function

$$f_{X_n, X_{n+1}}(t_1, t_2) = E\left(e^{it_1 X_n + it_2 X_{n+1}}\right)$$

$$= f_{e_n}(t_2)\left[\, p f_{X_n}(t_1) + (1-p) f_{X_n}(t_1 + t_2)\right]$$

$$= \frac{1+\tau\theta\ln\left(1+\lambda|t|^{\alpha}\right)}{1+\tau c\ln\left(1+\lambda|t|^{\alpha}\right)}\left[\, p\frac{1+tq\ \ln\left(1+l\ |t_1|^{a}\right)}{1+t\ \ln\left(1+l\ |t_2|^{a}\right)} + (1-p)\frac{1+tq\ \ln\left(1+l\ |t_1+t_2|^{a}\right)}{1+t\ \ln\left(1+l\ |t_1+t_2|^{a}\right)}\right].$$

This expression is not symmetric in t_1 and t_2.

REMARK 4.2.1

If $\{\varepsilon_n\}$ is a sequence of independent and identically distributed $TGeGL_1\left(a\,,\,l\,,t\,c,\frac{tq}{c}\right)$, where $\mathrm{c} = \mathrm{p} + (1-\mathrm{p})\theta$ then (4.2.2) is asymptotically stationary with $TGeGL_1(a\,,\,l\,,t\,,q)$ marginal distribution.

Consider the kth order autoregressive process

$$X_n = \begin{cases} \varepsilon_n & w.p. \quad p \\ X_{n-1} + \varepsilon_n & w.p. \quad p_1 \\ X_{n-2} + \varepsilon_n & w.p. \quad p_2 \\ \vdots & \\ X_{n-k} + \varepsilon_n & w.p. \quad p_k \end{cases} . \qquad (4.2.3)$$

If the process $\{X_n\}$ is stationary, then in terms of characteristic function (4.2.3) is

$$f_{\varepsilon_n}(t) = \frac{f_X(t)}{p + (1-p) f_X(t)},$$

where $1 - p = \sum_{i=1}^{k} p_i, \quad 0 < p_i < 1.$

Thus a necessary and sufficient condition for the model (4.2.3) defines a stationary AR(k) process with $TGeGL_1(\alpha, \lambda, \tau, \theta)$ marginal distribution is that $\{\varepsilon_n\}$ is distributed as $TGeGL_1\left(\alpha, \lambda, \tau c, \frac{\tau\theta}{c}\right)$.

Now we consider the type II generalized geometric Linnik distribution and study the tailed distribution generated by it.

DEFINITION 4.2.1

A random variable X is said to have tailed type II generalized geometric Linnik distribution and write $X \underset{=}{d} TGeGL_2(\alpha, \lambda, \tau, \theta)$ distribution if it has the characteristic function

$$f_X(t) = \frac{q\left[1+\ln\left(1+l\,|t|^a\right)\right]^t + (1-q)}{\left[1+\ln\left(1+l\,|t|^a\right)\right]^t}, 0 < a \le 2, 0 < q < 1,\ l, t > 0.$$

The tailed type II generalized geometric Linnik distribution being the tailed form of type II generalized geometric Linnik is infinitely divisible.

As in the case of $TGeGL_1$ distribution, we now obtain a representation of $TGeGL_2$ random variables in terms of tailed geometric gamma and stable random variables.

DEFINITION 4.2.2

A random variable X is said to have tailed geometric gamma distribution if it has Laplace transform

$$f_1(d) = q + (1-q)\frac{1}{\left[1+\ln\left(1+d\right)\right]^t}, \quad d, t > 0, 0 < q < 1.$$

THEOREM 4.2.2

Let X and Y be independent random variables such that X has Laplace transform $\theta + (1-\theta)\frac{1}{[1+\ln(1+\delta)]^{\tau}}$ and Y is stable with characteristic function

$e^{-l|t|^a}$, $0 < a \leq 2$. Then $U = X^{1/\alpha} Y$ has distribution $TGeGL_2(a, l, t, q)$.

Proof follows analogous to the proof of Theorem 4.2.1.

4.3 TAILED GENERALIZED GEOMETRIC ASYMMETRIC LINNIK DISTRIBUTION

Here we discuss the tailed distributions generated by generalized geometric asymmetric Linnik distributions.

DEFINITION 4.3.1

A random variable X is said to have tailed type I generalized geometric asymmetric Linnik distribution and write $U \underline{\underline{d}} TGeGAL_1(a, l, m, t, q)$ if it has characteristic function

$$f_U(t) = \frac{1 + tq \ln\left(1 + l\,|t|^a - imt\right)}{1 + t \ln\left(1 + l\,|t|^a - imt\right)}.$$

As in the case of $TGeGL_1$ random variables, we now consider the first order autoregressive model (4.2.2). We can prove that if $\{\varepsilon_n\}$ are independent and identically distributed $TGeGAL_1\left(a, l, m, t\,c, \frac{tq}{c}\right)$ random variables and $X_0 \underline{\underline{d}} TGeGAL_1(a, l, m, t, q)$, then the model (4.2.2) is stationary with

$TGeGAL_1(a, l, m, t, q)$ marginals. Also if the model (4.2.2) is stationary with $TGeGAL_1(a, l, m, t, q)$ marginals then it can be easily seen that the distribution of $\{\varepsilon_n\}$ is $TGeGAL_1\left(a, l, m, t\, c, \frac{tq}{c}\right)$ where $c = p + (1-p)q$. Based on this we can develop first order autoregressive models with tailed Type I generalized geometric asymmetric Linnik marginal distribution.

The model can be easily extended to higher order autoregressive model as in (4.2.3).

The type II generalized geometric asymmetric Linnik distribution can be defined using (4.1.1) with $\phi_X(t)$ replaced by

$$\frac{1}{\left[1+\ln\left(1+l\,|t|^{a}-imt\right)\right]^{t}}.$$

That is, a random variable U having tailed type II generalized geometric asymmetric Linnik distribution denoted by $TGeGAL_2(a, l, m, t, q)$ has characteristic function

$$q+(1-q)\frac{1}{\left[1+\ln\left(1+l\,|t|^{a}-imt\right)\right]^{t}}.$$

4.4 MARSHALL-OLKIN FORMS

Let X be a random variable having absolutely continuous distribution function F(x) in the support (a, b) where a can be $-\infty$ and b can be $+\infty$. Let $\overline{F}(x)=1-F(x)$ be the survival function of X. Marshall and Olkin (1997) introduced a flexible family of distributions by adding a new parameter in to the survival function. Starting with a survival function $\overline{F}(x)$, Marshall and Olkin (1997) defined a new class of distributions given by

$$\overline{G}(x)=\frac{g\,\overline{F}(x)}{1-(1-g)\overline{F}(x)},\ -\infty<x<\infty, 0<g<\infty. \qquad (4.4.1)$$

Sankaran and Jayakumar (2007) gave an interpretation for the family (4.4.1) using odds function. They showed that family (4.4.1) satisfies the proportional odds model. Jayakumar and Mathew (2006) considered a generalization of Marshall-Olkin scheme and discussed the application of this in Burr Type XII distribution. The generalized Marshall-Olkin scheme corresponding to the survival function $\overline{F}$ is defined as

$$\overline{H}(x)=\left[\frac{g\,\overline{F}(x)}{1-(1-g)\overline{F}(x)}\right]^{m}, -\infty<x<\infty, 0<g, m<\infty. \qquad (4.4.2)$$

Jayakumar and Kuttykrishnan (2007) defined the Marshall-Olkin schemes in terms of characteristic functions. Let X be a random variable on $(-\infty,\infty)$ with characteristic function $\phi(t)$. They defined the characteristic function $y(t)$ as

$$y(t)=\frac{gf(t)}{1-(1-g)f(t)},\ 0<g<\infty,\ -\infty<t<\infty.,\qquad (4.4.3)$$

We now introduce the Marshall-Olkin forms of the geometric Linnik characteristic function, its asymmetric versions and other extensions.

Suppose

$$f(t)=\frac{1}{1+\ln\left(1+l\,|t|^{a}\right)}.$$

The Marshall-Olkin form is

$$y(t)=\frac{1}{1+\frac{1}{g}\ln\left(1+l\,|t|^{a}\right)},\ 0<a\leq 2, g>0.\quad .$$

Note that the Marshall-Olkin form of $GL(\alpha,\lambda)$ distribution give rise to $GeGL_1\left(a,l,\frac{1}{g}\right)$ distribution.

Simiilarly, the Marshall-Olkin form of $GAL(\alpha,\lambda,\mu)$ distribution is the $GeGAL_1\left(a,l,m,\frac{1}{g}\right)$ distribution. It can be easily seen that if $X \stackrel{d}{=} GeGAL_1(a,l,m,t)$ then the Marshall-Olkin form of X is $GeGAL_1\left(a,l,m,\frac{t}{g}\right)$.

If $\phi(t) = \dfrac{1}{\left[1+\ln\left(1+ l\,|t|^{a} - imt\right)\right]^{t}}$, then

$$\psi(t) = \frac{g}{\left[1+\ln\left(1+ l\,|t|^{a} - imt\right)\right]^{t} - (1-g)}.$$

Also if $X \underset{=}{d}\, TGeGAL_1(a, l, m, t, q)$, then $\psi(t) = \dfrac{g\left[1+tq\ln\left(1+ l\,|t|^{a} - imt\right)\right]}{g + \left[1+(g-1)q\right]^{t}\ln\left(1+ l\,|t|^{a} - imt\right)}$

CHAPTER V

SOME BIVARIATE EXTENSIONS

5.1 INTRODUCTION

Heavy tailed bivariate distributions with different tail index are used for modeling bivariate data. Considering this, Kozubowski et al. (2005) introduced marginal Laplace and Linnik distributions. A random vector $\underline{X} = (X_1, X_2)$ is said to have marginal Laplace and Linnik distribution if it has the characteristic function

$$y\,(t,s) = \frac{1}{1 + l_1 t^2 + l_2 |s|^a},\ 0 < a \le 2,\ \ l_1, l_2 > 0,\ \ t, s \in R.$$

Note that,

$$\psi(t,0) = \frac{1}{1 + \lambda_1 t^2},$$

This Chapter is based on Mariamma Antony and Raju (2008b) and (2008c) and

$$\psi(0,s) = \frac{1}{1+\lambda_2 |s|^{\alpha}}.$$

Kozubowski et al. (2005) derived the representation of $\underline{X}$ as

$$\underline{X} \stackrel{d}{=} \left(U^{1/2}X_1, U^{1/\alpha}X_2\right)$$

where U is unit exponential, X_1 and X_2 are normal and α stable random variables with respective characteristic functions

$$\psi_1(t) = e^{-\lambda_1 t^2} \text{ and}$$

$$\psi_2(s) = e^{-\lambda_2 |s|^{\alpha}}.$$

The marginal Laplace and Linnik distribution can be generalized to include the asymmetry in the data as follows:

Consider a random vector $\underline{X} = (X_1, X_2)$ with characteristic function

$$\psi(t,s) = \frac{1}{1+\lambda_1 t^2 + \lambda_2 |s|^{\alpha} - i\mu t - i\nu s}, \; \lambda_1, \lambda_2 > 0, 0 < \alpha \le 2, \mu, \nu \in R.$$

Note that in this case, $\psi(t,0) = \dfrac{1}{1+\lambda_1 t^2 - i\mu t}$ and $\psi(0,s) = \dfrac{1}{1+\lambda_2 |s|^{\alpha} - i\nu s}$. We call the distribution with characteristic function $y(t,s)$ as marginal asymmetric Laplace and asymmetric Linnik distribution.

In Section 2, we introduce geometric marginal asymmetric Laplace and asymmetric Linnik distribution and study its properties. In Section 3, we consider geometric marginal asymmetric Linnik and asymmetric Linnik distribution and its extensions. Geometric bivariate semi-α-Laplace distribution are introduced and studied in Section 4

5.2 GEOMETRIC MARGINAL ASYMMETRIC LAPLACE AND ASYMMETRIC LINNIK DISTRIBUTION

Kozubowslki et al. (2005) introduced and studied a class of multivariate distributions called operator geometric stable laws by generalizing operator stable and geometric stable laws. As a particular case, they studied a new class of bivariate distributions namely marginal Laplace and Linnik distributions. Kutyikrishnan and Jayakumar (2005) generalized this class of distributions and introduced and studied a class of bivariate distributions that contains marginal Laplace and Linnik distributions. The resulting class of bivariate distributions namely generalized marginal asymmetric Laplace and asymmetric Linnik ($\mathrm{GeMALaAL}$) distributions have the characteristic function

$$f(t,s)=\left[\frac{1}{1+l_1t^2+l_2|s|^a-imt-ins}\right]^t, \qquad (5.2.1)$$

$$l_1, l_2>0, -\infty<m,n<\infty, t\geq 0, a\in(0,2].$$

Let $\psi(t,s)$ be the characteristic function of a geometrically infinitely divisible bivariate distribution given by the equation $\phi(t,s)=e^{\left(1-\frac{1}{\psi(t,s)}\right)}$ where $\phi(t,s)$ is the characteristic function of an infinitely divisible bivariate distribution.

Substituting (5.2.1) in the equation $\phi(t,s)=e^{\left(1-\frac{1}{\psi(t,s)}\right)}$, we obtain

$$y(t,s)=\frac{1}{1+t\ln\left(1+l_1t^2+l_2|s|^a-imt-ins\right)}, \qquad (5.2.2)$$

$$l_1, l_2>0, t\geq 0, -\infty<m,n<\infty, a\in(0,2].$$

Hence $y(t,s)=\frac{1}{1+t\ln\left(1+l_1t^2+l_2|s|^a-imt-ins\right)}$ is the characteristic function of a geometrically infinitely divisible bivariate distribution.

A bivariate distribution with characteristic function (5.2.2) is called Type I generalized geometric marginal asymmetric Laplace and asymmetric Linnik $\mathrm{GeGMALaAL}_1$ distribution with parameters a, m, n, l_1, l_2, t. Note that when

$m = n = 0$ and $t = 1$, this becomes geometric marginal Laplace and Linnik distribution.

If (X, Y) is bivariate random vector with characteristic function (5.2.2), we represent it as $(X, Y) \underline{\underline{d}}\, GeGMALaAL_1(a, l_1, l_2, m, n, t)$. An asymptotic property of $GeGMALaAL_1$ distribution is given in the following theorem.

THEOREM 5.2.1

The $\mathrm{GeGMALaAL}_1$ distribution is the limit distribution of the geometric sums of $GeMALaAL$ random variables.

The theorem can be proved using the argument similar to the Proof of Theorem 3.3.1.

Now it is useful to develop a bivariate time series model using the $GeGMALaAL_1$ marginal distribution. A one parameter autoregressive model equivalent to TEAR(1) structure of Lawrance and Lewis (1981) can be constructed corresponding to the set of bivariate time series data as follows:

Let $\{(e_n, h_n), n \geq 1\}$ be a sequence of independent and identically distributed bivariate random vectors and let $(X_0, Y_0) \underline{\underline{d}}\, GeGMALaAL_1(a, l_1, l_2, m, n, t)$.

Define $\{(X_n, Y_n), n \geq 1\}$ as

$$X_n = \begin{cases} e_n & w.p. \quad p \\ X_{n-1} + e_n & w.p. \quad 1-p \end{cases}$$

and

$$Y_n = \begin{cases} h_n & w.p. \quad p \\ X_{n-1} + h_n & w.p. \quad 1-p \end{cases} \tag{5.2.3}$$

where 0 < p < 1.

Let $\phi_{X_n, Y_n}(t,s)$ and $f_{e_n, h_n}(t,s)$ be the characteristic functions of (X_n, Y_n) and (ε_n, η_n) respectively. Then (5.2.3) gives

$$f_{(e_n, h_n)}(t,s) \quad = \quad \frac{f_{(X_n, Y)_n}(t,s)}{p + (1-p) f_{(X_{n-1}, Y_{n-1})}(t,s)}. \tag{5.2.4}$$

If $\{(X_n, Y_n)\}$ is a stationary sequence with $GeGMALaAL_1(a, l_1, l_2, m, n, t)$ marginal distribution, then from (5.2.4) we get

$$f_{(e_n h_n)}(t,s) = \frac{1}{1 + pt \ln\left(1 + l_1 t^2 + l_2 |s|^a - imt - ins\right)}. \tag{5.2.5}$$

Hence $(e_n, h_n) \underline{\underline{d}}\, GeGMALaAL_1(a, l_1, l_2, m, n, pt)$.

Also it can be verified that if $(X_0, Y_0) \underline{\underline{d}}\ GeGMALaAL_1(a, l_1, l_2, m, n, t)$ and $\{(e_n, h_n), n \geq 1\}$ is an independent and identically distributed sequence of bivariate random variables with characteristic function given by (5.2.5), the first order autoregressive process (5.2.3) is stationary with $GeGMALaAL_1(a, l_1, l_2, m, n, t)$ marginal distribution.

Hence, we have the following theorem.

THEOREM 5.2.3

Let $\{(e_n, h_n), n \geq 1\}$ be a sequence of independent and identically distributed $GeGMALaAL_1(a, l_1, l_2, m, n, pt)$ random vectors and $(X_0, Y_0) \underline{\underline{d}}$ $GeGMALaAL_1(a, l_1, l_2, m, n, t)$. Then the relation (5.2.3) defines a stationary bivariate time series with $GeGMALaAL_1$ marginal distribution.

5.3 GEOMETRIC MARGINAL ASYMMETRIC LINNIK AND ASYMMETRIC LINNIK DISTRIBUTION AND ITS EXTENSIONS

In practice we come across bivariate random vectors where the components of the vectors have heavy tails than normal distribution and component distributions are asymmetric with steep peak.

DEFINITION 5.3.1

Let $\underline{X}=(X_1, X_2)$ be a random vector with characteristic function

$$f(t,s)=\left[\frac{1}{1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins}\right]^t ,$$

$$l_1, l_2>0, ,t\ \geq 0, -\infty< m,n\ <\infty, 0<a_1, a_2\leq 2.$$

Then we say that $\underline{X}=(X_1, X_2)$ has generalized marginal asymmetric Linnik and asymmetric Linnik distribution and we denote $\underline{X}$ by $\underline{X}\ \underline{\underline{d}}\ \ GeMALAL(a_1, a_2, l_1, l_2, m, n, t)$.

The geometric version of this and its generalization are the subject of study in this Section.

DEFINITION 5.3.2

A random vector $\underline{X}=(X_1, X_2)$ is said to have geometric marginal asymmetric Linnik and asymmetric Linnik distribution and write $\underline{X}\ \underline{\underline{d}}\ GMALAL(a_1, a_2, l_1, l_2, m, n)$ distribution if it has the following characteristic function

$$f(t,s)=\frac{1}{1+\ln\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)},$$

$$l_1,l_2>0,,t\geq 0,-\infty<m,n<\infty,0<a_1,a_2\leq 2.$$

Note that when $a_1=2$ and $a_2=a$, the geometric marginal asymmetric Linnik and asymmetric Linnik distribution turns out to be geometric marginal asymmetric Laplace and asymmetric Linnik distribution studied in Section 2.

REMARK 5.3.1

As in Chapters II and III, the two generalizations of the geometric marginal asymmetric Linnik asymmetric Linnik distribution are Type I generalized geometric marginal asymmetric Linnik and asymmetric Linnik distribution defined by the characteristic function

$$f(t,s)=\left[\frac{1}{1+t\ \ln\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)}\right] \tag{5.3.1}$$

denoted by $\underline{X}=(X_1,X_2)\underline{\underline{d}}\,GeGMALAL_1(a_1,a_2,l_1,l_2,m,n,t)$ and Type II generalized geometric marginal asymmetric Linnik and asymmetric Linnik distribution defined by the characteristic function

$$f(t,s)=\left[\frac{1}{1+\ln\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)}\right]^t$$

denoted by $\underline{X}\,\underline{\underline{d}}\,GeGMALAL_2(a_1,a_2,l_1,l_2,m,n,t)$.

THEOREM 5.3.1

The $GeGMALAL_1(a_1,a_2,l_1,l_2,m,n,t)$ distribution is the limit distribution of geometric sums of $GeMALAL(a_1,a_2,l_1,l_2,m,n,\frac{t}{n})$ random variables.

PROOF

Let $f(t,s)$ be the characteristic function of $GeMALAL(a_1,a_1,l_1,l_2,m,n,\frac{t}{n})$

Then

$$f(t,s)=\left[\frac{1}{\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)}\right]^{\frac{t}{n}}.$$

Define $\Theta(t,s)=\dfrac{1}{f(t,s)}-1$

$$=\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)^{\frac{t}{n}}-1.$$

Consider $f_n(t,s) = \dfrac{1}{1+n\left\{\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)^{\frac{t}{n}}-1\right\}}$.

$$\lim_{n\to\infty} f_n(t,s) = \frac{1}{1+\lim\limits_{n\to\infty} n\left\{\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)^{\frac{t}{n}}-1\right\}}$$

$$= \frac{1}{1+t\ \ln\left(1+l_1|t|^{a_1}+l_2|s|^{a_2}-imt-ins\right)}.$$

Now we develop a bivariate time series model using $GeGMALAL_1$ marginal distribution.

Let $\{(e_n,h_n),n\geq 1\}$ be a sequence of independent and identically distributed bivariate random vectors and let $\left(\underline{X_0},\underline{Y_0}\right)\underline{\underline{d}}\, GeGMALAL_1(a_1,a_2,l_1,l_2,m,n,t)$ be a random vector with characteristic function (5.3.1). Define $\{(X_n,Y_n),n\geq 1\}$ as

$$X_n = \begin{cases} e_n & w.p.\quad p \\ X_{n-1}+e_n & w.p.\quad 1-p \end{cases} \qquad (5.3.2)$$

$$Y_n = \begin{cases} h_n & w.p.\quad p \\ X_{n-1}+h_n & w.p.\quad 1-p \end{cases}$$

where 0 < p < 1.

Let $f_{\left(X_n Y_n\right)}(t,s)$ and $f_{\left(e_n h_n\right)}(t,s)$ be the characteristic functions of $\{(\mathrm{X}_n, \mathrm{Y}_n)\}$, and $\{(\varepsilon_n, \eta_n)\}$ respectively.

Then (5.3.2) gives

$$f_{\left(e_n, h_n\right)}(t,s) = \frac{f_{\left(X_n, Y\right)_n}(t,s)}{p + (1-p) f_{\left(X_{n-1}, Y_{n-1}\right)}(t,s)}. \tag{5.3.3}$$

If $\{(X_n, Y_n)\}$ is a stationary sequence with $GeGMALAL_1(a_1, a_2, l_1, l_2, m, n, t)$ marginal distribution, then from (5.3.3) we get

$$f_{\left(e_n, h_n\right)}(t,s) = \frac{1}{1 + pt \ln\left(1 + l_1 |t|^{a_1} + l_2 |s|^{a_2} - imt - ins\right)}$$

Hence

$$\left(e_n, h_n\right) \underline{\underline{d}}\, GeGMALAL_1(a_1, a_2, l_1, l_2, m, n, pt). \tag{5.3.4}$$

Also it can be verified that if

$\left(X_0, Y_0\right) \underline{\underline{d}}\, GeGMALAL_1(a_1, a_2, l_1, l_2, m, n, t)$ and $\{(e_n, h_n), n \geq 1\}$ is independent and identically distributed sequence of bivariate random variables given by (5.3.4) then the first order autoregressive process (5.3.2) is stationary with $GeGMALAL_1(a_1, a_2, l_1, l_2, m, n, t)$ marginal distribution.

5.4 GEOMETRIC BIVARIATE SEMI α LAPLACE DISTRIBUTION

A bivariate semi α Laplace distribution $\underline{X}$ is defined by the characteristic function

$$\phi(t,s) = \frac{1}{1+\delta(t,s)} \qquad (5.4.1)$$

where $\delta(t,s)$ satisfies the functional equation

$$d(t,s) = \frac{1}{p} d\left(p^{1/a_1} t, p^{1/a_2} s \right), 0 < p < 1, 0 < a_1, a_2 \leq 2. \qquad (5.4.2)$$

A solution of the equation (5.4.2) is

$$d(t,s) = |t|^{a_1} d_1(t) + |s|^{a_2} d_2(s)$$

where $d_1(t)$ and $d_2(s)$ are periodic functions in $\ln|t|$ and $\ln|s|$ with periods $\frac{-2\pi\alpha_1}{\ln p}$ and $\frac{-2\pi\alpha_2}{\ln p}$ respectively.

The solution of (5.4.2) is not unique.

For example, the function

$$d(t,s) = \left[\tfrac{1}{2}(t,s)\Sigma(t,s)' \right]^{a/2}, 0 < a \leq 2 \qquad (5.4.3)$$

and Σ is any non negative definite matrix, satisfy the functional equation (5.4.3) with $\alpha_1 = \alpha_2 = \alpha$ and any $p \in (0,1)$. When $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho_1\sigma_1\sigma_2 \\ \rho_1\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$ in (5.4.3) with $a = 2$, then (5.4.1) becomes

$$f(t,s) = \frac{1}{1 + \frac{s_1^2}{2}t^2 + \frac{s_2^2}{2}s^2 + r s_1 s_2 ts}, s_1, s_2 > 0, 0 < r \le 1$$

is the characteristic function of a bivariate Laplace distribution (for details, see Kuttykrishnan and Jayakumar (2008) and Kotz et al. (2001)). Kuttykrishnan and Jayakumar (2008) studied the properties of bivariate semi α Laplace distribution in (5.4.1) and obtained some characterizations of the distribution.

In this Section, we introduce and study geometric bivariate semi α Laplace dsistribution. A random vector $\underline{X}$ is said to have geometric bivariate semi α Laplace distribution if it has characteristic function

$$f(t,s) = \frac{1}{1 + \ln\left(1 + d(t,s)\right)} \tag{5.4.4}$$

where $d(t,s)$ satisfies the equation (5.4.2). Note that the distribution we discuss here has the form

$$\phi(t,s)=\frac{1}{1+\ln\left(1+|t|^{\alpha_1}\delta_1(t)+|s|^{\alpha_2}\delta_2(t)\right)}$$

where $\delta_1(t)=\delta_1(p^{1/\alpha_1}t)$ and $d_2(s)=d_2(p^{1/a_2}s)$ for every $0<\alpha_1,\alpha_2\le 2$ and for some $p\in(0,1)$. When $\delta_1(t)=\delta_2(s)=1$ and $a_1=2$, the geometric bivariate semi-α Laplace distribution reduces to geometric marginal Laplace and Linnik distribution.

CHAPTER VI

SUMMARY AND CONCLUSION

Laplace distribution found and continues to find applications in a variety of disciplines that range from image and speech recognition (input distributions) and ocean engineering (distributions of navigation errors) to finance (distributions of log returns of a commodity). Laplace distribution was considered for modeling sizes of sand particles and diamonds. Now they are rapidly becoming distributions of first choice whenever 'something' with heavier than Gaussian tails is observed in the data. Laplace distribution has found applications in the fields: Engineering Sciences, Financial Data Analysis, Inventory Management and Quality Control, Astronomy and Biological and Environmental Sciences. Consequently, a large number of papers in diverse journals and monographs mention Laplace laws as the 'right' distribution.

The double exponential distribution was discovered by Pierre Laplace as the distribution form for which the likelihood function is maximized by setting the location parameter equal to the median of the observed values of an odd number of independent and identically distributed random variables. This result appeared in Laplace's fundamental paper on symmetric distribution for describing errors of measurement and is known as the first law of Laplace. Another mode of genesis for this distribution is as the distribution of difference of two independent and identically distributed exponential random variables. Laplace replaced the median by the arithmetic mean as the value maximizing the likelihood function and derived the corresponding distribution to be the normal distribution. This result is called as the Second Law of Laplace.

In studies with probabilistic content, the Laplace distribution serves as a tool for limiting theorems and representations with the emphasis on analyzing its difference from the classical theory based on the foundations of normality. Since the area under the normal and Laplace curve are the same, the peakedness of the Laplace distribution is counterbalanced by a corresponding distribution of frequencies in the tails. Generally, there is an over compensation so that the leptokurtic curve crosses the normal curve four times, first near the peak and then again at the tails and tends towards the x-axis by staying slightly above the normal curve. Empirical studies have shown that the data on financial time series and health sciences are heavy tailed than normal curve. Hence modeling such observations, the normality assumption will be insufficient.

The Laplace distribution is generalized by introducing one more parameter a and is named as a -Laplace (Linnik) distribution. Lin (1994) proved that Linnik distributions are self-decomposable and obtained a number of characterizations of the same (see also, Lin (1998)). George and Pillai (1987) obtained the density of the same in terms of Meijer's G- function. Jacques et al. (1999) proved that the generalized Linik laws belong to Paretian family. Pillai (1985) generalized Linnik distributions to semi-a -Laplace distributions and derived various properties of the same. For the applications of Linnik distributions in various fiels, see Kotz et al. (2001).

As noted in Chapter I, random summation has found applications in different fields such as Insurance Mathematics, Marketing, Reliability etc. For the applications of random sums in Markov Chain analysis, see Milne and Yeo (1989). The negative ageing property of random sum is investigated in Li et al. (2006). They showed that under certain circum stances the negative ageing property of the random sum $\sum_{i=1}^{N} X_i$, is solely determined by the negative property of the random count N, rather than that of X_i. Klebanov et al. (2006) gave a systematic account of applications of random summation, especially geometric summation.

From Chapter II, we have the geometric Linnik distribution is the limit distribution of geometric sum of Linnik random variables. Being the limit distribution random (geometric) sums, the geometric Linnik distribution, type I and type II generalized geometric Linnik distributions can be used for modeling data in diverse fields such as Engineering, Biology, Risk Theory etc.

Recently, a number of time series models with non-Gaussian marginal distribution have been introduced and studied by various authors. The need for such models arises from the fact that many naturally occurring time series are clearly non-Gaussian. Lawrance (1978), Andel (1983) and Dewald and Lewis (1985) developed and studied time series models using Laplace marginal distribution. Anderson and Arnold (1993) discussed the properties of Linnik distributions and developed Linnik processes to model time series data on stock price returns. Jayakumar et al. (1995) generalized Laplace processes of Lawrance (1978) and Dewald and Lewis (1985) and introduced a first order autoregressive a -Laplace process. Mariamma Antony and Raju (2005) introduced time series models with

Type I and Type II generalized geometric Linnik marginals. In Chapter II, autoregressive models with Type I and Type II generalized geometric Linnik marginals developed. The stationarity of the processes are established, along with other properties.

Ayebo and Kozubowski (2003) studied a class of skew continuous distributions on the real line that arises from symmetric exponential power laws by incorporating inverse scale factors into the positive and negative orthants. Skew and symmetric Laplace and normal laws are included in this class as special cases. Since exponential power laws including their special cases of normal and Laplace distribution are symmetric, they are not appropriate for modeling data with asymmetric empirical distributions. However various practical applications require models for unimodal but skew data. Ayebo and Kozubowski (2003) presented skew exponential power models of currency exchange rates. Although there is no agreement regarding the best theoretical model, a general consensus is that currency exchange rates are leptokurtic; their empirical distributions are fat- tailed with sharp peaks at the origin. It is also generally accepted that the currency exchange rates are increasingly leptokurtic with decreasing time intervals while daily changes have fat tails and quarterly changes are nearly normal. Some authors also believe that currency exchange rates are asymmetric.

Kozubowski and Podgorski (2000) studied asymmetric Laplace distributions, which arise as the limits of sums of independent and identically distributed random variables with finite second moment, where the number of terms summed is geometrically distributed, independently of the terms themselves. Ordinary symmetric Laplace laws are a subclass of the Asymmetric Laplace distributions. Their general characteristics include asymmetry, sharp peaks and heavier tails not unlike the properties of stable laws. However, the asymmetric Laplace distributions are much easier to work with in practice than stable or general geometric stable laws, because they have finite moments of all orders, explicit formulas for densities and distribution functions, natural extensions to the multivariate case, and also yield to classical estimation procedures. The concept of finite variance also agrees with the intuition of many financial analysts. Additionally asymmetric Laplace laws arise as limiting distributions of geometric summations, which provide natural models in finance, insurance, reliability and other fields.

Kozubowski and Podgorski (2000) presented an application of asymmetric Laplace distributions in modeling foreign currency exchange rates. Their model regarded an exchange rate change as a sum of a large number of small changes. However the sum is taken up to a random discrete time n_p, having a geometric distribution:

$$\text{Exchange Rate Change} = \sum_{i=1}^{n_p} (\mathit{Small\ Changes}).$$

Therefore the asymmetric Laplace distribution (provided the small changes have finite variance) can approximate the distribution of the exchange rate change. n_p is considered as the moment when the probabilistic structure governing the exchange rates breaks down. This can be due to new information, political or economic or to other events that affect the fundamentals of the exchange market. Kozubowski and Podgorski (2000) considered the currency exchange rates of the German Deutschmark versus the US dollar and the Japanese Yen versus the Us dollar from 1 January 1980 to 7 December 1990 (2853 data points) and they reached the conclusion that the asymmetric Laplace distributions model these data more correctly than normal distribution. Yu and Zhang (2005) introduced a three parameter asymmetric Laplace distribution which is useful for modeling the model errors of quantile regression models and applied the same to modeling a flood data

Julia and Vives-Rego (2005) establish the use of asymmetric Laplace distribution in modelling size distribution of bacteria. For the applications of skew distributions in biology, see also, Barrera et al. (2006). A comprehensive account of various classes of the skew Laplace distributions is given in Kozubowski and Podgorski (2007a). They clubbed the various classes into four types and the links between those types are established. Skew Laplace distributions are being widely applied in modeling of foreign currency exchange data, underreported data, interest rate data, share market return data, option pricing etc.

In Chapter III, a representation of the asymmetric Linnik distribution is obtained. Note that asymmetric Linnik distributions are generalizations of asymmetric Laplace laws described above. Also, Type I generalized geometric asymmetric Linnik distribution is introduced and is shown that this distribution arises as the limit distribution of the geometric sums of generalized asymmetric Linnik random variables. The stability property of type I generalized geometric asymmetric Linnik distribution is examined in Chapter III. Autoregressive models with type I generalized geometric asymmetric Linnik marginals are developed. Various forms of geometric asymmetric Laplace distributions are also introduced in Chapter III. The asymmetric Laplace/ Linnik distributions and the geometric asymmetric Linnik laws are applied in a variety of situations, where Laplace/ geometric Laplace distributions fail to describe the situations. The geometric asymmetric Laplace/ Linnik distribution and its generalizations are more realistic than their symmetric counterparts. There are many contexts such as gene sequence microarray data, bacterial colony size data etc. which follow asymmetric Laplace distributions. Julia and Vives-Rego (2005) report that the asymmetric Laplace distribution is an excellent fit to the sidelight scatter (SS) values in gram-negative bacterial sizes and to all microorganisms also. In the case of almost all financial data sets, geometric asymmetric Laplace/ Linnik distribution can prove as a more appropriate model. Laplace distribution is found useful for modeling data from genetics and molecular biology also. Elizabeth and Susan (2005) apply this distribution for modeling error in gene expression data. Thus the new distributions and processes discussed in Chapter III can be applied for modeling data from a rich variety of contexts.

Tailed distributions are found to be useful in the study of life testing experiments and clinical trials. These distributions can be used data which exhibit zeros, as in the case of stream flow data of rivers that are dry during part of the year. They are useful for modeling life times of devices, which have some probability for damage immediately when it is put to use. Tailed forms of type I and type II generalized geometric Linnik distribution and their asymmetric forms are studied in Chapter IV. By various methods new parameters can be introduced to expand families of distributions for added flexibility or to construct covariate model. Introduction of a scale parameter leads to accelerate life model and taking powers of a survival function introduces a parameter that leads to the proportional hazards model. Marshall and Olkin (1997) suggested a method of adding a parameter to a family of distributions. Jayakumar and Kuttykrishan (2007) applied Marshall-Olkin scheme to characteristic functions and studied the asymmetric Laplace model, so generated. In Chapter IV, Marshall – Olkin forms of type I and type II generalized geometric Linnik distributions are introduced and studied. A representation of tailed type I generalized geometric Linnik distribution is obtained. A first order autoregressive model with tailed type I generalized geometric Linnik distribution is introduced. It is shown that the process is not time reversible. The model is extended to higher order cases. The tailed type II generalized geometric Linnik distribution is also introduced and studied this Chapter. As a generalization of tailed type I and type II generalized geometric Linnik distributions, tailed type I and type II generalized geometric asymmetric Linnik distributions are introduced and studied in this Chapter. Marshall – Olkin scheme is applied to geometric Linnik characteristic function and its generalizations, and the distributions so generated are examined.

Empirical analysis of some bivariate data, especially in the fields of Biology, Mathematical Finance, Communication Theory, Environmental Science etc. shows that bivariate observations are asymmetric and heavy tailed with different tail behavior. Kozubowski *et al*. (2005) considered a bivariate distribution related to Laplace and Linnik distribution, namely marginal Laplace and Linnik distribution, which can be applied for modeling bivariate data with this character. In Chapter V, geometric marginal asymmetric Laplace and asymmetric Linnik distribution is introduced and studied. Note that, the geometric marginal asymmetric Laplace and asymmetric Linnik distribution arise as the limit distribution of geometric sums of asymmetric Laplace and asymmetric Linnik random variables. Time series models with geometric marginal asymmetric Laplace and asymmetric Linnik distributions are introduced. Also in this Chapter we study the properties of geometric marginal asymmetric Linnik - asymmetric Linnik distribution. A bivariate time series model with this marginal distribution is developed and studied. Geometric bivariate semi-a-Laplace distribution is also introduced and studied in this Chapter.

In short, in this Thesis, we have introduced some new classes of distributions that are useful for modeling skewed/symmetric data having heavy tailed nature. The properties of the members of these classes are obtained. Estimation of parameters of the distributions are done in some special cases. The time series models with members of this class as marginals are developed and their extensions to higher order cases are discussed. Bivaraiate distributions are also introduced and studied. These distributions can be used for modeling bivariate data sets with different tail indices.

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