Ph.D THESIS **MATHEMATICS**

A STUDY ON THE FORCING NUMBERS OF SOME GRAPHS

Thesis submitted to the

University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

in Mathematics
under the Faculty of Science

by

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APRIL 2021

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CERTIFICATE

I hereby certify that the thesis entitled "**A STUDY ON THE FORCING NUMBERS OF SOME GRAPHS**" submitted by **Premodkumar. K. P.** to St. Joseph's College (Autonomous), Devagiri, Kozhikode-673008 for the award of the Degree of **Doctor of Philosophy** is a bona-fide record of the research work carried out by him under my supervision and guidance. The contents of this thesis, in full or parts, have not been submitted and will not be submitted to any other Institute or University for the award of any degree or diploma.

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This is to certify that the corrections recommended by the adjudicator of the thesis entitled **" A Study on the Forcing Numbers of Some Graphs"** of **Mr. Premodkumar. K. P** have been incorporated in the thesis and that the contents in the thesis and the soft copy are one and the same.

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Yours truly,

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DECLARATION

I hereby declare that the thesis entitled "**A STUDY ON THE FORCING NUMBERS OF SOME GRAPHS**" is based on the original work done by me under the supervision of **Dr. Baby Chacko**, Research Supervisor, P.G Department and Research Centre of Mathematics, St. Joseph's College (Autonomous), Devagiri, Kozhikode-673008 and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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ACKNOWLEDGEMENT

I express my sincere gratitude to Dr. Baby Chacko, Research Supervisor, P.G. Department and Research Centre of Mathematics, St. Joseph's College (Autonomous), Devagiri, Kozhikode for accepting me as his research student and for all the help and guidance.

I acknowledge the help, encouragement and support received from Dr. Charles Dominic, Assistant Professor, CHRIST (Deemed to be University), Bangalore without whom this journey wouldn't have become easy.

Sincere thanks to Dr. Saji Mathew , Associate Professor and Head, and all other faculty members of the P.G.Department and Research Centre of Mathematics, St. Joseph's College (Autonomous), Devagiri, Kozhikode for their kind advices, assistance and for providing me facilities in the department.

It is my privilege to thank all my research friends of the department.

I am indebted to my family for the completion of this research work.

I express sincere gratitude to Dr. Anuradha. N, Assistant Professor, Govt.Brennan College, Thalasseri for the encouragement and support received.

I express my gratitude to University of Calicut and Dr. Preethi Kuttipulackal, Associate Professor and Head, Dr.P.T.Ramachandran, Rtd Professor and Former Head , Department of Mathematics, University of Calicut, for their help and the facilities provided.

Gratitude to Dr.Vineesh .K. P, Assistant Professor, Sree Narayana Guru College, Chelannur, Libeesh Kumar.K. B, Assistant Professor, Govt.College, Perambra, Dr.Mubeen.M, Assistant Professor, Govt. College Kondotty, Dr.Rajesh. K.P, Assistant Professor, Govt. College, Pattambi, Magi.P.M, Assistant Professor, Govt. College, Ckalakkudy for their advices and helps rendered to me.

I would like to express my gratitude to the Management, Principal and non teaching staff of St. Joseph's College (Autonomous), Devagiri for all the help and facilities provided to me. Finally, heartfelt thanks to the External Power, in which I believe, for making the circumstances favorable to me in my onward journey till date.

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Contents

Chapter 1

Introduction

Graph theory, an important and interesting branch of discrete mathematics, is believed to have originated in 1735 with the introduction of a solution to the famous Konigsberg bridge problem by Leonard Euler. Due to its diverse applications in almost all fields of real life, research has been flourishing in this area since it's inception.

Graph theory has numerous applications in day to day life. Applications of graph theory are very much useful in the field of artificial intelligence, biology, chemistry, quantum physics, computer science, operation research, economics, sociology etc.

Internet, the word which is familiar to every one and without which everyone's life become strenuous and worthless, is really a virtual graph. In internet, an individual page can be regarded as the vertex and a hyperlink between two pages can be treated as an edge.

In this thesis, our main goal is to study the forcing numbers of some graphs and splitting graphs. The forcing numbers which we consider in this thesis comprises of zero forcing number, k-forcing number, connected k-forcing number and 2-distance forcing number. Throughout the thesis, we consider only simple, connected and finite graphs.

Zero forcing number. The zero forcing number of a graph G, denoted by $Z(G)$, was found by the AIM Minimum Rank Special Graphs Group [19]. They used this graph parameter $Z(G)$ to bound the minimum rank for numerous families of graphs. The zero forcing number has wide range of applications in coding theory, logic circuits, in modelling the spread of diseases, power network monitoring, information in social networks etc.

In 2012, Minerva Catral et al. studied the relationship between the maximum nullity, the zero forcing number and the path cover number of a graph G [7] and obtained some useful results. They also included the relationship between the zero forcing number and the maximum nullity of edge subdivision graphs in their discussion. In 2015, Amos et al. [1] gave an upper bound of the zero forcing number of a connected graph G in terms order n and $\Delta \geq 2$, where Δ is the maximum degree of the graph G. In 2016, Michael Gentner et al. [14] verified this conjecture and proved that the result is true only for the graphs C_n, K_n and $K_{\Delta,\Delta}$. A lower bound for triangle free graphs were also provided by them. More useful studies regarding the zero forcing number were also carried out by Darren D. Row in the paper “A technique for computing the zero forcing number of a graph with a cut-vertex” [29]. In this paper, a technique for computing the zero forcing number of graph with a cut-vertex was established. Darren D. Row also focused his attention in finding graphs having very low or very high zero forcing numbers.

k**-forcing number**. The k-forcing number of a graph G, denoted by $Z_k(G)$, was introduced by D Amos, Y Caro, R Davila, and R Pepper in 2015 [1]. When $k = 1$, then this parameter is equivalent to the zero forcing number. Therefore, k-forcing number is a generalization of the zero forcing number. Thus, the introduction of the k-forcing number of a graph raised several questions which led to a more fruitful study of the topic in which new bounds had been studied and analysed. As a result, upper bounds on $Z_k(G)$ in terms of order (n), maximum degree (Δ), minimum degree (δ) and a positive integer $k \geq 1$ were provided. Apart from this, discussion on improved upper bounds on $Z_k(G)$ of connected graphs and upper bounds on $Z_k(G)$ of some special families of graphs such as Hamiltonian graph, cycle-tree graph, tree graph were given. A serious effort was also made by the authors to establish a relationship between the k-forcing number and the connected k-domination number of a graph G. As a conclusion of their work, they found that the sum of the zero forcing number and the connected domination number is at most the order for connected graphs.

Connected k**-forcing number**. Without knowing much about the structure of a forcing set, it is difficult to study $Z(G)$ of a graph G. This fact had actually motivated the study of the connected zero forcing set of a graph G, denoted by Z_c and the connected k-forcing set of G, denoted by Z_{ck}. In 2016, Boris Brimkov and Randy Davila [4] imposed a condition on the zero forcing set that the initially colored set of black vertices should form a connected induced subgraph. This restriction on the zero forcing set had led to the introduction of the concept of connected zero forcing set and thus the connected zero forcing number $Z_c(G)$.

Connected k-forcing number $Z_{ck}(G)$ of several families of graphs including trees, flower snarks, hyper cubes and graphs with a single maximal clique of size greater than two were studied and analysed. As a result of their work, the authors had arrived at the conclusion that the connected zero forcing number is a sharp upper bound to the maximum nullity, the path cover number and the leaf number of a graph. Further studies were carried out and in 2016, Randy Davila, Michael A. Henning, Colton Magnant, and Ryan Pepper [9] provided sharp lower and upper bounds on the connected forcing number of a graph based on the maximum degree (Δ), minimum degree (δ), girth and order (n) of a graph.

2-distance forcing number. In addition to the graph parameters discussed above, we introduced a new parameter namely 2-distance forcing number of a graph G. This forcing number is a generalization of the zero forcing number based on the distance in graphs. To define the 2-distance forcing number of a graph G, we have to familiarise with the following.

2-distance vertex. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If a vertex v of G lies at a distance at most two from the vertex u of G, then we say that v is a 2-distance vertex (or 2-distance neighbor) of u.

Color change rule. Let G be a graph with each vertex colored either black or white. If a black colored vertex has exactly one white colored 2-distance neighbor, then change the color of that white vertex to black. When the color change rule is applied to an arbitrary vertex v to change the color of the vertex u to black, then we say that the vertex v forces the vertex u to black and we denote it as $v \to u$ to black.

A **2-distance forcing set** of a graph G is a subset Z_{2d} of vertices of G such that if initially the vertices in Z_{2d} are colored black and the remaining vertices are colored white, the derived coloring of G is all black.

The 2-**distance forcing number** $Z_{2d}(G)$ is the minimum of $\mid Z_{2d} \mid$ over all 2-distance forcing sets $Z_{2d} \subseteq V(G)$.

In addition to the introductory chapter, the thesis is divided into seven other chapters and chapters into sections.

Chapter One is the introduction.

Chapter Two provides a brief description of the basic definitions of graph theory which will be very useful in the upcoming chapters.

In **Chapter Three**, we compute the zero forcing number of certain types of graphs. Also, this chapter investigates the zero forcing number of the splitting graph of a path P_n, cycle C_n, the star graph $K_{1,n}$ on $n+1$ vertices, the ladder graph, the wheel graph, the CP graph, the friendship graph etc. The chapter begins with some basic definitions, elementary results, facts and terminology which are also used in the subsequent chapters. This chapter is divided into **Five Sections**. In **Section 1**, we deal with some basic results regarding path and cycle. In **Section 2**, we provide some bounds on the zero forcing number of the splitting graph. An attempt is also made to incorporate the zero forcing number with its dominating number. **Section 3** discusses the families of graphs for which the equality $Z[S(G)] = 2Z(G)$ holds. We show that the result is true for the path P_n, the cycle C_n, the star graph $K_{1,n}$ on $n+1$ vertices and the ladder graph. The zero forcing number of family of CP graphs and their splitting graphs are also provided in this section. **Section 4** starts with an attempt to prove the inequality $Z[S(G)] \leq 2Z(G)$ for a connected graph G. We consider the CP graph C_3P_r first. This section also provides the zero forcing number of the friendship graph and its splitting graph. We also introduce a new family of graph, namely the ZP graph, in the last **Section**. **ZP** graph. A graph G

is said to be ZP if $Z(G) = P(G)$, where $P(G)$ is the path cover number of G. The relation between the path cover number and the zero forcing number of the splitting graph of path and star graph is also studied in this **Section.**

In **Chapter Four**, we focus mainly on determining the k-forcing number $Z_k(G)$ of some classes of graphs, especially the 2-forcing number $Z_2(G)$. This chapter contains **Four Sections.** Necessary definitions are provided in **Section 1** for the further development of this chapter. In **Section 2** of this chapter, we determine the 2-forcing number of graphs such as the Corona product $C_n \odot K_1$, the square graph of a path P_n, where $n \geq 3$, the cycle C_n, the graph $P_n \square P_m$, $(n, m \geq 2)$, the prism graph (or circular ladder graph), the wheel graph and the square graph of a cycle. Besides these findings, we also obtain the 2-forcing number of the splitting graph of path P_n, $n \geq 3$, cycle C_n, where $n \geq 4$, ladder graph, wheel graph and the prism graph $C_n \square K_2$. Again, an effort is taken to compute the k-forcing number of path and cycle. In the next **Section 3**, namely "Graphs for which $Z_2(G) \geq 4$", we consider the star graph and the generalized friendship graph F_p^k. **Section 4**, which is the concluding section of this chapter, describes the bounds on the 2-forcing number of a graph G. A relationship between $Z(G)$, $Z_2(G)$ and $Z[S(G)]$ is also given in this **Section**.

It is worth mentioning that **Chapter Five** is the continuation of **Chapter Four**. Throughout the **Chapter Five**, we address the problem of determining the connected k-forcing number of some graphs and splitting graphs. The concept of connected k-forcing number can be regarded as a generalization of the connected zero forcing number. In **Section 1**, we compute the connected zero forcing number of the splitting graph of path P_n, splitting graph of cycle C_n, friendship graph F_p, complete bipartite graph $K_{m,n}$, the Cartesian product

$C_n \Box P_m$ and the star graph $K_{1,n}$. We also compute the connected k-forcing number of the splitting graph of path and cycle in this section. Moreover, we provide an upper bound for connected zero forcing number of the corona product of two graphs, say G and H. **Section 2** describes the connected k-forcing number of rooted product of some graphs. In this section, we consider the rooted product of cycle and path, cycle and cycle, ladder graph and path, the grid graph and path, ladder graph and cycle, circular ladder graph and path, circular ladder graph and cycle, and path and path for determining this graph parameter. In addition to this, we also investigate the connected k-forcing number of the CP -graph C_3P_r and the circular ladder graph. In **Section 3**, we deal with the connected k-forcing number of square graph of path and cycle.

A new graph parameter, namely 2-distance forcing number $Z_{2d}(G)$, is introduced in **Chapter Six**. **Section 1** provides some preliminary definitions necessary for the further development of this chapter. In **Section 2**, we determine the exact value of this parameter for path, cycle, wheel graph, friendship graph, star graph, ladder graph and the complete bipartite graph. **Section 3** provides the 2-distance forcing number of graphs with diameter lying between 2 and 5. In this section, we compute the 2-distance forcing number of gear graph, jelly fish graph, helm graph and the sunflower graph.

Chapter Seven investigates the 2-distance forcing number of some special graphs with large diameter. In **Section 1**, we compute the 2-distance forcing number of shadow graph of path, middle graph of path, S^{th} Necklace graph, triangular snake graph, n-sunlet graph, comet graph, n- pan graph and the generalized friendship graph F_p^k. **Section 2** deals with the 2-distance forcing number of rooted product of graphs. Rooted product of path and path, path and cycle, cycle and path, and cycle and cycle are considered for discussion. In **Section 3**, we find the 2-distance forcing number of square graph of path and cycle. The 2-distance forcing number of the splitting graph of path P_n, where $n \geq 4$, is computed in **Section 4**. **Section 5** computes the 2-distance forcing number of Cartesian product of graphs such as the ladder graph $P_n \square P_2$, the grid graph $P_n \square P_m$ and the circular ladder graph $C_n \square K_2$. The final **Section 6** provides the 2-distance forcing number of complement of path and cycle.

Chapter Eight provides conclusion and further scope of research.

Chapter 2

Preliminaries

This chapter provides a brief account of the preliminary definitions from graph theory which are very useful for the upcoming chapters. For the notations and terminologies not defined directly in the thesis, we may refer to [12] and [16].

2.1 Basic Definitions

Definition 2.1.1. [3] A graph G is an ordered triple $G = (V(G), E(G), \psi(G))$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$ of edges disjoint from $V(G)$ and an incidence function $\psi(G)$ which associates with each element of $E(G)$, an unordered pair of vertices (not necessarily distinct) of G.

Definition 2.1.2. [3] A diagram of a graph represents the incidence relation between two edges and vertices.

Definition 2.1.3. [3] If two vertices are incident with a common edge, then that vertices are adjacent. Otherwise, they are non adjacent. If a vertex u is adjacent to a vertex v in a graph G, we denote it as $u \sim v$. If two edges are incident with a common vertex, then that edges are adjacent. Otherwise, non adjacent.

Definition 2.1.4. [3] An edge with identical end points is called a loop. An edge with distinct end points is known as a link. Edges joining the same pair of vertices are called multiple edges.

Definition 2.1.5. [3] A graph G is finite if both the vertex set $V(G)$ and the edge set $E(G)$ are finite. Otherwise, the graph G is said to be infinite.

Definition 2.1.6. [3] The number of vertices in a graph G is known as the order of the graph and it is denoted by $O(G)$.

Definition 2.1.7. [3] A graph which has no loops and multiple edges is called a simple graph. A graph is trivial if it has only one point.

Definition 2.1.8. [3] Two graphs G and H are identical if $V(G) = V(H), E(G) = E(H)$ and $\psi(G) = \psi(H)$.

Definition 2.1.9. [3] Two graphs G and H are said to be isomorphic if there exists bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$. If G and H are isomorphic, we write $G \approxeq H$. The pair (θ, ϕ) is called an isomorphism between the graphs G and H.

Definition 2.1.10. [3] A graph G is complete if every pair of distinct vertices is joined by an edge and G is simple. A complete graph on n vertices is denoted by K_n.

Definition 2.1.11. [3] An empty graph is a graph which has no edges.

Definition 2.1.12. [3] A graph G is called bipartite if the vertex set $V(G)$ can be partitioned into two subsets X and Y such that each edge of G has one end in X and the other end in Y. (X, Y) is called a partition of G.

Definition 2.1.13. [3] A graph G is said to be complete bipartite if G is simple, bipartite with bipartition (X, Y) and each vertex of X is joined to every vertex of Y. If $\mid X \mid= m, \mid Y \mid= n$, then G is denoted by $K_{m,n}$.

Definition 2.1.14. [3] A graph G is m-partite if the vertex set can be partitioned into m subsets $X_1, X_2, \ldots, X_m$ such that no edge of G has both ends in any one subset. $(X_1, X_2, \ldots, X_m)$ is called m-partition of G. Complete m-partite graph is a simple graph and a m-partite graph such that each vertex of each subset is joined to every vertex of other subsets.

Definition 2.1.15. [3] The complement of a simple graph G, denoted by G^c, is a simple graph with vertex set $V(G)$ and such that two vertices are adjacent in G^c if and only if they are non adjacent in G.

2.2 Subgraphs

Definition 2.2.1. [3] A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$.
If H is a subgraph of G, then it is denoted by $H \subseteq G$. G is the super graph of H. If H is a subgraph of G and $H \neq G$, then H is a proper subgraph of G $(H \subset G)$. H is a spanning subgraph of G if H is a subgraph and $V(H) = V(G)$. Let G be a graph. By deleting all loops and for every pair of adjacent vertices all except one link joining them, we obtain a simple spanning subgraph of G called

the underlying simple graph of G.

Definition 2.2.2. [3] Let V^1 be a non-empty subset of the vertex set V of G. The subgraph of G whose vertex set is V^1 and whose edge set is the set of those edges of G that have both ends in V^1 is called the subgraph of G induced by V^1 and is denoted by $G[V^1]$. We say $G[V^1]$ is the induced subgraph of G.

Definition 2.2.3. [3] Let G be a graph and v be it's vertex. Then, the degree of v is the number of edges of G incident with v, counting each loop as two edges. The degree of v is denoted as $deg(v)$. The vertex with zero degree is called an isolated vertex. A vertex with degree one is called an end vertex or pendant vertex.

We denote by $\delta(G)$ and $\Delta(G)$, the minimum degree and the maximum degree of vertices in G respectively.

Definition 2.2.4. [3] A graph G is k-regular if $deg(v) = k$ for all $v \in V$. A regular graph is a graph which is k-regular for some $k \geq 0$.

If G has vertices $v_1, v_2, \ldots, v_n$, then the sequence $deg(v_1), deg(v_2), \ldots, deg(v_n)$ is called a degree sequence of G.

2.3 Path and Connection

Definition 2.3.1. [3] A walk in a graph is a finite non-null or non-empty sequence $w = v_0e_1v_1e_2v_2, \ldots, e_kv_k$ whose terms are alternatively vertices and edges so that for $1 \leq i \leq k$, the ends of e_i are v_{i-1} and v_i. We say that w is a walk from v_0 to v_k or w is a (v_0, v_k) walk. v_0 is called the origin and v_k is called the terminus of w. $v_1, v_2, \ldots, v_{k-1}$ are called the internal vertices. The integer k is called the length of w.

Definition 2.3.2. [3] A walk in which every edge is distinct is called a trial.

Definition 2.3.3. [3] A walk in which every vertex is distinct (hence edges are distinct) is called a path.

Definition 2.3.4. [3] Two vertices u and v of G are said to be connected if there is a (u, v) path between them. Connectedness is an equivalence relation on the vertex set V. Thus, there is a partition $v_1, v_2, \ldots, v_w$ of $V(G)$. The induced subgraphs $G[v_1], G[v_2], \ldots, G[v_w]$ are called the components of G. The graph G is connected if G has exactly one component. Otherwise, G is disconnected. $w(G)$ is the number of components of G.

Definition 2.3.5. [3] The distance between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path connecting them.

Definition 2.3.6. [3] Diameter of a graph G is the maximum distance between the vertices of G.

Definition 2.3.7. [3] A cycle is a closed trial whose origin and internal vertices are different. A cycle of length k is called a k-cycle and it is denoted by C_k.

Definition 2.3.8. [3] A graph is called acyclic if it has no cycles.

Definition 2.3.9. [3] A tree is a acyclic graph which is connected.

Definition 2.3.10. [3] Forest is a acyclic graph.

2.4 Some Operations on Graphs

Definition 2.4.1. [12] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Then, the graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, is called the union of graphs G_1 and G_2 and is denoted by $G_1 \cup G_2$. If $V_1 \cap V_2 = \phi$, then $G_1 \cup G_2$ is usually denoted by $G_1 + G_2$, called the sum of the graphs G_1 and G_2.

Definition 2.4.2. [12] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 \neq \phi$. Then the graph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$, is called the intersection of graphs G_1 and G_2 and is denoted by $G_1 \cap G_2$.

Definition 2.4.3. [12] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 = \phi$. Then the join, $G_1 \vee G_2$, of G_1 and G_2 is the super graph of $G_1 + G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2.

Definition 2.4.4. [12] The Cartesian product of two simple graphs G_1 and G_2, commonly denoted by $G_1 \Box G_2$ or $G_1 \times G_2$, has the vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \Box G_2$ are adjacent if either $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$ or $v_1 = v_2$ and $u_1 u_2 \in E(G_1)$.

Definition 2.4.5. [12] The Composition or lexicographic product of two simple graphs G_1 and G_2, commonly denoted by $G_1[G_2]$ has the vertex set $V(G_1) \times V(G_2)$ and two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1[G_2]$ are adjacent if either u_1 is adjacent to u_2 or $u_1 = u_2$, and v_1 is adjacent to v_2.

Definition 2.4.6. [12] The Corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 by an edge to every vertex in the i^{th} copy of G_2.

Definition 2.4.7. [15] Rooted Product. Let G be a connected graph with vertices $v_1, v_2, \ldots, v_n$ and let H be a sequence of n rooted graphs $H_1, H_2, \ldots, H_n$. The rooted product of G and H is defined as the graph obtained by identifying the root of H_i, $1 \leq i \leq n$, with the i^{th} vertex of G for all i. This graph is denoted by $G(H)$ and is known as the rooted product of G by H.

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Chapter 3

On the Zero Forcing Number of Graphs and Their Splitting Graphs

In this Chapter, we address the problem of determining the zero forcing number of graphs and splitting graphs. First Section of this Chapter contains basic definitions and preliminary results. In Section 2, we give upper bounds on the zero forcing number of the splitting graph of a graph. In Section 3, we find several classes of graphs in which $Z[S(G)] = 2Z(G)$. Section 4 provides classes of graphs in which $Z[S(G)] < 2Z(G)$. In section 5, we provide more families of graphs with $Z(G) = P(G)$.

3.1 Introduction

Definition 3.1.1. [32] *The splitting graph of a graph G, denoted by $S(G)$, is the graph obtained by taking a vertex v' corresponding to each vertex $v \in G$ and join*

[1]This chapter has been published in the Journal *Algebra and Discrete Mathematics*, Volume 28, Number 1, 2019, Pages 29-43.

Figure 3.1: The graph K_3 and its splitting graph $S(K_3)$.

v' to all the vertices of G adjacent to v. If G contains p vertices and q edges, then $S(G)$ contains $2p$ vertices and $3q$ edges. For example, See figure 3.1.

For the further development of this chapter, it is necessary to define the following.

Definition 3.1.2. [19] (i) **Color change rule** : *If G is a graph with each vertex colored either black or white, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.*
(ii) *Given a coloring of G, the **derived coloring** is the result of applying the color change rule until no more changes are possible.*
(iii) *A **zero forcing set** for a graph G is a subset of vertices Z of G such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.*

The zero forcing number $Z(G)$ of a graph G can be defined as follows:

Definition 3.1.3. [19] *The zero forcing number $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.*

When the color change rule is applied to a vertex u to change the color of a vertex v, we say that u forces v and we write this as $u \to v$. The sequence $u_1 \to v_1$, $u_2 \to v_2$, ..., $u_k \to v_k$ is called a *forcing sequence* for the zero forcing set Z (See[29]).

Example 3.1.4. In Figure 3.2, the vertex set $\{1, 2\}$ generates a minimum zero forcing set for the house graph G. The order of applying the color change rule to the vertices of the house graph G are $1 \to 3$ and $2 \to 4, 3 \to 5$.

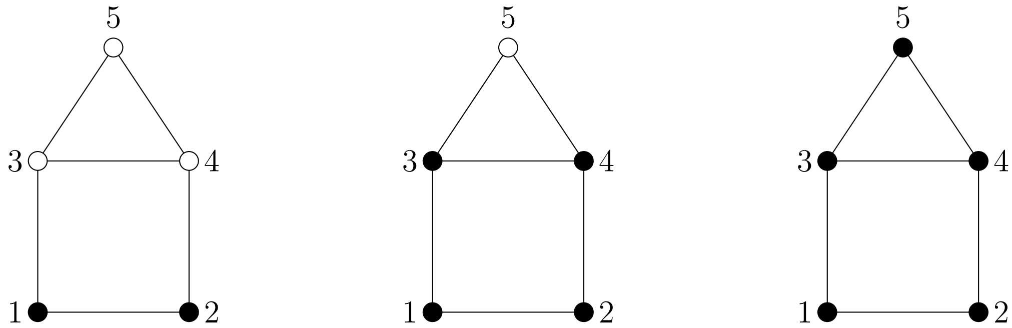

Figure 3.2: A minimum zero forcing set for the house graph marked by the black vertices $\{1, 2\}$ can been seen in the first figure. The forcing sequence $1 \to 3$ and $2 \to 4$ is shown in the second figure and the derived coloring of the house graph after applying the forcing $3 \to 5$ is depicted in the third figure.

The parameter namely zero forcing number was found by the AIM Minimum Rank Special Graphs Group (See[19]) and they used this parameter $Z(G)$ to bound the minimum rank for numerous families of graphs.

The zero forcing number of the splitting graph of a graph and some bounds besides finding the exact value of this parameter are discussed in this chapter. We prove that for any connected graph G of order $n \geq 2$, $Z[S(G)] \leq 2Z(G)$.

Also, many classes of graphs for which $Z[S(G)] = 2Z(G)$ are discussed. Further, classes of graphs in which $Z[S(G)] < 2Z(G)$ are also shown.

We start with some preliminary results. For more definitions on graphs, we refer to [12] and [16]. We can find the following observation in [9].

Observation 3.1.5. [9] *For any connected graph* $G = (V, E)$, $Z(G) = 1$ *if and only if* $G = P_n$ *for some* $n \geq 2$.

We note that if G is a connected graph with order $n \geq 3$, then $S(G)$ contains a cycle C_4. Therefore, by using the above observation we can determine the following.

Theorem 3.1.6. *Let* G *be a connected graph of order* $n \geq 3$. *Then,* $Z[S(G)] \geq 2$, *and this bound is sharp for the path* P_n, $n \geq 3$.

Proof. Since G is a connected graph with order $n \geq 3$, $S(G)$ contains a cycle C_4. Therefore, $Z[S(G)] \geq 2$ because $Z(C_n) = 2$ (See[19]). Next to prove the bound is sharp for the path P_n, $n \geq 3$.
Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n and $u'_1, u'_2, \ldots, u'_n$ be the corresponding vertices in $S(P_n)$. Assign black color to the vertices u_1 and u'_1. The remaining vertices are assumed to be white. Then, $u'_1 \rightarrow u_2$ and $u_1 \rightarrow u'_2$ to black. Again, $u'_2 \rightarrow u_3$ and $u_2 \rightarrow u'_3$ to black. After this stage, $u'_3 \rightarrow u_4$ and $u_3 \rightarrow u'_4$ to black and so on. Hence the set $Z = \{u_1, u'_1\}$ forms a zero forcing set for $S(P_n)$. The cardinality of the set Z is two. We can easily see that with only one black vertex, we cannot generate a zero forcing set for $S(P_n)$. Therefore, $Z[S(P_n)] = 2$. □

Theorem 3.1.7. *For any connected graph $G = (V, E)$, $Z[S(G)] = 1$ if and only if G is the path P_2.*

Proof. If $G = (V, E)$ is the path P_2, then $S(G)$ is the path P_4 and therefore $Z[S(G)] = 1$. The converse follows from Observation 3.1.5. □

3.2 Bounds on $Z[S(G)]$

In this section, we prove some bounds on the zero forcing number of $S(G)$.

Theorem 3.2.1. *Let G be a connected graph of order $n \geq 3$. Then, $Z[S(G)] \leq 2Z(G)$.*

Proof. Consider any minimum zero forcing set Z for G.
Let $Z = \{v_1, v_2, \ldots, v_k\}$, where $1 \leq k \leq n$, be a minimum zero forcing set for G. Now, consider the set

$$Z^{'} = \{v_1, v_2, \ldots, v_k\} \cup \{v_1', v_2', \ldots, v_k'\} \in V[S(G)]$$

where $v_1', v_2', \ldots, v_k'$ be the copies of the vertices $v_1, v_2, \ldots, v_k$ in $S(G)$. Color all vertices in $Z^{'}$ as black and the remaining vertices as white.

We show that the set $Z^{'}$ forms a zero forcing set for $S(G)$. Consider the vertices in G which has exactly one white neighbor in G. Let the vertices be $v_1, v_2, \ldots, v_l, l \leq k$ and $v_1', v_2', \ldots, v_l'$ be the corresponding vertices of $v_1, v_2, \ldots, v_l$ in $S(G)$. Now, we can see that in $S(G)$, each one of $N(v_1'), N(v_2'), \ldots, N(v_l')$ contains exactly one white vertex. Let it be $u_1, u_2, \ldots, u_l$. Clearly, $v_1' \to u_1$ to black, $v_2' \to u_2$ to black, …, $v_l' \to u_l$ to black. Again, consider the set $\{v_1, v_2, \ldots, v_l\}$ in $S(G)$. At this time, we can see that $v_1 \to u_1'$, $v_2 \to u_2'$, ..., $v_l \to u_l'$ to black. Consider the white vertices which are adjacent to $u_1, u_2, \ldots, u_l$ in G. Let it be $w_1, w_2, \ldots, w_l$. Clearly, $u_1' \to w_1$ to black, $u_1 \to w_1'$ to black and so on, where $w_1', w_2', \ldots, w_l'$ are the corresponding vertices of $w_1, w_2, \ldots, w_l$. Therefore, the set Z' forms a zero forcing set for $S(G)$. Hence the proof. □

Definition 3.2.2. [33] A subset $D \subseteq V(G)$ is called a dominating set if every vertex not in D is adjacent to at least one vertex in D. The size of a minimum dominating set D is called the domination number of G and is denoted by $\gamma(G)$. A dominating set which is connected is known as a connected dominating set and the number of vertices in a minimum connected dominating set is called the connected domination number $\gamma_c(G)$.

In [1], Amos et al. determined the following upper bound on the zero forcing number.

Corollary 3.2.3. [1] *For any connected graph G of order $n \geq 2$, $Z(G) \leq n - \gamma_c(G)$.*

From the Theorem 3.2.1 and the Corollary 3.2.3, we conclude the following upper bound.

Theorem 3.2.4. *For any connected graph G of order $n \geq 2$, $Z[S(G)] \leq 2[n - \gamma_c(G)]$, and this inequality is sharp.*

Proof. Theorem 3.2.1 gives

$$Z[S(G)] \leq 2Z(G) \tag{3.1}$$

whereas Corollary 3.2.3 implies

$$Z(G) \leq n - \gamma_c(G) \tag{3.2}$$

From (3.1) and (3.2), $Z[S(G)] \leq 2[n-\gamma_c(G)]$. To see the bound is sharp, consider cycles of order $n \geq 4$. □

3.3 Families of Graphs with $Z[S(G)] = 2Z(G)$

In this Section, we provide some familiar families of graphs for which the equality $Z[S(G)] = 2Z(G)$ holds. We start with path and cycle.

Theorem 3.3.1. *If G is the path P_n, $n \geq 3$, then $Z[S(G)] = 2 = 2Z(G)$.*

Proof. Proof follows from the Observation 3.1.5 and the Theorem 3.1.6. □

Theorem 3.3.2. *If G is the cycle C_n, where $n \geq 4$, then $Z[S(G)] = 4 = 2Z(G)$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of C_n and $v_1', v_2', ..., v_n'$ be the corresponding vertices of $v_1, v_2, ..., v_n$ in $S(C_n)$. Consider the set $Z = \{v_1, v_2, v_2', v_3'\}$. Color these vertices as black and the remaining vertices as white. Now, $v_2' \to v_3$ to black, $v_3' \to v_4$ to black, $v_3 \to v_4'$ to black and so on. Therefore, the set Z forms a zero forcing set for $S(G)$ and hence $Z[S(G)] \leq 4$. We can easily verify that with 3 black vertices, it is not possible to change the color of all other vertices of $S(G)$ to black. Therefore, $Z[S(G)] \geq 4$. Hence, $Z[S(G)] = 4 = 2Z(G)$, since $Z(G) = 2$. □

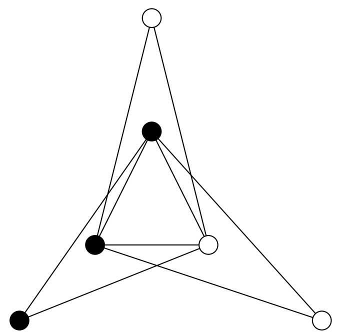

Figure 3.3: The splitting graph of C_3 with $Z[S(C_3)] = 3$.

If G is the graph C_3, then we can choose the above 3 black vertices for the zero forcing set of $S(G)$ as depicted in Figure 3.3. Hence, $Z[S(C_3)] = 3$.

Definition 3.3.3. [12] *The star graph $K_{1,n}$ is a tree having $n+1$ vertices with one vertex having degree n and all other vertices have degree one.*

Theorem 3.3.4. *If G is the star graph $K_{1,n}$ on $n+1$ vertices, then $Z[S(G)] = 2n-2 = 2(n-1) = 2Z(G)$.*

Proof. Assume that we have a zero forcing set Z consisting of $2n-3$ black vertices. Then, the number of white vertices in $S(G)$ is $2n+2-(2n-3) = 5$. Consider the five white vertices in $S(G)$. Consider the case when either two of them will be in A part or two of them will be in B part (See figure 3.4). We can easily verify that in this case the color change rule is not possible, a contradiction to our assumption. Therefore, we need at least $2n-2$ black vertices in any zero forcing set for $S(G)$ and hence

$$Z[S(G)] \geq 2n-2 \tag{3.3}$$

Conversely, consider the 4 white vertices as depicted in Figure 3.4. Consider one black vertex from A part. This black vertex forces the vertex u to black.

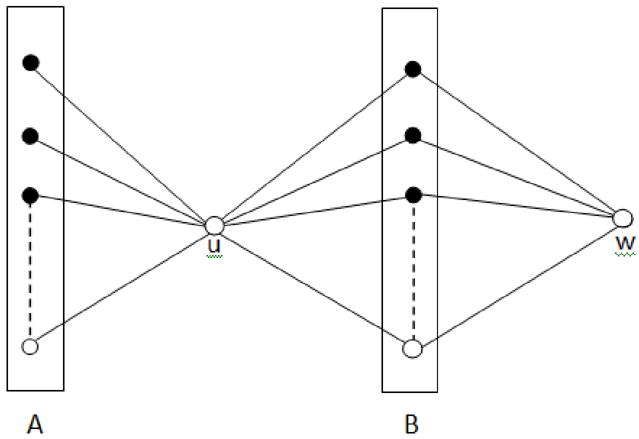

Figure 3.4

Now, consider one black vertex from B part. This black vertex forces the vertex w to black. Consider the vertex w. Then, the vertex w has exactly one white neighbor. Clearly, the vertex w forces this white vertex to black. In a similar manner, we can change the color of the white vertex in the A part to black. Thus, with $2n-2$ black vertices, we will get a derived coloring of $S(G)$. This implies,

$$Z[S(G)] \leq 2n-2 \tag{3.4}$$

From (3.3) and (3.4), we have $Z[S(G)] = 2n-2 = 2(n-1) = 2Z(G)$, since $Z(G) = n-1$ $(See[29])$. □

Theorem 3.3.5. [19] *Let G be any graph. Then, $Z(G) \geq \delta(G)$, where $\delta(G)$ denotes the minimum degree of the graph G.*

Theorem 3.3.6. *Let G be a connected graph with $Z(G) = k = \delta(G)$ and let $\widehat{G}$ be the graph obtained from G by adding a single vertex v and joining the vertex v to all vertices of G. Then, $Z(\widehat{G}) = Z(G) + 1$.*

Proof. Since G is a graph with $Z(G) = \delta(G)$ and we have from the Theorem 3.3.5, $\delta(\widehat{G}) \leq Z(\widehat{G})$. Let u be a vertex in G with $\delta(G) = k$. In $\widehat{G}$, $\delta(\widehat{G}) = k+1 = \delta(G) + 1$. Therefore, $\delta(G) + 1 \leq Z(\widehat{G})$. This implies,

$$Z(G) + 1 \leq Z(\widehat{G}) \tag{3.5}$$

Color the vertex v, which is connected to all vertices of G, as black. Then, $Z(G) \cup \{v\}$ forms a zero forcing set for $Z(\widehat{G})$. This implies,

$$Z(\widehat{G}) \leq Z(G) + 1 \tag{3.6}$$

From (3.5) and (3.6), the result follows. □

For a cycle C_n, $Z(C_n) = 2$. By using the Theorem 3.3.6, we can easily verify that if G is a wheel graph, then $Z(G) = 3$.

Theorem 3.3.7. *Let G be the wheel graph with $n \geq 5$ vertices obtained by connecting a single vertex to all vertices of the cycle C_{n-1}. Then, $Z[S(G)] = 6$.*

Proof. From the above observation $Z(G) = 3$ and by the Theorem 3.2.1 , we can conclude that

$$Z[S(G)] \leq 6 \tag{3.7}$$

To prove the reverse part, assume that $Z[S(G)] = 5$. Divide the graph $S(G)$ into three parts as shown in Figure 3.5.

Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices in $S(G)$ with $deg(v_i) = 6, 1 \leq i \leq n-1$, $v'_1, v'_2, \ldots, v'_{n-1}$ be the vertices in $S(G)$ with $deg(v'_i) = 3, 1 \leq i \leq n-1$ and let v_n be the vertex which is adjacent to $\{v_1, v_2, \ldots, v_{n-1}\} \cup \{v'_1, v'_2, \ldots, v'_{n-1}\}$ and v'_n be the vertex which is adjacent to $\{v_1, v_2, \ldots, v_{n-1}\}$ with $deg(v_n) = 2n-2$ and

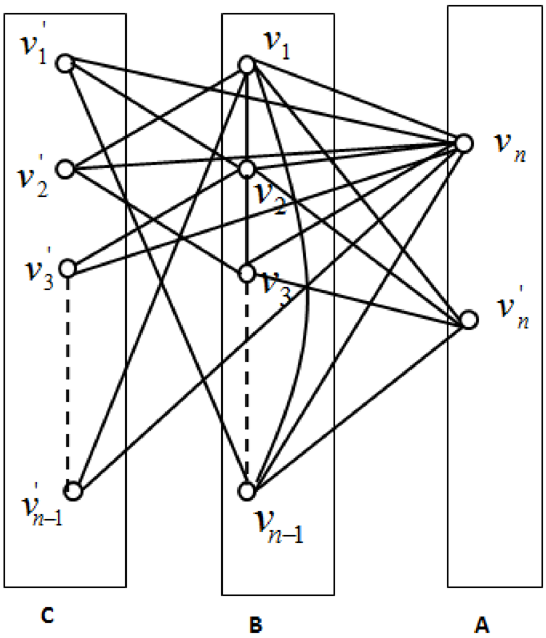

Figure 3.5

$deg(v_n') = n - 1$.

Case 1.

$\{v_n, v_n'\} \in Z$. Now, $S(G) - \{v_n, v_n'\} = S(C_n)$. We know that from the theorem 3.3.2, $Z[S(C_n)] = 4$. This implies that $Z[S(G)] = 4 + 2 = 6 \neq 5$, which is a contradiction to our assumption that $Z[S(G)] = 5$.

Case 2.

Suppose $v_n \in Z$ and $v_n' \notin Z$. Now, we have four black vertices remain in Z. If we use these four black vertices to begin the color change rule, then we can observe that with these 4 black vertices we can change the color of at most two vertices to black, not all. A contradiction to our assumption.

Case 3.

Suppose $v_n \notin Z$ and $v_n' \in Z$. Then, we have four black vertices remain in Z. If we use these four black vertices to begin the color change rule, we can observe that with these 4 vertices we can change the color of at most two vertices to black, not all. Again, a contradiction to our assumption that $Z[S(G)] = 5$. Hence,

$$Z[S(G)] \geq 6 \tag{3.8}$$

From (3.7) and (3.8), we have $Z[S(G)] = 6$. □

We prove one more family of graphs in which $Z[S(G)] = 2Z(G)$. The following definition can be found in [1].

Definition 3.3.8. [1] *A connected graph $G = (V, E)$ is defined as a cycle-path graph ($CP-graph$) if it contains r vertex disjoint cycles that are connected by $(r-1)$ edges of the path P_r. Thus, a CP-graph with n vertices contains $m = n + r - 1$ edges and each edge between two cycles is a cut edge.*

Example 3.3.9. *Let G be the graph depicted in Figure 3.6. Then, G represents the CP-graph with the cycle C_4 and the path P_3. That is, the graph G is the C_4P_3-graph.*

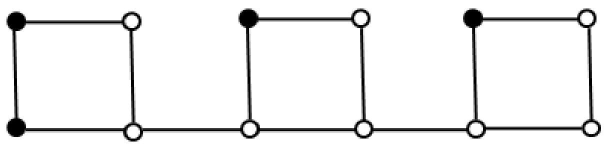

Figure 3.6

Theorem 3.3.10. *Let G be the CP-graph with r vertex disjoint cycles. Then, $Z(G) = r + 1$.*

Proof. We proceed by induction on the number of cycles r in G. Assume that $r = 1$. In this case, G is a cycle. Hence, $Z(G) = 2 = r + 1$. Therefore, the result is true for $r = 1$. Assume that the theorem is true for all CP-graphs with $r - 1$ cycles, where $r > 2$. Let C be an end-cycle, that is, a cycle connected to rest of the graph by an unique edge $e = \{u, v\}$, where $u \in V(G) - C$ and $v \in C$. The induced subgraph $< G[V - C] >$ is a CP-graph with $r - 1 < r$ cycles. Therefore by our assumption, $Z(< G[V - C] >) = r - 1 + 1 = r$. Let S be a minimum zero forcing set for $< G[V - C] >$ and let w be the black a neighbor of v on C. Consider the set $Z = S \cup \{w\}$. Since $\{u, v\}$ is the only cut edge between $< G[V - C] >$ and C, therefore, we can start the color change rule for the vertices of $< G[V - C] >$ with the vertices of the set S. Now, the vertex u is black. Since u is a black vertex and the only one white vertex which is adjacent to u is v, therefore, $u \to v$ to black. In the cycle C, we can see that the set $\{u, w\}$ forms a zero forcing set for the cycle C, where $u \in (< G[V - C] >)$. Therefore by induction hypothesis, $Z(G) = Z(< G[V - C] >) + |\{w\}| = r + 1$. Hence the proof. □

Theorem 3.3.11. *Let G be the CP-graph with r vertex disjoint cycles C_n, where $n \geq 4$. Then, $Z[S(G)] = 2(r + 1)$.*

Proof. We prove the result by induction on the number of cycles r on the CP-graph. Assume that $r = 1$. In this case, G is a cycle. We have from the Theorem 3.3.2, $Z[S(G)] = 4 = 2(1 + 1)$. Therefore, the result is true for $r = 1$. Assume that the result is true for all CP-graphs with $r - 1 < r$ cycles, where $r > 2$. Let C be an end-cycle, that is, a cycle connected to rest of the CP-graph by an unique edge $e = \{u, v\}$, where $u \in V(G) - C$ and $v \in C$ and let $S(C)$ be the splitting graph of the cycle C in $S(G)$. Now, $S(C)$ is connected to the rest of $S(G)$ by three edges. Let these edges be $X = \{u_1v_1, u_1v_2, u_2v_1\}$, where $\{u_1, u_2\} \in < V[S(G)] - V[S(C)] >$ (ie, the subgraph induced by $V[S(G)] - V[S(C)]$) and $\{v_1, v_2\} \in S(C)$. The induced subgraph $< V[S(G)] - V[S(C)] >$ is a CP graph with $r - 1$ cycles. Therefore by our assumption, $Z\{< V[S(G)] - V[S(C)] >\} = 2[(r - 1) + 1] = 2r$.

Let U be the minimum zero forcing set for $< V[S(G)] - V[S(C)] >$. Also, let w_1 be the neighbor of v_1 in $V[S(C)]$ and $w_1^{'}$ be the corresponding vertex of w_1 in $V[S(C)]$. Let w_1 and $w_1^{'}$ be black. Consider the set $Z = U \cup \{w_1, w_1^{'}\}$. Since X is a cut set between $S(G) - S(C)$ and $S(C)$, therefore, the set U forces the vertices v_1 and v_2 to black. The vertices w_1 and $w_1^{'}$ are in Z. Therefore, the set $\{v_1, v_2, w_1, w_1^{'}\}$ forms a zero forcing set for $S(C)$ in $S(G)$. Therefore by induction hypothesis, $Z[S(G)] = Z\{< V[S(G)] - V[S(C)] >\} + \mid \{w_1, w_1^{'}\} \mid = 2r + 2 = 2(r + 1)$. Hence the proof. □

Definition 3.3.12. [12] *The ladder graph L_n is the graph obtained by taking the Cartesian product of P_n with P_2.*

In [19], it was proved that if G is the ladder graph, then $Z(G) = 2$. Now, we prove one more family of graphs in which $Z[S(G)] = 2Z(G)$.

Theorem 3.3.13. *If G is the splitting graph of the ladder graph, then $Z(G) = 4$.*

Proof. Consider the graph G depicted in Figure 3.7. Clearly, the set of left black vertices of the graph G forms a zero forcing set for G. We can easily verify that with three black vertices, we cannot form a zero forcing set for G. Therefore, $Z(G) = 4$. □

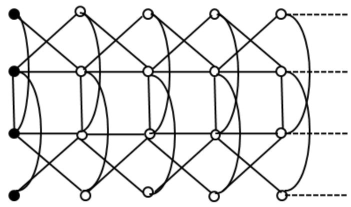

Figure 3.7: The splitting graph G of the ladder graph with $Z(G) = 4$.

3.4 Families of Graphs with $Z[S(G)] < 2Z(G)$

We start this section with a CP-graph family in which $Z[S(G)] < 2Z(G)$. Consider the CP-graph C_3P_r, where C_3 is a cycle on 3 vertices and P_r is the path on $r > 1$ vertices.

Theorem 3.4.1. *Let G be the C_3P_r graph. Then, $Z[S(G)] = r + 2 < 2(r + 1)$.*

Proof. Represent the r copies of the cycle C_3 in G by $C_3^{(1)}, C_3^{(2)}, \ldots, C_3^{(r)}$, where

$$V[C_3^{(1)}] = \{u_1^1, u_2^1, u_3^1\}$$

$$V[C_3^{(2)}] = \{u_1^2, u_2^2, u_3^2\}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$V[C_3^{(r)}] = \{u_1^r, u_2^r, u_3^r\}$$

Let $A_1 = \{v_1^1, v_2^1, v_3^1\}$, $A_2 = \{v_1^2, v_2^2, v_3^2\}$, $\ldots$, $A_r = \{v_1^r, v_2^r, v_3^r\}$ be the corresponding vertex sets of the cycles $C_3^{(1)}, C_3^{(2)}, \ldots, C_3^{(r)}$ in $S(G)$ respectively. Consider the set $Z = \{u_2^1, v_1^1, v_2^1, u_1^2, u_1^3, \ldots, u_1^r\}$. Color all vertices of the set Z as black and the remaining vertices in $S(G)$ as white. Then, clearly the black vertex v_1^1

forces the vertex u_3^1 to black, v_2^1 forces the vertex u_1^1 to black, u_1^1 forces the vertex v_3^1 to black, v_3^1 forces the vertex u_2^2 to black, u_3^1 forces the vertex v_2^2 to black, v_2^2 forces the vertex u_3^2 to black and so on. Thus, we get a derived coloring of the graph $S(G)$ using the set Z. Here $\mid Z \mid= 3 + r - 1 = r + 2$. Therefore,

$$Z[S(G)] \leq r + 2 \tag{3.9}$$

To prove the reverse inequality, consider $S[C_3^{(1)}]$ in $S(G)$. To obtain the derived coloring for $S[C_3^{(1)}]$, we need at least three black vertices. To proceed further, we have to assign black color to at least one vertex of each of the remaining cycles $C_3^{(2)}$,$C_3^{(3)}, \dots, C_3^{(r)}$ in $S(G)$. Hence to get the derived coloring for $S(G)$, we need at least $3 + (r - 1) = r + 2$ black vertices. Therefore,

$$Z[S(G)] \geq r + 2 \tag{3.10}$$

Hence from (3.9) and (3.10), $Z[S(G)] = r + 2$. Therefore, $Z[S(G)] = r + 2 < 2(r + 1) = 2Z(G)$. That is, $Z[S(G)] < 2Z(G)$. □

Definition 3.4.2. [12] *The friendship graph F_n can be obtained by joining n copies of the cycle C_3 with a common vertex.*

Now, we compute the zero forcing number of the friendship graph F_n. The following Lemma can be found in [29].

Lemma 3.4.3. [29] *Let $G = (V, E)$ be a graph with cut-vertex $v \in V(G)$. Let $X_1, \dots, X_k$ be the vertex sets of the connected components of $G - v$ and for $1 \leq i \leq k$, let $G_i = G[X_i \cup \{v\}]$. Then, $Z(G) \geq \sum_{i=1}^{k} Z(G_i) - k + 1$.*

Theorem 3.4.4. *For a friendship graph F_n with n copies of the cycle C_3, $Z(F_n) = \lfloor p/2 \rfloor + 1$, where p is the order of the friendship graph F_n.*

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices of F_n, where v_p is the common vertex. The cycle C_3 is a complete graph of order three. Therefore, $Z(C_3) = 2$. Since v_p is a cut vertex, $G - v_p$ will have $\lfloor p/2 \rfloor$ components. Hence Lemma 3.4.3 gives

$$Z(F_n) \geq 2\lfloor p/2 \rfloor - \lfloor p/2 \rfloor + 1 = \lfloor p/2 \rfloor + 1 \tag{3.11}$$

To establish the reverse inequality, consider the following set of black vertices

$$Z = \{v_1, v_3, \ldots, v_{p-2}\} \cup \{v_p\}.$$

Clearly, the vertices v_1 and v_p are black, therefore, the vertex $v_1 \to v_2$ to black. The vertices v_3 and v_p are black, therefore, the vertex $v_3 \to v_4$ to black and so on. Similarly, the vertices v_{p-2} and v_p are black, therefore, the vertex $v_{p-2} \to v_{p-1}$ to black. Thus, we get a derived coloring of the graph F_n with the set Z. The cardinality of the set Z is $\lfloor p/2 \rfloor + 1$ and hence

$$Z(F_n) \leq \lfloor p/2 \rfloor + 1 \tag{3.12}$$

Therefore from (3.11) and (3.12), $Z(F_n) = \lfloor p/2 \rfloor + 1$. □

Lemma 3.4.5. *Let $S(F_n)$ be the splitting graph of the friendship graph F_n and let*

$$A_l = \{v_i, v_i', v_j, v_j'\}, 1 \leq l \leq \lfloor p/2 \rfloor$$

(v_iv_j is an edge in F_n and $i, j \neq p$) be the set of vertices of $S(F_n)$ obtained by deleting the vertices v_p and v_p' from $S(F_n)$. Then, at least one vertex from the set A_l will be in any optimal zero forcing set of $S(F_n)$.

Proof. On the contrary, assume that none of them belongs to any zero forcing set Z. That is, $v_i, v_i', v_j, v_j' \notin Z$. In any color change rule the vertices v_p and v_p'

will never force the vertices in A_l to black, since $N(v_p)$ and $N(v'_p)$ have two white neighbors in A_l. Therefore, at least one vertex from the set A_l will be in Z. □

Theorem 3.4.6. *For a friendship graph F_n, $Z[S(F_n)] = \lfloor p/2 \rfloor + 2$, where p is the order of the friendship graph F_n.*

Proof. Let $v_1, v_2, \ldots, v_p$ be the vertices of F_n, where v_p is the common vertex. Let $v'_1, v'_2, \ldots, v'_p$ be the copies of the vertices $v_1, v_2, \ldots, v_p$ in $S(F_n)$. Now, consider the set $Z = \{v_p, v'_p, v'_1, v'_3, v'_5, \ldots, v'_{p-2}\}$ of black vertices. The remaining vertices are assumed to be white. Also, let T_1 be the triangle in F_n with $V(T_1) = \{v_1, v_2, v_p\}$ and $V(T'_1) = \{v'_1, v'_2, v'_p\}$, where v'_1, v'_2, v'_p are the corresponding vertices of v_1, v_2, v_p in $S(F_n)$.

We can see that in the color change rule, the vertex v'_1 forces the vertex v_2 to black and then the vertex v_2 forces the vertex v_1 to black and then the vertex v_1 forces the vertex v'_2 to black. Clearly, the set $\{v_p, v'_p, v'_1\}$ forms a zero forcing set for $V(T_1) \cup V(T'_1)$. In a similar manner, we can prove that the set $\{v_p, v'_p, v'_1, v'_3\}$ forms a zero forcing set for $[V(T_1) \cup V(T'_1)] \cup [V(T_2) \cup V(T'_2)]$ and so on, where $V(T_2) = \{v_3, v_4, v_p\}$ and $V(T'_2) = \{v'_3, v'_4, v'_p\}$. Therefore, the set $Z = \{v_p, v'_p, v'_1, v'_3, v'_5, \ldots, v'_{p-2}\}$ forms a zero forcing set for $S(F_n)$. Cardinality of the set Z is $\lfloor p/2 \rfloor + 2$. Therefore,

$$Z[S(F_n)] \leq \lfloor p/2 \rfloor + 2 \tag{3.13}$$

To prove the reverse part, assume that there exists a zero forcing set Z consisting of $\lfloor p/2 \rfloor + 1$ black vertices for $S(F_n)$. Now, we consider the following cases.

Case 1.

The vertex $v'_p \notin Z$. Now, $deg(v_i) = 4$ for $i \neq p$ and in $S(F_n)$ the vertex

v'_p is adjacent to all vertices of the friendship graph F_n, except the vertex v_p. Therefore, in any color change rule to force the vertex v'_p, we need two more black vertices from the set A_l (Refer Lemma 3.4.5), a contradiction to our assumption that there exists a zero forcing set consisting of $\lfloor p/2 \rfloor + 1$ black vertices . If we take two more black vertices from the set A_l, then we get a zero forcing set. Therefore, it is clear from the Lemma 3.4.5 that $Z[S(F_n)] \geq \lfloor p/2 \rfloor + 2$.

Case 2.

The vertex $v'_p \in Z$ and v'_p is black. We have from the Lemma 3.4.5 that we need at least one vertex from the set A_l to get a zero forcing set. Without loss of generality, assume that $B = \{v'_1, v'_3, \ldots, v'_{p-2}\}$ is the set of black vertices of $S(F_n)$. Then, $B \cup \{v'_p\}$ will never force the vertex v_p to black, a contradiction to our assumption. Therefore, we need at least one more black vertex from the set A_l to get a zero forcing set for $S(F_n)$. Hence, $Z[S(F_n)] \geq \lfloor p/2 \rfloor + 2$. In both cases,

$$Z[S(F_n)] \geq \lfloor p/2 \rfloor + 2 \tag{3.14}$$

Therefore from (3.13) and (3.14), $Z[S(F_n)] = \lfloor p/2 \rfloor + 2$. □

Definition 3.4.7. [12] *The generalized friendship graph F_p^k is the graph obtained by joining k copies of the cycle C_p, $p \geq 4$ and $k \geq p$, with a common vertex v_p.*

The following theorem provides the zero forcing number of the generalized friendship graph F_p^k.

Theorem 3.4.8. *For a generalized friendship graph F_p^k, where $p \geq 4$ and $k \geq p$, with a common vertex v_p, $Z(F_p^k) = k + 1$.*

Proof. We know that for a cycle C_n, $Z(C_n) = 2$. Now, Lemma 3.4.3 gives

$$Z(F_p^k) \geq \sum_{i=1}^{k} Z(C_i) - k + 1 = 2k - k + 1 = k + 1 \tag{3.15}$$

Conversely, consider one vertex from each cycle C_p, which is adjacent to the common vertex v_p. Denote the k copies of the cycle C_p in F_p^k as $C_p^{(1)}, C_p^{(2)}, \ldots, C_p^{(k)}$, where

$$\begin{aligned} C_p^{(1)} &= v_1^1, v_2^1, \ldots, v_p \\ C_p^{(2)} &= v_1^2, v_2^2, \ldots, v_p \\ \vdots \quad & \quad \vdots \qquad \vdots \\ C_p^{(k)} &= v_1^k, v_2^k, \ldots, v_p \end{aligned}$$

Consider the set of black vertices $Z = \{v_1^1, v_1^2, \ldots, v_1^k\} \cup \{v_p\}$. The remaining vertices in F_p^k are assumed to be white. Now, we can see that $N(v_1^1)$ contains only one white vertex v_2^1. Therefore, $v_1^1 \to v_2^1$ to black, $v_2^1 \to v_3^1$ to black and so on. Similarly, we can see that $N(v_1^2)$ contains only one white vertex v_2^2. Therefore, $v_1^2 \to v_2^2$ to black, $v_2^2 \to v_3^2$ to black and so on. In the cycle $C_p^{(k)}$, we can see that $N(v_1^k)$ contains only one white vertex v_2^k. Therefore, $v_1^k \to v_2^k$ to black, $v_2^k \to v_3^k$ to black and so on. Thus, the set Z forms a zero forcing set for F_p^k. The cardinality of the set Z is $k + 1$. Hence,

$$Z(F_p^k) \leq k + 1 \tag{3.16}$$

Therefore from (3.15) and (3.16), $Z(F_p^k) = k + 1$. □

Theorem 3.4.9. *Let F_p^k be the generalized friendship graph with $p \geq 4$ and $k \geq p$, with a common vertex v_p. Then, $Z[S(F_p^k)] \leq 2k + 2$.*

Proof. First we note that the Theorem 3.4.8 gives $Z(F_p^k) = k + 1$, and the Theorem 3.2.1 implies $Z[S(F_p^k)] \leq 2\ Z(F_p^k)$. Combining these two results, we get

$$Z[S(F_p^k)] \leq 2k + 2.$$

□

3.5 $Z(G)$ and $P(G)$ of the Splitting Graph of a Graph.

Definition 3.5.1. [22] *A path cover of a graph G is a set of vertex disjoint paths of G containing all the vertices of G. The minimum number of paths in any path cover of G is called the path cover number of G and is denoted by $P(G)$.*

Theorem 3.5.2. [22] *For any connected graph $G, P(G) \leq Z(G)$.*

Theorem 3.5.3. *If G is the splitting graph of the path P_n on $n \geq 3$ vertices, then $Z(G) = 2 = P(G)$.*

Proof. The proof is obvious. □

Theorem 3.5.4. *If G is the splitting graph of the star graph $K_{1,n}$, then $Z(G) = 2n - 2 = P(G)$.*

Proof. Let G be the splitting graph of the star graph $K_{1,n}$. By Theorem 3.3.4, we have $Z(G) = 2n-2$. Now, we prove $P(G) = 2n-2$. We consider the following three cases.

Case 1.

Suppose, if we take two vertex disjoint paths of length 1 (that is the complete graph K_2) to cover the graph G, then it must include the vertices u and w (Refer figure 3.4). If we include u and w in these vertex disjoint paths, then there remains $2n-2$ uncovered vertices. To count these vertices in the path covering, we have to choose each of them as independent paths. In this case, the total number of paths we need to cover the entire vertices in G is $2n-2+2=2n$.

Case 2.

Suppose, if we take two vertex disjoint paths of length 2 (i.e the graph P_3) to cover the graph G, take two vertices from part A and the vertex u as the path P_1. Similarly, take any two vertices from part B and the vertex w as the path P_1 (Refer figure 3.4). As in Case 1, the total number of paths we need to cover the entire vertices in G is $2n+2-6+2=2n-2$.

Case 3.

Suppose, if we consider a path of length 3 (i.e the path P_4) as a path to cover the graph G, then it is not possible to choose another path of length 2 or 3 as a path to cover the vertices. Now as in Case 1, the total number of paths we need to cover the entire vertices in G is $2n+2-4+1=2n-1$. From the above three cases, we can conclude that the minimum number of vertex disjoint paths possible to cover the vertices in G is occurred in Case 2 and it is $2n-2$. Therefore, $P(G)=2n-2$.

This completes the proof.

Chapter 4

k-Forcing Number of Some Graphs and Their Splitting Graphs

In the first Section of this Chapter, we introduce the concept of k-forcing number $Z_k(G)$ of a graph G. We investigate the 2-forcing number $Z_2(G)$ of some graphs in Section 2. In this Section, we find some simple graphs for which $1 \leq Z_2(G) \leq 4$. Then, we consider some more graph classes for which $Z_2(G) > 4$ in Section 3. In Section 4, we provide some bounds on $Z_2(G)$.

4.1 Introduction

Definition 4.1.1. [1] *A* ***k-forcing set*** *of a graph G is a subset Z_k of vertices of G such that if initially the vertices in Z_k are colored black and the vertices in $V(G) - Z_k$ are colored white, the whole graph G is colored black by applying the k-color change rule. The* ***k-forcing number*** *$Z_k(G)$ is the minimum of $\mid Z_k \mid$*

[1]This chapter has been published in *International Journal of Scientific Research in Mathematical and Statistical Sciences,* Volume 6, Issue 3, 2019, Pages 121-127.

over all k-forcing sets $Z_k \subseteq V(G)$.

When $k = 1$, *this is equivalent to the zero forcing number, denoted by* $Z(G)$ (*See*[19]). *When* $k = 2$, *this is known as the* 2*-forcing number and is denoted by* $Z_2(G)$.

Definition 4.1.2. **k-Color change rule**. [1] *Let G be a graph with each vertex colored either black or white. If a black colored vertex has at most k white colored neighbors, then change the color of k white neighbors to black.*

When the k-color change rule is applied to an arbitrary vertex v to change the color of some vertices $w_1, w_2, \ldots, w_k$ to black, then we say that the vertex v "k-forces" the vertices $w_1, w_2, \ldots, w_k$ and we denote it as $v \to w_1, v \to w_2, \ldots, v \to w_k$. If it is zero forcing, then we say that the vertex v forces w.

The following definitions are necessary for the further development of this chapter.

Definition 4.1.3. [34] *Three vertices* u, v *and* w *in a graph* G *are said to be* 3*-consecutive if* uv *and* vw *are edges in* G.

Definition 4.1.4. [27] *The square of a graph* G, *represented by* $G^{(2)}$, *is the graph with the vertex set same as that of the vertex set of* G $(ie, V(G))$ *and such that two vertices are adjacent in* $G^{(2)}$ *if their distance in* G *is either 1 or 2.*

4.2 Graphs for which $1 \leq Z_2(G) \leq 4$

In this section, we obtain some simple graphs for which $1 \leq Z_2(G) \leq 4$. We start with path and cycle. The following theorems are easy to observe.

Theorem 4.2.1. *Let G be a connected graph. If $\Delta(G) \leq 2$, then $Z_2(G) = 1$.*

Theorem 4.2.2. *If G is a connected graph with minimum degree $\delta \geq 3$, then $Z_2(G) \geq 2$.*

Next we consider some classes of graphs for which $Z_2(G) = 1$.

Theorem 4.2.3. *Let G be the corona product $C_n \odot K_1$ of a cycle C_n with the graph K_1. Then, $Z_2(G) = 1$.*

Proof. Color any pendant vertex of G as black. Then, clearly this black vertex generates a 2-forcing set for G. This completes the proof. □

Theorem 4.2.4. *Let G be the square graph of a path P_n, where $n \geq 3$. Then, $Z_2(G) = 1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the graph G. We generate a 2-forcing set for G as follows. Color the vertex v_1 as black. The remaining vertices are assumed to be white. Clearly, the vertex v_1 is adjacent to the vertices v_2 and v_3. So the black vertex v_1 2-forces the vertices v_2 and v_3 to black. Again, consider the black vertex v_2. The vertex v_2 is adjacent to the vertices v_1, v_3 and v_4 of which the vertices v_1 and v_3 are already black. Therefore, the vertex v_2 forces the vertex v_4 to black and the process continues till we get a derived coloring for G. Hence the set $Z_2 = \{v_1\}$ forms a 2-forcing set for G. The cardinality of the set Z_2 is 1. Therefore, $Z_2(G) = 1$. □

Theorem 4.2.5. *Let G be the graph $C_n \odot K_1$. Let $G_1, G_2, \ldots, G_m, m \geq n$, be the m copies of G and let G^* be the graph obtained by identifying a pendant vertex of G_1 with a pendant vertex of G_2, a pendant vertex of G_2 with a pendant* *vertex of G_3 etc, a pendant vertex of G_{n-1} with a pendant vertex of G_n. Then, $Z_2(G^*) = 1$.*

Proof. Color any pendant vertex of G_1 as black. Then, clearly this black vertex gives a derived coloring for the graph G^*. Hence, $Z_2(G^*) = 1$. □

Definition 4.2.6. [20] *The prism graph or the circular ladder graph is the graph obtained by taking the Cartesian product of a cycle C_n with the complete graph K_2. The circular ladder graph can be denoted as $C_n \square K_2$.*

Theorem 4.2.7. *Let G be the prism graph $C_n \square K_2$ and $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n. Let $v'_1, v'_2, \ldots, v'_n$ be the vertices corresponding to $v_1, v_2, \ldots, v_n$ in $C_n \square K_2$. Let H be the graph obtained by sub dividing the edges $v_1v'_1, v_2v'_2, \ldots, v_nv'_n$ in $C_n \square K_2$ exactly once. Then, $Z_2(H) = 1$.*

Proof. Let w be the vertex which subdivides the edge $v_1v'_1$ of G exactly once. Color the vertex w as black and the remaining vertices as white. Then, the black vertex w 2-forces the vertices v_1 and v'_1 to black. Again, the black vertex v'_1 2-forces the vertices v'_2 and v'_n to black, the black vertex v_1 2-forces the vertices v_2 and v_n to black and so on. Thus, the set $Z_2 = \{w\}$ forms a 2-forcing set for H. Hence, $Z_2(H) = 1$. □

Theorem 4.2.8. *Let G be the graph $P_n \square P_m$, where P_n and P_m are two paths, $n \geq 2$ and $m \geq 2$. Then, $Z_2(G) = 1$.*

Proof. Consider the vertex u of G with $\delta(u) = 2$. Assign black color to the vertex u. Then, we can easily observe that the set $Z_2 = \{u\}$ generates a 2-forcing set for G. Therefore, $Z_2(G) = 1$. □

Now we consider some splitting graphs.

Theorem 4.2.9. *Let G be a path, where $n \geq 3$ and $2 \leq k \leq \Delta$ be a positive integer. Then, the 2-forcing number of the splitting graph of G is 1. That is, $Z_2[S(G)]$=1.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of G and $v'_1, v'_2, \ldots, v'_n$ be the corresponding neighbors of the vertices $v_1, v_2, \ldots, v_n$ in $S(G)$. Consider the set $Z_2 = \{v'_2\}$. Color the vertex v'_2 as black and all other vertices as white. Clearly, the vertex v'_2 is adjacent to only two white vertices v_1 and v_3. Therefore, the black vertex v'_2 2-forces the vertices v_1 and v_3 to black. Then, the black vertex v_1 forces the vertex v_2 to black. Now, the vertex v_2 2-forces the vertices v'_1 and v'_3 to black and the process continues till we get the derived coloring for the graph G. Thus, the set $Z_2 = \{v'_2\}$ forms a 2-forcing set for $S(G)$. Hence, $Z_2[S(G)] = 1$. We can easily verify that if $k > 2$, then $Z_k[S(G)] = 1$. □

Next we consider the splitting graph of a cycle C_n

Theorem 4.2.10. *Let G be the cycle C_n, where $n \geq 4$. Then, $Z_2[S(G)] = 2$. Moreover, $Z_k[S(G)] = 1$, If $3 \leq k \leq 4$.*

Proof. Let $A = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of the cycle C_n in $S(G)$ and $B = \{u'_1, u'_2, \ldots, u'_n\}$ be the set of corresponding vertices of $A = \{u_1, u_2, \ldots, u_n\}$ in $S(G)$. We observe that the degree of each vertex in A is 4 and the degree of each vertex in B is 2 in $S(G)$. Assume that there exists a 2-forcing set with cardinality one. Let $v \in S(G)$ be the black vertex.

Case 1.

Assume $v \in B$. Clearly, the vertex v is black and all other vertices in $S(G)$ are white. Now, the vertex v can 2-forces two more white vertices to black, not all because each vertex of B is adjacent to two vertices in A and each vertex in A is of degree 4. A contradiction to our assumption that there exists a 2-forcing set with cardinality one.

Case 2.

Assume $v \in A$. We can easily observe that the color change rule is not possible, since all the vertices in A have degree 4. Therefore,

$$Z_2[S(G)] \geq 2 \tag{4.1}$$

To prove the reverse part, we proceed as follows.

Let $Z_2 = \{u_1, u_1'\}$. Color u_1, u_1' as black and all other vertices as white. Clearly, the black vertex u_1' 2-forces the vertices u_2 and u_n to black. Now, the vertex u_1 2-forces the vertices u_2' and u_n' to black. Consider the black vertex u_2. The vertex u_2 is adjacent to u_1, u_1', u_3, u_3' and the vertices u_1, u_1' are already colored black. So the vertex u_2 2-forces the vertices u_3 and u_3' to black, and the process continues till we obtain the derived coloring for $S(G)$. Therefore, the set $Z_2 = \{u_1, u_1'\}$ forms a 2-forcing set for $S(G)$. Hence,

$$Z_2[S(G)] \leq 2 \tag{4.2}$$

Therefore from (4.1) and (4.2), $Z_2[S(G)] = 2$.

Case 3.

If $3 \leq k \leq 4$, then any black vertex forms a 2-forcing set for $S(G)$. Hence, $Z_k[S(G)] = 1$. □

Corollary 4.2.11. *If G is the cycle C_3, then $Z_2[S(G)] = 1$.*

Proof. Let u_1, u_2, u_3 be the vertices of G and u_1', u_2', u_3' be the corresponding vertices of u_1, u_2, u_3 in $S(G)$. Color the vertex u_1' as black and the other vertices as white. Clearly, the vertex u_1' 2-forces the vertices u_2 and u_3 to black. Then, the vertex u_2 2-forces the vertices u_1 and u_3' to black, the vertex u_3 forces u_2' to black. Thus, the set $Z_2 = \{u_1'\}$ generates a 2-forcing set for $S(G)$. Therefore, $Z_2[S(G)] = 1$. □

Theorem 4.2.12. *Let G be the ladder graph $P_n \square P_2$. Then, the 2-forcing number of the splitting graph of G is 2. That is, $Z_2[S(G)] = 2$.*

Proof. Let $A = \{u_1, u_2, \ldots, u_n\}$ and $B = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets of the copies of the path P_n in $S(G)$. Then, $C = \{u_1', u_2', \ldots, u_n'\}$ and $D = \{v_1', v_2', \ldots, v_n'\}$ be the sets of the corresponding vertices of the copies A and B of the path P_n in $S(G)$ respectively. We can easily assert that it is not possible to obtain a 2- forcing set for $S(G)$ with only one black vertex. Therefore,

$$Z_2[S(G)] \geq 2 \tag{4.3}$$

We generate a derived coloring for $S(G)$ by taking 2 black vertices and all other vertices are colored as white. Let $Z_2 = \{u_1, u_1'\}$ be a set of black vertices in $S(G)$. Since u_1' is adjacent to v_1 and u_2, therefore, the vertex u_1' 2-forces the vertices v_1 and u_2 to black. Now, the vertex u_1 is adjacent to the vertices u_2, u_2', v_1 and v_1'. The vertices v_1 and u_2 are already colored black. Therefore, the vertex u_1 2-forces the vertices v_1' and u_2' to black. Apply the color change rule iteratively, all the remaining white vertices in $S(G)$ will be colored as black. Thus, the set Z_2 generates a 2-forcing set for $S(G)$. The cardinality of the set Z_2 is 2. Hence,

$$Z_2[S(G)] \leq 2 \tag{4.4}$$

From the above two inequalities, we have $Z_2[S(G)] = 2$. □

Theorem 4.2.13. *Let G be the prism graph $C_n \square K_2$. Then, $Z_2(G) = 2$.*

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertex set of G, where $u_1, u_2, \ldots, u_n$ are the vertices of the inner cycle. We construct a 2-forcing set for G as follows. Without loss of generality, take two vertices u_1 and u_2. Assign them the black color. The remaining vertices are considered to be white. Clearly, the black vertex u_1 2- forces the vertices v_1 and u_n to black. Also, the black vertex u_2 2- forces the vertices v_2 and u_3 to black. Now, the vertex u_3 2-forces the vertices v_3 and u_4 to black. Repeatedly apply these steps, we can see that the set $Z_2 = \{u_1, u_2\}$ generates a 2-forcing set for G. Here the cardinality of the set Z_2 is 2.

Therefore,

$$Z_2(G) \leq 2 \tag{4.5}$$

On the other hand, it is clear that we cannot form a 2-forcing set for G with only one black vertex. Therefore,

$$Z_2(G) \geq 2 \tag{4.6}$$

The above two inequalities conclude the result.

Definition 4.2.14. *The graph $C_4 \square K_2$ is known as the 3-regular cube Q_3.*

Theorem 4.2.15. *Let G be the 3-regular cube graph $C_4 \square K_2$. Then, $Z_2(G) = 2$.*

Proof. Proof is obvious by the Theorem 4.2.13. □

We recall the following Theorem from [1] to prove the next result.

Theorem 4.2.16. [1] *For any connected graph G with minimum degree δ, $Z_2(G) \geq \delta - 1$.*

Theorem 4.2.17. *Let G be the wheel graph on $n \geq 5$ vertices. Then, $Z_2(G) = 2$.*

Proof. Since the minimum degree $\delta = 3$ in a wheel graph, we have from the Theorem 4.2.16,

$$Z_2(G) \geq 2 \tag{4.7}$$

Conversely, let $u, u_1, u_2, \ldots, u_{n-1}$ be the vertices of G, where u is the central vertex. Consider two adjacent vertices, say u_1 and u_2, on the outer cycle C_{n-1} of the wheel graph G and color these vertices as black. Let $Z_2 = \{u_1, u_2\}$. Clearly, the set Z_2 forms a 2-forcing set for the graph G and therefore

$$Z_2(G) \leq 2 \tag{4.8}$$

From (4.7) and (4.8), $Z_2(G) = 2$. □

Next Theorem deals with a particular class of graph for which $Z_2[S(G)] = 3$.

Theorem 4.2.18. *For a wheel graph G on $n \geq 5$ vertices, $Z_2[S(G)] = 3$.*

Proof. In $S(G)$, it is clear that with 2 black vertices we can 2-force the maximum of two more vertices to black, not all. Therefore,

$$Z_2[S(G)] \geq 3 \tag{4.9}$$

Let $v_1, v_2, \ldots, v_{n-1}$ be the vertices on the cycle C_{n-1} and v_n be the central vertex of the wheel graph. Also, let $v'_1, v'_2, \ldots, v'_{n-1}$ be the corresponding vertices of $v_1, v_2, \ldots, v_{n-1}$ in $S(G)$ and v'_n be the vertex corresponds to the vertex v_n in $S(G)$. Consider the set $Z_2 = \{v'_1, v'_2, v_n\}$ of black vertices in $S(G)$ and assume that the remaining vertices are white. Clearly, the vertex $v'_1 \rightarrow \{v_2, v_{n-1}\}$ to black and $v'_2 \rightarrow \{v_1, v_3\}$ to black. Now, in $N(v_1)$, we have exactly two white neighbors v'_{n-1} and v'_n. Therefore, $v_1 \rightarrow \{v'_{n-1}, v'_n\}$ to black. The remaining white vertices forms a splitting graph of the cycle and hence we can force all the vertices as black. Since the cardinality of the set Z_2 is 3,

$$Z_2[S(G)] \leq 3 \tag{4.10}$$

Hence, the result follows from (4.9) and (4.10). □

Next Theorem deals with another class of graph G for which $Z_2(G) = 3$.

Theorem 4.2.19. *Let G be the cycle C_n, where $n \geq 5$. Then, $Z_2[G^{(2)}] = 3$.*

Proof. Clearly, the graph $G^{(2)}$ is a 4 regular graph. Therefore, $\Delta[G^{(2)}] = \delta[G^{(2)}] = 4$. So by using the Theorem 4.2.16, we have

$$Z_2[G^{(2)}] \geq 4 - 1 = 3 \tag{4.11}$$

Conversely, let the vertices of G be $v_1, v_2, \ldots, v_{n-1}, v_n$. Color the vertices v_1, v_2, v_3 as black and the remaining vertices as white. Clearly, the vertex v_1 2-forces the vertices v_n and v_{n-1} to black. Now, $v_2 \rightarrow v_4$ as black, $v_3 \rightarrow v_5$ to black and so on. Hence the set $Z_2 = \{v_1, v_2, v_3\}$ forms a 2-forcing set for $G^{(2)}$. The cardinality of the set Z_2 is 3. Therefore,

$$Z_2[G^{(2)}] \leq 3 \tag{4.12}$$

Therefore from the above two inequalities, we have $Z_2[G^{(2)}] = 3$. □

Theorem 4.2.20. *Let G be the prism graph $C_n \Box K_2$, where $n \geq 4$. Then, $Z_2[S(G)] = 4$.*

Proof. Clearly, with two black vertices we can force the maximum of two more vertices to black in $S(G)$. Therefore, color change rule is not applicable with two black vertices. This implies that $Z_2[S(G)] \neq 2$. Assume that $Z_2[S(G)] = 3$. Since the vertices in $S(G)$ are of degree 3 or 6, we have the following cases. Suppose that u, v, w are the three black vertices in $S(G)$. The remaining vertices in $S(G)$ are assumed to be white.

Case 1.

Suppose u, v and w are mutually non adjacent. Now, $\mid N(u) \mid= 3$ or 6, $\mid N(v) \mid= 3$ or 6 and $\mid N(w) \mid= 3$ *or* 6. Since $N(u)$, $N(v)$ and $N(w)$ contain 3 or 6 white vertices, the color change rule is not applicable. This implies that there exists at least two adjacent black vertices in the 2- forcing set. Therefore, $deg(u) = deg(v) = deg(w) = 3$ is not possible, since the vertices with degree three are independent.

Case 2.

Suppose that $deg(u) = deg(v) = deg(w) = 6$. The number of white vertices adjacent to u, v or w is at least 4. Therefore, these black vertices will never force any of the other vertices, a contradiction to our assumption that $Z_2[S(G)] = 3$.

Case 3.

Suppose that $deg(u) = deg(v) = 6$ and $deg(w) = 3$. In this case, at least two vertices must be adjacent by Case 1. Assume that $u \sim v$ and $deg(w) = 3$. We observe that to force any other vertex, the vertices u, v and w must form a path of length 2. That is, they must form 3-consecutive vertices. Since the graph is triangle free, therefore in this case, we can force the maximum of two more vertices to black, a contradiction to our assumption.

Case 4.

$deg(u) = deg(v) = 3$ and $deg(w) = 6$. Since $deg(u) = deg(v) = 3$, this implies that u is not adjacent to v. Now, we have the following two Subcases.

Subcase 1.

Assume that u is adjacent to w and v is not adjacent to w. Then, the vertex u will force two white vertices of degree 6 to black and these two vertices will have at least 4 white vertices adjacent to it. Therefore, further forcing is not possible, a contradiction to our assumption that $Z_2[S(G)] = 3$.

Subcase 2.

Assume that $w \sim u$ and $w \sim v$. In this case u and v together can force a maximum of 4 more vertices to black, not all. By considering the above cases, we can conclude that

$$Z_2[S(G)] \geq 4 \tag{4.13}$$

To prove the reverse inequality, consider the graph depicted in Figure 4.1. In

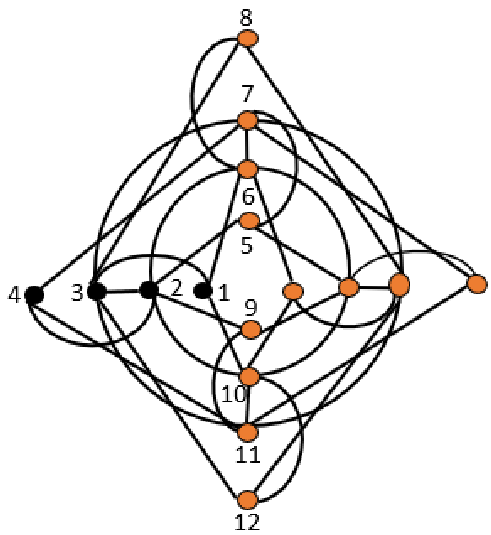

Figure 4.1: The splitting graph of the prism graph $C_4 \Box K_2$

this figure, the orange vertices represent the white vertices. Consider the set of black vertices $\{1, 2, 3, 4\}$. The vertices 6 and 10 are 2-forced to black by the vertex 1. Consider the vertex 2. Since the vertex 6 and 10 are black and they are in the open neighborhood of the vertex 2, therefore, the vertices 5 and 9 are 2-forced to black by the vertex 2. Since there are two white neighbors in the open neighborhood of the vertex 4, therefore, the vertices 7 and 11 are 2-forced to black by the vertex 4. Now, if we consider the vertex 3, it can 2-forces the vertices 8 and 12 to black. If we color the vertex set $\{5, 6, 7, 8\}$ or $\{9, 10, 11, 12\}$ as black, then also we can force the remaining vertices to black. The same argument is true for the splitting graph of the prism graph, where $n \geq 5$. The cardinality of the set $\{1, 2, 3, 4\}$ is 4. Hence,

$$Z_2[S(G)] \leq 4 \tag{4.14}$$

Therefore from (4.13) and (4.14), $Z_2[S(G)] = 4$. □

4.3 Graphs with $Z_2(G) > 4$

In this section, we consider some more graph classes in which $Z_2(G) > 4$. We start with the splitting graph of the star graph.

Theorem 4.3.1. *Let G be the star graph $K_{1,n}$ on $n+1$ vertices, where $n \geq 5$. Then, $Z_2[S(G)] = 2n - 4$.*

Proof. Assume that we have a 2- forcing set consisting of $2n - 5$ black vertices. Then, the number of white vertices in $S(G)$ is 7 , that is, $2n + 2 - (2n - 5) = 7$. We divide the vertex set of $S(G)$ into four sets $A = \{u'\}, B = \{u_1, u_2, \ldots, u_n\}$,

$C = \{u\}$ and $D = \{u'_1, u'_2, \ldots, u'_n\}$ as depicted in Figure 4.2.

Case 1.

Assume that the vertices in A and C are white. Consider the remaining 5 white vertices in $S(G)$. Then, we have the following Subcases.

Subcase 1.

Assume that 3 of them will be in B and 2 of them will be in D or vice versa. We can see that the color change rule is not applicable in this case, because three vertices will remain as white either in B or in D.

Subcase 2.

Assume that 4 of them will be in B and 1 will be in D or vice versa. In this case the color change rule is not possible, since four vertices will remain as white either in B or in D.

Subcase 3.

Assume that there exists no white vertex in the set B and all 5 white vertices are in the set D or vice versa. Here also we cannot apply the color change rule.

Case 2.

Assume that the vertices in A and C are black. The remaining 7 white vertices can be distributed in the sets B and D as follows.

B	D
7	0
6	1
5	2
4	3

We note that the number of vertices in B and D can also be interchanged. From the above partition we can observe that in each case, at least 3 white vertices will be either in B or in D. Therefore, the color change rule is not possible.

Case 3.

Assume that the vertex $u' \in A$ is black or the vertex $u \in C$ is black. Consider the following distribution of the 7 white vertices among the sets A, B, C and D.

B	D	A and C
6	0	1
0	6	1
1	5	1
5	1	1
4	2	1
2	4	1
3	3	1

From the above partition, we can see that at least 3 white vertices will be there in B or in D. Therefore, the color change rule is not possible in this case. From the above cases we conclude that with $2n-5$ black vertices, we cannot obtain a derived coloring for $S(G)$. Therefore,

$$Z_2[S(G)] \geq 2n - 4 \tag{4.15}$$

On the other hand, let $A = \{u'\}$, $B = \{u_1, u_2, \ldots, u_{n-1}, u_n\}, C = \{u\}, D = \{u'_1, u'_2, \ldots, u'_{n-1}, u'_n\}$. Consider the 6 white vertices as $u, u', u_n, u_{n-1}, u'_n, u'_{n-1}$. Consider one black vertex, say u_1, in B. Clearly, the vertex u_1 2-forces the vertices u and u' to black. Consider the vertex u'. Then, the black vertex u'

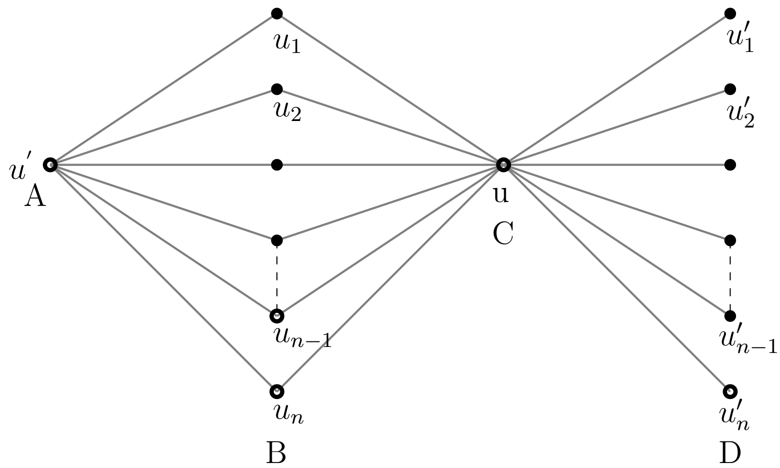

Figure 4.2: The splitting graph of the graph $K_{1,n}$

2-forces the vertices u_{n-1} and u_n to black. Again, the vertex u 2-forces the vertices u'_n and u'_{n-1} to black. Therefore, the set $Z_2 = \{u_1, u_2, \ldots, u_{n-2}\} \cup \{u'_1, u'_2, \ldots, u'_{n-2}\}$ of black vertices forms a 2-forcing set for the graph $S(G)$. Thus, we will get a derived coloring for $S(G)$ with all vertices colored black. Hence, $Z_2[S(G)] \leq n - 2 + n - 2 = 2n - 4$. Therefore,

$$Z_2[S(G)] \leq 2n - 4 \tag{4.16}$$

From (4.15) and (4.16), $Z_2[S(G)] = 2n - 4$. □

Remark 19 . If G is the star graph $K_{1,2}$, then $Z_2[S(G)] = 1$.

Theorem 4.3.2. *Let G be the generalized friendship graph F_p^k, $p \geq 6, k \geq p$. Then, $Z_2(G) = k - 1$.*

Proof. Represent the k cycles in F_p^k as $C_p^{(1)}, C_p^{(2)}, \ldots, C_p^{(k)}$, where

$$V[C_p^{(1)}] = \{v_1^1, v_2^1, \ldots, v_p\}$$

$$V[C_p^{(2)}] = \{v_1^2, v_2^2, \ldots, v_p\}$$

$$\vdots \quad \vdots \quad \vdots$$

$$V[C_p^{(k)}] = \{v_1^k, v_2^k, \ldots, v_p\}$$

Consider the set of black vertices $Z_2 = \{v_1^1, v_1^2, \ldots, v_1^{k-1}\}$. The remaining vertices in G are assumed to be white. We can see that $N(v_1^1)$ contains only two white vertices v_p and v_2^1. Therefore, the black vertex v_1^1 2-forces the vertices v_p and v_2^1 to black. The vertex v_2^1 forces v_3^1 to black and so on. Similarly, we can observe that $N(v_1^2)$ contains vertices v_p and v_2^2 of which v_p is already black. Now, $v_1^2 \to v_2^2$ to black, $v_2^2 \to v_3^2$ to black and so on. In a similar way, $v_1^{k-1} \to v_2^{k-1}$ to black, $v_2^{k-1} \to v_3^{k-1}$ to black and so on. Consider the cycle $C_p^{(k)}$ in G. The vertex set of the cycle $C_p^{(k)}$ is $\{v_1^k, v_2^k, \ldots, v_p\}$. The black vertex v_p is adjacent to two white vertices v_1^k and v_{p-1}^k. So the black vertex v_p 2- forces the vertices v_1^k and v_{p-1}^k to black. Now, $v_1^k \to v_2^k$ to black , $v_2^k \to v_3^k$ to black and so on in $C_p^{(k)}$. Therefore, the set Z_2 forms a 2- forcing set for F_p^k. The cardinality of the set Z_2 is $(k-1)$. We can easily observe that with $(k-2)$ black vertices it is not possible to form a 2-forcing set for G, because there exists at least 4 white vertices adjacent to the vertex v_p. Hence, $Z_2(G) = k - 1$. □

4.4 Bounds on $Z_2(G)$

In this section, we consider some bounds on $Z_2(G)$.

Theorem 4.4.1. *For any connected graph G of order $n \geq 3$, $Z_2[S(G)] \leq 2Z_2(G)$.*

Proof. Assume that $Z_2 = \{u_1, u_2, \ldots, u_m\}, 1 \leq m \leq n$, be a minimun 2-forcing set for G. Now, consider the set

$$Z_2^{'} = \{u_1, u_2, \ldots, u_m\} \cup \{u_1', u_2', \ldots, u_m'\} \in V[S(G)]$$

where $u_1', u_2', \ldots, u_m'$ be the copies of the vertices $u_1, u_2, \ldots, u_m$ in $V[S(G)]$. Color all the vertices in $Z_2^{'}$ as black. We prove that the set $Z_2^{'}$ will form a 2-forcing set for $S(G)$. Assume that G is colored with black and white vertices and the vertices in Z_2 are black. Consider the vertices in G which have exactly two white neighbors in G. Let it be $u_1, u_2, \ldots, u_l$, where $l \leq m$ and $u_1', u_2', \ldots, u_l'$ be the vertices corresponds to $u_1, u_2, \ldots, u_l$ in $S(G)$. We see that in $S(G)$, each one of $N(u_1'), N(u_2'), \ldots, N(u_l')$ contains exactly two white vertices. Let it be $V_1, V_2, \ldots, V_l$, where each one of $V_1, V_2, \ldots, V_l$ represents the set of two white vertices. Clearly, $u_1' \to V_1$ to black, $u_2' \to V_2$ to black, …, $u_l' \to V_l$ to black. Consider the set $\{u_1, u_2, \ldots, u_l\}$ in $S(G)$. At this point of time, we can see that $u_1 \to V_1'$ to black, $u_2 \to V_2'$ to black,…, $u_l \to V_l'$ to black, where $V_1', V_2', \ldots, V_l'$ are the corresponding vertices of $V_1, V_2, \ldots, V_l$. Consider the white vertices which are adjacent to $V_1, V_2, \ldots, V_l$ in G. Let it be $w_1, w_2, \ldots, w_l$. Clearly, $V_1' \to w_1$ to black, $V_1 \to w_1'$ to black and so on, where $w_1', w_2', \ldots, w_l'$ denote the corresponding vertices of $w_1, w_2, \ldots, w_l$ respectively. Therefore, the set $Z_2^{'}$ forms a 2- forcing set for $S(G)$. Hence, $Z_2[S(G)] \leq 2Z_2(G)$. □

Now, we consider the following results from [1] and [8] to prove a relationship between $Z(G)$, $Z_2(G)$ and $Z[S(G)]$.

Theorem 4.4.2. [8] *Let G be a connected graph of order $n \geq 3$. Then, $Z[S(G)] \leq 2Z(G)$.*

Theorem 4.4.3. [1] *Let $G = (V, E)$ be a connected graph. Then, $Z(G) \geq Z_2(G)$.*

Next we prove a relationship between $Z(G)$, $Z_2(G)$ and $Z[S(G)]$.

Theorem 4.4.4. *Let G be a connected graph of order $n \geq 3$. Then, $Z_2(G) + Z[S(G)] \leq 3Z(G)$, and the bound is sharp if G is the path P_3.*

Proof. Theorem 4.4.2 and Theorem 4.4.3 imply that $Z_2(G) + Z[S(G)] \leq Z(G) + 2Z(G) = 3Z(G)$. This bound is sharp for P_3, since $Z_2(P_3) = 1, Z[S(P_n)] = 2$ and $Z(G) = 1$. □

Chapter 5

Connected k -Forcing Number of Graphs and Splitting Graphs

In this Chapter, we address the problem of determining the connected k-forcing number $Z_{ck}(G)$ of certain graphs. In Section 1, we find the connected zero forcing number and the connected k-forcing number of some familiar graphs and splitting graphs. We also provide an upper bound of $Z_{ck}(G \odot H)$ for the corona product of two graphs G and H. In Section 2, we investigate the connected zero forcing number and the connected k-forcing number of rooted product of some graphs, the CP-graph C_3P_r and the circular ladder graph. Section 3 deals with the connected k-forcing number of square graph of path and cycle.

5.1 Introduction

Definition 5.1.1. [1] **k-color change rule**: *Let G be a graph with each vertex colored either black or white. If a black colored vertex has at most k white*

[1]This chapter has been published in the *Journal of Mathematical and Computational Science*, Volume 10, Number 3, 2020, Pages 656-680.

neighbors, then change the color of k *white neighbors to black. When the* k*-color change rule is applied to an arbitrary vertex* v *to change the color of some vertices* $w_1, w_2, \ldots, w_k$ *to black, then we say that the vertex* v *"*k*-forces" the vertices* $w_1, w_2, \ldots, w_k$ *and we denote it as* $v \to w_1, v \to w_2, \ldots, v \to w_k$.

Definition 5.1.2. [1] *A* ***k-forcing set*** *of a graph* G *is a subset* Z_k *of vertices of* G *such that if initially the vertices in* Z_k *are colored black and the vertices in* $V(G) - Z_k$ *are colored white, the whole graph* G *is colored black by continuously applying the* k*-color change rule. The* ***k-forcing number*** $Z_k(G)$ *is the minimum of* $\mid Z_k \mid$ *over all* k*-forcing sets* $Z_k \subseteq V(G)$*. If the subgraph induced by the vertices in* Z_k *(ie,* $\langle Z_k \rangle$*) is connected, then* Z_k *is known as the* ***connected k-forcing set****, denoted by* Z_{ck}.

Definition 5.1.3. *The* ***connected k-forcing number*** $Z_{ck}(G)$ *is the minimum of* $\mid Z_{ck} \mid$ *over all connected* k*- forcing sets* $Z_{ck} \subseteq V(G)$.

When $k = 1$*, this is known as the connected zero forcing number and is denoted by* $Z_c(G)$.

From the definitions above, we have the following Theorem.

Theorem 5.1.4. *Let* P_n *be a path, where* $n \geq 3$*. Then,*

$$Z_{ck}[S(P_n)] = \begin{cases} 3 \text{ if } k = 1 \\ 1 \text{ if } k \geq 2. \end{cases}$$

Proof. **Case 1.**

Assume that $k = 1$. We can easily observe that if we color any two adjacent vertices as black, it is not possible to obtain a derived coloring for $S(P_n)$. Therefore,

$$Z_c[S(P_n)] \geq 3 \tag{5.1}$$

Let $u_1, u_2, \ldots, u_n$ be the vertices of P_n and $u_1^{'}, u_2^{'}, \ldots, u_n^{'}$ be the corresponding vertices in $S(P_n)$. Color the vertices u_1, u_2 and $u_1^{'}$ as black and the remaining vertices as white. Clearly, the vertex u_1 forces $u_2^{'}$ to black. Then, the vertex $u_2^{'}$ forces u_3 to black, the vertex u_2 forces $u_3^{'}$ to black and so on. Therefore, the set $Z_c = \{u_1, u_2, u_1^{'}\}$ forms a connected zero forcing set for $S(P_n)$. Therefore,

$$Z_c[S(P_n)] \leq 3 \tag{5.2}$$

Hence from (5.1) and (5.2), $Z_c[S(P_n)] = 3$.

Case 2.

Assume that $k \geq 2$. In this case if we color the vertex u_1 as black, then the black vertex u_1 forms a connected zero forcing set and hence the result follows. □

Theorem 5.1.5. *For any simple graph G, $Z(G) \leq Z_{ck}(G)$ for any fixed positive integer k.*

Proof. Proof is obvious. □

We consider the next Theorem from [8] to prove the result for the splitting graph of a cycle C_n.

Theorem 5.1.6. [8] *If G is the cycle C_n, where $n \geq 4$, then $Z[S(G)] = 4$.*

In the succeeding Theorem, we consider the splitting graph of a cycle $C_n, n \geq 4$.

Theorem 5.1.7. *Let $S(C_n)$ be the splitting graph of a cycle C_n, $n \geq 4$. Then,*

$$Z_{ck}[S(C_n)] = \begin{cases} 4 \ \mathit{if} \ k = 1 \\ 3 \ \mathit{if} \ k = 2 \\ 1 \ \mathit{if} k \geq 3. \end{cases}$$

Proof. **Case 1.**

Assume that $k = 1$. From the Theorem 5.1.5 and the Theorem 5.1.6, we have

$$Z_c[S(C_n)] \geq 4 \tag{5.3}$$

To prove the reverse part, consider the vertices of the cycle C_n as $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ be the corresponding vertices in $S(C_n)$. Consider the set $Z_c = \{v_1, v'_1, v_2, v'_2\}$. Color the vertices of the set Z_c as black and the remaining vertices as white. Then, the vertex $v'_2 \to v_3$ to black, $v_2 \to v'_3$ to black, $v'_3 \to v_4$ to black, $v_3 \to v'_4$ to black and so on. Therefore, we can find a derived coloring for $S(C_n)$ with the set $Z_c = \{v_1, v'_1, v_2, v'_2\}$. The cardinality of the set Z_c is 4. Therefore, we have

$$Z_c[S(C_n)] \leq 4 \tag{5.4}$$

Hence from (5.3) and (5.4), $Z_c[S(C_n)] = 4$.

Case 2.

Assume that $k = 2$ and $Z_{c2}[S(C_n)] = 2$. Consider a connected 2-forcing set consisting of two black vertices. Let u and v be the two adjacent black vertices in the connected 2-forcing set for $S(C_n)$. Then, we have the following Subcases.

Subcase 1.

$deg(u) = deg(v) = 4$. Since u and v are adjacent to three white neighbors, the color change rule is not applicable in this case, a contradiction to our assumption that $Z_{c2}[S(C_n)] = 2$.

Subcase 2.

$deg(u) = 2$ and $deg(v) = 4$. In this case, the black vertex u can force one adjacent vertex of degree 4 to black. Therefore in this case, it is not possible to obtain a derived coloring. Hence from the Subcases 1 and 2, we have

$$Z_{c2}[S(C_n)] \geq 3 \tag{5.5}$$

We can easily observe that the set $Z_{c2} = \{v_1, v_1', v_2\}$ of black vertices forms a connected 2-forcing set for $S(C_n)$ and hence

$$Z_{c2}[S(C_n)] \leq 3 \tag{5.6}$$

Therefore from (5.5) and (5.6), $Z_{c2}[S(C_n)] = 3$.

For $k = 3$, the proof is obvious. □

Theorem 5.1.8. *Let F_p denotes the friendship graph with $p \geq 2$ triangles. Then,*

$$Z_{ck}(F_p) = \begin{cases} p + 1 - \frac{k}{2} \ if\ k \ is\ even\ and\ k < \Delta - 2 \\ \frac{2p-k+3}{2} \ if\ k\ is\ odd\ and\ k < \Delta - 2 \\ 1\ if\ k \geq \Delta - 2. \end{cases}$$

Proof. **Case 1.**

Assume that k is even and $k < \Delta - 2$. Let v be the vertex with maximum degree Δ. We can observe that v should be a member of any connected k-forcing set. Otherwise, the k- forcing set will not be connected. Therefore, assume that the vertex v is there in any connected k-forcing set for F_p. If we take one black vertex from each of the $p - \frac{k}{2} - 1$ triangles, then it is not possible to obtain a derived coloring since $deg(v) = 2p$ and by using the color change rule we get $2(p - \frac{k}{2} - 1) = 2p - k - 2$ black vertices which are adjacent to the vertex v. Now, we have $2p - (2p - k - 2) = k + 2$ white vertices remain. It is not possible to force these $k + 2$ vertices to black by using the black vertex v. Therefore, we must take one black vertex from each of the $p - \frac{k}{2}$ triangles. Since the vertex v is black, we have

$$p + 1 - \frac{k}{2} \leq Z_{ck}(F_p) \tag{5.7}$$

Take one vertex from each of the $p - \frac{k}{2}$ triangles as black. Since the vertex v is black, these $p - \frac{k}{2}$ vertices will force the remaining vertices in the $p - \frac{k}{2}$ triangles as black. Then, we have $2(p - \frac{k}{2})$ black vertices together with the black vertex v in the connected k-forcing set. We can observe that at this stage we have $2p - (2p - k) = k$ white vertices adjacent to the vertex v. Now, the vertex v can force these k-vertices as black. Therefore, we get a derived coloring with $p + 1 - \frac{k}{2}$ black vertices. Hence,

$$Z_{ck}(F_p) \leq p + 1 - \frac{k}{2} \tag{5.8}$$

Therefore, $Z_{ck}(F_p) = p + 1 - \frac{k}{2}$.

Case 2.

Assume that k is odd and $k < \Delta - 2$. Let v be the vertex with maximum degree Δ. We can see that v should be a member of any connected k- forcing set. Otherwise, the k-forcing set will not be connected. Therefore, assume that the vertex v is there in any connected k- forcing set for F_p. Suppose that there exists a k- forcing set consisting of $\frac{2p-k+1}{2}$ black vertices. Since the vertex v is black, we can distribute the remaining $\frac{2p-k-1}{2}$ black vertices among the triangles. To force the maximum number of vertices as black, we need to distribute one black vertex for each of $\frac{2p-k-1}{2}$ triangles. Now, we have $2(\frac{2p-k-1}{2}) + 1 = 2p - k$ black vertices and $2p+1-(2p-k) = k+1$ white vertices. All these white vertices are adjacent to v. So the color change rule is not applicable since $(k+1)$ white vertices are adjacent to the black vertex v. Therefore with $\frac{2p-k+1}{2}$ black vertices, we cannot obtain a derived coloring for F_p. Hence,

$$Z_{ck}(F_p) \geq \frac{2p-k+3}{2} \tag{5.9}$$

We take one vertex from each of the $\frac{2p-k+3}{2}-1$ triangles as black. Since the vertex v is back, these $\frac{2p-k+3}{2}-1$ black vertices will force the remaining vertices in the $\frac{2p-k+3}{2}-1$ triangles as black. At this stage, we have $2(\frac{2p-k+3}{2}-1)+1 = 2p-k+2$ black vertices remain. Therefore, the total number of white vertices remain at this stage is $2p+1-(2p-k+2) = k-1$. All these $(k-1)$ white vertices are adjacent to v. Therefore, the vertex v forces all these $(k-1)$ white vertices as black. Hence,

$$Z_{ck}(F_p) \leq \frac{2p-k+3}{2} \tag{5.10}$$

Therefore from (5.9) and (5.10), $Z_{ck}(F_p) = \frac{2p-k+3}{2}$.

We can easily observe that if $k \geq \Delta - 2$, then $Z_{ck}(F_p) = 1$. □

Theorem 5.1.9. *Let G be a connected graph with $\mid V(G) \mid = p_1$ and let H be another connected graph with $Z_{ck}(H) = p_2$. Let $\mathcal{G}$ be the graph obtained by taking the corona product of G and H, that is, $\mathcal{G} = G \odot H$. Then, $Z_{ck}(\mathcal{G}) \leq p_1(1+p_2)$.*

Proof. Without loss of generality, assume that G is connected with $\mid V(G) \mid = p_1$ and $Z_{ck}(H) = p_2$. Color all the vertices of G as black. To form the k-forcing set for the sub graph induced by $\{v_1\} \cup H_1$, we need a maximum of $1+p_2$ black vertices. That is, $Z_k(\langle\{v_1\} \cup H_1\rangle) \leq 1+p_2$, where H_1 is the first copy of H corresponds to the vertex v_1 in $\mathcal{G}$. Also, $Z_k(\langle\{v_2\} \cup H_2\rangle) \leq 1+p_2$, where H_2 is the second copy of H corresponds to the vertex v_2 in $\mathcal{G}$. Proceeding like this, we can observe that $Z_k(\langle\{v_{p_1}\}\cup H_{p_1}\rangle) \leq 1+p_2$. Now, the graph $\mathcal{G} = \langle\{v_1\}\cup H_1\rangle\cup\langle\{v_2\}\cup H_2\rangle\cup\ldots\cup\langle\{v_{p_1}\}\cup H_{p_1}\rangle$. Therefore, $Z_k(\mathcal{G}) \leq (1+p_2)+(1+p_2)+\cdots+(1+p_2)$-$p_1$ times. This follows that $Z_k(\mathcal{G}) \leq p_1(1+p_2)$. Since each vertex in G is connected to the vertices of all copies of H, the k-forcing set thus obtained forms a connected k-forcing set. Therefore, $Z_{ck}(\mathcal{G}) \leq p_1(1+p_2)$. □

Theorem 5.1.10. *Let G be the complete bipartite graph $K_{m,n}$, where $n \geq 2$, $m \geq 2$. Then, the connected zero forcing number of G is $m+n-2$. That is, $Z_c(G) = m+n-2$.*

Proof. Since G is a complete bipartite graph, therefore, the vertex set of G can be partitioned into two sets X and Y. Let $u_1, u_2, \ldots, u_m$ be the vertices in X and $v_1, v_2, \ldots, v_n$ be the vertices in Y. Clearly, the vertices in X are non adjacent. The vertices in Y are also non adjacent. To start the color change rule, color any vertex, say u_1, in X as black. The remaining vertices are considered to be white. Since each vertex in X is connected to every vertex in Y, we have to color $(n-1)$ vertices in Y as black. Let the only one white vertex in Y be v_n. Now, the vertex $u_1 \to v_n$ to black. In X, there are $(m-1)$ white vertices remain. Each vertex in Y is joined to $(m-1)$ white vertices in X. Assign black color to $(m-2)$ white vertices in X. Then, any black vertex in Y, say v_1, forces the remaining white vertex in X as black. Now, the zero forcing set consists of $1+m-2+n-1$ black vertices, which are connected. Thus, with $1+m-2+n-1 = m+n-2$ black vertices, we can obtain a derived coloring for G. We can easily observe that with $m+n-3$ black vertices, forming a connected zero forcing set for G is not possible. Hence the connected zero forcing number of G is $m+n-2$. □

We use the following results from [4] and [19] to prove the next result.

Theorem 5.1.11. [4] *For any connected graph* G, $Z(G) \leq Z_c(G)$.

Theorem 5.1.12. [19] *Let* G *be the graph obtained by taking the Cartesian product of a cycle* C_n *with the path* P_m. *Then,* $Z(G) = min\{n, 2m\}$

Theorem 5.1.13. *Let* G *be the graph obtained by taking the Cartesian product of a cycle* C_n *with the path* P_m *and let* $n \geq 2m$. *Then,* $Z_c(G) = 2m$.

Proof. Let v_1 and v_2 be the two adjacent vertices in the cycle C_n. Let $A = \{v_1^1, v_1^2, \ldots, v_1^m\}$ be the vertices corresponding to the vertex v_1 in G and let $B = \{v_2^1, v_2^2, \ldots, v_2^m\}$ be the vertices corresponding to the vertex v_2 in G, where $v_1 = v_1^1$ and $v_2 = v_2^1$. Consider the $2m$ vertices in the set $A \cup B$ and color these vertices as black in G. The remaining vertices are assumed to be white. Then, these $2m$ black vertices in $A \cup B$ forces the remaining vertices in G as black. Clearly, these vertices are connected in G and thus form a connected zero forcing set for G. Hence,

$$Z_c(G) \leq 2m \tag{5.11}$$

Also, by the Theorem 5.1.11 and the Theorem 5.1.12

$$Z_c(G) \geq 2m \tag{5.12}$$

From (5.11) and (5.12), $Z_c(G) = 2m$. □

Theorem 5.1.14. *Let* G *be the star graph* $k_{1,n}$ *on* $n + 1$ *vertices,* $n \geq 3$. *Then,* $Z_c(G) = n$. *In general, if* $2 \leq k \leq n$, *then* $Z_{ck}(G) = n - k + 1$.

Proof. Let $v, u_1, u_2, \ldots, u_n$ be the vertices of the star graph $k_{1,n}$, where v is the vertex having degree n. Assume that the vertex v is black. We generate a connected zero forcing set as follows.

Since $deg(v) = n$, to apply the color change rule, we have to color $(n-1)$ vertices in G adjacent to v as black. Then, the vertex v forces the remaining white vertex to black. Thus, with $n - 1 + 1 = n$ black vertices, we get a derived coloring for G and we can easily observe that with $(n - 1)$ black vertices we cannot generate a connected zero forcing set for G. Therefore, $Z_c(G) = n$.

If $k = 2$, we can easily show that the connected 2-forcing number of G is $n-2+1 = n - 1$. Again, if $k = 3$, then we can easily observe that the connected 3-forcing number of G is $n-3+1 = n-2$. Proceeding like this, we obtain $Z_{ck}(G) = n-k+1$ for any positive integer $2 \leq k \leq n$. □

5.2 Connected k-Forcing Number of Rooted Product of Graphs

In this section, we deal with the connected k-forcing number of rooted product of cycle and path, cycle and cycle, ladder graph and path, grid graph and path, ladder graph and cycle, circular ladder graph and path, circular ladder graph and cycle, and path and path.

Theorem 5.2.1. *Let G be the graph obtained by taking the rooted product of the cycle C_n and the rooted path P_t rooted with the pendant vertex of P_t, where $t \geq 2$. Then,*

$$Z_{ck}(G) = \begin{cases} n \; if \; k = 1 \\ 1 \; if \; 2 \leq k \leq \Delta(G) = 3. \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n in G and $P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(n)}$ be the n copies of the path P_t rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Represent the vertex set of the paths $P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(n)}$ in G as follows.

$$V[P_t^{(1)}] = \{p_1^1, p_2^1, \ldots, p_t^1\}$$

$$V[P_t^{(2)}] = \{p_1^2, p_2^2, \ldots, p_t^2\}$$

$$\ldots$$

$$\ldots$$

$$V[P_t^{(n)}] = \{p_1^n, p_2^n, \ldots, p_t^n\}$$

Let u_1 be the vertex identified with the vertex p_1^1 in G, u_2 be the vertex identified with the vertex p_1^2 in G, ..., u_n be the vertex identified with the vertex p_1^n in G.

Case 1.

Assume that $k = 1$. This case is similar to that of the connected zero forcing number of G. Color the vertices $u_1, u_2, \ldots, u_n$ in G as black and the remaining vertices are assumed to be white. We can easily infer that

$$Z_c(G) \leq n \tag{5.13}$$

It is worth mentioning that if we start the color change rule with the vertices of $P_t^{(i)}$, where $1 \leq i \leq n$, other than the vertices identified with the vertices $u_1, u_2, \ldots, u_n$ of C_n, we cannot obtain a connected zero forcing set with n black vertices. Therefore, we need to consider the vertices in the cycle to force the remaining vertices in G.

Assume that we have a connected zero forcing set consisting of $(n-1)$ black vertices. From above, we can assert that these vertices must be from the cycle

C_n. Without loss of generality, assume that the vertices $u_1, u_2, \ldots, u_{n-1}$ are black. Clearly, the vertex u_2 can force the vertices of the path $P_t^{(2)}$ to black, the vertex u_3 can force the vertices of the path $P_t^{(3)}$ to black, ..., the vertex u_{n-2} can force the vertices of the path $P_t^{(n-2)}$ to black. Since the black vertex u_1 is adjacent to two white vertices u_n and p_2^1, the vertex u_1 cannot force the vertices u_n and p_2^1. Similarly, the black vertex u_{n-1} is adjacent to two white vertices u_n and p_2^{n-1}. Therefore, the vertex u_{n-1} cannot force u_n and p_2^{n-1}. This contradicts our assumption that $Z_c(G) = n - 1$. Therefore,

$$Z_c(G) \geq n \tag{5.14}$$

Hence from (5.13) and (5.14), $Z_c(G) = n$.

Case 2.

Assume that $k \geq 2$. In this case, if we consider any pendant vertex of G as a black vertex , then it can force the remaining white vertices of G as black. Hence, $Z_{ck}(G) = 1$. □

Theorem 5.2.2. *Let G be the graph representing the rooted product of a cycle C_n and the rooted cycle C_m. Then,*

$$Z_{ck}(G) = \begin{cases} 2n \ if \ k = 1 \\ n \ if \ k = 2 \\ 1 \ if \ 3 \leq k \leq \Delta(G) = 4. \end{cases}$$

Proof. Let $u_1, u_2 \ldots, u_n$ be the vertices of the cycle C_n in G and $C_m^{(1)}, C_m^{(2)}, \ldots, C_m^{(n)}$ be the n copies of the cycle C_m rooted at $u_1, u_2, \ldots, u_n$ respectively. Represent the vertex set of the cycles $C_m^{(1)}, C_m^{(2)}, \ldots,$ $C_m^{(n)}$ in G by

$$V[C_m^{(1)}] = \{d_1^1, d_2^1, \ldots, d_m^1\}$$

$$V[C_m^{(2)}] = \{d_1^2, d_2^2, \ldots, d_m^2\}$$

$$\ldots$$

$$\ldots$$

$$V[C_m^{(n)}] = \{d_1^n, d_2^n, \ldots, d_m^n\}$$

Assume that the vertex d_1^1 be rooted at u_1, the vertex d_1^2 be rooted at u_2, ..., the vertex d_1^n be rooted at u_n.

Case 1.

Suppose that $k = 1$. This case is similar to that of the connected zero forcing number of G. Color the vertices $u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ldots, d_2^n$ as black and the remaining vertices as white. We can easily see that these black vertices form a connected zero forcing set for G. Hence,

$$Z_c(G) \leq 2n \tag{5.15}$$

We can easily observe that to form a minimum connected zero forcing set for G, we need to color the vertices $u_1, u_2, \ldots, u_n$ as black and color at least one vertex from each of the cycles $C_m^{(i)}$, $1 \leq i \leq n$, adjacent to each u_i $(1 \leq i \leq n)$ as black. Otherwise, we cannot form a minimum connected zero forcing set. Clearly,

$$Z_c(G) \geq 2n \tag{5.16}$$

Hence from (5.15) and (5.16) $Z_c(G) = 2n$.

Case 2.

Assume that $k = 2$. Color all vertices of C_n in G as black and the remaining vertices as white. Each vertex $u_i, 1 \leq i \leq n$, is adjacent to exactly two white vertices of $C_m^{(i)}, 1 \leq i \leq n$ and $k = 2$. Therefore, these vertices form a 2-forcing set for G. The sub graph induced by these black vertices is connected and hence form a connected 2-forcing set for G. Therefore,

$$Z_{c2}(G) \leq n \tag{5.17}$$

We can easily observe that to form a minimum connected 2-forcing set for G, we need to color the vertices $u_1, u_2, \ldots, u_n$ as black, otherwise we cannot form a minimum connected 2-forcing set. Clearly,

$$Z_{c2}(G) \geq n \tag{5.18}$$

Therefore from (5.17) and (5.18), the result follows.

Case 3.

Suppose that $k \geq 3$. In this case, an arbitrary black vertex with degree 2 from the cycle $C_m^{(i)}$, $1 \leq i \leq n$, will k-forces the remaining vertices in G as black. Therefore, $Z_{ck}(G) = 1$. Hence the result. □

Theorem 5.2.3. *Let G be the rooted product of the ladder graph $P_n \square P_2$ and the rooted path P_t rooted with the pendant vertex of P_t, where $n \geq 3$, $t \geq 2$. Then,*

$$Z_{ck}(G) = \begin{cases} 2n \ \ if \ k = 1 \\ \\ 1 \ \ if \ 2 \leq k \leq 4 = \Delta(G). \end{cases}$$

Proof. Represent the vertices of the graph $P_n \square P_2$ by u_1, u_2, $\ldots$, u_n and v_1, v_2, $\ldots$, v_n. Let $P_t^{(1)}$, $P_t^{(2)}$,$\ldots$,$P_t^{(n)}$ be the copies of the path P_t rooted at the vertices u_1, u_2,$\ldots$, u_n respectively. Also, let $Q_t^{(1)}$, $Q_t^{(2)}$, $\ldots$, $Q_t^{(n)}$ be the copies of the path P_t rooted at the vertices $v_1, v_2, \ldots, v_n$ respectively. The vertex set of the paths $P_t^{(1)}$, $P_t^{(2)}$, $\ldots$, $P_t^{(n)}$ and $Q_t^{(1)}$, $Q_t^{(2)}$, $\ldots$, $Q_t^{(n)}$ in G can be named as follows.

$$V[P_t^{(1)}] = \{p_1^1, p_2^1, \ldots, p_t^1\}, \quad V[Q_t^{(1)}] = \{q_1^1, q_2^1, \ldots, q_t^1\}$$

$$V[P_t^{(2)}] = \{p_1^2, p_2^2, \ldots, p_t^2\}, \quad V[Q_t^{(2)}] = \{q_1^2, q_2^2, \ldots, q_t^2\}$$

$$\cdots \qquad \cdots$$

$$\cdots \qquad \cdots$$

$$V[P_t^{(n)}] = \{p_1^n, p_2^n, \ldots, p_t^n\}, \quad V[Q_t^{(n)}] = \{q_1^n, q_2^n, \ldots, q_t^n\}$$

Color the vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ as black (Refer figure 5.1). The remaining vertices are assumed to be white. Clearly, these black vertices form a connected zero forcing set for G and hence

$$Z_c(G) \leq 2n \tag{5.19}$$

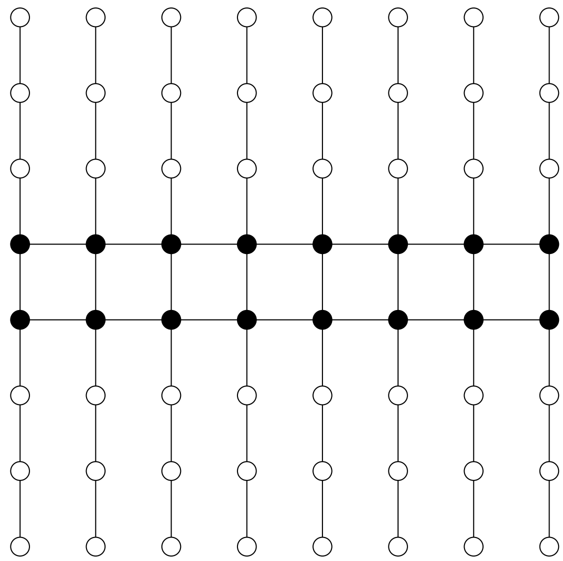

Figure 5.1: Rooted product of $P_8 \square P_2$ and P_4.

There exists three types of minimum connected zero forcing sets with $Z_c(G) = 2n$. Consider these three sets as follows. We denote them as A, B and C.

$$A = \{u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_n\}$$

$$B = \{u_1, u_2, \ldots, u_n, p_2^1, p_2^2, \ldots, p_2^n\}$$

$$C = \{v_1, v_2, \ldots, v_n, q_2^1, q_2^2, \ldots, q_2^n\}$$

We can easily observe that if we take a set of $2n$ vertices other than these three sets, then it will not form a minimum connected zero forcing set. Assume that there exists a connected zero forcing set consisting of $(2n-1)$ black vertices. Consider the following cases.

Case 1.

Consider the black vertices as depicted in Figure 5.2. Assume that the black vertices are from the set A, except the vertex u_n. Consider the vertex $u_n = u_8$ as white. The blue colored vertices represent the vertices which are forced by the black vertices. In this case if we consider G, then there are $3t-2$ vertices remain as white. Therefore, we cannot obtain a derived coloring, a contradiction to our assumption that there exists a connected zero forcing set consisting of $(2n-1)$ black vertices. The case is similar if we consider u_1, v_1 and $v_n = v_8$ as white vertices.

Case 2.

Consider the black vertices as depicted in Figure 5.1. If we choose any black vertex other than $u_1, v_1, u_n = u_8, v_n = v_8$ as white, then we can observe that there are $4t-3$ white vertices remain in G, a contradiction to our assumption that $Z_c(G) = 2n-1$.

Case 3.

Consider the black vertices as depicted in Figure 5.3. Assume that the black vertices are from the set B, except the vertex p_2^1. Consider the vertex p_2^1 as white. The blue colored vertices represent the vertices which are forced by the

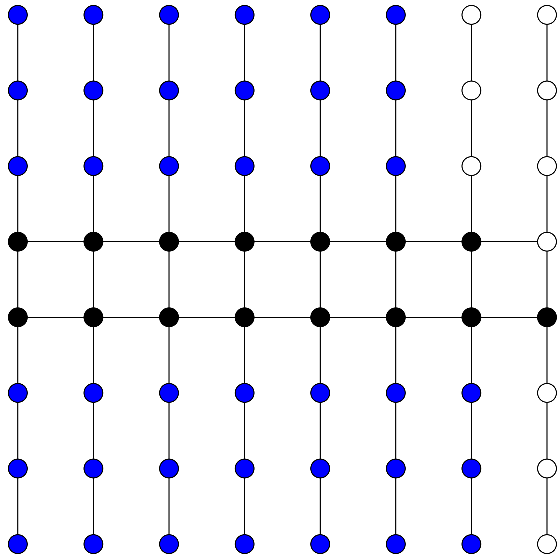

Figure 5.2: Rooted product of $P_8 \square P_2$ and P_4.

black vertices. In this case if we consider the graph G, then there are $3t-2$ vertices remain as white. Therefore, we cannot obtain a derived coloring with $(2n-1)$ black vertices. A contradiction to our assumption. The case is similar if we consider the vertex p_2^n as white.

Subcase 3.1.

Consider the black vertices as depicted in Figure 5.3. Assume that the black vertices are from the set B, except the vertex p_2^i, $2 \leq i \leq n-1$. Consider the vertex p_2^i as white. In this case if we consider G, then there are $4t-3$ vertices remain as white. The color change rule is not applicable at this stage, a contradiction to our assumption that $Z_c(G) = 2n-1$.

Subcase 3.2.

Assume that the black vertices are from the set B, except the vertex u_i, where $1 \leq i \leq n$. In this case, we loose the connectivity of the zero forcing set. That is, the zero forcing set is not connected. Again, a contradiction to our assumption

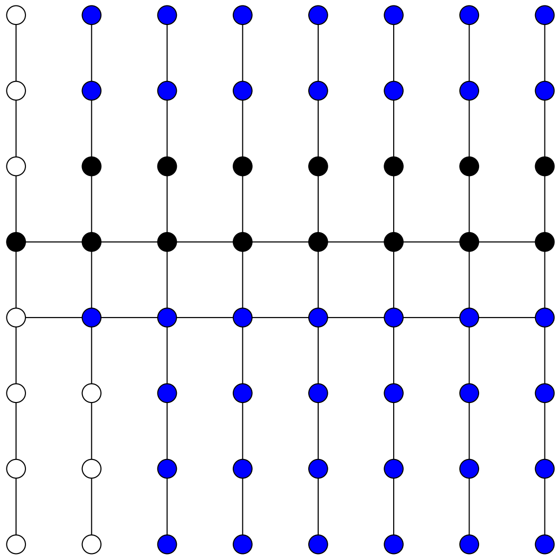

Figure 5.3: Rooted product of $P_8\Box P_2$ and P_4.

that $Z_c(G) = 2n - 1$.

Case 4.

Assume that the black vertices are from the set C, except one. This case is similar to that of Case 3, since the sub graph induced by the connected zero forcing sets B and C are isomorphic. Combining the Cases $1, 2, 3$ and 4, we have

$$Z_c(G) \geq 2n \tag{5.20}$$

From (5.19) and (5.20), $Z_c(G) = 2n$.

Case 5.

Let $2 \leq k \leq 4$. If we color any one of the pendant vertices of G as black, then this pendant vertex forms a connected k- forcing set for G. Hence, $Z_{ck}(G) = 1$ if $2 \leq k \leq 4$. □

Theorem 5.2.4. *Let G be the rooted product of the grid graph $P_n\Box P_n$ and the* *rooted path P_t rooted with the pendant vertex of P_t, where $t \geq 2$. Then,*

$$Z_{ck}(G) \begin{cases} \leq n^2 \ if \ k = 1 \\ \leq \ n \ if \ k{=}2 \\ = 1 \ if \ 3 \leq k \leq \Delta(G) = 5. \end{cases}$$

Proof. **Case 1.**

Assume that $k = 1$. In this case, color all vertices of the Cartesian product $P_n \square P_n$ in G as black. We can easily observe that these n^2 black vertices form a connected zero forcing set for G. Thus, $Z_c(G) \leq n^2$.

Case 2.

Assume that $k = 2$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n in $P_n \square P_n$ of G. Color these vertices as black in G. Now, we can easily verify that these vertices form a connected 2- forcing set for G. Therefore, $Z_{c2}(G) \leq n$.

Case 3.

Assume that $3 \leq k \leq 5$. Let P_t be the path rooted at the vertex u_1 in G. Color the pendant vertex of the path P_t in G as black. Let it be the vertex v. Clearly, we can form a derived coloring for G with this black vertex v. Thus, $Z_{ck}(G) = 1$, as desired. □

We strongly believe that the bounds in the above theorem are sharp.

Theorem 5.2.5. *Let G be the rooted product of $P_n \square P_2$ and the rooted cycle C_m, where $n \geq 3$. Then,*

$$Z_{ck}(G) \begin{cases} \leq 4n \ if \ k = 1 \\ \leq 2n \ if \ k = 2 \\ = 1 \ if \ 3 \leq k \leq \Delta(G) = 5. \end{cases}$$

Proof. **Case 1.**

Assume that $k = 1$. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n in G and let $v_1, v_2, \ldots, v_n$ be the vertices corresponding to the copy of the path P_n in G. We observe that $deg(u_1) = deg(v_1) = deg(u_n) = deg(v_n) = 4$. The remaining vertices in G have degree 5. We note that any connected k-forcing set for G must contain all the vertices of $P_n \Box P_2$. Otherwise, the k- forcing set will be disconnected. Without loss of generality, assume that we have a set consisting of $2n$ connected black vertices from $P_n \Box P_2$ in G. The remaining vertices are considered to be white. To force the white vertices in each cycle, we must select a black vertex from each cycle rooted at the vertices $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and each black vertex should be adjacent to the respective vertex at which each cycle is rooted. Therefore, we need to choose $2n$ number of black vertices from the $2n$ copies of the cycle C_m. Now, we have a set of $4n$ black vertices which forces the remaining vertices of G as black and this set of $4n$ black vertices is connected. Therefore, $Z_c(G) \leq 4n$.

Case 2.

Suppose that $k = 2$. We can observe that the connected 2-forcing set for G must contain all the vertices of $P_n \Box P_2$. Otherwise, the zero forcing set will be disconnected. If we take the $2n$ black vertices from $P_n \Box P_2$ in G, then these vertices will 2-forces the remaining white vertices as black. Hence, $Z_{c2}(G) \leq 2n$.

Case 3.

Let $3 \leq k \leq 5$. Consider the cycle rooted at the vertex u_1, say $C_m^{(1)}$. Choose a vertex from $C_m^{(1)}$ of degree 2 as black. Clearly, this black vertex will give a derived coloring for G. Hence, $Z_{ck}(G) = 1$.

We strongly believe that the above bounds are sharp. $\square$

Theorem 5.2.6. *Let G be the rooted product of the circular ladder graph $C_n \Box K_2$ and the rooted path P_t, $t \geq 2$, rooted with the pendant vertex of P_t. Then, $Z_c(G) = 2n$.*

Proof. Represent the vertices of $C_n \Box K_2$ in G as $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ and $P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(n)}$ be the copies of the path P_t rooted at $v_1, v_2, \ldots, v_n$. Let $v_1 = p_1^1$, $v_2 = p_1^2, \ldots, v_n = p_1^n$, where $p_1^1, p_1^2, \ldots, p_1^n$ are the pendant vertices of the paths rooted with the vertices $v_1, v_2, \ldots, v_n$ respectively, where

$$V[P_t^{(1)}] = \{p_1^1, p_2^1, \ldots, p_t^1\}$$

$$V[P_t^{(2)}] = \{p_1^2, p_2^2, \ldots, p_t^2\}$$

$$\ldots$$

$$\ldots$$

$$V[P_t^{(n)}] = \{p_1^n, p_2^n, \ldots, p_t^n\}$$

We examine the different possibilities for forming a connected zero forcing set as follows.

Case 1.

Assume that we have a connected zero forcing set consisting of $(2n-1)$ black vertices $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_{n-1}\}$ for G. Then, the black vertex u_n has two white neighbors v_n and a vertex of the path rooted at u_n. So further forcing from the black vertex u_n is not possible. which is a contradiction to our assumption.

Case 2.

Suppose that $Z_c = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots u_{n-1}\}$ is a connected zero forcing set for G. Then, we can easily observe that further forcing from the black vertex v_n is not possible, because it has two white neighbors, a contradiction to our assumption.

Case 3.

The case for forming a connected zero forcing set by taking $(2n-1)$ black pendant vertices of the paths is ruled out, since the subgraph induced by these black vertices is not connected.

Case 4.

Consider a connected zero forcing set of $(2n-1)$ black vertices having the following combinations.

Subcase 4.1.

Combination of the vertices u_i and the vertices of the path $Q_t^{(i)}$, where $Q_t^{(i)}$ is the path rooted at the vertex u_i and $i = 1, 2, \ldots, n$.

Subcase 4.2.

Combination of the vertices v_i and the vertices of the path $P_t^{(i)}$, where $i = 1, 2, \ldots, n$.

Subcase 4.3.

Combination of the vertices u_i, v_i and the vertices of the path $P_t^{(i)}$, $i = 1, 2, \ldots, n$.

Subcase 4.4.

Combination of the vertices u_i, v_i and the vertices of the path $Q_t^{(i)}$, where $i = 1, 2, \ldots, n$.

Subcase 4.5.

Combination of the vertices u_i, v_i, the vertices of $P_t^{(i)}$ and the vertices of $Q_t^{(i)}$, $i = 1, 2, \ldots, n$.

We can observe that none of the above combinations form a connected zero forcing set for G. Hence from the above cases, we can conclude that

$$Z_c(G) \geq 2n \tag{5.21}$$

To claim $Z_c(G) \leq 2n$, we proceed as follows.

Select $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ as the $2n$ black vertices. The remaining vertices are considered to be white. Then, the black vertex $v_1 \to p_2^1$ to black, the black vertex $p_2^1 \to p_3^1$ to black, $\ldots, p_{t-1}^1 \to p_t^1$ to black. Similarly, all the vertices of the paths rooted at the black vertices $v_2, v_3, \ldots, v_n$ will be colored as black. The same argument holds good for the vertices of the paths rooted at the black vertices $u_1, u_2, \ldots, u_n$. So the set $Z_c = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ generates a connected zero forcing set for G. The cardinality of the set Z_c is $2n$. Hence,

$$Z_c(G) \leq 2n \tag{5.22}$$

From (5.21) and (5.22), $Z_c(G) = 2n$. □

Theorem 5.2.7. *Let G be the rooted product of a circular ladder graph $C_n \square K_2$ and the rooted cycle C_k. Then, $Z_c(G) \leq 4n$.*

Proof. Let $A = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertex set of the circular ladder graph $C_n \square K_2$ in G, where $u_1, u_2, \ldots, u_n$ are the vertices of the inner cycle. Let $C_k^{(1)}, C_k^{(2)}, \ldots, C_k^{(n)}$ be the copies of the cycle C_k rooted at the vertices $v_1, v_2, \ldots, v_n$ respectively and $D_k^{(1)}, D_k^{(2)}, \ldots, D_k^{(n)}$ be the copies of the cycle C_k rooted at the vertices $u_1, u_2, \ldots, u_n$ respectively. Let the vertex set of the cycles $C_k^{(1)}, C_k^{(2)}, \ldots, C_k^{(n)}$ and $D_k^{(1)}, D_k^{(2)}, \ldots, D_k^{(n)}$ be as follows.

$$V[C_k^{(1)}] = \{c_1^1, c_2^1, \ldots, c_k^1\}$$

$$V[C_k^{(2)}] = \{c_1^2, c_2^2, \ldots, c_k^2\}$$

$$\ldots$$

$$\ldots$$

$$V[C_k^{(n)}] = \{c_1^n, c_2^n, \ldots, c_k^n\}$$

$$V[D_k^{(1)}] = \{d_1^1, d_2^1, \ldots, d_k^1\}$$

$$V[D_k^{(2)}] = \{d_1^2, d_2^2, \ldots, d_k^2\}$$

$$\ldots$$

$$\ldots$$

$$V[D_k^{(n)}] = \{d_1^n, d_2^n, \ldots, d_k^n\}$$

Let $v_1 = c_1^1, v_2 = c_1^2, \ldots, v_n = c_1^n$ and $u_1 = d_1^1, u_2 = d_1^2, \ldots, u_n = d_1^n$.

We generate a connected zero forcing set for the graph G as follows. Consider the set $Z_c = \{v_1, c_2^1, v_2, c_2^2, \ldots, v_n, c_2^n, u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ldots, d_2^n\}$. Color the vertices in Z_c as black and the remaining vertices as white. Then, the vertices in Z_c can force the remaining white vertices of the cycles $C_k^{(1)}, C_k^{(2)}, \ldots, C_k^{(n)}$ and $D_k^{(1)}, D_k^{(2)}, \ldots, D_k^{(n)}$ as black by repeatedly applying the color change rule. Thus, the set $Z_c = \{v_1, c_2^1, v_2, c_2^2, \ldots, v_n, c_2^n, u_1, u_2, \ldots, u_n, d_2^1, d_2^2, \ \ldots, d_2^n\}$ generates a connected zero forcing set for G. The cardinality of the set Z_c is $4n$. Hence, $Z_c(G) \leq 4n$. □

We strongly believe that the above bound is sharp.

Theorem 5.2.8. *Let G be the rooted product of a path P_n and the rooted path P_t rooted with the pendant vertex of P_t, where $n \geq 3, t \geq 2$. Then,*

$$Z_{ck}(G) = \begin{cases} n \ if \ k = 1 \\ 1 \ if \ k \geq 2. \end{cases}$$

Proof. Denote the vertices of the path P_n in G by $u_1, u_2, \ldots, u_n$. Let $u_1 = p_1^1$, $u_2 = p_1^2, \ldots,$ $u_n = p_1^n$, where $p_1^1, p_1^2, \ldots, p_1^n$ are the vertices of copies of the path P_t rooted at $u_1, u_2, \ldots, u_n$ respectively.

Claim.

Any set consisting of $(n-1)$ black vertices will never form a connected zero forcing set for the graph G. For, consider the following cases.

Case 1.

Select the pendant vertex of each path rooted at the vertices $u_1, u_2, \ldots, u_{n-1}$. Clearly, they cannot form a connected zero forcing set for G.

Case 2.

Form a set of $(n-1)$ black vertices from the vertices of the paths rooted at the vertices $u_1, u_2, \ldots, u_n$. Then, we can easily observe that this set will not be connected and hence will not form a connected zero forcing set for G.

Case 3.

Assume that $Z_c = \{u_1, u_2, \ldots, u_{n-1}\}$. Color the vertices in the set Z_c as black and the vertices in $V(G) - Z_c$ as white. Then, we can see that the vertices of the paths rooted at $u_1, u_2, \ldots, u_{n-2}$ can be colored as black by applying the color change rule. We observe that the forcing from the black vertex u_{n-1} is not possible, because u_{n-1} has two white neighbors. Therefore, the set Z_c cannot generate a connected zero forcing set for G. In view of the above cases, we have

$$Z_c(G) \geq n \tag{5.23}$$

To prove the reverse part, let $Z_c = \{u_1, u_2, \ldots, u_n\}$. Assign black color to the vertices in the set Z_c. Then, we can see that the set Z_c generates a connected zero forcing set for G. Therefore,

$$Z_c(G) \leq n \tag{5.24}$$

Hence from (5.23) and (5.24), $Z_c(G) = n$.

Case 4.

When $k = 2$. Let u be an arbitrary vertex of G such that $deg(u) \neq 3$. Color the vertex u as black. Clearly, the vertex u gives a derived coloring for G. Hence, $Z_{c2}(G) = 1$.

When $k = 3$, any black vertex of G forms a connected zero forcing set. Hence the result follows.

Next we compute the connected zero forcing number of the CP- graph considered in [8].

Theorem 5.2.9. *Let G be the CP-graph C_3P_r. Then,*

$$Z_{ck}(G) = \begin{cases} 2r \ if \ k = 1 \\ \\ 1 \ if \ k = 2, 3. \end{cases}$$

Proof. Denote the cycles in G by $C_3^{(1)}, C_3^{(2)}, \ldots, C_3^{(r)}$. Let the vertex set of the cycles in C_3P_r be

$$V[C_3^{(1)}] = \{c_1^1, c_2^1, c_3^1\}$$

$$V[C_3^{(2)}] = \{c_1^2, c_2^2, c_3^2\}$$

$$\ldots$$

$$\ldots$$

$$V[C_3^{(r)}] = \{c_1^r, c_2^r, c_3^r\}$$

Case 1.

Assume that $k = 1$. We prove the result by mathematical induction on the number of cycles r in the CP-graph. Assume that $r = 1$. In this case, G is the cycle C_3. Therefore, $Z_c(C_3) = 2$ and the result is true for $r = 1$.

Assume that the result is true for all C_3P_r graphs with $(r-1)$ cycles C_3, where $r > 2$. Let C be the end cycle connected to the rest of the C_3P_r graph by an edge $e = ab$, where $a \in [V(C_3P_r) - V(C)]$ and $b \in V(C)$. Let $Y = \{a, b\}$ be the cut set where $a \in \langle V(C_3P_r) - V(C)\rangle$ and $b \in V(C)$.

Now, the induced sub graph $\langle V(C_3P_r) - V(C)\rangle$ is a C_3P_r graph with $(r-1)$ cycles. Therefore by our assumption, $Z_c(\langle V(C_3P_r) - V(C)\rangle) = 2(r-1) = 2r-2$. Let W be the minimum zero forcing set of $\langle V(C_3P_r) - V(C)\rangle$ with $\mid W \mid = 2r-2$. Let u_1 and u_2 be two white neighbors of the vertex b in C. Since the vertex a is black, it forces the vertex b to black. Since the black vertex b has two white neighbors, further forcing from the vertex b is not possible. In order to make the zero forcing set connected, we have to include the black vertex b in the zero forcing set. Therefore, our new connected set is $W \cup \{b\}$. The set $W \cup \{b\}$ cannot force the remaining two white vertices (u_1 and u_2) adjacent to b. Therefore, we need to include either u_1 or u_2 into the set $W \cup \{b\}$. Let it be u_1 and color u_1 as black. Hence by induction

$$Z_c(G) = \mid W \cup \{b, u_1\} \mid$$

$$= 2r - 2 + 2 = 2r.$$

Therefore, $Z_c(G) = 2r$, if $k = 1$.

Case 2.

Assume that $k = 2, 3$. In this case, any black vertex of degree 2 will form a connected k-forcing set. □

Theorem 5.2.10. [9] *Let G be a graph with girth at least 4 and minimum degree $\delta(G) \geq 3$. Then, $Z_c(G) \geq \delta(G) + 1$.*

Theorem 5.2.11. *Let G be the circular ladder graph $C_n \Box K_2$, where $n \geq 4$. Then,*

$$Z_{ck}(G) = \begin{cases} 4 \ if \ k = 1 \\ 2 \ if \ k = 2 \\ 1 \ if \ k = 3 = \Delta(G). \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n$ be the vertices of the circular ladder graph G, where $u_1, u_2, \ldots, u_n$ are the vertices of the inner circle.

Case 1.

Assume that $k = 1$. By the Theorem 5.2.10, since $\Delta(G) = \delta(G) = 3$ and the girth is at least 4 ,we have

$$Z_c(G) \geq 3 + 1 = 4 \tag{5.25}$$

To establish the reverse inequality, we proceed as follows.

Without loss of generality, choose four adjacent vertices u_1, u_2, u_3 and v_1 in G. Color these vertices as black and assume that the remaining vertices are white. Then, clearly the black vertex $u_2 \rightarrow v_2$ to black. Now, the black vertex $v_2 \rightarrow v_3$ to black. Again, the black vertex $u_3 \rightarrow u_4$ to black, $v_3 \rightarrow v_4$ to black. Proceeding like this, the black vertex $u_{n-1} \rightarrow u_n$ to black and $v_{n-1} \rightarrow v_n$ to black. Therefore, the set $Z_c = \{u_1, u_2, u_3, v_1\}$ forms a connected zero forcing set for G. Here the cardinality of the set Z_c is 4. Hence,

$$Z_c(G) \leq 4 \tag{5.26}$$

Therefore from (5.25) and (5.26), $Z_c(G) = 4$.

Case 2.

Assume that $k = 2$. In this case , clearly a set consisting of any two adjacent black vertices forms a connected 2-forcing set for G and with one black vertex, obtaining a connected 2-forcing set is not possible. Hence, $Z_{c2}(G) = 2$.

Case 3.

Suppose that $k = 3$. It is obvious that any black vertex gives a derived coloring for G. Therefore, the result follows. □

5.3 Connected k-Forcing Number of Square of Graphs

In this section, we deal with the connected k-forcing number of square graph of path $P_n, n \geq 3$ and the cycle $C_n, n \geq 5$.

Theorem 5.3.1. *Let G denotes the square graph of a path P_n, where $n \geq 3$. Then, the connected zero forcing number of G is 2.*

Proof. Represent the vertices of G by $u_1, u_2, \ldots, u_n$. It is obvious that with one black vertex, we cannot obtain a derived coloring for G. Therefore,

$$Z_c(G) \geq 2 \tag{5.27}$$

On the other hand, without loss of generality, color the vertices u_1 and u_2 as black and the remaining vertices as white. Then, the black vertex u_1 forces u_3 to black, the vertex u_2 forces u_4 to black, the vertex u_3 forces u_5 to black and so on till all vertices of G are colored black. Thus, the set $Z_c = \{u_1, u_2\}$ forms a connected zero forcing set for G. Here $\mid Z_c \mid= 2$. Hence,

$$Z_c(G) \leq 2 \tag{5.28}$$

From (5.27) and (5.28), $Z_c(G) = 2$. □

Theorem 5.3.2. *The connected zero forcing number of the square graph of a cycle C_n is 4, where $n \geq 5$.*

Proof. Let G denotes the square graph of the cycle C_n, $n \geq 5$. It is clear that G is a 4-regular graph. Therefore $\Delta(G) = \delta(G) = 4$. Since $Z_c(G) \geq Z(G)$ and $Z(G) \geq \delta(G)$, we have

$$Z_c(G) \geq 4 \tag{5.29}$$

In order to establish the reverse inequality, choose any four adjacent vertices of G. Let the vertices be u_1, u_2, u_3 and u_n. Color these four vertices as black and the remaining vertices as white. In G, the vertices adjacent to the vertex u_1 are u_2, u_3, u_n and u_{n-1}, of which u_2, u_3, u_n are black. So the black vertex u_1 forces the vertex u_{n-1} to black. Consider the black vertex u_2. The adjacent vertices of u_2 are u_1, u_n, u_3, and u_4. Of these vertices, u_1, u_3, u_n are already black. Therefore, the vertex u_2 forces u_4 to black. Again, consider the black vertex u_3. At this stage, the vertex u_3 has only one white neighbor u_5. Hence u_3 forces u_5 to black, u_4 forces u_6 to black and so on. Finally, consider the black vertex u_{n-4}. The vertex u_{n-4} has 4 neighbours $u_{n-5}, u_{n-6}, u_{n-3}, u_{n-2}$ of which u_{n-2} is the only white vertex. Therefore, the vertex u_{n-4} forces u_{n-2} to black. The vertex u_{n-3} is already colored black by the vertex u_{n-5}. Therefore, the set $Z_c = \{u_1, u_2, u_3, u_n\}$ generates a connected zero forcing set for the graph G. Hence we have

$$Z_c(G) \leq 4 \tag{5.30}$$

From (5.29) and (5.30), $Z_c(G) = 4$. □

Chapter 6

2-Distance Forcing Number of a Graph

This Chapter introduces the notion of 2*-distance forcing number* $Z_{2d}(G)$ *of a graph* G*. Necessary definitions for the development of this chapter are given in Section 1. In Section* 2*, we find the exact value of* $Z_{2d}(G)$ *of some graphs such as path, cycle, wheel graph, Petersen graph, star graph, friendship graph, pineapple graph and the fan graph. In Section* 3*, we focus on investigating the* 2*-distance forcing number of some classes of graphs with diameter lies between* 2 *and* 5.

6.1 Introduction

Here we discuss a generalization of the zero forcing set based on the distance in graphs. We use the following definitions for the further development of this chapter.

[1]This chapter has been published in the *Journal of Mathematical and Computational Science*, Volume 10, Number 6, 2020, Pages 2233-2248

Definition 6.1.1. [2] *Open neighborhood and closed neighborhood. The set of all vertices adjacent to a vertex v excluding the vertex v is called the open neighborhood of v and is denoted by $N(v)$. The set of all vertices adjacent to a vertex v including the vertex v is called the closed neighborhood of v and is denoted by $N[v]$. i.e, $N[v] = \{v\} \cup N(v)$.*

Definition 6.1.2. [17] *The length of a $u - v$ path in a graph G is the number of edges in the path $u - v$. The distance between two vertices u and v in a graph G is the length of the shortest path between u and v.*

Definition 6.1.3. *The* ***2-distance open neighborhood*** *of a vertex u in a graph G is the set of all vertices which are at a distance at most two from u excluding the vertex u and is denoted by $N_{2d}(u)$. The 2- distance closed neighborhood of a vertex u is denoted as $N_{2d}[u]$, and is defined as $N_{2d}[u] = \{u\} \cup N_{2d}(u)$.*

Definition 6.1.4. ***The 2-distance vertex or 2-distance neighbor****. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. If a vertex v of G lies at a distance at most two from the vertex u of G, then we say that the vertex v is a 2-distance vertex or 2-distance neighbor of u.*

For example, consider the graph G depicted in Figure 6.1. In Figure 6.1, the vertices p, q, u, v are the 2-distance neighbors of the vertex r. Therefore, the 2-distance closed neighborhood of the vertex r is $N_{2d}[r] = \{r, p, q, u, v\}$. Hence $\mid N_{2d}[r] \mid = 5$.

Definition 6.1.5. ***Color change rule****. Let G be a graph with each vertex colored either black or white. If a black colored vertex has exactly one white colored 2-distance neighbor, then change the color of the white vertex to black.*

When the color change rule is applied to an arbitrary vertex v to change the

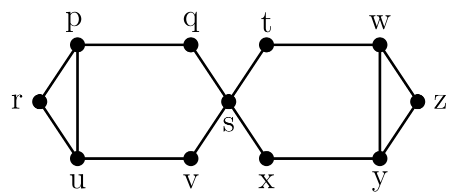

Figure 6.1: The graph G

color of the vertex u to black, then we say that the vertex v forces the vertex u to black and we denote it as $v \to u$ to black.

Definition 6.1.6. *A* 2***-distance forcing set*** *for a graph G is a subset Z_{2d} of vertices of G such that if initially the vertices in Z_{2d} are colored black and the remaining vertices are colored white, the derived coloring of G is all black.*

Definition 6.1.7. ***The*** 2***-distance forcing number*** $Z_{2d}(G)$ *is the minimum of $\mid Z_{2d} \mid$ over all 2-distance forcing sets $Z_{2d} \subseteq V(G)$.*

6.2 Exact Values of $Z_{2d}(G)$

In this section, we consider some simple graphs such as path, cycle, wheel graph, friendship graph, star graph, ladder graph and the complete bipartite graph for computing the 2-distance forcing number $Z_{2d}(G)$. We begin with the path P_n.

Theorem 6.2.1. *The 2-distance forcing number of a path P_n is 2, where $n \geq 3$. That is, $Z_{2d}(P_n) = 2$.*

Proof. Represent the vertices of the path P_n as $u_1, u_2, \ldots, u_n$. The pendant vertices u_1 and u_n have the least number of 2-distance neighbors. Note that the vertices u_2 and u_3 are the only 2-distance neighbors of u_1. We start with the vertices u_1 and u_2. Assign black color to the vertices u_1 and u_2. The remaining vertices in P_n are assumed to be white. Now, the vertex $u_1 \to u_3$ to black. Consider the black vertex u_2. The 2-distance neighbors of u_2 are u_1, u_3 and u_4 of which the vertices u_1, u_3 are already black. Hence the black vertex $u_2 \to u_4$ to black and so on. Therefore, the set $Z_{2d} = \{u_1, u_2\}$ forms a 2-distance forcing set for the path P_n. The number of vertices in the set Z_{2d} is two. We wish to say that with only one black vertex, obtaining a 2-distance forcing set for the path P_n is not possible. Hence, $Z_{2d}(P_n) = 2$. □

Next we find the 2-distance forcing number of a cycle C_n.

Theorem 6.2.2. *Let G be the cycle C_n, where $n \geq 5$. Then, $Z_{2d}(G) = 4$.*

Proof. Let the vertices of G be denoted by $u_1, u_2, \ldots, u_n$. Each vertex u_i, where $i = 1, 2, \ldots, n$, in G has four 2-distance neighbors. From the cycle C_n we can easily observe that with any three black vertices we cannot obtain a derived coloring for G, because each black vertex has at least two 2-distance white neighbors. Therefore,

$$Z_{2d}(G) \geq 4 \tag{6.1}$$

On the other hand, take four arbitrary adjacent vertices u_{n-1}, u_n, u_1 and u_2 of G. Assign them the black color. The remaining vertices are considered to be white. Consider the black vertex u_1. The vertices u_{n-1}, u_n, u_2 and u_3 are the 2-distance neighbors of the vertex u_1, of which the three vertices u_{n-1}, u_n, u_2 are already black. So the vertex $u_1 \to u_3$ to black. Again, the vertices u_n, u_1, u_3 and u_4 are at a distance at most two from the vertex u_2. Since the vertices u_1, u_n

and u_3 are black, the black vertex $u_2 \to u_4$ to black. Proceeding like this, we will get a derived coloring for G. Hence the set $Z_{2d} = \{u_{n-1}, u_n, u_1, u_2\}$ generates a 2-distance forcing set for G. Therefore,

$$Z_{2d}(G) \leq 4 \tag{6.2}$$

Therefore from (6.1) and (6.2), $Z_{2d}(G) = 4$. □

Definition 6.2.3. [17] *The maximum distance between any pair of vertices in a graph G is called the diameter of G and is denoted by diam(G).*

Theorem 6.2.4. *Let G be a connected graph of order $n \geq 3$ with $diam(G) = 2$. Then, $Z_{2d}(G) = n - 1$.*

Proof. Without loss of generality, assume that G is a connected graph of order $n \geq 3$ and $diam(G) = 2$. We can easily assert that in G if we color exactly one vertex as white and the remaining vertices as black, then any black vertex will force the white vertex to black. Therefore,

$$Z_{2d}(G) \leq n - 1 \tag{6.3}$$

To prove the reverse inequality, assume that there exists a 2- distance forcing set consisting of $(n-2)$ black vertices. Let the black vertices be $v_1, v_2, \ldots, v_{n-2}$. Since the diameter of the graph G is 2, each black vertex v_i, where $1 \leq i \leq n-2$, will have two white vertices in it's 2-distance open neighborhood. Therefore, derived coloring for G is not possible. A contradiction to our assumption. Hence,

$$Z_{2d}(G) \geq n - 1 \tag{6.4}$$

From (6.3) and (6.4), $Z_{2d}(G) = n - 1$. □

Corollary 6.2.5. *The following are true.*

(i) Let G be the wheel graph with n vertices. Then, $Z_{2d}(G) = n - 1$.

(ii) If G is the Petersen graph with n vertices, then $Z_{2d}(G) = n - 1$.

(iii) For the star graph $K_{1,n}$, where $n \geq 2$, $Z_{2d}(K_{1,n}) = n$.

(iv) Let G be the friendship graph with n vertices. Then, $Z_{2d}(G) = n - 1$.

Theorem 6.2.6. *For a complete graph K_n of order $n \geq 3$, $Z_{2d}(K_n) = n - 1$.*

Proof. The proof is obvious. □

More graphs with $Z_{2d}(G) = n - 1$ can be obtained as follows.

Definition 6.2.7. [10] *The pineapple graph, denoted by K_m^k, is the graph formed by coalescing any vertex of the complete graph K_m with the star graph $K_{1,k}$ ($m \geq 3$, $k \geq 2$). The number of vertices in K_m^k is $m + k$, the number edges in K_m^k is $\frac{m^2-m+2k}{2}$ and the diameter of K_m^k is 2.*

Theorem 6.2.8. *Let G be the pineapple graph K_m^k. Then, $Z_{2d}(G) = m + k - 1$.*

Proof. Since the graph G is connected with order $n \geq 3$ and diam(G)=2, the proof follows by the Theorem 6.2.4. □

Definition 6.2.9. [12] *The fan graph, denoted by F_m, is the graph obtained by the join $K_1 \vee P_n$, where P_n is a path on n vertices and K_1 is the empty graph. The order of the fan graph F_m is $n + 1$ and the diam(F_m) $= 2$.*

Theorem 6.2.10. *For a fan graph F_m, the 2-distance forcing number is n.*

Proof. Since the diam(F_m)=2 and the graph F_m is connected with order $n \geq 3$, the proof follows by the Theorem 6.2.4. □

6.3 Graphs for which $2 < diam(G) < 5$

We start this section by computing the 2-distance forcing number of a graph having diameter 3.

Theorem 6.3.1. *If G is the graph obtained by appending one pendant edge to each vertex of a complete graph K_m, where $m \geq 3$, then $Z_{2d}(G) = 2(m-1)$.*

Proof. Let $u_1, u_2, \ldots, u_m$ be the vertices of the complete graph K_m and let $v_1, v_2, \ldots, v_m$ be the vertices attached to the vertices $u_1, u_2, \ldots, u_m$ respectively in G. It suffices to construct a 2-distance forcing set for G consisting of $2(m-1)$ black vertices. Without loss of generality, take the vertex v_1. Clearly, the vertex v_1 has m 2-distance neighbors $u_1, u_2, \ldots, u_m$. Assign black color to the vertex v_1. To start the color change rule from the vertex v_1, we have to color at least $(m-1)$ vertices out of $u_1, u_2, \ldots, u_m$ as black. Let $u_1, u_2, \ldots, u_{m-1}$ be the black vertices. The remaining vertices are colored as white. Then, $v_1 \to u_m$ to black. Now, consider the black vertex u_1. The vertices $v_2, v_3, \ldots, v_m$ are the 2-distance white neighbors of u_1. Let $v_2, v_3, \ldots, v_{m-1}$ be black. Then, the black vertex $u_1 \to v_m$ to black. Therefore, the set $Z_{2d} = \{v_1, u_1, u_2, \ldots, u_{m-1}, v_2, v_3, \ldots, v_{m-1}\}$ forms a 2-distance forcing set for G. The cardinality of the set Z_{2d} is $1+m-1+m-2 = 2(m-1)$. Moreover, we can easily observe that any set consisting of less than $2(m-1)$ black vertices will never give a derived coloring for G. Hence, $Z_{2d}(G) = 2(m-1)$. □

Next we find the 2-distance forcing number of some graphs G with $diam(G) = 4$.

Definition 6.3.2. [12] *A gear graph or the bipartite wheel graph, denoted by G_n, is the graph derived from the wheel graph W_n by attaching a vertex between* *every pair of adjacent vertices of the n cycle. The gear graph G_n contains $2n+1$ vertices and $3n$ edges.*

Theorem 6.3.3. *For a gear graph G_n, $Z_{2d}(G_n) = n+2$, where $n \geq 4$.*

Proof. Let $V(G_n) = \{u, u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$, where u is the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the n cycle of the wheel graph W_n and $v_1, v_2, \ldots, v_n$ are the vertices with $u_iv_i, v_iu_{i+1} \in E(G_n)$, where $i+1$ is taken modulo n. We generate a 2-distance forcing set as follows. Consider the vertices $u_1, u_2, \ldots, u_n, u, v_1$ of G_n. Color these vertices as black and the remaining vertices in G_n as white. Now, we can easily verify that the vertex $u_2 \to v_2$ to black, $u_3 \to v_3$ to black and so on. Finally, $u_n \to v_n$ to black. Therefore, the set $Z_{2d} = \{u_1, u_2, \ldots, u_n, u, v_1\}$ forms a 2-distance forcing set for G_n and $\mid Z_{2d} \mid = n+2$. Hence,

$$Z_{2d}(G_n) \leq n+2 \tag{6.5}$$

Conversely, note that each vertex v_i, where $i = 1, 2, \ldots, n$, has five 2-distance neighbors. Therefore, we can observe that $Z_{2d}(G_n) \geq 5$. Since $Z_{2d}(G_n) \geq 5$, we can construct a 2-distance forcing set starting with five black vertices. If we start to construct a 2-distance forcing set consisting of five black vertices, then two cases arise for applying the color change rule.

Case 1.

Assume that the five black vertices are distributed among the outer cycle of the graph G_n and they are connected. We can easily verify that the central vertex u will be forced to black. To start further forcing, we have to include at least one black vertex from the cycle into the 2-distance forcing set. Then, we can force a maximum of one more white vertex to black. This process continues and at each step we can see that we have to add at least one black vertex from the cycle into the 2-distance forcing set. If $n = 3$, we can observe that we need at least 5 black vertices to form a 2-distance forcing set. If $n = 4$, then we need at least 6 black vertices to form a 2-distance forcing set. Similarly if $n = 5$, then we need at least 7 black vertices to form a 2-distance forcing set. Therefore, we need at least $(n + 2)$ black vertices to form a 2-distance forcing set for G_n. Hence,

$$Z_{2d}(G_n) \geq n + 2 \tag{6.6}$$

Case 2.

Assume that four black vertices are distributed among the outer cycle of the graph G_n and they are connected and the vertex u is black. We can easily verify that with these five black vertices, we can force a maximum of one more vertex to black. Now, to start further forcing, we have to include at least one black vertex from the cycle into the 2-distance forcing set. Then, we can force a maximum of one more white vertex to black. This process continues and at each step we can note that we have to add at least one vertex from the cycle into the 2-distance forcing set. If $n = 3$, then we can verify that we need at least 5 black vertices to form a 2-distance forcing set. If $n = 4$, then we need at least 6 black vertices to form a 2-distance forcing set. Similarly, if $n = 5$, then we need at least 7 black vertices to form a 2-distance forcing set. Therefore, we need at least $(n+2)$ black to form a 2-distance forcing set for G_n. Hence,

$$Z_{2d}(G_n) \geq n + 2 \tag{6.7}$$

In both Cases,

$$Z_{2d}(G_n) \geq n + 2 \tag{6.8}$$

Therefore from (6.5) and (6.8), $Z_{2d}(G_n) = n + 2$. □

Definition 6.3.4. [12] *The jelly fish graph, denoted by $J(m,n)$, is the graph obtained from a 4-cycle wxyzw by joining the vertex w and the vertex y by an edge and attaching the central vertex of $K_{1,m}$ to x and attaching the central vertex of $K_{1,n}$ to z.*

Theorem 6.3.5. *Let G be the jelly fish graph. Then, $Z_{2d}(G) = m + n + 1$.*

Proof. Our aim is to construct a 2-distance forcing set consisting of $m + n + 1$ black vertices. For, we proceed as follows.
Color all the m vertices $v_1, v_2, \ldots, v_m$ of $K_{1,m}$ as black. Also, we color the vertices w and y as black. The remaining vertices of G are assumed to be white. Then, any vertex v_i, where $i = 1, 2, \ldots, m$, can force the vertex x to black. Now, consider the black vertex x. The vertex x has only one 2-distance white neighbor z. So $x \to z$ to black. Again, the 2-distance white neighbors of z are $u_1, u_2, \ldots, u_n$, where $u_1, u_2, \ldots, u_n$ are the n vertices of $K_{1,n}$. If we color the vertices $u_1, u_2, \ldots, u_{n-1}$ as black, then $z \to u_n$ to black. Therefore, the set $Z_{2d} = \{v_1, v_2, \ldots, v_m, w, y, u_1, u_2, \ldots, u_{n-1}\}$ forms a 2-distance forcing set for G. The cardinality of the set Z_{2d} is $m + n + 1$. Hence,

$$Z_{2d}(G) \leq m + n + 1 \tag{6.9}$$

To establish the reverse inequality, we proceed as follows:
We can see that in any 2-distance forcing set for G, it is compulsory to include $(m - 1)$ black vertices from the sub graph $K_{1,m}$ and $(n - 1)$ black vertices from the sub graph $K_{1,n}$ of G. Now, we have a set from G consisting of $m+n-2$ black vertices and these vertices are from the sub graphs $K_{1,m}$ and $K_{1,n}$ of G. We can choose the vertices w, x, y and z to form a 2-distance forcing set for G. We claim that we need to choose at least 3 more vertices from w, x, y and z, otherwise we

arrive at a contradiction as follows:

If we choose any two vertices from w, x, y and z as black, then we can observe that each black vertex of G will have at least two 2-distance white neighbors. Hence further forcing is not possible. Therefore, it is not possible to form a 2-distance forcing set for G with $m-1+n-1+2 = m+n$ black vertices. Hence,

$$Z_{2d}(G) \geq m+n+1 \tag{6.10}$$

Therefore from (6.9) and (6.10), $Z_{2d}(G) = m+n+1$. □

Definition 6.3.6. [12] *The helm graph, denoted by H_n, is the graph constructed from the wheel graph W_n by appending a pendant edge to each vertex of the outer n cycle.*

Theorem 6.3.7. *Let G be the helm graph H_n, where $n \geq 4$. Then, $Z_{2d}(G) \leq n+1$.*

Proof. Let $u, u_1, u_2, \ldots, u_n$ be the vertices of the wheel graph W_n in G, where u is the central vertex. Also, let $v_1, v_2, \ldots, v_n$ be the pendant vertices of the graph G. Consider the set $Z_{2d} = \{v_1, v_2, \ldots, v_{n-2}, u_1, u_2, u_n\}$ of black vertices. The vertices in $V(G) - Z_{2d}$ are assumed to be white. We claim that the set Z_{2d} forms a 2-distance forcing set for G. For, we start with the black vertex v_1. Clearly, the vertex $v_1 \to u$ to black. Then, the vertex $v_2 \to u_3$ to black. Again, consider the black vertex v_3. The vertex $v_3 \to u_4$ to black and the process continues. Consider the black vertex v_{n-2}. We can see that the vertex $v_{n-2} \to u_{n-1}$ to black. Now, the black vertex $u_{n-2} \to v_{n-1}$ to black. Consequently, the vertex v_n will be colored as black. Thus, with the set Z_{2d}, we can force all the vertices of G to black. The cardinality of the set Z_{2d} is $n+1$. Therefore, $Z_{2d}(G) \leq n+1$. □

Definition 6.3.8. [12] *The sunflower graph, denoted by SF_n, is the graph obtained by taking a wheel graph W_n with central vertex u and the outer n cycle $u_1, u_2, \ldots, u_n$ and additional vertices $v_1, v_2, \ldots, v_n$, where v_i is joined by edges to u_i,u_{i+1}, where $i+1$ is taken modulo n.*

Theorem 6.3.9. *Let G be the sunflower graph SF_n. Then,*

$$Z_{2d}(G) \begin{cases} = n+3 & if \quad 3 \leq n \leq 5 \\ \\ \leq n+2 & if \quad n > 5. \end{cases}$$

Proof. Without loss of generality, divide the vertex set of SF_n into three sets, namely $A = \{u\}$, $B = \{u_1, u_2, \ldots, u_n\}$ and $C = \{v_1, v_2, \ldots, v_n\}$, where u is the central vertex, $u_1, u_2, \ldots, u_n$ are the vertices of the outer n cycle and $v_1, v_2, \ldots, v_n$ are the additional vertices.

Consider the following cases.

Case 1.

Assume that $n = 3$. Then, G is a graph with diameter 2. We have from the Theorem 6.2.4, $Z_{2d}(G) = O(G) - 1 = 7 - 1 = 6 = n + 3$.

Case 2.

Assume that $n = 4$. Let $A = \{u, u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ be the vertex set of the sunflower graph SF_4, where u is the central vertex, u_1, u_2, u_3, u_4 are the vertices of the outer cycle and v_1, v_2, v_3, v_4 are the additional vertices. Consider the set $Z_{2d} = \{v_1, v_2, v_4, u_1, u_2, u_3, u_4\}$ as the set of black vertices in SF_4. The remaining vertices in SF_4 are considered to be the white vertices. Clearly, the vertex $v_1 \to u$ to black. Then, $v_2 \to v_3$ to black and hence the set Z_{2d} forms a 2-distance forcing set for SF_4. Therefore,

$$Z_{2d}(G) \leq 7 = n + 3 \tag{6.11}$$

Conversely, suppose that there exists a 2-distance forcing set for SF_4 consisting of 6 black vertices. Then, there will be 3 vertices remain as white colored in SF_4. At least two of these white vertices will be there in the 2-distance neighborhood of each of the six black vertices. Since $\mid N_{2d}[u_i] \mid= 9$, where $1 \leq i \leq 4$, $\mid N_{2d}[u] \mid= 9$ and $\mid N_{2d}[v_i] \mid= 8$, where $1 \leq i \leq 4$, further forcing is not possible. Therefore with six black vertices, obtaining a 2-distance forcing set for SF_4 is not possible. Hence,

$$Z_{2d}(G) \geq 7 = n + 3 \tag{6.12}$$

Hence from (6.11) and (6.12), $Z_{2d}(G) = n + 3$.

Case 3.

Suppose that $n = 5$. Let $A = \{u, u_1, u_2, u_3, u_4, u_5, v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of the sunflower graph SF_5, where u is the central vertex, u_1, u_2, u_3, u_4, u_5 are the vertices of the outer cycle and v_1, v_2, v_3, v_4, v_5 are the additional vertices. Consider the set $Z_{2d} = \{v_1, v_2, v_4, v_5, u_1, u_2, u_3, u_5\}$ of black vertices. The remaining vertices are considered to be white. Clearly, the black vertex $v_1 \to u$ to black. Then, the black vertex $v_5 \to u_4$ to black. Consequently, the vertex v_3 will be colored as black. Thus, the set Z_{2d} generates a 2-distance forcing set for SF_5. The cardinality of the set Z_{2d} is $8 = n + 3$. Therefore,

$$Z_{2d}(G) \leq 8 = n + 3 \tag{6.13}$$

Conversely, assume that there exists a 2-distance forcing set for SF_5 consisting of 7 black vertices. Then, we have the following Subcases.

Subcase 1.

Assume that the vertex u is black.

Subcase 1.1.

Take five black vertices from the set B and one black vertex from the set C. In this case, further forcing is not possible since each black vertex has at least two 2-distance white neighbors. A contradiction to our assumption that there exists a 2-distance forcing set for SF_5 with 7 black vertices.

Subcase 1.2.

Choose one black vertex from the set B and five black vertices from the set C. Here also further forcing is not possible, a contradiction to our assumption.

Subcase 1.3.

Select four black vertices from the set B and two black vertices from the set C. In this case, we can force a maximum of one more white vertex to black, not all. Therefore, the color change rule is not possible. Again, a contradiction to our assumption.

Subcase 1.4.

Select two black vertices from the set B and four black vertices from the set C. In this case we cannot form a derived coloring for G, a contradiction to our assumption.

Subcase 1.5.

Select three black vertices from both the sets B and C. In this case also, we can force one more white vertex to black, not all. A contradiction to our assumption.

Subcase 2.

Assume that the vertex u is not black. Then, we have the following Subcases.

Subcase 2.1.

Select five black vertices from the set B and two black vertices from the set C. In this case the derived coloring is not possible, since forcing is not possible because each black vertex has at least two 2-distance white neighbors. A contradiction to our assumption.

Subcase 2.2.

Choose two black vertices from the set B and five black vertices from the set C. In this case also further forcing is not possible, a contradiction to our assumption.

Subcase 2.3.

Select four black vertices from the set B and three black vertices from the set C. Here also the derived coloring is not possible, because in this case we can force only one more white vertex to black, not all. Again, a contradiction to our assumption.

Subcase 2.4.

Choose three black vertices from the set B and four black vertices from the set C. We can easily observe that the color change rule is not applicable in this case. A contradiction to our assumption.

Hence from the above Subcases, we can conclude that

$$Z_{2d}(G) \geq 8 = n + 3 \tag{6.14}$$

Therefore from (6.13) and (6.14), $Z_{2d}(G) = n + 3$.

Case 4.

In this case, we assume that $n > 5$. Consider the set

$$Z_{2d} = \{u_1, u_2, \ldots, u_{n-2}, u_n, v_1, v_2, v_n\}$$

of black vertices and the remaining vertices in $V(G) - Z_{2d}$ are assumed to be white. Clearly, the black vertex $v_1 \to u$ to black. Then, the black vertex $v_2 \to v_3$ to black, since v_3 is the only 2-distance white vertex of v_2. Again, consider the black vertex v_3. Then, $v_3 \to v_4$ to black, $v_4 \to v_5$ to black, $v_5 \to v_6$ to black and so on. Now, the black vertex $v_{n-4} \to v_{n-3}$ to black. The white vertex u_{n-1} can be colored as black by any one of the black vertices $u_2, u_3, \ldots, u_{n-4}$. Then, the black vertex $v_{n-3} \to v_{n-2}$ to black, the black vertex $v_{n-2} \to v_{n-1}$ to black. Thus, the set $Z_{2d} = \{u_1, u_2, \ldots, u_{n-2}, u_n, v_1, v_2, v_n\}$ forms a 2-distance forcing set for G. The cardinality of the set Z_{2d} is $n + 2$. Hence, $Z_{2d}(G) \leq n + 2$. □

Chapter 7

Graphs with Large Diameter and Their 2-Distance Forcing Number

Necessary definitions and some preliminary results are given in Section 1. In Section 2, we investigate the 2-distance forcing number of the shadow graph of path, the middle graph of path, the S^{th} necklace, the n-sunlet graph, the triangular snake graph, the generalized friendship graph, the comet graph and the n-pan graph. The 2-distance forcing number of the rooted product of some graphs are discussed in the Section 3. Section 4 deals with the 2-distance forcing number of the square graph of path and cycle. Section 5 provides the 2-distance forcing number of the splitting graph of path. In Section 6, the ladder graph, the grid graph $P_n \square P_m$, the circular ladder graph are taken into discussion. The final Section deals with the 2-distance forcing number of the complement of path and cycle.

[1]This chapter has been published in the *Journal of Mathematical and Computational Science*, Volume 11, Number 2, 2021, Pages 1810-1837.

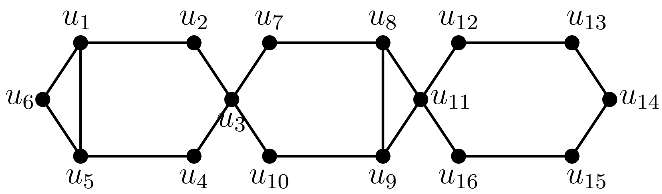

Figure 7.1: The graph G

7.1 Introduction

This chapter provides a generalization of the zero forcing set based on the distance in graphs. The following definitions are essential for the further development of this chapter.

Definition 7.1.1. *Let u be an arbitrary vertex in G. The 2-distance degree of u is defined as the number of vertices which are at a distance at most two from u including the vertex u. The 2-distance degree of the vertex u is denoted by $deg_{2d}(u)$.*

For example, consider the graph depicted in Figure 7.1. In Figure, $\deg_{2d}(u_1) = 6, \deg_{2d}(u_2) = 8, \deg_{2d}(u_3) = 9$, etc.

Definition 7.1.2. *Consider the 2-distance degree of all vertices in a graph G. The minimum among them is called the minimum 2-distance degree of the graph G and it is denoted by $\delta_{2d}(G)$. Similarly, the maximum among them is called the maximum 2-distance degree of the graph G. The maximum 2-distance degree of G is denoted by $\Delta_{2d}(G)$.*

We start with the following preliminary result.

Theorem 7.1.3. *Let G be a connected graph of order $n \geq 3$. Then, $Z_{2d}(G) \geq \delta_{2d}(G) - 1$.*

Proof. The proof is obvious. □

For a connected graph G of order $n \geq 3$, any Z_{2d} set forms a zero forcing set Z of G. Therefore, we have the following.

Theorem 7.1.4. *Let G be a connected graph of order $n \geq 3$. Then, $Z(G) \leq Z_{2d}(G)$.*

Proof. The proof is obvious. □

7.2 Certain Graph Classes and Their $Z_{2d}(G)$

In this section, we first find the 2-distance forcing number of the shadow graph of a path with large diameter.

Definition 7.2.1. [35] *The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2, and join each vertex u_1 in G_1 to the neighbors of the corresponding vertex u_2 in G_2.*

Theorem 7.2.2. *The 2-distance forcing number of the shadow graph $D_2(P_n)$ of a path P_n is $n+2$, where $n \geq 3$.*

Proof. Let G_1 and G_2 be the two copies of the path P_n. Denote the vertices of G_1 by $u_1, u_2, \ldots, u_n$ and that of G_2 by $v_1, v_2, \ldots, v_n$. Consider the set $Z_{2d} = \{u_1, u_2, \ldots, u_n, v_1, v_2\}$ of black vertices. The remaining vertices are assumed to be white. We shall show that this set Z_{2d} generates a 2-distance forcing set for $D_2(P_n)$. For, consider the black vertex u_1. We observe that v_3 is the only 2-distance white vertex of u_1. So the vertex $u_1 \to v_3$ to black. Again, the black vertex $u_2 \to v_4$ to black. Also, $u_3 \to v_5$ to black, $u_4 \to v_6$ to black, $\ldots$, $u_{n-2} \to v_n$ to black. Hence the set Z_{2d} generates a 2-distance forcing set for $D_2(P_n)$. The cardinality of the set Z_{2d} is $n+2$. Therefore,

$$Z_{2d}[D_2(P_n)] \leq n+2 \tag{7.1}$$

To prove the reverse inequality, we assume that there exists a 2-distance forcing set consisting of $(n+1)$ black vertices and we arrive at a contradiction. We consider the following cases.

Case 1.

Consider the set $Z_{2d} = \{u_1, u_2, \ldots, u_n, v_1\}$ of black vertices. The vertices in $V[D_2(P_n)] - Z_{2d}$ are assumed to be white. Then, further forcing is not possible because each black vertex has at least two 2-distance white neighbors. Therefore, the set Z_{2d} will never form a 2-distance forcing set for $D_2(P_n)$, a contradiction to our assumption.

Case 2.

Let $Z_{2d} = \{v_1, v_2, \ldots, v_n, u_1\}$ be a set of $(n+1)$ black vertices. By applying the same argument as in Case 1, we can easily observe that the set Z_{2d} cannot give a derived coloring for $D_2(P_n)$, again a contradiction to our assumption.

Case 3.

Let P_n be a path on n black vertices and Z_{2d} be it's vertex set. The vertices in V$[D_2(P_n)] - Z_{2d}$ are assumed to be white. Then, each black vertex will have at least three white neighbors in it's 2-distance open neighborhood. If we add one more black vertex to the set Z_{2d} to form a 2-distance forcing set , then all the black vertices will have at least two 2-distance white neighbors. Therefore, we cannot start the color change rule from any one of these black vertices. Hence from the above cases, we can conclude

$$Z_{2d}[D_2(P_n)] \geq n + 2 \tag{7.2}$$

From (7.1) and (7.2), $Z_{2d}[D_2(P_n)] = n + 2$. □

Definition 7.2.3. [36] *The middle graph of a graph G, denoted by $M(G)$, is the graph with vertex the set $V(G) \cup E(G)$ and such that two vertices in $M(G)$ are adjacent if and only if, either they are adjacent edges in G or one is a vertex of G and the other is an edge incident with it.*

Theorem 7.2.4. *Let G represents the middle graph of a path P_n, $n \geq 4$. Then, $Z_{2d}(G) = n$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n, v_1^1, v_2^1, v_{n-1}^1\}$, where $v_1, v_2, \ldots, v_n$ are the vertices of the path P_n and v_1^1, v_2^1, v_{n-1}^1 are the vertices corresponds to the edges $e_1, e_2, \ldots, e_{n-1}$ of the path P_n in G. Our aim is to generate a 2-distance forcing set for G consisting of n black vertices. For, color the vertex v_1 as black and the remaining vertices as white. Then, the 2-distance white neighbors of v_1 are v_1^1, v_2^1 and v_2. To begin the color change rule, assign black color to at least two of these vertices . Let v_1^1, v_2 to be black. Then, $v_1 \rightarrow v_2^1$ to black. Now, consider the black vertex v_2. The white colored 2-distance vertices of v_2 are v_3 and v_3^1. Color v_3 to black. Then, $v_2 \rightarrow v_3^1$ to black. Again, consider the black vertex v_3. Color v_4 to black. Then, $v_3 \rightarrow v_4^1$ to black and so on. Repeatedly apply this process, consider the black vertex v_{n-2}. Clearly, the vertex v_{n-2}^1 is black at this stage. The 2-distance white neighbors of v_{n-2} are v_{n-1} and v_{n-1}^1. Color v_{n-1} to black. Then, the vertex $v_{n-2} \rightarrow v_{n-1}^1$ to black, $v_{n-1}^1 \rightarrow v_n$ to black. Thus, we will obtain a derived coloring for G using the set $Z_{2d} = \{v_1, v_2 \ldots, v_{n-1}, v_1^1\}$ of black vertices. The cardinality of the set Z_{2d} is n. Hence,

$$Z_{2d}(G) \leq n \tag{7.3}$$

To prove the result, it suffices to show that $Z_{2d}(G) \geq n$. For this, we claim that a set having $(n-1)$ black vertices will never form a 2-distance forcing set for G. Consider the following cases. In each case, the vertices in $V(G) - Z_{2d}$ are considered to be white.

Case 1.

Let $Z_{2d} = \{v_1, v_2, \ldots, v_{n-1}\}$ be a set of black vertices. Then, the color change rule is not possible, since each black vertex v_i, $i = 1, 2, \ldots, n-1$, contains at least two 2-distance white neighbors.

Case 2.

Suppose that $Z_{2d} = \{v_1^1, v_2^1, \ldots, v_{n-1}^1\}$ be a set of black vertices. In this case, each black vertex has at least three 2-distance white neighbors. Therefore, further forcing is not possible.

Case 3.

Let P_{n-1} be a path on $(n-1)$ black vertices. Let Z_{2d} be it's vertex set. Then, P_{n-1} will be of the following types.

Type 1.

Choose the path P_{n-1} as $v_1^1 v_2^1 .... v_{n-1}^1$. In this case, it is obvious that the derived coloring for G is not possible since each black vertex has at least three 2-distance white neighbors.

Type 2.

Assume that n is odd. Select a path P_{n-1} of the type
$v_1 v_1^1 v_2 v_2^1 \ldots v^1_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor} v^1_{\lfloor \frac{n}{2} \rfloor}$. Note that we can consider only the last three black vertices of this path for further forcing. But, forcing from these vertices is not

possible since each black vertex has at least two 2-distance white neighbors. Again, choose a path $v_1^1 v_2 v_2^1 v_3, \ldots, v_{\lfloor \frac{n}{2} \rfloor} v^1_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor + 1}$. In this case, it is obvious that we cannot force all the remaining vertices of G to black.

Type 3.

Suppose that n is even.

Consider the paths $v_1 v_1^1 v_2 v_2^1, \ldots, v_{\lfloor \frac{n}{2} \rfloor - 1} v^1_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor}$ and $v_1^1 v_2 v_2^1 v_3, \ldots, v^1_{\lfloor \frac{n}{2} \rfloor - 1} v_{\lfloor \frac{n}{2} \rfloor} v^1_{\lfloor \frac{n}{2} \rfloor}$. We observe that neither of these paths generates a 2-distance forcing set for G. From the above mentioned cases, we can conclude that a set consisting of $(n-1)$ black vertices will never generates a 2-distance forcing set for G. Therefore,

$$Z_{2d}(G) \geq n \tag{7.4}$$

Hence from (7.3) and (7.4), $Z_{2d}(G) = n$. □

Definition 7.2.5. [10] *The S^{th} Necklace graph N_s is defined as a 3-regular graph that can be constructed from a $3s$-cycle by appending s central vertices. Each extra vertex is adjacent to 3-sequential cycle vertices. The order of a S^{th} Necklace graph N_s is $4s$ and the diameter is $\lfloor \frac{3s}{2} \rfloor$.*

Theorem 7.2.6. *If G is the S^{th} Necklace graph N_S, then $Z_{2d}(G) \leq s + 4$.*

Proof. We generate a 2-distance forcing set for G as follows. First we color all s vertices of G as black and the remaining vertices as white. Each s vertex will have five 2-distance vertices. Without loss of generality, start the color change rule from any one of the s vertices. To begin the color change rule from any one of the s vertices, we have to color at least four 2-distance vertices of that s vertex to black. Then, clearly these $(s + 4)$ black vertices generate a 2-distance forcing set for G. Hence, $Z_{2d}(G) \leq s + 4$. □

Definition 7.2.7. [28]. *The triangular snake graph T_n can be viewed as the graph formed by replacing each edge of the path P_n by a triangle C_3, thus adding $(n-1)$ vertices and $2(n-1)$ edges.*

Theorem 7.2.8. *Let G be a triangular snake graph with at least 2 triangles. Then, $Z_{2d}(G) = k+2$, where k is the number of triangles in G.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n in G. Represent the $(n-1)$ additional vertices in G by $v_1, v_2, \ldots, v_{n-1}$. We prove the result by induction on the number of triangles in G.

Assume that $k = 2$. Represent the vertices in G by u_1, u_2, u_3, v_1, v_2. Let $Z_{2d} = \{u_1, u_2, u_3, v_1\}$ be a set of black vertices and the remaining vertices are white. Then, any black vertex, say u_1, forces the vertex v_2 to black. So the set Z_{2d} forms a 2-distance forcing set for G. Here $| Z_{2d} |= 4$. We can easily observe that with 3 black vertices forming a 2-distance forcing set for G is not possible, because each black vertex has two 2-distance black neighbor. Therefore, $Z_{2d}(G) = 4 = k+2$.

Again, assume that $k = 3$. Let $u_1, u_2, u_3, u_4, v_1, v_2, v_3$ be the vertices of G. Suppose that $Z_{2d} = \{u_1, u_2, u_3, u_4, v_1\}$ be a set of black vertices. Clearly, the vertex $u_1 \to v_2$ to black. Then, the vertex $u_2 \to v_3$ to black. Thus, the set Z_{2d} generates a 2-distance forcing set for G. Here the cardinality of the set Z_{2d} is 5 and it is clear that with 4 black vertices, we cannot form a 2-distance forcing set for G. Hence, $Z_{2d}(G) = 5 = k+2$.

Assume that the result is true for the graph G with $(k-1)$ triangles, where $k \geq 5$. Let $A = \{u_n, v_{n-1}\}$. The induced subgraph $< G[V-A] >$ is a triangular snake graph with $k-1 < k$ triangles. Therefore by mathematical induction,

$Z_{2d}(< G[V-A] >) = k-1+2 = k+1$.

Let W be a minimum 2-distance forcing set for $< G[V-A] >$ with $\mid W \mid = k+1$. Color the vertex u_n in A as black. Then, it is easy to observe that the vertex v_{n-1} will be colored as black. Therefore by mathematical induction, $Z_{2d}(G) = Z_{2d}(< G[V-A] >) + \mid \{u_n\} \mid = k+1+1 = k+2$. □

Definition 7.2.9. [31]. *The n-sunlet graph S_n is the graph on $2n$ vertices got by attaching n pendant edges to a cycle C_n. Sunlet graphs are also called Crown graphs. The diameter of a sunlet graph is $\lfloor \frac{n}{2} \rfloor + 2$.*

Theorem 7.2.10. *Let G denotes the sunlet Graph S_n. Then,*

$$Z_{2d}(G) = \begin{cases} 4 & if \quad n = 3 \\ n & if \quad n \geq 4. \end{cases}$$

Proof. **Case 1.**

Assume that $n = 3$. Let $u_1, u_2, u_3, v_1, v_2, v_3$ be the vertices of the sunlet graph G, where v_1, v_2, v_3 are the vertices joined to u_1, u_2, u_3 of the cycle C_3 respectively. Let $Z_{2d} = \{u_1, u_2, v_1, v_2\}$ be a set of black vertices. The vertices in $V(G) - Z_{2d}$ are considered to be white. Then, clearly the black vertex $v_1 \to u_3$ to black. Now, the vertex $u_2 \to v_3$ to black. Thus, the set Z_{2d} forms a 2-distance forcing set for G. Here $\mid Z_{2d} \mid = 4$. Also, we can easily observe that with three black vertices the derived coloring is not possible because in this case we can force a maximum of one more vertex to black. Hence, $Z_{2d}(G) = 4$.

Case 2.

Assume that $n \geq 4$. Let $V(G) = \{u_1, u_2, \ldots, u_n, v_1, v_2 \ldots, v_n\}$, where $u_1, u_2, \ldots, u_n$ are the vertices of the cycle C_n and $v_1, v_2 \ldots, v_n$ are the vertices joined to

$u_1, u_2. \dots, u_n$ respectively. We show that the set $Z_{2d} = \{u_1, u_n, v_1, v_2, \dots, v_{n-2}\}$ of black vertices generates a 2-distance forcing set for G. The vertices in $V(G) - Z_{2d}$ are assumed to be white. The pendant vertex v_1 has three 2-distance vertices u_1, u_n and u_2, of which u_1 and u_n are black. So the black vertex $v_1 \rightarrow u_2$ to black. Consider the black vertex v_2. The 2-distance vertices of v_2 are u_1, u_2 and u_3. Hence $v_2 \rightarrow u_3$ to black, since u_1 and u_2 are black. Again, consider the black vertex v_3. Then, $v_3 \rightarrow u_4$ to black, since u_4 is the only 2-distance white neighbor of v_3. Proceeding like this, consider the black vertex v_{n-2}. Clearly, the vertex $v_{n-2} \rightarrow u_{n-1}$ to black. Also, $u_{n-2} \rightarrow v_{n-1}$ to black, $u_n \rightarrow v_n$ to black. Hence we obtain a derived coloring for G with the set Z_{2d}. Here $| Z_{2d} | = n$. Therefore,

$$Z_{2d}(G) \leq n \tag{7.5}$$

In order to prove the reverse part, we assert that any set Z_{2d} containing $(n-1)$ black vertices will not form a 2-distance forcing set for the graph G. For this, we consider the following cases. In each case, the vertices in $V(G) - Z_{2d}$ are white.

Case 1.

Let $Z_{2d} = \{v_1, v_2, \dots, v_{n-1}\}$ be a set of $(n-1)$ black vertices. Then further forcing is not possible, since each black vertex will have three 2-distance white neighbors.

Case 2.

Let P_{n-1} be a path on $(n-1)$ black vertices. Let Z_{2d} be the vertex set of P_{n-1} . Then, the path P_{n-1} will be of the following types.

Assume that n is even.

Type 1.

Consider the path $u_1 u_2 \dots u_{n-1}$. Then, further forcing is not possible since each black vertex has at least three 2-distance white neighbors.

Type 2.

Select a path $u_1u_2 \ldots u_{n-3}u_{n-2}v_{n-2}$. Then, only one forcing is possible from the vertex v_{n-2} because all other black vertices have at least two 2-distance white neighbors. So in this case, the derived coloring for G is not possible.

Type 3.

Choose a path $v_1u_1u_2 \ldots u_{n-4}u_{n-3}u_{n-2}$. Clearly, only one forcing from the vertex v_1 is possible because all other black vertices have at least two white neighbors in their 2-distance open neighborhood. Hence we cannot generate a 2-distance forcing set for G.

We can also observe that a set of $(n-1)$ black vertices constructed in any way other than what we mentioned above will never form a 2-distance forcing set for G. We also note that the cases are the same when n is odd.

Therefore from the above cases, we can conclude that a set of $(n-1)$ black vertices cannot form a 2-distance forcing set for G. Hence,

$$Z_{2d}(G) \geq n \tag{7.6}$$

From (7.5) and (7.6), $Z_{2d}(G) = n$. □

Definition 7.2.11. [28] *A comet graph is the graph obtained by appending m pendant edges to one end of a path P_n, where $n \geq 3$. The order of a comet graph is $m + n$ and the diameter is n.*

Theorem 7.2.12. *Let G be a comet graph. Then, $Z_{2d}(G) = m + 1$.*

Proof. Let $u_1, u_2, \ldots, u_n$ be the vertices of the path P_n. Also, let $v_1, v_2, \ldots v_m$ be the m vertices appended to the vertex u_n of the path P_n in G. By coloring the vertices u_1 and u_2 as black, we can color the remaining vertices of the path P_n as black. Since all the m pendant vertices are the 2-distance vertices of u_n or u_{n-1}, by coloring $(m-1)$ of these vertices, say $v_1, v_2, \ldots, v_{m-1}$, as black we can obtain a derived coloring for G. Thus, with the set $Z_{2d} = \{u_1, u_2, v_1, v_2, \ldots, v_{m-1}\}$ of black vertices, we can generate a 2-distance forcing set for G and the cardinality of the set Z_{2d} is $m+1$. Therefore,

$$Z_{2d}(G) \leq m+1 \tag{7.7}$$

Conversely, we claim that with m black vertices we cannot generate a 2-distance forcing set for G. For, we consider the following cases.

Case 1.

Start the coloring process with any one of the vertices v_i, where $1 \leq i \leq m$. Without loss of generality, color the vertex v_1 as black and the remaining vertices as white. Since the vertex v_1 has $m-1+2 = (m+1)$ 2-distance white neighbors, to apply the color change rule we have to assign black color to at least m of these 2-distance white vertices. So we must have at least $(m+1)$ black vertices to start forcing from the vertex v_1. Therefore, we cannot form a 2-distance forcing set for G with m black vertices if we start from any one of the vertices v_i, where $1 \leq i \leq m$.

Case 2.

Consider the vertices $u_1, u_2 \ldots, u_{n-2}$, where $N_{2d}(u_i) = 4$, $i = 3, 4, \ldots, n-2$. Consider the vertex u_3. Color the vertex u_3 as black. To proceed further, we have to color the vertices u_1, u_2 and u_4 as black. Then, the black vertex $u_3 \to u_5$ to black. Now, the black vertex $u_4 \to u_6$ to black, $u_5 \to u_7$, $\ldots$, $u_{n-3} \to u_{n-1}$, $u_{n-2} \to u_n$ to black. Then, to start the color change rule from the vertex u_n, we have to color at least $(m-1)$ pendant vertices, say $v_1, v_2, \ldots, v_{m-1}$, as black. Thus, we have a set $Z_{2d} = \{u_1, u_2, u_3, u_4, v_1, v_2, \ldots, v_{m-1}\}$ of black vertices. The cardinality of the set Z_{2d} is $m+3$.

Case 3.

We observe that the color change rule is not possible with m black vertices if we start the forcing process either from u_{n-1} or from u_n, since $N_{2d}(u_{n-1}) = m+3$ and $N_{2d}(u_n) = m+2$.

From the above three cases, we can conclude that the minimum number of black vertices required to obtain a derived coloring for G is $m+1$. Hence,

$$Z_{2d}(G) \geq m+1 \tag{7.8}$$

Therefore from (7.7) and (7.8), $Z_{2d}(G) = m+1$. □

Definition 7.2.13. [27]. *The n-pan graph is the graph formed by joining the cycle C_n to a singleton graph K_1 with a bridge. The order of the n-pan graph is $n+1$ and the diameter is $\lfloor \frac{n}{2} \rfloor + 1$.*

Theorem 7.2.14. *Let G be a n-pan graph, where $n \geq 6$. Then, $Z_{2d}(G) = 4$.*

Proof. Represent the vertices of the cycle C_n in G by $u_1, u_2, \ldots, u_n$, $n \geq 6$, and the singleton graph K_1 by u. Let u be joined to the vertex u_1 of C_n in G. Color any four adjacent vertices of C_n as black, except the vertex u_1. Clearly, these four vertices give a derived coloring for the cycle C_n in G. Consequently, the singleton graph K_1 (that is, the vertex u) will be colored as black. Also, we observe that any set of three black vertices never give a derived coloring for G, because in this case each black vertex has at least two 2-distance white vertices. Therefore, $Z_{2d}(G) = 4$. Hence the proof. □

Theorem 7.2.15. *Let G be a connected graph with maximum degree $\Delta(G) = 2$ and order $n \geq 5$. Then, $Z_{2d}(G) = 2$ or 4.*

Proof. Suppose that the maximum degree of G is 2. Then, G is either a path P_n or a cycle C_n. Now, we have from [26] that if G is a path P_n, then $Z_{2d}(G) = 2$ and if G is a cycle C_n, then $Z_{2d}(G) = 4$. This completes the proof. $\square$

Theorem 7.2.16. *Let G be the generalized friendship graph F_p^k, $p \geq 5$ and $k \geq p$. Then, $Z_{2d}(G) = 3k$.*

Proof. Represent the k copies of the cycle C_p in F_p^k as $C_p^{(1)}, C_p^{(2)}, \ldots, C_p^{(k)}$, where

$$V[C_p^{(1)}] = \{v_1^1, v_2^1, \ldots, v_p\}$$

$$V[C_p^{(2)}] = \{v_1^2, v_2^2, \ldots, v_p\}$$

$$\cdots$$

$$\cdots$$

$$V[C_p^{(k)}] = \{v_1^k, v_2^k, \ldots, v_p\}$$

We have the 2-distance forcing number of a cycle C_n is 4, where $n \geq 5$ (See[26]). Consider the cycle $C_p^{(1)}$ in G. Let $A = \{v_p, v_1^1, v_2^1, v_3^1\}$ be a set of four adjacent black vertices of the cycle $C_p^{(1)}$). Clearly, the set A generates a 2-distance forcing set for $C_p^{(1)}$. Therefore, $Z_{2d}[C_p^{(1)}] = 4$. We observe that the vertex v_p is also a black vertex of the cycle $C_p^{(2)}$. Assume that $B = \{v_p, v_1^2, v_2^2, v_3^2\}$ is a set of four adjacent black vertices of the cycle $C_p^{(2)}$. Then, the set B forms a 2-distance forcing set for the cycle $C_p^{(2)}$. Again, suppose $C = \{v_p, v_1^3, v_2^3, v_3^3\}$ be a set of four adjacent black vertices of the cycle $C_p^{(3)}$. Clearly, the set C generates a 2-distance forcing set for the cycle $C_p^{(3)}$. The case is similar for other cycles $C_p^{(4)}, C_p^{(5)} \ldots, C_p^{(k-1)}$. Hence to get the derived coloring for the cycles $C_p^{(2)}, C_p^{(3)}, \ldots, C_p^{(k-1)}$, we need in total $3(k-2)$ black vertices. Now, consider the cycle $C_p^{(k)}$. Also, consider the vertex v_1^k in $C_p^{(k)}$ which is adjacent to the vertex v_p. Color the vertex v_1^k as black. Then, the vertex v_1^1 in $C_p^{(1)}$ forces the vertex v_{p-1}^k in $C_p^{(k)}$ to black. Hence we need one more black vertex v_2^k in $C_p^{(k)}$ to get the derived coloring for the cycle $C_p^{(k)}$. Thus, with $4+3(k-2)+2 = 3k$ black vertices, we can force all the vertices of G to black. Therefore,

$$Z_{2d}(G) \leq 3k \tag{7.9}$$

To prove the converse, claim that with $(3k-1)$ black vertices we cannot obtain a derived coloring for G. For, assume that we have a 2-distance forcing set for G consisting of $(3k-1)$ black vertices. Since the vertex v_p will be colored as black after getting the derived coloring for the cycle $C_p^{(1)}$ using four adjacent black vertices and the vertex v_p is common for all cycles, we need at least 3 adjacent black vertices (These black vertices together with the vertex v_p should form four adjacent black vertices) for each of the cycles $C_p^{(2)}, C_p^{(3)}, \ldots, C_p^{(k-1)}$ to get the derived coloring for them. So in total, we need at least $3(k-2) = 3k-6$ black vertices to form the derived coloring for the cycles $C_p^{(2)}, C_p^{(3)}, \ldots, C_p^{(k-1)}$. Now, we have used $4+3k-6 = 3k-2$ black vertices to obtain the derived coloring for the cycles $C_p^{(1)}$, $\mathrm{C}_p^{(2)}, C_p^{(3)}, \ldots, C_p^{(k-1)}$. Then, there remains $[(3k-1)-(3k-2)] = 1$ black vertex in the 2-distance forcing set. Using this black vertex together with the black vertex v_p, we cannot form a derived coloring for the cycle $C_p^{(k)}$, a contradiction to our assumption. Therefore,

$$Z_{2d}(G) \geq 3k \tag{7.10}$$

Hence from (7.9) and (7.10), $Z_{2d}(G) = 3k$. □

7.3 2-Distance Forcing Number of Rooted Product of Graphs

In this section, we deal with the 2-distance forcing number of rooted product of some graphs.

Theorem 7.3.1. *Let G be the rooted product $P_n(P_m)$ of a path P_n and the rooted path P_m rooted with the pendant vertex of P_m, where $n \geq 2$, $m \geq 4$. Then, $Z_{2d}(G) \leq 2(n-1)$.*

Proof. Let u_1,u_2,...,u_n be the vertices of the path P_n and $P_m^{(1)}$, $P_m^{(2)}$,..., $P_m^{(n)}$ be the n copies of the path P_m rooted at the vertices u_1, u_2,..., u_n in G respectively. Denote the vertex set of the paths $P_m^{(1)}$, $P_m^{(2)}$, ..., $p_m^{(n)}$ in G as follows.

$$V[P_m^{(1)}] = \{p_1^1, p_2^1, \ldots, p_m^1\}$$

$$V[P_m^{(2)}] = \{p_1^2, p_2^2, \ldots, p_m^2\}$$

$$\cdots$$

$$\cdots$$

$$V[P_m^{(n)}] = \{p_1^n, p_2^n, \ldots, p_m^n\}.$$

Root the vertex p_1^1 of the path $P_m^{(1)}$ at u_1, the vertex p_1^2 of the path $P_m^{(2)}$ at u_2, ..., the vertex p_1^n of the path $P_m^{(n)}$ at u_n. That is, $p_1^1 = u_1, p_1^2 = u_2, \ldots, p_1^n = u_n$. We generate a 2-distance forcing set for the graph G as follows.

Let $A_1 = \{p_{m-1}^1, p_m^1\}$, $A_2 = \{p_{m-1}^2, p_m^2\}$, ..., $A_{n-1} = \{p_{m-1}^{n-1}, p_m^{n-1}\}$. Suppose that $Z_{2d} = A_1 \cup A_2 \cup \ldots \cup A_{n-1}$. Assign black color to the vertices of the set Z_{2d}. The vertices in $V(G) - Z_{2d}$ are considered to be white. Since $Z_{2d}(P_t) = 2$ for a path P_t, $t \geq 3$ (See [26]), clearly the set A_1 generates a 2-distance forcing set for the path $P_m^{(1)}$. In a similar manner, the set A_2 forms a 2-distance forcing set for the path $P_m^{(2)}$. Proceeding like this, the set A_{n-1} forms a 2-distance forcing set for the path $P_m^{(n-1)}$. Consider the black vertex p_2^{n-1} of the path $P_m^{(n-1)}$. We can easily see that the vertex p_2^{n-1} forces the vertex u_n to black. Then, the black vertex u_{n-1} forces the vertex p_2^n of the path $P_m^{(n)}$ to black. Now, the set $\{u_n, p_2^n\}$ of black vertices generates a 2-distance forcing set for the path $P_m^{(n)}$. Thus, the set Z_{2d} generates a 2-distance forcing set for the graph G. Since the cardinality of the set Z_{2d} is $2(n-1)$, we have $Z_{2d}(G) \leq 2(n-1)$. This completes the proof. □

We strongly believe that the above bound is sharp.

Theorem 7.3.2. *Let G denotes the rooted product $P_n(C_m)$ of a path P_n and the rooted cycle C_m, where $n \geq 2, m \geq 5$. Then, $Z_{2d}(G) \leq 3n$.*

Proof. Represent the vertices of the path P_n by $u_1, u_2, \ldots, u_n$. Let $C_m^{(1)}$, $C_m^{(2)}$, …, $C_m^{(n)}$ be the n copies of the cycle C_m rooted at the vertices $u_1, u_2, \ldots, u_n$ in G respectively. Denote the vertex set of the cycles $C_m^{(1)}, C_m^{(2)}, \ldots, C_m^{(n)}$ in G as follows.

$$V[C_m^{(1)}] = \{v_1^1, v_2^1, \ldots, v_m^1\}$$

$$V[C_m^{(2)}] = \{v_1^2, v_2^2, \ldots, v_m^2\}$$

$$\cdots$$

$$\cdots$$

$$V[C_m^{(n)}] = \{v_1^n, v_2^n, \ldots, v_m^n\}.$$

Let the vertex v_1^1 of the cycle $C_m^{(1)}$ be rooted at the vertex u_1 of P_n, the vertex v_1^2 of the cycle $C_m^{(2)}$ be rooted at the vertex u_2 of P$_n$, …, the vertex v_1^n of the cycle $C_m^{(n)}$ be rooted at the vertex u_n of P_n. That is, $u_1 = v_1^1, u_2 = v_1^2, \ldots, u_n = v_1^n$. Let $A_1 = \{u_1, v_2^1, v_3^1, v_4^1\}, A_2 = \{v_2^2, v_3^2, v_4^2\}, A_3 = \{v_2^3, v_3^3, v_4^3\}, \ldots, A_{n-1} = \{v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\}$, $\mathrm{A}_n = \{v_2^n, v_3^n\}$. Also, let $Z_{2d} = A_1 \cup A_2 \cup \ldots \cup A_n$. Color all vertices of the set Z_{2d} as black and the remaining vertices as white. Then, we claim that the set Z_{2d} generates a 2-distance forcing set for G.

For, Since $Z_{2d}(C_n) = 4$ for $n \geq 5$ (See[26]), clearly the set A_1 generates a 2-distance forcing set for the cycle $C_m^{(1)}$ in G. Then, the black vertex v_m^1 of the cycle $C_m^{(1)}$ forces the vertex u_2 of the path P_n to black, since u_2 is the only 2-distance white neighbor of v_m^1. Since u_2 is also a black vertex of the cycle $C_m^{(2)}$, we can observe that the set A_2 together with the black vertex u_2 forms a 2-distance forcing set for the cycle $C_m^{(2)}$ in G. Then, the black vertex v_m^2 of $C_m^{(2)}$ forces the vertex u_3 of the path P_n to black, because u_3 is the only 2-distance white neighbor of the vertex v_m^2. Consider the cycle $C_m^{(3)}$ in G. Since u_3 is also a black vertex of the cycle $C_m^{(3)}$, we can see that the set A_3 together with the black vertex u_3 generates a 2-distance forcing set for the cycle $C_m^{(3)}$ in G. Proceeding like this, Consider the cycle $C_m^{(n-1)}$. In $C_m^{(n-1)}$, the vertex u_{n-1} is already colored as black by the vertex v_m^{n-2} of the cycle $C_m^{(n-2)}$. Therefore, the set $A_{n-1} = \{v_2^{n-1}, v_3^{n-1}, v_4^{n-1}\}$ together with the black vertex u_{n-1} will form a 2-distance forcing set for the cycle $C_m^{(n-1)}$ in G. Finally, consider the cycle $C_m^{(n)}$ in G. Clearly, the vertex u_n of the path $P_m^{(n)}$ will be forced to black by the black vertex v_m^{n-1} of the cycle $C_m^{(n-1)}$. Then, the black vertex u_{n-1} forces the vertex v_m^n of the cycle $C_m^{(n)}$ to black because the vertices u_n and v_2^n are black vertices in $C_m^{(n)}$. Now, We can easily observe that the set A_n together with the black vertices u_n and v_m^n generates a 2-distance forcing set for the cycle $C_m^{(n)}$. Thus, the set Z_{2d} generates a 2-distance forcing set for G. The cardinality of the set Z_{2d}

is $4 + 3(n-2) + 2 = 3n$. Therefore, we have $Z_{2d}(G) \leq 3n$. Hence the proof. □

We strongly believe that the above bound is sharp.

Theorem 7.3.3. *Let G be a graph representing the rooted product $C_n(P_t)$ of a cycle C_n and the rooted path P_t rooted with the pendant vertex of P_t, where $n, t \geq 4$. Then, $Z_{2d}(G) \leq 2n - 4$.*

Proof. Let $P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(n)}$ be the n copies of the path P_t rooted at the vertices $u_1, u_2, \ldots, u_n$ of the cycle C_n in G respectively. Denote the vertex set of the paths $P_t^{(1)}, P_t^{(2)}, \ldots, P_t^{(n)}$ in G as follows.

$$V[P_t^{(1)}] = \{p_1^1, p_2^1, \ldots, p_t^1\}$$

$$V[P_t^{(2)}] = \{p_1^2, p_2^2, \ldots, p_t^2\}$$

$$\ldots$$

$$\ldots$$

$$V[P_t^{(n)}] = \{p_1^n, p_2^n, \ldots, P_t^n\}.$$

Let the vertex p_1^1 of the path $P_t^{(1)}$ be rooted at the vertex u_1, p_1^2 of the path $P_t^{(2)}$ at u_2, $\ldots$, the vertex p_1^n of the path $P_t^{(n)}$ at u_n. That is, $p_1^1 = u_1$, $p_1^2 = u_2$, $\ldots$, $p_1^n = u_n$. Let $A_1 = \{p_t^1, p_{t-1}^1\}$, $A_2 = \{p_t^2, p_{t-1}^2\}$, $\ldots$, $A_{n-2} = \{p_t^{n-2}, p_{t-1}^{n-2}\}$. Also, let $Z_{2d} = A_1 \cup A_2 \cup \ldots \cup A_{n-2}$. Color all vertices of the set Z_{2d} as black. The remaining vertices are treated as white. Then, we assert that the set Z_{2d} forms a 2-distance forcing set for G.

For, consider the path $P_t^{(1)}$ rooted at the vertex u_1. Since $Z_{2d}(P_n) = 2$ $(n \geq 3)$ (See[26]), clearly the set A_1 generates a 2-distance forcing set for the path $P_t^{(1)}$. We observe that after getting the derived coloring for the path $P_t^{(1)}$, the color of the vertex u_1 is forced to black. Similarly, the set A_2 forms a 2-distance forcing set for the path $P_t^{(2)}$ rooted at the vertex u_2. Then, the black vertex p_2^1 of the path $P_t^{(1)}$ forces the vertex u_n of the cycle C_n to black. Again, the set A_3 generates a 2-distance forcing set for the path $P_t^{(3)}$ rooted at the vertex u_3. Proceeding like this, consider the path $P_t^{(n-2)}$ rooted at the vertex u_{n-2}. We can observe that the set A_{n-2} will form a 2-distance forcing set for the path $P_t^{(n-2)}$. Now, the black vertex p_2^{n-2} of the path $P_t^{(n-2)}$ forces the vertex u_{n-1} to black. Then, the black vertex u_{n-2} forces the vertex p_2^{n-1} of the path $P_t^{(n-1)}$ to black, the black vertex u_1 forces the vertex p_2^n of the path $P_t^{(n)}$ to black. Now, we can see that the remaining white vertices of the paths $P_t^{(n-1)}$ and $P_t^{(n)}$ will be colored as black. Thus, the set Z_{2d} generates a 2-distance forcing set for G. Here $\mid Z_{2d} \mid = 2n-4$. Therefore, $Z_{2d}(G) \leq 2n-4$, as we desired. □

We strongly believe that the above bound is sharp.

Theorem 7.3.4. *Let G be the rooted product $C_n(C_m)$ of a cycle C_n and the rooted cycle C_m, where $m, n \geq 5$. Then, $Z_{2d}(G) \leq 3n$. We strongly believe that this bound is sharp.*

Proof. Denote the vertices of the cycle C_n in G by $u_1, u_2, \ldots, u_n$ and the n copies of the cycle C_m in G by $\mathrm{C}_m^{(1)}, C_m^{(2)}, \ldots, C_m^{(n)}$, where

$$V[C_m^{(1)}] = \{v_1^1, v_2^1, \ldots, v_m^1\}$$

$$V[C_m^{(2)}] = \{v_1^2, v_2^2, \ldots, v_m^2\}$$

$$\ldots$$

$$\ldots$$

$$V[C_m^{(n)}] = \{v_1^n, v_2^n, \ldots, v_m^n\}$$

Let the vertex v_1^1 of the cycle $C_m^{(1)}$ be rooted at the vertex u_1, the vertex v_1^2 of the cycle $C_m^{(2)}$ be rooted at the vertex u_2, ..., the vertex v_1^n of the cycle $C_m^{(n)}$ be rooted at the vertex u_n. That is, $u_1 = v_1^1, u_2 = v_1^2, \ldots, u_n = v_1^n$. Suppose that $A_1 = \{u_1, v_2^1, v_3^1, v_4^1\}$, $A_2 = \{u_2, v_2^2, v_3^2, v_4^2\}$, $A_3 = \{v_2^3, v_3^3, v_4^3\}$, $A_4 = \{v_2^4, v_3^4, v_4^4\}$, $A_5 = \{v_2^5, v_3^5, v_4^5\}, \ldots$, $A_{n-2} = \{v_2^{n-2}, v_3^{n-2}, v_4^{n-2}\}$, $A_{n-1} = \{v_2^{n-1}, v_3^{n-1}\}$, $A_n = \{v_2^n, v_3^n\}$. Also, let $Z_{2d} = A_1 \cup A_2 \cup \ldots \cup A_n$. Assign black color to all vertices of the set Z_{2d}. The vertices in $V(G) - Z_{2d}$ are considered to be white. We claim that the set Z_{2d} forms a 2-distance forcing set for the graph G.

For, consider the set A_1. Clearly, the set A_1 generates a 2-distance forcing set for the cycle $C_m^{(1)}$ rooted at the vertex u_1. Similarly, the set A_2 generates a 2-distance forcing set for the cycle $C_m^{(2)}$ rooted at the vertex u_2. Now, since u_n is the only 2-distance white neighbor of the vertex v_2^1 of the cycle $C_m^{(1)}$, the black vertex v_2^1 forces the vertex u_n to black. In a similar way, the black vertex v_2^2 of the cycle $C_m^{(2)}$ forces the vertex u_3 to black. Again, consider the set A_3. Clearly, the set A_3 together with the black vertex u_3 generates a 2-distance forcing set for the cycle $C_m^{(3)}$ rooted at the vertex u_3. Proceeding like this, consider the cycle $C_m^{(n-2)}$ rooted at the vertex u_{n-2}. We observe that at this stage the vertex u_{n-2} is forced to black by the vertex v_2^{n-3} of the cycle $C_m^{(n-3)}$. Now, we can see that the set A_{n-2} together with the vertex u_{n-2} generates a 2-distance forcing set for the cycle $C_m^{(n-2)}$ rooted at the vertex u_{n-2}. In the cycle $C_m^{(n-1)}$, clearly, the vertex u_{n-1} is forced to black by the vertex v_2^{n-2} of the cycle $C_m^{(n-2)}$. Then, the vertex u_{n-2} forces the vertex v_m^{n-1} of the cycle $C_m^{(n-1)}$ to black, the vertex u_1 forces the vertex v_m^n of the cycle $C_m^{(n)}$ to black. Now, the set A_{n-1} together with the black vertices u_{n-1} and v_m^{n-1} forms a 2-distance forcing set for the cycle $C_m^{(n-1)}$. Similarly, the

set A_n together with the black vertices u_n and v_m^n generates a 2-distance forcing set for the cycle $C_m^{(n)}$. Thus, the set Z_{2d} generates a 2-distance forcing set for the graph G. The cardinality of the set Z_{2d} is $4+4+3(n-4)+2+2=3n$.
Hence, $Z_{2d}(G) \leq 3n$. This completes our proof. □

7.4 2-Distance Forcing Number of Square Graph of a Graph

In this section, we consider the square of path and cycle.

Theorem 7.4.1. *Let G denotes the square graph of a path P_n, where $n \geq 5$. Then, $Z_{2d}(G) = 4$.*

Proof. In G, $\delta_{2d}(G) = 5$. So by the Theorem 7.1.3, we have

$$Z_{2d}(G) \geq 4 \tag{7.11}$$

To establish the reverse inequality, we proceed as follows.
Let $Z_{2d} = \{u_1, u_2, u_3, u_4\}$ be a set of black vertices. The remaining vertices are colored as white. We observe that u_5 is the only 2-distance white neighbor of the vertex u_1. So the black vertex $u_1 \to u_5$ to black. Now, consider the black vertex u_2. The 2-distance neighbors of the vertex u_2 are u_1, u_3, u_4, u_5 and u_6, out of which u_6 is the only white vertex. Therefore, the vertex $u_2 \to u_6$ to black. Again, consider the black vertex u_3. The 2-distance neighbors of u_3 are u_1, u_2, u_4, u_5, u_6 and u_7, of which u_7 is the only white vertex. Hence, the vertex $u_3 \to u_7$ to black. Continue this process, we can obtain a forcing sequence $u_1 \to u_5$, $u_2 \to u_6$, $u_3 \to u_7, \ldots$, $u_{n-5} \to u_{n-1}$, $u_{n-4} \to u_n$. Thus, all vertices of G will be colored

as black using the set Z_{2d}. Therefore, the set $Z_{2d} = \{u_1, u_2, u_3, u_4\}$ forms a 2-distance forcing set for G. Here the cardinality of the set Z_{2d} is 4. Therefore,

$$Z_{2d}(G) \leq 4 \tag{7.12}$$

From the above two inequalities, $Z_{2d}(G) = 4$. □

Theorem 7.4.2. *Let G represents the square graph of a cycle C_n, $n \geq 9$. Then, $Z_{2d}(G) = 8$.*

Proof. We observe that G is a 4-regular graph. Denote the vertices of the cycle C_n in G by u_1,u_2,...,u_n. Each vertex u_i, where $i = 1, 2, 3, ..., n$, has eight vertices in it's 2-distance open neighborhood. The 2-distance vertices of u_1, u_2, u_3, u_4,...,u_{n-1}, u_n are displayed in the following table.

Vertex	**2-distance vertices**
u_1	$u_2, u_3, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}$
u_2	$u_1, u_3, u_4, u_5, u_6, u_n, u_{n-1}, u_{n-2}$
u_3	$u_1, u_2, u_4, u_5, u_6, u_7, u_n, u_{n-1}$
u_4	$u_1, u_2, u_3, u_5, u_6, u_7, u_8, u_n$
...	....
...	....
...	....
u_{n-1}	$u_1, u_2, u_3, u_n, u_{n-2}, u_{n-3}, u_{n-4}, u_{n-5}$
u_n	$u_1, u_2, u_3, u_4, u_{n-1}, u_{n-2}, u_{n-3}, u_{n-4}$

Since $\delta_{2d}(G) = 9$ in G, by the Theorem 7.1.3, we have

$$Z_{2d}(G) \geq 8 \tag{7.13}$$

To prove the reverse inequality, we proceed as follows.

Start the color change rule by coloring the vertex u_1 as black and the remaining vertices as white. Since the vertex u_1 has eight 2-distance white neighbors, to proceed further, we have to color at least seven of these white vertices as black. Let the black vertices be u_{n-3}, u_{n-2}, u_{n-1}, u_n, u_2, u_4 and u_5. Then, $u_1 \to u_3$ to black. Now, consider the black vertex u_2. The vertices $u_1, u_3, u_4, u_5, u_6, u_n, u_{n-1}, u_{n-2}$ are the 2-distance vertices of the vertex u_2. Out of these vertices, u_1, u_n, u_{n-1}, u_{n-2}, u_3, u_4, u_5 are already black. So the vertex $u_2 \to u_6$ to black. Then, the vertex $u_3 \to u_7$ to black, since the other 2-distance vertices u_1, u_2, u_n, u_{n-1}, u_4, u_5 and u_6 of the vertex u_3 are already black. In a similar manner, the black vertex u_4 forces the vertex u_8 to black. Proceeding like this, the black vertex u_{n-8} forces the vertex u_{n-4} to black. We observe that the vertices u_{n-3}, u_{n-2}, u_{n-1}, u_n are already coloured black. Hence with the black vertices u_1, u_2, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}, we can force all vertices of the graph G to black. Thus, the set $Z_{2d} = \{u_1, u_2, u_4, u_5, u_n, u_{n-1}, u_{n-2}, u_{n-3}\}$ of black vertices is a 2-distance forcing set for the graph G. The cardinality of the set Z_{2d} is 8. Therefore,

$$Z_{2d}(G) \leq 8 \tag{7.14}$$

Hence from the above two inequalities, $Z_{2d}(G) = 8$. □

7.5 2-Distance Forcing Number of Splitting Graph

In this section, we consider the splitting graph of a path.

Theorem 7.5.1. *Let G be the splitting graph of a path P_n, where $n \geq 4$. Then, $Z_{2d}(G) = n$.*

Proof. Denote the vertices of the path P_n in G by $u_1, u_2, \ldots, u_n$. Represent the corresponding vertices of $u_1, u_2, \ldots, u_n$ in G by $u'_1, u'_2, \ldots u'_n$. In this graph, the vertices u'_1 and u'_n have the least number of 2-distance vertices and the number of such 2-distance vertices is 4.

We form a 2-distance forcing set for G as follows.

Begin with the vertex u'_1. Assign black color to u'_1. The vertex u'_1 has four 2-distance neighbors u_1, u_2, u_3 and u'_3. To begin the color change rule, color at least three of these 2-distance neighbors as black. Let the black vertices be u_1, u_2, u_3 and the remaining vertices are treated as white vertices.. Then, the black vertex u'_1 forces the vertex u'_3 to black. Consequently, the black vertex u_1 forces u'_2 to black. Now, the black vertex u_2 has only two 2-distance white neighbors u_4 and u'_4. Assign black color to the vertex u_4. Then, the vertex u'_4 is forced to black by the vertex u_2. Similarly, the black vertex u_3 has two 2-distance white vertices u_5 and u'_5. Color the vertex u_5 to black. Then, the black vertex u_3 forces the vertex u'_5 to black. Now, consider the vertex u_4. The vertices u_6 and u'_6 are the 2-distance white neighbors of u_4. By assigning black color to the vertex u_6, we can force the vertex u'_6 to black by the vertex u_4. Proceeding like this, consider the black vertex u'_{n-1} (we observe that the vertex u'_{n-1} is forced to black by the vertex u_{n-3}). Now, the black vertex u'_{n-1} has only one 2-distance white neighbor u_n. So u'_{n-1} forces u_n to black. Then, the black vertex u_{n-1} forces the vertex u'_n to black. Thus, the set $Z_{2d} = \{u'_1, u_1, u_2, u_3, u_4, u_5, \ldots, u_{n-1}\}$ forms a 2-distance forcing set for G. Clearly, the cardinality of the set Z_{2d} is n. Therefore,

$$Z_{2d}(G) \leq n \tag{7.15}$$

To prove the converse, we claim that no set with $(n-1)$ black vertices generates a 2-distance forcing set for G. For, we consider the following cases.

Case 1.

Let $Z_{2d} = \{u_1, u_2, \ldots, u_{n-1}\}$ be a set of $(n-1)$ black vertices. The remaining vertices are assumed to be white. Then, we can observe that further forcing is not possible because each black vertex has at least three 2-distance white neighbors.

Case 2.

Assume that $Z_{2d} = \{u'_1, u'_2, \ldots, u'_{n-1}\}$ be a set of black vertices. The vertices in $V(G) - Z_{2d}$ are assumed to be white. Here also the color change rule is not possible since each black vertex has at least three 2-distance white neighbors.

Case 3.

Let P_{n-1} be a path on $(n-1)$ black vertices and the set Z_{2d} be the vertex set of P_{n-1}. The vertices in $V(G) - Z_{2d}$ are treated as white. Then, we consider the following Subcases.

Subcase 1.

Assume that n is odd.

Let P_{n-1} be the path $u_1u'_2u_3u'_4...u'_{n-3}u_{n-2}u'_{n-1}$. Then, further forcing is not possible since each black vertex has at least two 2-distance white neighbors.

Subcase 2.

Consider the path $u'_1u_2u'_3u_4...u_{n-3}u'_{n-2}u_{n-1}$. Here also the derived coloring is not possible because each black vertex has at least two white vertices in it's 2-distance open neighborhood.

Subcase 3.

Suppose that n is even.

Consider the paths $u_1u'_2u_3u'_4...u_{n-3}u'_{n-2}u_{n-1}$ and $u'_1u_2u'_3u_4...u'_{n-3}u_{n-2}u'_{n-1}$.

In this case also color change rule is not possible, because each black vertex has at least two 2-distance white neighbors.

It is worth mentioning that a set of $(n-1)$ black vertices generated in any way other than what we mentioned above cannot form a 2-distance forcing set for G. Therefore from the above cases, we can observe that

$$Z_{2d}(G) \geq n \tag{7.16}$$

Hence from (7.15) and (7.16), we get $Z_{2d}(G) = n$. □

7.6 2-Distance Forcing Number of Cartesian Product of Graphs

This section investigates the 2-distance forcing number of Cartesian product such as the ladder graph $P_n \square P_2$, the grid graph $P_n \square P_m$ and the circular ladder graph $C_n \square K_2$. We start with the ladder graph $P_n \square P_2$.

Theorem 7.6.1. *Let G be the ladder graph $P_n \square P_2$, where $n \geq 3$. Then, $Z_{2d}(G) = 4$.*

Proof. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be the vertices of the graph $G = P_n \square P_2$. Here $\delta_{2d}(G) = 5$. Since $Z_{2d}(G) \geq \delta_{2d}(G) - 1$ (See Theorem 7.1.3), we have

$$Z_{2d}(G) \geq 4 \tag{7.17}$$

In order to prove the reverse inequality, consider the set $Z_{2d} = \{u_1, u_2, v_1, v_2\}$. Assign black color to the vertices of the set Z_{2d} and white color to the remaining vertices. Take the black vertex v_1. Since v_3 is the only 2-distance white neighbor of v_1, the black vertex $v_1 \to v_3$ to black. Clearly, $u_1 \to u_3$ to black. Repeatedly apply the color change rule, we get the forcing sequence $v_2 \to v_4$, $u_2 \to u_4$, $v_3 \to v_5$, $u_3 \to u_5$,..., $v_{n-3} \to v_{n-1}$, $u_{n-3} \to u_{n-1}$, $v_{n-2} \to v_n$, $u_{n-2} \to u_n$ to black. Thus, all vertices of G will be colored as black. Therefore, the set $Z_{2d} = \{u_1, u_2, v_1, v_2\}$ generates a 2-distance forcing set for G. The cardinality of the set Z_{2d} is 4. Hence,

$$Z_{2d}(G) \leq 4 \tag{7.18}$$

From (7.17) and (7.18), $Z_{2d}(G) = 4$. □

Theorem 7.6.2. *Let G be the Cartesian product $P_n \square P_m$ of a path P_n and a path P_m, $m \geq 5$, where $n \geq m$. Then, $Z_{2d}(G) = 2m$.*

Proof. Denote the vertices of the graph G by

$$p_1^1, p_2^1, \ldots, p_n^1$$

$$p_1^2, p_2^2, \ldots, p_n^2$$

$$p_1^3, p_2^3, \ldots, p_n^3$$

$$\ldots$$

$$\ldots$$

$$p_1^m, p_2^m, \ldots, p_n^m$$

We can easily infer that the vertices p_1^1, p_n^1, p_1^m and p_n^m have the least number of 2-distance vertices and the number of such vertices is 5. We generate a 2-distance forcing set for G as follows.

Consider the set $Z_{2d} = \{p_1^1, p_2^1, p_1^2, p_2^2, p_1^3, p_2^3, p_1^4, p_2^4, \ldots, p_1^{m-1}, p_2^{m-1}, p_1^m, p_2^m\}$ of $2m$ black vertices and all other vertices are colored as white. Then, clearly the black vertex p_1^1 forces the vertex p_3^1 to black, since p_3^1 is the only 2-distance white neighbor of the vertex p_1^1. Similarly, the black vertex p_1^2 forces the vertex p_3^2 to black because p_3^2 is the only 2-distance white neighbor of the vertex p_1^2. Continue like this, we can see that the vertex p_1^3 forces p_3^3 to black, p_1^4 forces p_3^4 to black,$\ldots$, p_1^m forces the vertex p_3^m to black. Again, the black vertex p_2^1 forces the vertex p_4^1 to black, p_2^2 forces the vertex p_4^2 to black,$\ldots$, p_2^m forces the vertex p_4^m to black. Apply this process step by step, finally the black vertex p_{n-2}^1 forces the vertex p_n^1 to black, p_{n-2}^2 forces the vertex p_n^2 to black,$\ldots$, p_{n-2}^m forces the vertex p_n^m to black. Thus, the set Z_{2d} generates a 2-distance forcing set for G. Clearly, the cardinality of the set Z_{2d} is $2m$. Hence,

$$Z_{2d}(G) \leq 2m \tag{7.19}$$

To establish the reverse inequality, we proceed as follows.

Case 1.

Omit the black vertex p_1^1 (or p_1^m) from the set Z_{2d}. Then after possible forcings, there exists only $2 + \frac{m(m-1)}{2}$ black vertices in the graph G. So in this case, the derived coloring is not possible for G.

Case 2.

Delete the black vertex p_1^2 (or p_1^{m-1}) from the 2-distance forcing set Z_{2d}. In this case, after possible number of forcings, there exists only $4+\frac{(m-1)(m-2)}{2}$ black vertices in the graph G. Therefore, the set Z_{2d} will never form a 2-distance forcing set for G.

Case 3.

If we exclude the vertex p_2^1 (or p_2^m) from the set Z_{2d}, then after possible forcings there exists only $\frac{m(m+1)}{2}$ black vertices in the graph G. Hence, the set Z_{2d} will not form a 2-distance forcing set for G.

Case 4.

If we omit the vertex p_2^2 (or p_2^{m-1}) from the set Z_{2d}, then clearly there are only $2 + \frac{(m-1)m}{2}$ black vertices in G. In this case also, the set Z_{2d} cannot form a 2-distance forcing set for G.

Case 5.

Delete the black vertex p_1^j, $j = 3, 4, \ldots, m-2$, from the set Z_{2d}. Then, we can easily assert that obtaining a derived coloring for G is impossible.

Case 6.

Remove the black vertex p_2^j, $j = 3, 4, \ldots, m-2$, from the set Z_{2d}. Then, we can observe that we cannot change the color of all vertices of G to black.

Hence from the above cases, we can conclude that a set with $(2m-1)$ black vertices will not form a 2-distance forcing set for G. Therefore,

$$Z_{2d}(G) \geq 2m \tag{7.20}$$

Hence from (7.19) and (7.20), $Z_{2d}(G) = 2m$.

□

Theorem 7.6.3. [26] *Let G be a connected graph of order $n \geq 3$ with $diam(G) = 2$. Then, $Z_{2d}(G) = n - 1$.*

Theorem 7.6.4. *Let G represents the circular ladder graph $C_n \square K_2$. Then,*

$$Z_{2d}(G) = \begin{cases} 5 & \textit{if} \quad n = 3 \\ 6 & \textit{if} \quad n = 4 \\ 7 & \textit{if} \quad n = 5 \\ 8 & \textit{if} \quad n \geq 6. \end{cases}$$

Proof. We consider the following cases.

Case 1.

Assume that $n = 3$. Let $\{u_1, u_2, u_3, v_1, v_2, v_3\}$ be the vertex set of G, where u_1, u_2, u_3 are the vertices of the inner cycle. In this case since the diameter of G is 2, $Z_{2d}(G) = O(G) - 1 = 6 - 1 = 5$, by the Theorem 7.6.3.

Case 2.

Assume that $n = 4$. Represent the vertex set of G by $\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$, where u_1, u_2, u_3, u_4 are the vertices of the inner cycle. Here, $\Delta_{2d}(G) = \delta_{2d}(G) = 7$. By the Theorem 7.1.3, we have

$$Z_{2d}(G) \geq 6 \tag{7.21}$$

On the other hand, consider the set $Z_{2d} = \{v_1, v_2, v_3, v_4, u_1, u_2\}$ of black vertices. The remaining vertices are assumed to be white. Then, clearly the vertex $v_1 \to u_4$ to black, since u_4 is the only 2-distance white neighbor of the black vertex v_1. Consequently, the vertex u_3 will be colored as black. Therefore, the set Z_{2d} forms a 2-distance forcing set for G. The cardinality of the set Z_{2d} is 6. Hence we have,

$$Z_{2d}(G) \leq 6 \tag{7.22}$$

Therefore from (7.21) and (7.22), $Z_{2d}(G) = 6$.

Case 3.

Assume that $n = 5$. Denote the vertex set of G by $\{u_1, u_2, \ldots, u_5, v_1, v_2, \ldots, v_5\}$, where u_1, u_2, u_3, u_4, u_5 are the vertices of the inner cycle. In G, $\Delta_{2d}(G) = \delta_{2d}(G) = 8$. Therefore by the Theorem 7.1.3, we have

$$Z_{2d}(G) \geq 7 \tag{7.23}$$

Conversely, Let $Z_{2d} = \{v_1, v_2, v_3, v_5, u_1, u_2, u_5\}$ be a set of seven black vertices. Then, we can easily see that the black vertex $v_1 \to v_4$ to black. Now, the vertex

$v_5 \to u_4$ to black. Consequently, the vertex u_3 will be colored as black. Thus, the set Z_{2d} generates a 2-distance forcing set for G. The cardinality of the set Z_{2d} is 7. Therefore,

$$Z_{2d}(G) \leq 7 \tag{7.24}$$

Hence from (7.23) and (7.24), $Z_{2d}(G) = 7$.

Case 4.

Assume that $n \geq 6$. Let $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ be the vertex set of the circular ladder graph G, where $u_1, u_2, \ldots, u_n$ are the vertices of the inner cycle. We construct a 2-distance forcing set for G as follows.

Clearly, $\delta_{2d}(G) = 8$ in G. Since $Z_{2d}(G) \geq \delta_{2d}(G) - 1$ for a connected graph G, we have $Z_{2d}(G) \geq 7$. But it is obvious that with 7 black vertices we cannot form a 2-distance forcing set for G, because with 7 black vertices the maximum number of further forcing possible is only one. Hence we can conclude that

$$Z_{2d}(G) \geq 8 \tag{7.25}$$

To claim the reverse part, we proceed as follows.

Let $Z_{2d} = \{u_1, u_2, u_n, u_{n-1}, v_1, v_2, v_n, v_{n-1}\}$ be a set of black vertices and all the remaining vertices are assumed to be white. Then, the black vertex $u_1 \to u_3$ to black, since u_3 is the only 2-distance white vertex of u_1. Similarly, the black vertex $v_1 \to v_3$ to black, since v_3 is the only 2-distance white vertex of v_1. Then, clearly $u_2 \to u_4$ to black, $v_2 \to v_4$ to black and so on. Proceeding like this, we get the forcing sequence $u_1 \to u_3$, $v_1 \to v_3$, $u_3 \to u_5$, $v_3 \to v_5$, $u_4 \to u_6$, $v_4 \to v_6$, $\ldots$, $u_{n-4} \to u_{n-2}$, $v_{n-4} \to v_{n-2}$ to black. Thus, we get a derived coloring for G using the set Z_{2d}. Therefore, the set Z_{2d} forms a 2-distance forcing set for G.

Here $| Z_{2d} |= 8$. Hence,

$$Z_{2d}(G) \leq 8 \tag{7.26}$$

Therefore from the above two inequalities, $Z_{2d}(G) = 8$. □

Theorem 7.6.5. *Let G be the complete bipartite graph K_{mn}, where $m, n \geq 2$. Then, $Z_{2d}(G) = m + n - 1$.*

Proof. Since the complete bipartite graph K_{mn} is a graph with $diam(K_{mn}) = 2$ and having more than two vertices, the proof follows by the Theorem 7.6.3. □

7.7 2-Distance Forcing Number of Complement of Graphs

In this section, we compute the 2-distance forcing number of complement of path and cycle.

Theorem 7.7.1. *Let G denotes the complement of a path P_n, where $n \geq 5$. Then, $Z_{2d}(G) = n - 1$.*

Proof. Since the graph G is connected with $n \geq 3$ and $diam(G) = 2$, the proof follows by the Theorem 7.6.3. □

Theorem 7.7.2. *Let G represents the complement of a cycle C_n, where $n \geq 5$. Then, $Z_{2d}(G) = n - 1$.*

Proof. Here the graph G is connected with $n \geq 3$ and having $diam(G) = 2$. Therefore, the proof follows by the Theorem 7.6.3. □

Chapter 8

Conclusion and Further Scope of Research

8.1 Summary of the Thesis

First Chapter is the Introductory Chapter.

In the Second Chapter, we provided the basic definitions from the graph theory which are very useful in the forthcoming chapters.

In the Third Chapter, we addressed the problem of determining the zero forcing number of graphs and their splitting graphs. In Section 2, we gave upper bounds on the zero forcing number of splitting graph of a graph. In Section 3, we found several classes of graphs in which $Z[S(G)] = 2Z(G)$. Section 4 provided classes of graphs in which $Z[S(G)] < 2Z(G)$. In section 5, we provided more families of graphs with $Z(G) = P(G)$.

In the Fourth Chapter, we dealt with the characterization of graphs G for which $1 \leq Z_2(G) \leq 4$. Also, we determined the k-forcing number of splitting graph of some graphs. In fact there are many graph classes for which $1 \leq Z_2(G) \leq 4$.

In the Fifth Chapter, we addressed the problem of determining the connected

k-forcing number of certain graphs. In Section 1, we found the exact value of connected zero forcing number of some classes of graphs. In Section 2, we investigated the value of the connected k-forcing number of rooted product of some graphs. Section 3 dealt with the connected k-forcing number of square graph of path and cycle.

In Chapter Six, we introduced the notion of 2-distance forcing number $Z_{2d}(G)$ of a graph G. In Section 2, we found the exact value of $Z_{2d}(G)$ of path, cycle, wheel graph, Petersen graph, star graph, friendship graph, pineapple graph and the fan graph. Also, we proved that if G is a connected graph of order $n \geq 3$ and $diam(G) = 2$, then $Z_{2d}(G) = n - 1$. In Section 3, we focused on some classes of graphs with diameter lies between 2 and 5 and their 2-distance forcing number was determined. Finding an exact value of the 2-distance forcing number of helm graph and the sunflower graph is open.

Chapter Seven provided the 2-distance forcing number of some special graphs with large diameter. In Section 1, we computed the 2-distance forcing number of some graphs. The upper bound for the 2-distance forcing number of rooted product of some graphs was provided in Section 2. Section 3 studied the 2-distance forcing number of square graph of path and cycle. The 2-distance forcing number of splitting graph of path was found in Section 4. Section 5 introduced the 2-distance forcing number of the Cartesian product of some graphs. The 2-distance forcing number of complement of path and cycle was discussed in the Section 6.

8.2 Further Scope of Research

(i) Characterize the graphs G for which $2Z(G) = Z[S(G)]$.

(ii) Characterize the graphs G with $P[S(G)] = Z[S(G)]$.

(iii) Find the families of graphs G for which $Z(G) = n - \gamma_c(G)$.

(iv) Examine some more families of graphs G in which $Z(G) = P(G)$.

(v) Investigate the zero forcing number of splitting graph of some more graphs.

(vi) Characterize $S(G)$ and G for which $Z_2[S(G)] = 1$, $Z_2(G) = 1$, $Z_2[S(G)] = 2$, $Z_2(G) = 2$, $Z_2[S(G)] = 3$, $Z_2(G) = 3$, $Z_2[S(G)] = 4$ and $Z_2(G) = 4$.

(vii) Characterize graph G for which $Z_2(G) + Z[S(G)] = 3Z(G)$.

(viii) Investigate the exact value of the 2-distance forcing number for the rooted product of path and path, path and cycle, cycle and path, cycle and cycle, and some more graphs.

(ix) Compute the 2-distance forcing number of some more families of splitting graphs.

(x) Determine the 2-distance forcing number of some other graph products.

(xi) Find the 2-distance forcing number of square graph of some more graphs.

(xii) Investigate the 2-distance forcing number of more classes of graphs with large diameter.

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APPENDIX

List of Publications

1. B. Chacko, C. Dominic, and K. P. Premodkumar, *On the Zero Forcing Number of Graphs and Their Splitting Graphs*, Algebra and Discrete Mathematics, Volume 28 (2019). Number 1, pp. 29 – 43.

2. K. P. Premodkumar, C. Dominic, B. Chacko, *K- Forcing Number of Some Graphs and Their Splitting Graphs*, International Journal of Scientific Research in Mathematical and Statistical Sciences, Volume 6, Issue 3, (2019), pp. 121 – 127.

3. K. P. Premodkumar, C. Dominic, and B. Chacko, *Connected k- Forcing Sets of Graphs and Splitting Graphs*, Journal of Mathematical and Computational Science, 10 (2020), Number 3, pp. 656 – 680.

4. K. P. Premodkumar, C. Dominic, B. Chacko, *Two Distance Forcing Number of a Graph* , Journal of Mathematical and Computational Science, 10 (2020), Number 6, pp. 2233 – 2248.

5. K. P. Premodkumar, C. Dominic, B. Chacko, *Graphs with Large Diameter and Their Two Distance Forcing Number*, Journal of Mathematical and Computational Science, 11 (2021), Number 2, pp. 1810 – 1837.