MATHEMATICAL AND STATISTICAL ANALYSIS OF SOME STOCHASTIC MODELS IN QUEUES, INVENTORY AND RELIABILITY

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DEDICATED TO MY BELOVED PARENTS

DECLARATION

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma of any other university or institution and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference made in the text of the thesis.

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CERTIFICATE

This is to certify that the work reported in this thesis entitled MATHEMATICAL AND STATISTICAL ANALYSIS OF SOME STOCHASTIC MODELS IN QUEUES, INVENTORY AND RELIABILITY that is being submitted by Sri. Jenson, P.O. for the award of Doctor of Philosophy, to the University of Calicut, is based on the bonafide research work carried out by him under my supervision and guidance in the Department of Statistics, University of Calicut. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma of any other university or institution.

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CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Stochastic Modeling

Many real world phenomena require the analysis of systems in stochastic settings. So stochastic models have become increasingly important for understanding or making performance evaluation of physically realizable complex systems in various branches of science and in almost all walks of life. Recently there have been striking advances in both mathematical and statistical aspects of stochastic models. For dynamic system with a given probabilistic law of motion, the simple Markov model is often found to be appropriate. One of the greatest success stories for stochastic models is the development of computer communication networks. There are several interesting scenarios in the internet and the emerging next generation networks where stochastic modeling is applicable.

Tijims (1994) gave an algorithmic approach to stochastic modeling with a wide variety of realistic examples. A. A. Markov at the beginning of twentieth century developed a probability model in which the outcomes of the successive trials are allowed to be independent on each other such that each trial depends on its predecessor.

Even Kolmogorov's axiomatic approach to probability may be considered as modeling of the real world concept of chance by a set of axioms so that probability measure can be studied mathematically. Since then several scientists have proposed stochastic models to describe different phenomena. Some of such models are – insurance risk models, dam models, inventory and queueing models, reliability models, model for population growth and epidemic models. This thesis is devoted to the study of some interesting models in queues, inventory and reliability, which are physically realizable though complex.

A model may be iconic giving shape to an abstraction, symbolic, involving the mathematical relationships or statements underlying the various aspects of the phenomenon under observation. It provides an approximation to the real world situation. A model involving a random variable or a chance factor is called a stochastic or a probability model. Deterministic models may be viewed as special cases of stochastic models. As we believe in the philosophy – 'every thing in the universe is random', the stochastic models become our prime concern in any scientific study.

In model building, the first step in the construction of a model is to identify the essential characteristics of the phenomenon and their inter relationships. Then we will analyze the model deducing the various observable characteristics, which would follow if the model is satisfactory. The next step in the model building is to fit a suitable model from the family of models, by choosing the value of one or more parameters of the model. If the fit is not satisfactory, one tries to modify the model. Once we arrive at the satisfactory model, it can be used for the purpose of prediction and control. Decision theory can be used to decide on the optimal action from the model.

Janssen et. al. (1995) cited a variety of stochastic models in finance, insurance, economic forecasting, marketing etc. He described these models as an interface between stochastic modeling and data analysis and their applications in various fields. Krishna Reddy et. al. (1994) gave an elaborate account of stochastic models and optimization techniques to upgrade and modernize the existing technology. Software reliability and computer communication systems are closely related to optimization and stochastic modeling. Guttierrez (1994) explained some stochastic models in operations research, earth and life sciences and information theory, from an application perspective. Brandt et. al. (1990) gave detailed account of stationary stochastic models with a special reference to queueing models. Kijima (1997) explained the transient behavior of Markov chains through monotone processes, reversibility and rate of convergence that provide good help for stochastic modeling. Wolf (1989) explained many stochastic models – most of them are queueing models – with concrete basic knowledge of the theory and a very good literature of the models. For a systematic exposition of the basic theory and applications of stochastic models one may refer to Bhat (2000). More recently Latouche and Taylor (2000) explained some algorithmic methods for stochastic models. For a wide variety of stochastic models, the steady state and occasionally the transient measure of the underlying process can be expressed in terms of a matrix. That matrix is a minimal non-negative solution to a non-linear equation. M. F. Neuts first proposed such matrix solutions to stochastic models in the early 1970's. Chakravarthy and Alfa (1997) obtained a number of stochastic models in which matrix analytic methods employed as a main tool.

In stochastic modeling, the ideas in queue, inventory and reliability theory have considerable importance. Recently many articles are available by applying one of these theoretical concepts in the areas of the other or both. Several authors have studied these theories simultaneously. For example one can see Kalashnikov (1994), Atkinson (1995), Kusaka and Mori (1995), Park et. al. (1996), Sharma and Chauhan (1996), Li Wei et. al. (1997), Wang (1997), Karaesmen and Gupta (1997), Gopalan (1999), Ke and Wang (1999) and Perry and Posner (1999).

Each chapter in this thesis is provided with self-introduction, notations and some important references. The next three sections of this chapter review briefly the earlier developments of the theories of queue, inventory and reliability. Some analytical tools used in this work supplemented in section 1.5. A brief account of PH class of distributions is given in section 1.6. The final section presents an outline of the results obtained in this thesis is given.

1.2 Queueing Models - An Outline

Queueing theory is a well-developed subject of applied probability theory. The systematic study of the queueing theory was pioneered by A. K. Erlang at the beginning of twentieth century whose work initially started with telephone exchange problems. Since then many researchers throughout the world have continued his work. Amongst the known books available in the literature are those of Saaty (1961), Takacs (1962), Prabhu (1965), Cohen (1969), Gross and Haris (1998) and Asmussen (1987). By now, there are many thousands of articles on queueing and related topics. The flow of new theories and methodologies has become very hard to keep up with.

We have all experienced waiting for service some way or another. A queue is formed when customers arrive at a service facility and demand service. Clearly in different situations, there are different service facilities, service discipline, customer behavior etc. Thus a queueing model has the following basic characteristics:

1) Arrival pattern of customers.
2) Service pattern of servers.
3) Number of service channels.
4) System capacity.
5) Queue discipline.

The arrival pattern describes the manner in which the customers arrive and join in the queue. Usually the process of arrivals is stochastic and it is thus necessary to know the probability distribution of the inter arrival time, the time between the successive arrivals. The arrival can be either one at a time or in batches and if it is in batches, we should know the size of the batch. We have to examine the reaction of the customer upon entering the system. If a customer decides not to enter the queue upon the arrival the customer is said to have balked. The reneged customers are those who enter the queue and after some time loss patience and decide to leave. Besides these the customers may switch from one queue to another, if there is more than one parallel waiting lines, and this is known as jockey for position. These three situations are examples of queue with impatient customers. The arrival pattern independent of time is called stationary and not independent of time is called non-stationary.

Most often, service pattern is described by a probability distribution of the average service time of a customer. Service may also be single or batch. Unlike from the arrival pattern service depends on the number of customers in the queue is referred as state-dependent service. Service can also be stationary or non-stationary. When we speak about service time, the system must be non-empty. If the system is empty, the service facility is idle. Thus we may say that queue length would be the result of two separate patterns – arrivals and services – which are generally assumed to be independent.

Service channel refers the capacity of the servicing facility. That is the number of servers available. System capacity is the capacity of the waiting room, may be finite or infinite. In the case of finite system capacity, those customers who arrive after the room is full will be lost. Queue discipline refers to the manner in which customers are selected for service when a queue has formed. The most common discipline that can be observed in every day life is first come, first served FCFS. Some others are last come first served (LCFS), Random service selection (RSS) and Priority (PR) schemes. There are two general priority schemes, pre-emptive and non-pre-emptive. In the first, the customer with highest priority is allowed to enter service facility immediately even if a customer with lower priority is already in service. But in the second, the highest priority customer goes to the head of the queue and gets into service immediately after the customer presently in service served out.

When we come to the notation of a queueing model, Kendall described a model by A/B/X/Y/Z where A indicates the inter arrival time distribution, B the service time distribution, X number of service channels, Y the system capacity and Z the queue discipline. Dshalalow (1995) included server capacity and busy period discipline in this notation by introducing two more abbreviations. If there is no limit on the system capacity and the queue discipline is FIFO, A/B/X can be taken as the notation of the queueing model. For example, M/M/1 means arrival and service time follows exponential, there is only one server, the system capacity is infinite and the queue discipline is FCFS.

Analysis of a queueing model mainly involves the probability distribution of the following random variables. 1) The system size – the number of customers in the system. 2) The waiting time in the system – the duration of the time a customer has to spend in the system before leaving. 3) Busy period – the period of time the server is continuously busy.

The behavior of a queue at time t depends on t. However, most results arrived at in the literature are for the behavior of the system in equilibrium. That is as $t \to \infty$. In such cases, the system is unaffected by the shift of time and is said to be in the steady state or the system is stationary. It should be noted that besides the fact that the study of a system in equilibrium is mathematically simpler, it is also considered to be more important in applications.

The time dependent solution of M/M/1 queue was first given by Bailey (1954) using the method of generating function. Even now so many articles aimed at the transient solutions of system size of M/M/1 queueing models are available in standard journals.

For non-Markovian queues, Palm (1943) and Kendall (1953) have used the method of regeneration points and imbedded Markov chain techniques respectively. The study of bulk queues is originated with the pioneering work of Bailey. For details of bulk queues, one may refer Medhi (1984) and Chaudhry and Templeton (1984). Queueing models in which the service process is subject to interruptions resulting from unscheduled breakdowns of servers, scheduled off periods, arrival of customers with pre-emptive and non-pre-emptive priorities or the server working on primary and secondary customers arise naturally. A detailed analysis of single server queues with server failure is given in Gnedenko and Kovalenko (1968). When the on-periods are not exponential, the model becomes complicated and one such model is studied by Federgruen and Green (1986).

The time dependent behavior as well as steady state behavior of M/G/1 and G/M/1 queueing system first studied by Bhat (1968) in which bulk arrival and bulk service are considered and the behavior of the waiting time process is obtained. A review of papers based on the generalizations of the classical M/G/1 system is available in Medhi (1994). Atkinson (1995) considered an M/G/1/0 queue and obtained the transient analysis, which has application in reliability model for a single repairable component.

In the case of vacation model, Miller (1964) analyzed a system in which the server goes for a vacation of random duration whenever it becomes idle. A queueing system in which, the server taking exactly one vacation at the end of each busy period, is called a single vacation system. When the system becomes empty, server starts a vacation and he keeps on taking vacations until on return from vacation at least one customer is present. Several authors analyzed vacation systems. For more details on queueing systems with vacations one may refer to Doshi (1986, 1990).

Models of retrial queues, G/G/1 queues, tandem queues, queue with infinite servers, and queue with finite system capacity played a prominent role in the recent research. Kasahara et. al. (1996) considered an M/G/1/K system with push-out scheme and multiple vacations and derived the Laplace - Stieltjes transform of waiting time distribution for the customer which is eventually served. Langaris and Katsaros (1997) considered an M/G/1/N model multiple server vacations and gated service policy. They analyzed the steady state probabilities and the customers waiting time. Frey and Takahashi (1997) considered an M/G/1/N queue with vacation time and exhaustive service discipline and analyzed the queue length distribution at an arbitrary time as well as for the waiting time distribution. In 1998, they analyzed it using imbedded Markov chain technique, which makes analysis simple.

Bogoyavlenskaya (1997) considered M/G/1/m queue and analyzed using Erlangian phase method and obtained the steady state distribution of the queue length. Chaudhry and Gupta (1999) considered a single server finite capacity queue with general bulk service rule where customers arrive according to Poisson process and service times of the batches are arbitrarily distributed. Using supplementary variable and imbedded Markov chain technique they presented the relations between state dependent probabilities at departure and arbitrary epochs. Pekoz (1999) derived formulas for moments of the number of refused customers in a busy period in M/GI/1/n and GI/M/1/n queueing systems. Frey and Takahasi (1999) considered an M^X/GI/1/N finite capacity queue with close down time, vacation time and exhaustive service discipline under the partial acceptance strategy as well as under the whole batch acceptance strategy.

Earlier we have revealed five basic characteristics of a queue for the case when there is only one stage of service. It is also possible in many situations that a customer has to go through several stages of service. Thus, he is in a series of queues so that his departure from one stage be his arrival at the next. Therefore the departure processes of different queues are of paramount importance in the study of queues in tandem. A natural question, in cases where the only observables from the system are the departures, is whether or not from the behavior of the departures of queue one can identify or gain some information about the unknown input process or service system. There is a vast literature on output processes from queues. Much of it is reviewed in Daley (1976) and that review is updated later by Gnedenko and Konig (1983).

Usually in queueing models the server stops service when there is no customer in the queue and resumes service when a new customer joins the queue next time. This model fails to describe many real life situations where there is a control limit for shutting down and resuming of service. This may be treated as a model with removable server – the server being removed depending on the demand level. Queueing systems with removable servers were considered by Yadin and Naor (1963), Bell (1975) and Rhee and Sivazlian (1990) and other researchers.

There are many real life queueing situations in which service is rendered with control limit policies. In these situations one can use single and batch service in the same queueing model, which will be profitable than either could provide. For example one can see Baburaj and Manoharan (1997a) and Baburaj (2000).

Apart from this, one can see some remarkable achievements of queueing theory in the various field of applied mathematics and other applied sciences. For some advanced developments in queueing one can refer Dshalalow (1995, 1997). He cited spatial queues, queues with Markovian arrival processes and some other major research areas in queueing theory by providing an elaborated list of references.

1.3 Inventory Models – An out line.

The study of inventory models started with the work of Harris in 1915, which formulated and optimized a simple inventory situation. The formula derived by Harris is an optimal production lot size given as a square root function of a fixed cost, an investment or holding cost, and demand. This formula is known as the economic order quantity (EOQ) formula. Most of the individual elements of current inventory models are available before 1950.

An inventory is an amount of material stored for the purpose of sale or production. Inventory management of physical goods, or other products or elements is an integral part of logistic systems common to all sectors of the economy, such as business, industry, agriculture and defence. In an economy that is perfectly predictable, inventory may be needed to take advantage of the economic features of a particular technology, or to synchronize human tasks or to regulate the production process to meet the changing trends in demand. When uncertainty is present, inventories are used as protection against risk or stock-out.

Inventory theory is concerned with the analysis of several types of decisions relating primarily to the problem of when and how much to buy of a given item. An analysis of an inventory consists of the following steps: 1) Determination of the properties of the system, 2) Formulation of the problem, 3) Development of a model of the system and 4) Derivation of a solution of the problem.

The movement or flow of a single commodity into and out of a single storage point characterized the simplest type of inventory system.

Inflow (Procurement) --------------→ Inventory -----------------→ Outflow (Demand)

Several policies may be used to control an inventory system; of these the most important policy is the two-bin or (s, S) policy. Under this particular policy, whenever the position inventory (sum of on-hand inventory and outstanding orders) is equal to or less than a value s, procurement is made to bring its level to S. Under a continuous review system, the (s, S) policy will usually imply the procurement of a fixed quantity $Q = S - s$ of the commodity, while in periodic review systems the procurement quantity will vary. The (s, S) policy incorporates two decision variables s and S. The variable s is known as the reorder level, which identifies when to order, while both variables s and S identify how much to order.

In an inventory problem, the objective function may take several forms, and these usually involve the minimization of a cost function or the maximization of a profit function. The cost function in general, consist of the additive contribution of the procurement, the holding cost, and the shortage cost. Under the (s, S) policy, in a number of situations, it is more convenient to determine the optimal values of the pair (s, Q) or (S, Q) instead of optimizing the objective function by determining the values of s and S.

The inventory models are usually characterized by the demand pattern and the policy for replenishing the stock in the store. The replenishments ordered may arrive after a time lag L, which may be fixed or a random variable. This time lag L is called the lead-time. The time during which the inventory is empty is termed as a dry period. Asymptotic account of probabilistic treatment in the study of inventory systems using renewal theoretic argument is given in Arrow et. al. (1958). A review of the problems in the probability theory of storage systems is given by Gani (1957).

Most of the inventory models in the literature are concerned with determining the optimal stocking policies at a single installation. But nowadays the models, which determines optimal inventory management with more than a single installation attracts considerable attention. Scarf et. al. (1963) gave a detailed account of multistage inventory models. When demand is discrete the (S – 1, S) policy calls for the placement of a replenishment order after each demand equal in magnitude to the size of the demand. This policy has often advocated for controlling inventory of expensive and slow moving items.

A detailed review of the work carried out in (s, S) inventory systems up to 1966 can be found in Veinote (1966). The earliest work on the decay problem is due to Ghare and Schrader (1963) who considered the generalization of the standard EOQ model without shortages. The view of Nahmias (1982) provides the state of art on perishable inventory models until the beginning of eighties. Manoharan and Krishnamoorthy (1989) considered an inventory problem with all items subject to decay and derived the limiting probability distribution. Kalpakam and Arivarignan (1989) analyze a perishable inventory model in which the inventoried items have lifetimes with negative exponential distribution with demands forming a Poisson process. Krishnamoorthy and Varghese (1995) extended the above model, subject to disasters. Kalpakam and Sapna (1996) analyzed a one-to-one ordering perishable inventory model with renewal demands and exponential life times. Aggoun et. al. (1997) deals with a parametric multi-period integer valued inventory model for perishable items. In 1999 the same authors proposed a single-product, discrete time inventory model for perishable items. Williams and Patuwo (1999) derived the necessary equations to determine the single period, periodic review and optimal incoming quantity for a single product with a useful lifetime of two periods, subject to a known positive order lead time and a lost sales policy.

Stidham (1974) has introduced and studied wide class of stochastic input–output systems. Berg et. al. (1994) considered a production/inventory system with unreliable machines. A production/inventory system consisting of a single processor producing three product types and a warehouse is considered by Altoik and Shiue (1995). Glasserman and Tayur (1996) developed a simple approximation for multi stage production inventory system with limited production capacity and variable demands. Arreola-Risa (1996) studied an integrated multi-item production inventory system with stochastic demands and capacitated production and derived analytical expressions that lead to the optimal base stock levels. Gullu (1996) explored how total system costs and inventory positions are affected when forecasts are incorporated explicitly in production/inventory systems.

Aviv and Federgruen (1997) considered a single item, periodic review inventory model with uncertain demands in which each period production volume is limited by a capacity level. Bar-Lev et. al. (1996) analyzed a stochastic production/inventory problem with compound Poisson demand and state (inventory level) dependent production rates. Ha (1997) considered the problem of production control and stock rationing in a make – to – stock production system with two priority customer classes and back ordering. Inder Furth (1997) addressed a problem of product recovery management where a single product is stocked in order to fulfil a stochastic demand of customers who may return products after usage. Van-Houtum and Zijm

(1997) studied incomplete convolutions of continuous distribution functions as they appear in the analysis of multistage production and inventory systems.

Gullu (1998) considered a single item, stochastic demand production/inventory problem where the maximum amount that can be produced (ordered) in any given period is assumed to be uncertain. Kijima and Takimoto (1999) considered a single item single location, (T, S) inventory/production model with uncertain demands in which a production capacity is limited per period. Apart from stationary and time dependent behavior of system characteristics the effects of limited production capacity and the order-up-to-level on performance characteristics are also discussed.

1.4 Reliability Models - An Outline

The formation of reliability theory as a discipline started about half a century ago. Before that time the concept of reliability was looked at from an intuitive, subjective and qualitative point of view. The mathematical theory of reliability has grown out of the demands of modern technology and particularly out of the experiences in World War II with complex military systems.

The term "reliability" encompasses a wide class of research topics, ranging from transistor reliability to the reliability of complex systems. The problems of failures, repairs, maintenance etc. are to be viewed seriously in the use of any equipments or systems, because of future performance or future life is entirely depending on such reliability characteristics. Today design and production engineers discuss methods of improving reliability in design and production stages. Nowadays, a new science was even invented with the aim of discussing how reliability is achieved by nature and of trying to initiate such methods in building similar electronic or mechanical devices.

An equipment may be unreliable due to various reasons. They are infant mortalities, random failures and wear outs. One of the principal methods of improving system reliability is the introduction of redundancy. Repair and preventive maintenance are other methods of increasing system reliability.

Reliability as a human attribute has been praised for a very long time. In reliability, we are mainly concerned with devices or systems that fail at an unpredictable random age of $T > 0$. This random variable is assumed to have a distribution F, $F(t) = P[T \leq t]$, $t \in R$, with a density f. Then reliability R(t) of the system is defined as

$$P[T > t] = 1 - F(t) = \bar{F}(t) .$$

This function is also known as survival function. The failure rate λ is defined on the support of the distribution by

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}.$$

The failure rate $\lambda(t)$ measures the proneness to failure at time t and we have the relation

$$\bar{F}(t) = e^{-\int_0^t \lambda(s)\,ds} \quad [\textit{See Aven and Jensen } (1999)].$$

Epstein and Sobel (1953) began the work in the field of life testing, which marked the beginning of wide spread research using exponential distribution. Until 1960 reliability was defined as the probability that an item will perform a required function under stated conditions for a stated period of time. Research on coherent structures (general system reliability) began with the paper by Birnbaum et. al. (1961). Barlow and Proschan (1965) stimulated the research work on the failure rate function and classes of life distributions defined in terms of this function. Barlow and Proschan (1975) can be viewed as first milestone in lifetime analysis, complex systems and maintenance models.

Osaki and Nakagawa (1976) obtained a brief review of stochastic models on system reliability using stochastic processes such as Markov chains, Markov processes, renewal processes and semi - Markov processes. Ashok Kumar and Manju Agarwal (1980) reviewed different type of systems discussed in literature and various methods employed by different researchers in the analysis. Sherif and Smith (1981) provided an extensive bibliography of 524 references followed by Valdez-Flores and Feldman (1989) with 129 references on reliability models.

The earlier works in the multi state coherent systems are available in Barlow and Wu (1978). Joseph and Manoharan (1997) obtained the transient and steady state probabilities of a repairable multi state system having different failure modes.

Manoharan (1998) considered a multi state system that deteriorates over time and obtained the optimum performance level at which replacement is to be made. Shey – Huei (1999) proposed a generalized replacement model where a deteriorating system has two types of failures and is replaced at the N^{th} type I failure (minor failure) or first type II failure (Catastrophic failure) or at the working age T, whichever occurs first.

Shock models are described in Barlow and Proschan (1975). Perez – ocon and Gamiz – Perez (1995) derive a condition for a correlated cumulative shock model under which the system failure time is harmonically new better than used in expectation (HNBUE). Gasemyr and Natwig (1995) considered a binary monotone system whose component states are dependent such that shocks that destroy several components at once. They showed how the minimal path and cut sets of the two systems are related, and use it to examine the relation between the bounds on the reliability based on the two systems. Abouammoh et.al. (1995) considered a device subjected to shocks which arrive according to a counting process $\{N(t), t \geq 0\}$ and the probability that the device survives beyond t, and their ageing properties are studied for $\bar{H}(t)$ when $\{N(t), t \geq 0\}$ is a non-homogenous Poisson process.

Lee and Lee (1995) introduced a random shock model for a linearly deteriorating system and obtained the characteristic function (c.f) of X(t), the state of the system at the time t. Thangaraj and Stanley (1995) considered a system subject to shocks occurring randomly in time according to a general pure birth process and studied the rate of generation of information about the failure times of systems for different processes.

Mitra and Basu (1996) considered the life distribution of a device subject to a sequence of shocks occurring randomly in time according to a homogenous Poisson process. Thangaraj and Sundararajan (1997) proposed some multivariate replacement policies based on a number of shocks, cumulative damage caused due to shocks and the working age of the system subject to shocks. Again Gasemyr and Natwig (1998) presented a Bayesian approach based on autopsy data.

Shei – Huei and Griffith (1996) considered a system subject to shocks that arrive according to a non-homogeneous Poisson process and describe two type of failures. Shei – Huei (1997) proposed and analyzed a generalization of the block replacement policy for a system subject to shocks and the average cost rate of replacement of failed items is obtained using renewal reward theory. Shei – Huei (1998) considered the same system and derived the expression for the expected long run cost per unit time. Sing et. al. (1998) dealt with the effect of the shocks on the different characteristics of a system having two main units and one protective unit and obtained several characteristics of the system using a regenerative point technique.

Ebrahimi (1999) described a model in which each shock to the system causes a random damage that grows in time and system fails if the total damage exceeds a certain capacity or threshold. He obtained various properties of this model and derived sufficient conditions for the future rate order. Skoulakis (2000) studied a reliability system subject to shocks generated by renewal process. He derived closed expressions for the Laplace-Stieltjes transform and the expectation of the time to system failure for a more general system, which has very important special cases, such as k-out-of-n: F system.

Sreehari (2000) developed a model in which a system may fail if two shocks occur with very little gap in between or the operating environment comprises of several stresses/shocks whose accumulated damage effect on the failure rate. Touzi (2000) studied the optimal insurance demand problem of an agent whose wealth is subject to shocks produced by some marked point process and concluded that below the critical value, agents prefer to be completely insured whereas above the upper critical value they take no insurance.

Apart from these so many reliability models are developed under the premise that the operating environment is static. Now new models have been developed when the environment under which the items operate is dynamic. One of such type is a software reliability model on which plenty of articles are available in standard journals.

1.4.1 Intermittently Used systems

Generally we can classify a system as priority system, system with imperfect switchover, parallel system, system with preventive maintenance (PM) and intermittently used systems. In all the systems except the last there is a basic assumption that the system is required continuously. But there are situations where the systems are not required continuously. Such systems are known as intermittently used systems. These systems are 'needed' or 'not needed' during alternative periods of time, which are governed by a pair of random variables.

Gaver (1963) laid stress on the point event called a disappointment characterized by the entry of the system to either the down state during a usage period or the need arising for the system when it is already in the down state. Srinivasan (1966) extended Gaver's result and later Srivastava et. al. (1971) considered the stochastic behavior of an intermittently working system with standby redundancy. Nakagawa et. al. (1976) have considered the preventive maintenance policy and obtained the mean time to first disappointment.

Srinivasan et. al. (1979) discussed intermittently used two dissimilar unit system and obtained various reliability measures. Kapoor and Kapoor (1980) derived the explicit expressions for joint distribution of first uptime and disappointment time of an intermittently used two unit cold standby redundant system. Subramanian et. al. (1981) have analyzed an intermittently used n-unit standby redundant system. Sharma and Natarajan (1982) considered a general model where the random variable describing the need and no-need periods are independent and identically distributed, which are not necessarily exponential.

In all the models considered so far it is usually assumed that the need is always for only one unit and the need and no-need periods alternate. But there may be situations that may warrant the use of one or more units for the satisfactory performance of the system. In such situations the number of units needed at any time may be described by a stochastic process called the "need process". Sharafali et. al. (1988) introduced this concept for a two unit system with stochastic demand.

1.4.2 Aspects of Ageing

The notion of aging is first discussed in Bryson and Siddique (1969). The concept of ageing for life distributions has been found very useful in reliability theory. Based on these notions, results may be derived concerning the behavior of systems, bounds for survival functions, moment inequalities etc. It has been found useful for arriving at efficient algorithms for use in maintenance policies.

Concerning the lifetime random variable T, the mean of T is denoted by μ. For each t with $\bar{F}(t) > 0$, $\bar{F}(x/t) = \frac{\bar{F}(t+x)}{\bar{F}(t)} = \Pr\{T \geq t + x/T \geq t\}$ represents the survival function of a unit of age t. The remaining (residual) life, at age t, is $E(T - t/T > t)$, which may be shown to be $\int_0^\infty \bar{F}(x/t)dx$. When F'(t) = f(t) exists, we can define the failure rate as

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}, \text{ for t such that } \bar{F}(t) > 0.$$

It follows that, if $\lambda(t)$ exists,

$$-\log \bar{F}(x) = \int_0^x \lambda(t)dt$$, which represents the cumulative failure rate.

The following notions of positive (adverse effect) ageing are common in reliability theory. We say F is said to be

(i) Increasing failure rate (IFR) if $\bar{F}(x/t)$ is decreasing in $0 \le t < \infty$ for each $t \ge 0$.

When the density exists, this is equivalent to $\lambda(t) = \frac{f(t)}{\bar{F}(t)}$ being non-decreasing in $t \ge 0$.

(ii) Increasing failure rate average (IFRA) if $-\frac{1}{x}\log \bar{F}(x)$ is non-decreasing in x.

(iii) Decreasing mean residual life (DMRL) if the mean remaining life function $\int_0^\infty \bar{F}(x/t)dx$ is non-increasing in x.

(iv) New better than used (NBU) if $\bar{F}(x/t) \le \bar{F}(x)$ for $t \ge 0$, $x \ge 0$.

(v) New better than used in expectation (NBUE) if $\int_0^\infty \bar{F}(x/t)dx \le \mu$ for $t \ge 0$.

(vi) Harmonically new better than used in expectation (HNBUE) if $\int_t^\infty \bar{F}(x)dx \le \mu \exp\left(\frac{-t}{\mu}\right)$ for $t \ge 0$.

(vii) L – distribution if $\int_0^\infty \exp(-st)\bar{F}(t)dt \ge \frac{\mu}{1+s}$ for $s \ge 0$.

(viii) New better than used in failure rate (NBUFR) if $\lambda(t) \ge \lambda(0)$ for $t \ge 0$.

(ix) New better than used in average failure rate (NBAFR) if

$$\lambda(0) \le \frac{1}{t}\int_0^{\infty} \lambda(x)dx = \frac{-\log \bar{F}(t)}{t}.$$

It is verified that among the positive ageing properties the following and only the following implications hold.

$$\begin{array}{ccccccccc} IFR & \Rightarrow & IFRA & \Rightarrow & NBU & \Rightarrow & NBUFR & \Rightarrow & NBAFR \\ \Downarrow & & & & \Downarrow & & & & \\ DMRL & \Rightarrow & NBUE & \Rightarrow & HNBUE & \Rightarrow & L. & & \end{array}$$

Note that we have a parallel discussion of the dual classes of life distributions, which can be similarly defined based on negative (beneficial effect) ageing. There exist a vast amount of literature examining the relevance of the above notions in reliability theory and survival analysis. Closure of classes of distributions possessing the above properties under formation of coherent structures, convolutions, mixtures etc. has been investigated. The preservation of the classes under various shock models has been extensively studied in the literature.

1.5 Some Analytical Tools for the Analysis

1.5.1 Renewal Process

Renewal theory was first introduced by Feller (1941). Later Smith (1954) demonstrated its uses in the study of general theory of stochastic processes. Let $\{X_n, n \ge 1\}$ be a sequence of non-negative independent and identically distributed random variables with a common distribution function (d. f.) F(.). Again let $S_0 = 0$, and $S_n = X_1 + X_2 + \ldots + X_n$, $n \geq 1$. Then the process $\{N(t), t \geq 0\}$, where $N(t) = \text{Sup}\{n \mid S_n \leq t\}$ is called a renewal process.

Define M(t) to be the expected value of N(t). Then M(t) is called the renewal function. The derivative of M(t), if it exists, is denoted by m(t) and is called the renewal density. The quantity m(t)dt represents the probability that a renewal occurs in (t, t + dt), which is important in practical applications.

Let $F_n(x) = P\{S_n \leq x\}$ be the d.f. of S_n. Since X_i's are independent and identically distributed, $F_n(.)$ is the n – fold convolution of F with itself and for n = 0 we interpret it to be unity. It is easy to see that $N(t) \geq n \Leftrightarrow S_n \leq t$.

Thus the distribution of N(t) is given by

$$P\{N(t) = n\} = F_n(t) - F_{n+1}(t) \text{ and}$$

$$M(t) = \sum_{n=1}^{\infty} F_n(t).$$

The renewal density satisfies the integral equation

$$m(t) = f(t) + \int_0^t f(u)\, m(t-u)\, du \quad \text{where } f(t) = F'(t).$$

A similar equation satisfied by the renewal function M(t) is given by

$$M(t) = F(t) + \int_0^t F(u) M(t-u) dF(u).$$

The above equation is called the renewal equation.

Now, suppose that X_1 has a distribution G(.) which is different from the common distribution F(.) of the remaining $X_2, X_3, \ldots$, then the process $\{N_D(t), t \geq 0\}$ is called a delayed renewal process where $N_D(t) = \text{Sup}\{n \mid S_n \leq t\}$.

Theorem 1.1. (Elementary Renewal Theorem) Let $\mu = E(X_n)$ with convention,

$$\frac{1}{\mu} = 0, when \mu = \infty. \qquad Then \ \lim_{t \to \infty} \frac{M(t)}{t} = \frac{1}{\mu}.$$

Theorem 1.2. (Key Renewal Theorem) If H(t) is a non-negative function of t such that

$$\int_0^\infty H(t)\,dt < \infty, \ then \ \lim_{t \to \infty} \int_0^t H(t-u)dM(u) = \frac{1}{\mu} \int_0^\infty H(t)dt.$$

1.5.2 Stochastic Point Process

The theory of point processes introduced by Von Mises (1936). Srinivasan (1974) obtained a detailed treatment of stochastic point processes with special reference to its applications in management problems.

Let $\{T_n\}$, n = 0, ±1, ±2, ... be a discrete parameter stochastic process associated with the probability space $\{\Omega, A, P\}$ where each T_n is finite valued and non-negative with probability one.

If T_n is specified as above and if

$$t_n = \begin{cases} \sum_{k=0}^{n} T_k & ; \text{ if } n \geq 0 \\ T_0 - \sum_{k=n}^{-1} T_k & ; \text{ if } n < 0 \end{cases}$$

then the sequence $\{t_n\}$ is called a stochastic point process.

1.5.3 Product Densities

One of the ways of characterizing a general stochastic point process is through the product densities (Srinivasan (1974)).

Let us introduce the following notation.

$N(t, x)$ = the number of events in $(t, t + x]$ which is a random variable, $d_x N(t, x)$ = the number of events in $(t + x, t + x + dx]$ and $P_n(t, x) = P[N(t, x) = n]$.

A point process defined on the real line is said to be orderly regular if

$$\sum_{n=2}^{\infty} P_n(t,\Delta) = P[N(t,\Delta) \geq 2] = o(\Delta). \qquad (1.5.1)$$

An interesting consequence of (1.5.1) is given by the relation

$$\lim_{\Delta \to 0} \frac{E[N(t,\Delta)]}{\Delta} = \lim_{\Delta \to 0} \frac{P[N(t,\Delta) \geq 1]}{\Delta}. \qquad (1.5.2)$$

Note:-

$$\lim_{\Delta \to 0} \frac{E[N(t,\Delta)]}{\Delta} = \lim_{\Delta \to 0} \frac{P[N(t,\Delta) = 1]}{\Delta}.$$

It can be shown that, E [N(t, Δ)] = Var [N(t, Δ)] for small Δ.

Now the densities of the factorial moment measures can be conveniently defined by

$$h_n(x_1, x_2, \cdots, x_n) = \lim_{\Delta_1, \Delta_2, \cdots, \Delta_n \to 0} \frac{E\left(\prod_{i=1}^{n} N(x_i, \Delta_i)\right)}{\Delta_1 \Delta_2 \cdots \Delta_n} \quad ; \quad x_1 \neq x_2 \neq \cdots \neq x_n$$

$$= \lim_{\Delta_1, \Delta_2, \cdots, \Delta_n \to 0} \frac{P[N(x_i, \Delta_i) \geq 1 \quad ; \quad i = 1, 2, \cdots, n]}{\Delta_1 \Delta_2 \cdots \Delta_n} \quad ; \quad x_1 \neq x_2 \neq \cdots \neq x_n .$$

Since $h_n(.)$ is a product of density of expectation measures at different points, the density is aptly called product density.

We can introduce the product densities corresponding to the counting process, which is defined by

$h_n(t, x_1, x_2, \ldots, x_n)dx_1, dx_2, \ldots, dx_n = E[d_{x_1} N(t, x_1) \quad d_{x_2} N(t, x_2) \quad \ldots \quad d_{x_n} N(t, x_n)]$.

Even though the functions $h_n(.)$ are called the densities it is important to note that their integration will not give probabilities, but will yield the factorial moments.

1.5.4 Markov Renewal Process

Markov renewal process is a generalization of both Markov process and renewal process and is a blend of the two. It creates tools that are more powerful than those either could provide. A detailed study of Markov renewal process (MRP) has been made by Cinlar (1975).

Suppose a particle or a system move from one state to another state with random sojourn times between the states. The successive states visited form a Markov chain and a sojourn time has a distribution, which depends on the state being visited as well as the next state to be entered. Such a process becomes a Markov process if the distributions of sojourn time are all exponential, independent of next state. The process becomes renewal process if there is only one state in the state space 'S'.

Let (Ω, A, P) be a probability space, N be the set of non-negative integers and R_+ be the non-negative real numbers. On this probability space, the random variables $X_n: \Omega \rightarrow E$ and $T_n: \Omega \rightarrow R_+$ are defined for each $n \in N$ so that $0 = T_0 \leq T_1 \leq \ldots$ where E is a finite set. These elements are said to form a MRP (X,T) with state space E if

$$P[X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_0, X_1, \ldots, X_n; T_0, T_1, \ldots, T_n] = P[X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_n]$$

$$\forall\ n \in N,\ j \in E, \text{ and } t \in R_+.$$

The semi Markov kernel Q has its $(i, j)^{th}$ entry

$$Q(i, j, t) = P[X_{n+1} = j,\ T_{n+1} - T_n \leq t \mid X_n = i]$$

so that

$\sum_j Q(i,j,t)$ *is the distribution function of the sojourn time in i and* $P(i,j) = Q(i,j,\infty)$

is the transition matrix of the Markov chain $\{X_n, n \geq 0\}$.

The Markov renewal function is

$$R(i,j,t) = \sum_{n=0}^{\infty} Q^n(i,j,t) \quad where$$

$$Q^{n+1}(i,j,t) = \sum_{k} \int_0^t Q(i,k,du) Q^n(k,j,t-u)du, \ for\ n \geq 0 \quad and\ Q^0(i,j,t) = I(i,j) = \begin{cases} 1\ if\ i=j \\ 0\ if\ i \neq j \end{cases}.$$

The functions

R(i, j, t) = E[the number of transitions into state j in (0, t) | X_0 = i; i, j ∈ E] are called Markov renewal functions and {R(i, j, t); i, j ∈ E, t ∈ R_+} is known as Markov renewal kernel.

There are processes, which are non-Markovian and posses strong Markov property at certain selected random times. Then, imbedded at such intervals, one finds a MRP.

1.5.5 Regenerative process

Bellman and Harris (1948) first introduced regeneration points in the literature. Many stochastic processes have the property of regenerating themselves at certain points in time so that the behavior of the process after the regenerating epoch is a probabilistic replica of the behavior starting at time zero and is independent of the behavior before the regeneration epoch. That is, a stochastic process {X(t), t ∈ T} with state space S and time index set T is said to be regenerative if there exists a random epoch S_1 such that (i) {X(t + S_1), t ∈ T} is independent of {X(t), 0 ≤ t ≤ S_1}and (ii) {X(t + S_1), t ∈ T} has the same distribution as {X(t), t ∈ T}. It is assumed that the index set T is either [0, ∞) or T is equal to 0, 1, 2, ... When T = [0, ∞) we have a continuous time regenerative process and in the other case a discrete time regenerative process.

The existence of the regeneration epoch S_1 implies the existence of further regeneration epochs S_2, S_3, … having the same property as S_1. That is regenerative process can be split into independent and identically distributed renewal cycles. A cycle is defined as the time interval between two consecutive regeneration epochs.

In many practical situations a reward structure is imposed on the regenerative process $\{X(t), t \in T\}$. Let R_n = the total reward earned in the n^{th} renewal cycle, n = 1, 2, … and assumed that R_1, R_2, … are independent and identically distributed random variables. Note that R_n typically depends on L_n = the length of the n^{th} renewal cycle , that is, $L_n = S_n - S_{n-1}$ with $S_0 = 0$. Define R(t) = the cumulative reward earned up to time t, for any $t \in T$. Then the process $\{R(t), t \geq 0\}$ is called a renewal reward process.

Theorem 1.3. (Renewal reward theorem)

$$\lim_{t\to\infty}\frac{R(t)}{t}=\frac{E(R_1)}{E(L_1)} \quad \textit{with probability } 1$$

$$\textit{or} \quad \lim_{t\to\infty}\frac{E(R(t))}{t}=\frac{E(R_1)}{E(L_1)}.$$

Theorem 1.4. For a continuous time regenerative process $\{X(t), t \geq 0\}$,

$$\lim_{t\to\infty}\frac{1}{t}\int I_B(u)du=\frac{E(T_B)}{E(L_1)} \quad \textit{with probability } 1.$$

For a discrete time regenerative process $\{X(n), n = 0, 1, \ldots\}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} I_B(k)=\frac{E(T_B)}{E(L_1)} \quad \textit{with probability } 1$$

where I_B is the indicator function and T_B is the amount of time the process spends in the set B of states during one cycle.

Theorem 1.5. For the regenerative process $\{X(t), t \in T\}$

$$\lim_{t \to \infty} P\{X(t) \in B\} = \frac{E(T_B)}{E(L_1)}$$

provided that the probability distribution of the cycle length has a continuous part in the continuous case and is aperiodic in the discrete time case.

1.6 PH - Class of Distributions

The class of phase type (PH) distributions represents the natural family to which Erlang's "method of stages" extends. The PH-class contains commonly used distributions like exponential, Erlangian, sum and mixture of exponential distributions. This family has gained widespread acceptance in recent years because of its computational properties in applied stochastic modeling. A non-negative random variable (or its distribution) is said to be of phase type if it is the time until absorption in a finite state, time-homogeneous Markov chain. This family was introduced by Marcel F. Neuts (Neuts (1975)) as a tool for unifying a variety of stochastic models and for constructing new models that yield to algorithmic analysis. These distributions are important in their own respects and have found useful in various fields such as queueing, reliability, survival analysis, branching process, telecommunications, computer science etc.

There are two parallel discussions of PH- distributions. One corresponding to distributions on the non-negative integers obtained from absorption times in discrete parameter Markov chains, the other to distributions on the non-negative real line obtained from absorption times in continuous parameter Markov chain. We briefly present them here.

1.6.1 Discrete Phase Type Distribution (DPH)

We require the following properties of finite absorbing Markov chain. Consider a finite Markov chain $\{Y_n, n \geq 0\}$ on the state space $\Omega = \{1, 2, \ldots, m, m+1\}$. Let its transition probability matrix be of the form

$$P = \begin{bmatrix} T & T^0 \\ 0 & 1 \end{bmatrix}$$

where

$$T = (T_{ij})_{1 \leq i,j \leq m} \quad , \quad T^0 = (T_1^0 \quad T_2^0 \quad \cdots \quad T_m^0)^T \quad , \quad 0 = (0 \quad 0 \quad \cdots \quad 0)_{1 \times m}$$
$$and \quad Te + T^0 = e = (1 \quad 1 \quad \cdots \quad 1)^T.$$

Let A_i^* $(1 \leq i \leq m)$ denote the probability of ultimate absorption into any state m+1 starting from given initial state i. Then we have the following elementary result.

Theorm 1.6. $A_i^* = 1, 1 \leq i \leq m$ if, and only if, $(I - T)^{-1}$ exists.

We shall assume that $(I - T)^{-1}$ exists so that the states $\{1, 2, \ldots, m\}$ are transient and absorption into the state m + 1 starting from each one of them is certain.

Also from the theory of Markov chains it is easy to prove the following result on absorption times.

Theorem 1.7. If the initial state of the chain is chosen according to initial probability vector $(\alpha, \alpha_{m+1}) = (\alpha_1, \alpha_2, \ldots, \alpha_m, \alpha_{m+1})$, the probability p_n of absorption at time n is given by $p_0 = \alpha_{m+1}$ and $p_n = \alpha T^{n-1} T^0$, $n \geq 1$.

So we define

Definition 1.1. A probability density $\{p_k\}$ on the set of non-negative integers is called a DPH if it is the density of the time until absorption in a finite state Markov chain with stationary transition probability matrix given by

$$P = \begin{bmatrix} T & T^0 \\ 0 & 1 \end{bmatrix} \tag{1.1}$$

and initial probability vector (α, α_{m+1}). Here T is an m×m sub-stochastic matrix such that $Te + T^0 = e$ and $(I - T)$ is non-singular.

The DPH density is given by

$$p_0 = \alpha_{m+1}$$

$$p_n = \alpha T^{n-1} T^0, \text{ for } n \geq 1.$$

The pair (α, T) is called the representation of DPH and m is called the order.

The probability generating function (pgf) of DPH density $\{p_k\}$ can be obtained as

$$P(z) = \sum_{n=0}^{\infty} p_n z^n = \alpha_{m+1} + z\alpha(I - T)^{-1}T^0, \quad |z| \le 1,$$

which exists since $(I - T)^{-1}$ is non-singular.

Hence the mean and variance of DPH can be obtained as

$$\begin{aligned} \mu_1' &= P'(1) = \alpha(I - T)^{-1} e \\ \sigma^2 &= P''(1) + P'(1) - [P'(1)]^2 \\ &= 2\alpha(I - T)^{-2} e - \mu_1' - \left(\mu_1'\right)^2. \end{aligned}$$

Also the j^{th} factorial moments can be obtained as

$$\mu^{(j)} = \left.\frac{d^j P(z)}{dz^j}\right)_{z=1} = j!\alpha T^{j-1}(I - T)^{-j} e \quad , \quad j \ge 1.$$

Note that all raw moments of DPH density exists.

Examples of DPH:-

1.Geometric distribution $p_k = pq^k$, $k \ge 0$ $(0 < p < 1,\ p + q = 1)$ has a DPH (α, T) of order 1 where $\alpha = q$ and $T = q$.

2.Generalized negative binomial distribution (distribution of sum of geometric random variables) has a DPH (α, T) of order m where $\alpha = (1, 0, \ldots, 0)_{1\times m}$,

$$T = \begin{bmatrix} q_1 & p_1 & 0 & \cdots & 0 \\ 0 & q_2 & p_2 & \cdots & o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_m \end{bmatrix} \quad and \quad T^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ p_m \end{bmatrix}$$ where $q_i + p_i = 1$; $i = 1, 2, \ldots, m$.

1.6.2 Continuous Phase Type Distribution (CPH)

Consider a continuous time Markov chain $\{X(t), t \geq 0\}$ on $\{1, 2, \ldots, m+1\}$ where m +1 is absorbing and others transient. Its infinitesimal generator Q by virtue of the absorbing state, has the block partition as

$$Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$$

where T^0 is called the exist vector, giving the conditional intensities of absorption and the m×m dimensional matrix T is called the phase type generator and it is always non-singular. Further since each raw in Q sums to zero, $Te + T^0 = 0$.

The transition probability matrix **P(t)** $=(P_{ij}(t))$ of the Markov chain X(t) satisfies the matrix differential equation

$$P'(t) = P(t)\,Q \qquad (1.6.2)$$

with initial condition P(0) = I. The matrix solution of (1.6.2) is

$$\mathbf{P(t)} = \exp\,(Qt) = \sum_{n=0}^{\infty} \frac{Q^n t^n}{n!}$$

which has the corresponding Block partitioned form as

$$P(t) = \begin{bmatrix} \exp(Tt) & e - \exp(Tt)e \\ 0 & 1 \end{bmatrix}.$$

Let $Y = \inf\{t > 0: X(t) = m + 1\}$, the absorption time and (α, α_{m+1}) be the initial probability vector. Then the probability F(t) that absorption occurs no later than time t, ie, the distribution of Y is given by,

$$F(t) = 1 - \alpha \exp(Tt)e, \quad t \geq 0.$$

This yields the continuous version as

Definition 1.2. A distribution F on $[0, \infty)$ is a CPH, if it is the distribution of time until absorption in a finite state Markov chain with generator

$$Q = \begin{bmatrix} T & T^0 \\ 0 & 0 \end{bmatrix}$$

and initial probability vector (α, α_{m+1}), $T = (T_{ij})$ is a non-singular matrix of order m and satisfies $T_{ii} \leq 0$, $1 \leq i \leq 0$, $T_{ij} \geq 0$ for $i \neq j$. The distribution F is given by

$$F(x) = 1 - \alpha \exp(Tx)e, \; x \geq 0.$$

And we say F has representation $(\alpha, T)_m$.

Some basic analytical characteristics of CPH:-

(i) If $0 < \alpha_{m+1} < 1$, then F(.) has a jump of height α_{m+1}, at 0. On $(0, \infty)$, it has a density f(x) given by

$$f(x) = \alpha \exp(Tx)T^0 \quad , \quad x > 0.$$

(ii) The survival function $\bar{F}(x)$ has the simple form,

$$\bar{F}(x) = \alpha \exp(Tx)e.$$

(iii) The Laplace Stieltjes transform is

$$f^*(s) = \int_0^\infty \exp(-sy)dF(y) = \alpha_{m+1} + \alpha(sI - T)^{-1} T^0, \; Re(s) \geq 0.$$

(iv) The n^{th} moment $\mu_n' = \int_0^\infty y^n dF(y) = (-1)^n n! \, \alpha T^{-n} e, \; n \geq 1.$

In particular mean $\mu_1' = -\alpha T^{-1} e$.

Examples of CPH:-

1.The exponential distribution $F(x) = 1 - e^{-\lambda x}$, $x \geq 0, \lambda > 0$ has CPH representation $(\alpha, T)_m$ with $\alpha = 1$, $T = -\lambda$ and $m = 1$.

2.The generalized Erlang distribution of order m (convolution of m exponential distributions with parameters $\lambda_1, \lambda_2, \ldots, \lambda m$) has CPH $(\alpha, T)_m$ where $\alpha = (1, 0, \ldots, 0)_{1\times m}$ so that $\alpha_{m+1} = 0$,

$$T = \begin{bmatrix} -\lambda_1 & \lambda_1 & 0 & \cdots & 0 \\ 0 & -\lambda_2 & \lambda_2 & \cdots & o \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\lambda_m \end{bmatrix} \quad and \quad T^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_m \end{bmatrix}.$$

3. Hyper exponential distribution $F(x) = \sum_{i=1}^{m} \beta_i \left(1 - e^{-\lambda_i x}\right)$, $\beta_i, \lambda_i > 0, \sum \beta_i = 1$ has CPH $(\alpha, T)_m$ where $\alpha = (\beta_1, \beta_2, \ldots, \beta_m)$, $T = -diag(\lambda_1, \lambda_2, \ldots, \lambda_m)$, so that $\alpha_{m+1} = 0$ and $T^0 = (\lambda_1, \lambda_2, \ldots, \lambda_m)^T$.

When applied to PH- distributions many standard constructions of probability theory again yield PH- distributions, we say that the family of PH- distributions is closed under these constructions. Closure properties of great generality are known, but the practical useful ones are those for which a convenient representation for the resulting distribution can be written down. Neuts (1981) discusses the results under finite convolution, mixtures, maximum, minimum and other useful operations arising in stochastic models.

1.7 An Overview of the Main Contributions of the Thesis

In the literature we have many interesting stochastic models in queues, inventory and reliability describing various complex situations. In this thesis, we consider some models in queueing theory, inventory theory and reliability theory, which describe real life situations. It is important to have some knowledge of the structural aspects of distributions frequently encountered in the theory of queues. We shall have the investigation based on the properties such as unimodality and infinite divisibility, which could provide us an insight about the behavior of distributions.

Keilson (1971) and Roster (1980) discuss the unimodality properties of distributions occurring in some situations. Unimodality and infinite divisibility are two of the most important concepts in distribution theory. Here we shall study these properties with regard to the stationary population size distribution $\{p_n; n \geq 0\}$ of birth and death processes.

In the classical queuing model the server stops service when there is no customer in the queue and resumes service when a new customer joins the queue next time. This model fails to describe many real life situations where there is a control limit for shutting down and resuming of service. An appropriate model in such a situation may the one with (a, b) policy. The stationary distribution of the system and its evaluation becomes important in such a study.

Neuts (1967), Cohen (1969) etc. considered Poisson arrival queues with single as well as batch service. Recently Baburaj and Manoharan (1997a) and Baburaj (2000) have considered a Markovian queueing system with single and batch services. This model with single control limit c on the batch size are found to be important in many real life situations, but if the control limit c is very large, the model suffers from the risk of perpetual change over from batch service to single service. We have developed a model by introducing a secondary limit a (< c).

Inventory systems of (S - 1, S) type had been studied quite extensively in the past. Recently Schults (1990) established that (S - 1, S) policy is optimal when the renewal function of demand sizes is concave. Further he has shown that if the ratio of the re-order cost to the expected time between demands is smaller than a specified function of the lead time demand distribution and the holding and penalty costs, then it is optimal to order after each demand. So we have developed an (S – 1, S) policy in which the demand time and production time both follows PH – distributions F(.) and G(.) respectively and obtained the stationary distribution of the inventory level. The reason why the PH distributions are chosen in this work is that many researchers have been investigating these distributions and developing procedures to fit these distributions to given data sets (See for example Bobbio and Telek (1994) and Asmussen et al (1996)).

Gaver (1963) first discussed intermittently used system and introduced "disappointment time". In most of the intermittently used systems one or more units may be required for the satisfactory performance of the system. In such situations the number of units needed at any time may be described by a stochastic process called the need process. Sharafali (1988) considered a two-unit system with stochastic demand and obtained various reliability measures. We consider a more general system with n-units and obtain analogous results.

The question of preservation of various ageing properties under shock models attracted many researchers in the past. Lately Klefsjo (1981) proved that $\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\} \bar{P}_k$ is HNBUE if $\bar{P}_k$ has the discrete HNBUE property and Abouammoh et al (1988) have proved the NBUFR and NBAFR preservation results. We prove the corresponding theorems for the increasing likelihood ratio (ILR) class of life distributions under a generalized Poisson shock model. We also get analogous results for the dual class - decreasing likelihood ratio (DLR) class.

This thesis consists of seven chapters including this introductory chapter. We have reviewed some developments in queue, inventory and reliability after giving a brief introduction on stochastic models in the first four sections of this chapter. In section 5, we give a brief description about the main basic tools used in the thesis. In section 6 we obtain a brief account on PH – distribution. In this last section we present an overview of the thesis.

Our work starts with some structural aspects of distributions frequently encountered in the theory of queues. In chapter 2, we study the properties unimodality and infinite divisibility with regard to the stationary population size distribution of birth and death processes and aim at some characterization results on certain queueing models. After giving a brief review of the preliminary concepts and results, we obtain a criterion for strong unimodality of the stationary population size distribution of a birth – death process.

In light of this criterion we have, when the population size distribution $\{p_n\}$ exists, then it is log convex, if and only if $\{a_n\}$ is non-decreasing where $a_n = \frac{\lambda_n}{\mu_{n+1}}$, $n \geq 0$. Using this criterion we have examined the strong unimodality of M/M/∞ queues, M/M/s; $1 \leq s \leq \infty$ queueing system and linear growth model with immigration. Further we have proved that the stationary queue length distribution in M/M/s queueing system is infinitely divisible if and only if either $s = 1$ or $s = \infty$.

In chapter 3, we consider a finite system capacity M/G/1/K queue with removable server. Here the server being removed depending on the demand level. Recently Baburaj and Manoharan (1997) considered a controllable queueing system having two servers with bulk service rate and with all underlying distributions as exponential. Here we consider single server queueing system with a finite capacity K and the service times are independent and identically distributed random variables with distribution function G(.) with finite mean μ.

The server employs an (a, b) – policy for the closing down and the starting up of the station. That is we assume that customers arrive according to a Poisson process and there is only one server in the system. The server depending on the queue length serves the customers. Specifically, if the number of customers in the waiting line is greater than or equal to ' b ', the server serves the customers according to first come first served (FCFS) basis. When the number of customers in the waiting line reaches ' a ', then the server stops the service and the station is closed down. It is resumed only when a queue of ' b ' customers have accumulated where $0 \leq a < b \leq K$.

For M/G/1/K model Cohen (1969) established the relationship

$$\pi_j = \frac{p_j}{1 - p_K}, \; 0 \leq j \leq K\text{-}1.$$

Using the regenerative process, we derive the same equation for M/G/1/K (a, b) – policy (a general form of M/G/1/K model).

Expected busy period of this model obtained in terms of the expected busy period of standard M/G/1/K model. An expression for the expected busy period of standard M/G/1/K model is derived and a particular case when service time follows exponential is also worked out. We also investigate stationary probabilities in terms of expected busy period and then to find out the steady state probabilities by the relation $\pi_j = \frac{p_j}{1 - p_K}$. The method has been illustrated through numerical examples.

In chapter 4, another queueing model with single and batch service is considered. Baburaj and Manoharan (1997a) and Baburaj (2000) have considered a Markovian queueing system with single and batch services and with a single control limit c on the batch service. But in many real life situations, when the control limit ' c ' is very large the model suffers from the risk of perpetual change over from batch service to single service. This problem may be overcome by the introduction of a secondary limit ' a ' (< c), so that when a batch service has been on, the server may continue the batch service even with a size less than ' c ' but up to the limit a + 1 and when the number of customers drops below a + 1 single service commences.

We analyze the model in detail and obtain the expression for steady state and transient state distributions. Expected queue length and expected busy period are considered. The distribution of busy period is also attempted. We consider a numerical example illustrating the results.

In chapter 5, we are considering an inventory model, which is of (S – 1, S) type. In this model there is a production facility and a finished product warehouse for a single commodity. For this model, the demand time as well as production time follows independent PH – distributions. We analyze this general model and investigate the stationary distribution of the inventory level.

In chapter 6, we consider an intermittently used complex n-unit system with stochastic demand. For this model, we obtain the expressions for the stationary distribution, time to the first disappointment and mean number of disappointments. The first order product density and sojourn time are also discussed.

In the last chapter, we study a generalized Poisson shock model and prove some results for the ILR class of life distributions. We consider a device subjected to shocks occurring randomly over time according to a counting process **N** = {N(t): t ≥ 0}. Let $\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\} \bar{P}_k$ be the survival function then we prove the preservation of ILR (DLR) properties when **N** is (i) homogeneous, (ii) non-homogeneous and (iii) mixed Poisson processes.

The computations mentioned in the study are performed using computer programs developed in FORTRAN and using the utility MATHCAD.

CHAPTER 2

STRUCTURAL ASPECTS OF STATIONARY POPULATION SIZE DISTRIBUTIONS OF THE BIRTH AND DEATH QUEUES

2.1 Introduction

It is important to have some knowledge of the structural aspects of distributions frequently encountered in the theory of queues. The properties such as unimodality and infinite divisibility could provide us an insight about the behavior of distributions. We shall have the investigation based on these properties. Several of the distributions in the standard queuing models, such as those of queue length, waiting times and inter departure times are of well known form like Poisson, exponential, geometric or mixtures of these. Structural aspects of these distributions are already well established in the existing literature. However there also exist cases in which one does not know much about these distributions and may have very few sketch papers to learn about them.

Keilson (1971) and Roster (1980) discuss the unimodality properties of distributions occurring in some situations. Keilson establishes certain results connected with log-concavity and log-convexity and hence unimodality of passage time densities of diffusion and birth death processes whereas Roster asserts that the first passage time densities in state free Markov processes are unimodal. Here we shall study these properties relative to the stationary population size distribution $\{p_n; n \geq 0\}$ of the birth and death processes. The concept of unimodality and related topics are often the underlying assumptions in estimation theory and decision theory and are employed in many smoothing operations via convolutions, mixing and other suitable transformations. Further, they are of substantial importance in optimization theory and mathematical programming.

Many of the well-known distributions belong to either or both of the classes of unimodal and infinitely divisible distributions. Unimodality and infinite divisibility are two of the most important concepts in distribution theory. These properties as well as related topics like strong unimodality, log convexity, self-decomposability, stability etc. are of potential importance in both theoretical and practical problems in statistics and allied fields. Indeed, these properties determine certain structural aspects of probability distributions.

In this chapter we study these properties with regard to the stationary population size distribution of birth and death processes and aim at some characterization results on certain queuing models. Section 2.2 gives a brief view of the preliminary concepts and results. In section 2.3 we consider a birth and death process and obtain conditions for strong unimodality of its stationary population size distribution. Some special cases are also investigated here. In the last section we characterize the M/M/s, $1 \leq s \leq \infty$ Queuing system via infinite divisibility properties of their stationary queue length distribution.

2.2 Preliminaries

(a) Unimodality of distributions:- The notion of unimodality exists for both lattice as well as non-lattice distributions each one having its own interpretation. However, one can always construct a sequence of unimodal lattice distributions whose weak limit is a given unimodal non-lattice distribution. In view of this fact, so much effort has been devoted to the class of unimodal lattice distributions during the past few decades. It should be noted here that the converse approach does not usually work and several of the properties of the unimodal non-lattice distribution are not preserved by the unimodal lattice distributions. We shall mention the following:

Definition 2.1. A d.f. F(x) is called unimodal if there exists at least one value $x = x_0$ such that F(x) is convex on $(-\infty, x_0)$ and concave on (x_0, ∞). The point x_0 is called vertex of F or alternatively F is said to have a mode at (about) $x = x_0$. If the terms

' convex ' and ' concave ' in definition 2.2.1 are replaced by ' strictly convex ' and ' strictly concave ' then F is said to be strictly unimodal about x_0.

The following are known in the literature on unimodality.

Result 2.1. (Khinchin's characterization) A function ϕ on the real line, is the c.f. of a unimodal d.f. with mode at x_0, if, and only if,

$$\phi(t) = \frac{e^{i x_0 t}}{t} \int_0^t \psi(u)\, du = e^{i x_0 t} \int_0^1 \psi(ut)\, du \quad ; \ -\infty < t < \infty$$

where ψ is some c.f.

It means that a random variable (r.v.) X is unimodal about x_0 if and only if it can be expressed as $X \overset{d}{=} YU + x_0$ where Y is some random variable and U is a U(0,1) random variable independent of Y ($\overset{d}{=}$ implies equality in distribution).

Result 2.2. (Olstein and Savage characterization) A random variable X is unimodal about some point x_0, if, and only if, the function $tE[g(t(X-x_0))]$ is non-decreasing in t, for $t > 0$, for every bounded non-negative real Borel measurable function g.

Result 2.3. If a sequence of unimodal d.f.'s converge weakly to a d.f., then the latter is also unimodal.

Result 2.4. Convolution of two unimodal distribution is not necessarily again unimodal.

Examples:- Most of the common distributions like uniform, normal, gamma (and hence exponential), chi-square, beta, cauchy, F and student's t-distribution are unimodal.

Remark 2.1. According to the definition (2.2.1), the only unimodal discrete distributions are the degenerate ones. This necessitates the following definition to deal with the corresponding structural aspects of lattice distributions.

Definition 2.2. A discrete distribution $\{p_k\}$ with support on the lattice of integers is said to be unimodal if there exists at least one integer n_0 such that

$$p_n \geq p_{n-1} \quad \forall\, n \leq n_0$$

$$\text{and } \; p_{n+1} \leq p_n \quad \forall\, n \geq n_0.$$

The point $n = n_0$ is called a vertex of $\{p_n\}$ or $\{p_n\}$ is said to be unimodal with mode at (about) $n = n_0$.

We state below some important results concerning lattice unimodality.

Result 2.5. A lattice distribution $\{p_n\}_{-\infty}^{\infty}$ is unimodal about n_0, if, and only if,

$q_n = (n_0 - n + \alpha)(p_n - p_{n-1})$ with $\alpha \in (0,1)$ arbitrary, is some lattice distribution with $q_{n_0+1} > 0$.

Result 2.6. A lattice distribution $\{p_n\}_0^\infty$ is unimodal about some point n_0 (≥ 0), if, and only if, its probability generating function P(.) has the form

$$P(z) = \frac{z_0^{n_0+k}}{1-z} \int_z^1 \frac{Q(u)}{u^{\alpha+n_0+1}} du \quad ; \ 0 < z < 1$$

where $\alpha \in (0,1)$ is arbitrary, Q(u) is the probability generating function of a lattice distribution $\{q_n\}_0^\infty$ for which $q_{n_0+1} > 0$.

Result 2.7. If $\{p_n\}_0^\infty$ is a lattice distribution and for some $\alpha \in (0,1)$,

$$p_n \geq \alpha p_{n-1} + (1-\alpha) p_{n+1} \quad , \quad n = 1, 2, \ldots$$

then $\{p_n\}_0^\infty$ is unimodal.

Result 2.8. If $\{p_n\}_0^\infty$ with $p_0 > 0$ is a lattice distribution and for some $\alpha > 0$, $\beta > 0$

$$\left(\frac{p_{n-1}}{p_n}\right)^\alpha \leq \left(\frac{p_n}{p_{n+1}}\right)^\beta \quad , \ n = 1, 2, \ldots$$

then it is unimodal.

Result 2.9. The weak limit of a sequence of unimodal lattice distribution is again unimodal.

Examples of lattice unimodal distributions:- Discrete uniform, Poisson, binomial, negative binomial (and hence geometric) distributions are all unimodal.

Remark 2.2. Many lattice analogues of non-lattice unimodality results are valid. However, there are some exceptions. For example, the analogues of Result 2.2.1 & Result 2.2.2 are not valid except for trivial cases.

Strongly Unimodal distributions:- The fact that the convolution of two unimodal distributions is not necessarily unimodal brought forth the notion of strong unimodality which is given below.

Definition 2.3. A distribution is called strongly unimodal if its convolution with every unimodal distribution is unimodal.

Since a degenerate distribution at the origin is trivially unimodal, it follows that any strongly unimodal distribution is unimodal.

We have the results of Ibragimov(1956).

Result 2.10. The class of strongly unimodal distributions is closed under convolutions and is also closed relative to weak convergence.

Result 2.11. A (non-degenerate) unimodal d.f. F is strongly unimodal if and only if, F is absolutely continuous and there exists a version of the probability density function of F which is log-concave.

Remark 2.3. In view of Result 2.2.11, it is evident that the class of strongly unimodal distributions are precisely that of Polya frequency densities of order 2.(PF_2).

Also we have

Result 2.12. A lattice distribution $\{p_n;\ n = 0, \pm 1, \pm 2, \ldots\ldots\}$ is strongly unimodal if and only if it is log-concave.

That is, $p_n^2 \geq p_{n-1}\, p_{n+1}\ \ \forall\, n.$

Note:- Strongly unimodal lattice distributions have all moments and they are uniquely determined by their moments.

Further extensions of unimodality concepts are α-unimodality, total unimodality and also of multivariate random variables and they are omitted in this context.

(b) Infinitely divisible distributions:- A random variable X is said to be decomposable if it can be written as $X \stackrel{d}{=} X_1 + X_2$ where X_1 and X_2 are two independent non-degenerate random variables.

If for every $\alpha \in (0,1)$ there exists a random variable X_α independent of X such that $X = \alpha X + X_\alpha$, then the random variable X (or its distribution) is said to be self-decomposable (s.d.).

In terms of c.f.it may be written as:

Definition 2.4. A c.f. $\phi(t)$ on the real line is s.d., if, and only if, for $\alpha \in (0, 1)$,

$$\phi(t) = \phi(\alpha t)\, \phi_\alpha(t) \quad , \quad -\infty < t < \infty.$$

Note that for $\alpha \geq 1$, the above relation forces ϕ and ϕ_α to degenerate characteristic functions.

Stable distributions are special cases of s.d. distributions. They are defined as:

Definition 2.5. A c.f. $\phi(t)$ on the real line is said to be stable if for every positive b_1 and b_2 there exists some positive b such that, for every $-\infty < t < \infty$,

$$\phi(b_1 t)\phi(b_2 t) = \phi(bt)\, e^{i\mu t} \quad \text{with } \mu \text{ real.}$$

When ϕ satisfies the above equation with $\mu = 0$ then it is called strictly stable.

The class of s.d. distribution is a subset of the class of infinitely divisible distributions.

This class is defined as:

Definition 2.6 A c.f. $\phi(t)$ on the real line is infinitely divisible if for every $n \in N$ (the set of positive integers), there exist a c.f. $\phi_n(t)$ such that

$$\phi(t) = (\phi_n(t))^n, \qquad t \in R.$$

The properties of this class of distributions can be found in Lukacs (1970). For the special case, when the random variable X takes only non-negative integer values, the reader is referred to Van Harn (1978).

2.3 Characterization of Birth-Death Model Via Unimodality

Consider a birth and death process with the birth rate $\{\lambda_n\}$ $n \geq 0$ and death rate $\{\mu_n\}$ $n \geq 1$. It is well known that the stationary population size distribution of the process $\{p_n\}$ is given by

$$p_n = \frac{\lambda_{n-1}}{\mu_n} p_{n-1} \quad , \qquad n \geq 1. \tag{2.3.1}$$

Also it is clear that $\{p_n\}$ exists as a distribution, if, and only if,

$$\sum_{n=0}^{\infty} \frac{p_n}{p_0} < \infty . \tag{2.3.2}$$

Denoting $a_n = \frac{\lambda_n}{\mu_{n+1}}$, $n \geq 0$, from (2.3.1) we have,

$$p_n = \left(\prod_{j=0}^{n-1} a_j \right) p_0 \qquad ; \; n \geq 1. \tag{2.3.3}$$

We now have the following theorem.

Theorem 2.1. The stationary population size distribution $\{p_n;\, n \geq 0\}$ of a birth-death process exists and is strongly unimodal with a mode at $n_0 = \min\{n\colon a_n < 1\}$ if and only if, the sequence $\{a_n\}$; $n \geq 0$ is non-increasing and $a_i < 1$ for at least one integer i.

Proof:-

Suppose that $\{a_n\} n \geq 0$ is non-increasing and $a_i < 1$ for some i, then,

$$\sum_{n=0}^{\infty} \frac{p_n}{p_0} = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} a_j$$

$$= 1 + \sum_{n=1}^{i} \prod_{j=0}^{n-1} a_j + \sum_{n=i+1}^{\infty} \prod_{j=0}^{n-1} a_j$$

$$\leq 1 + \sum_{n=1}^{i} a_0^n + a_0^i \sum_{n=i+1}^{\infty} a_i^{n-i}$$

$$= 1 + \frac{1-a_0^i}{1-a_0} + a_0{}^i \cdot \frac{a_i}{1-a_i}$$

$$< \infty.$$

We have $p_n = a_{n-1}\, p_{n-1}$ and $p_{n+1} = a_n\, p_n$

$$\therefore \frac{p_n}{p_{n+1}} = \frac{a_{n-1}}{a_n} \cdot \frac{p_{n-1}}{p_n}.$$

That is, $p_n^2 = \dfrac{a_{n-1}}{a_n} \cdot p_{n-1} p_{n+1}$.

So we have $p_n^2 > p_{n-1}\, p_{n+1} \Leftrightarrow \{a_n\}$; $n \geq 0$ is non-increasing.

Now, since for non-increasing $\{a_n\}$,

$N_0 = \min \{n: a_n < 1\}$ if and only if $p_n = a_{n-1}\, p_{n-1} \geq p_{n-1}$ for $n \leq n_0$

and $p_{n+1} = a_n\, p_n < p_n$ for $n \geq n_0$.

We have n_0 is the vertex of $\{p_n\}$; $n \geq 0$.

The existence of some integer $i \geq o$ for the only if part such that $a_i < 1$ is evident from (2.3.3) and the existence of $\{p_n\}$. □

Remark 2.4. When $\{p_n\}$ in the above theorem exists then it is a log-convex, if, and only if, $\{a_n\}$ is non-decreasing. In this case, $a_n \leq 1 \; \forall \; n = 0, 1, \ldots$. However, in this situation $\{p_n\}$ may not be a distribution.

In the light of theorem (2.1), we examine strong unimodality of the population size distribution in equilibrium of different birth-death processes.

2.3.1 The Immigration-Death Processes (M/M/∞ queues)

For this process we have $\lambda_n = \lambda$ and $\mu_n = n\mu$, $0 < \lambda, \mu < \infty$, $n = 0, 1, 2, \ldots$.

Here $a_n = \dfrac{\lambda}{(n+1)\mu}$, $n \geq 0$.

Therefore $\{a_n\}$ is decreasing. Further, it is obvious that there always exist an integer i such that $a_i < 1$. Hence the stationary population size distribution (the queue length distribution) $\{p_n\}$ exists and it is strongly unimodal with mode at $n_0 = \dfrac{\lambda}{\mu}$. In fact, we have $\{p_n\}$ is Poisson distribution with parameter $\dfrac{\lambda}{\mu}$, which is known to be a strongly unimodal lattice distribution.

2.3.2 The M/M/s, 1 ≤ s < ∞ Queuing System

In this case $\lambda_n = \lambda$, $n \geq 0$

$$\mu_n = \begin{cases} n\mu, & n < s \\ s\mu, & n \geq s \end{cases}, \qquad \text{with } 0 < \lambda, \mu < \infty .$$

Thus $a_n = \begin{cases} \dfrac{\lambda}{(n+1)\mu} & ; \quad n < s \\ \dfrac{\lambda}{s\mu} & ; \quad n \geq s. \end{cases}$

In view of theorem (3.1), it is clear that $\{p_n\}$ exists and is strongly unimodal, if, and only if, $\dfrac{\lambda}{s\mu} < 1$. The vertex will be at $n_0 = \dfrac{\lambda}{\mu}$.

Explicitly we have,

$$p_n = \begin{cases} \dfrac{p_0}{n!}\left(\dfrac{\lambda}{\mu}\right)^n & ; 0 \leq n \leq s \\ \dfrac{p_0}{s^{n-s}s!}\left(\dfrac{\lambda}{\mu}\right)^n & ; n > s \end{cases}$$

where p_0 is determined by the condition $\sum_{n=0}^{\infty} p_n = 1$.

2.3.3 The Linear Growth Model with Immigration

Here we have $\lambda_n = \lambda_0 + n\lambda$ and $\mu_n = n\mu$, $n \geq 0$, where $0 < \lambda, \lambda_0, \mu < \infty$.

Clearly, in this process $a_n = \dfrac{\lambda_0 + n\lambda}{(n+1)\mu}$; $n \geq 0$, which is non-increasing, if, and only if, $\lambda_0 \geq \lambda$.

In this case, if $\lambda_0 \geq \mu$ we have $a_n \geq 1$ for some $n \geq 0$.

Since $\mu < \lambda_0 \Rightarrow a_i < 1$ for every $i > \dfrac{\lambda_0 - \mu}{\mu - \lambda}$

and $\mu > \lambda_0 \Rightarrow a_i < 1$ for every $i = 0, 1, 2, \ldots$, we have in view of theorem (2.3.1), that $\{p_n\}$ exists and is strongly unimodal, if, and only if, $\lambda \leq \lambda_0$ and $\lambda < \mu$.

Clearly, then the vertex n_0 will be at zero if $\mu > \lambda_0$ and at $\left[\dfrac{\lambda_0 - \mu}{\mu - \lambda}\right]$ if $\mu \leq \lambda_0$.

2.4 A Characterization Result via Infinite Divisibility

Infinite divisibility properties of different stochastic processes have been studied by several researchers in the past. This property has been employed as a tool for characterizing certain queuing systems as seen in connection with the departure processes in the works of Daley (1972), Shanbhag (1973), Natvig (1975) etc. We aim at characterizing the M/M/s, $1 \leq s \leq \infty$ queuing system via infinite divisibility properties of their stationary queue length distribution $\{p_n\}$. We know that in this model the distribution in question is geometric when $s = 1$ and Poisson when $s = \infty$, which are both infinitely divisible. The infinite divisibility properties of $\{p_n\}$ for $1 < s < \infty$ has not been considered earlier. We have the following result.

Theorem 2.2. The stationary queue length distribution in an M/M/s queuing system is infinitely divisible if, and only if, either $s = 1$ or $s = \infty$ in which case the distribution is self decomposable (it is stable only when $s = \infty$).

Proof:-

Recall that $\{p_n\}$ is given by

$$p_n = \begin{cases} \dfrac{p_0}{n!}\left(\dfrac{\lambda}{\mu}\right)^n, & 0 \le n < s \\ \dfrac{p_0}{s^{n-s}s!}\left(\dfrac{\lambda}{\mu}\right)^n, & n \ge s \end{cases} \qquad (2.4.1)$$

$\{p_n\}$ exists if $\dfrac{\lambda}{\mu s} < 1$.

Let P(.) denote the probability generating function of $\{p_n\}$. Following Van Harn (1978) we know that P is infinitely divisible if, and only if, it has the form

$P(z) = c.e^{Q(z)}$ where c is the norming constant and $Q(z) = \sum_{n=1}^{\infty} q_n z^n$, $q_n \ge 0 \ \forall \ n \ge 1$.

Here $c = p_0$ and thus

$$P(z) = p_0[1+zq_1+\frac{(zq_1)^2}{2!}+........][1+z^2q_2+\frac{(z^2q_2)^2}{2!}+...] .... \qquad (2.4.2)$$

On comparing the coefficients on both sides we have,

$q_1=\dfrac{\lambda}{\mu}$ and $q_2 = q_3 = ...= q_s = 0$.

Then we can write,

$$p_{s+1} = p_0\left[\frac{q_1^{s+1}}{(s+1)!} + q_{s+1}\right]$$

$$= p_0\left[\frac{(\lambda/\mu)^{s+1}}{(s+1)!} + q_{s+1}\right].$$

Comparing this with (2.4.1) we get,

$$q_{s+1} = \frac{(\lambda/\mu)^{s+1}}{s(s+1)!}.$$

Hence (2.4.2) implies,

$$p_{s+2} \geq p_0\left[\frac{q_1^{s+2}}{(s+2)!} + q_1 q_{s+1}\right]$$

$$= p_0\left[\frac{(\lambda/\mu)^{s+2}(2s+2)}{s(s+2)!}\right].$$

In view of p_{s+2} given by (2.4.1), this means

$$\frac{1}{s^2 . s!} \geq \frac{2(s+1)}{s(s+2)!}, \text{ which is impossible when } s > 2.$$

Therefore $\{p_n\}$ cannot be infinitely divisible when $s > 2$.

Note that for $s = 2$, when we compute q_n's it turns out that $q_6 = -\frac{15}{1152}\left(\frac{\lambda}{\mu}\right)^6$ (Negative!) which contradicts our assumptions. Therefore $\{p_n\}$ cannot be infinitely divisible (and hence self decomposable or stable) for any $1 < s < \infty$.

□

CHAPTER 3

M/G/1/K QUEUE WITH REMOVABLE SERVER

3.1 Introduction

In the classical queuing model the server stops service when there is no customer in the queue and resumes service when a new customer joins the queue next time. This model fails to describe many real life situations where there is a control limit for shutting down and resuming of service. For instance, the manufacturing of certain products like motor cars and other vehicles the demand of the customers are processed with a control limit policy on service. The firm may stop production of the product when the demand is less than a pre-assigned level and restart the production as soon as the demand reaches a specific level. This may be treated as a model with removable server – the server being removed depending on the demand level. Queuing systems with removable servers were considered by Yadin and Naor (1963) , Bell (1975), Rhee and Sivazlian (1990) and other researchers.

Recently Wang, Kuo-Hsiung and Huang, Hui-Mei (1995) studied the optimal operation of an $M/E_K/1$ queueing system with a removable service station under steady state conditions and derived analytic closed form solutions of the controllable $M/E_K/1$ queueing system. Feinberg and Kim, Dong (1996) studied bicriterion optimization of an M/G/1 queue with a server that can be switched on and off. Here one criterion is an average number of customers in the system and another criterion is an average operating cost per unit time where operating cost consists of switching and running costs. Baburaj and Manoharan (1997) considered a controllable queuing system having two servers with bulk service rule and studied the general characteristics of the system in transient as well as the steady state, the busy period of the two servers and also waiting time in the queue.

Here we consider a single service queuing system with a finite capacity K. Customers arrive at the system according to a Poisson process with parameter λ and enters the system if the total number of the customers present in the system is less than or equal to K-1. Otherwise they leave the system without receiving service. Customers are served according to FCFS basis. The successive service times are independent and identically distributed random variables with distribution function G(.) with finite mean μ .

The server employs an (a,b) – policy for the closing down and the starting up of the station. That is we assume that customers arrive according to a Poisson process and there is only one server in the system. The customers are served by the server depending on the queue length. Specifically, if the number of customers in the waiting line is greater than or equal to ' b ', the server serves the customers according to FCFS basis. When the number of customers in the waiting line reaches ' a ', then the server stops the service and the station is closed down. It is resumed only when a queue of ' b ' customers have accumulated where $0 \leq a < b \leq K$.

In section 3.2, the model is analyzed and the relationship between the limiting distribution of process at arbitrary time epoch and the limiting distribution of the process at the departure epoch is established. An expression for busy period is obtained in section 3.3. In section 3.4, the stationary probabilities are obtained in terms of expected busy period. This provides a simple method for calculating these limiting probabilities. Some numerical examples are given in section 3.5.

3.2 Analysis of the Model

Let X(t) = the number of customers in the system at an arbitrary time instant $t \geq 0$. Clearly $\{X(t), t \geq 0\}$ is not a Markov Process. Let $t_1, t_2, \ldots, t_n, \ldots$ be the service completion of the successive customers. Define $X_n = X(t_n+)$, $n \geq 1$. That is, the number of customers in the system immediately after the departure of the n^{th} served customer.

Assume X_0 = a (That is, a server terminates at t = 0 leaving ' a ' customers in the system.).

Let $p_j(t) = \Pr[X(t) = j]$; $a \le j \le K$ and $\pi_j(n) = \Pr[X_n = j]$; $a \le j \le K - 1$.

We are interested in the following probability distributions:

$$p_j = \lim_{t\to\infty} p_j(t) \qquad \text{and} \qquad \pi_j = \lim_{n\to\infty} \pi_j(n).$$

For standard M/G/1/K model (treated as (0,1) - policy), it is easy to calculate the above distributions and to observe that the two distributions are related by (Refer Cohen (1969))

$$\pi_j = \frac{p_j}{1 - p_K}, \quad 0 \le j \le K\text{-}1. \tag{3.2.1}$$

This may be obtained by successively calculating the two distributions and comparing them. It is of interest to know the relation for the general model considered here.

3.2.1 Relation between $\{p_j\}$ and $\{\pi_j\}$

$\{X(t), t \ge 0\}$ is a regenerative process for which the regeneration cycles are the busy cycles $T_1, T_2, \ldots, T_n, \ldots$ where T_i = the duration between the i^{th} shutting down and $(i+1)^{th}$ shutting down, $i \ge 1$. Let T be a random variable distributed as T_j, $j \ge 1$.

It can be proved that by the theory of regenerative process (ref. Stidham (1972))

$$p_j = \frac{E(T(j))}{E(T)}, \quad j = a, a+1, \ldots, K. \tag{3.2.2}$$

where T(j) = The amount of time (in a busy cycle) during which the number of customers in the system is equal to j.

Also $\{X_n, n \geq 0\}$ is a discrete time regenerative process for which the 1st regenerative cycle consists of the epochs $(0, 1, 2, \ldots, N_1 - 1)$ the second cycle consists of the epochs $(N_1, N_1 + 1, \ldots, N_1 + N_2 - 1)$ etc. where N_i $(i \geq 1)$ is the number of customers served during the i^{th} busy period. Let N be a random variable distributed as N_i, $i \geq 1$. Since the distribution of N is non-periodic, it follows from the theory of regenerative process that,

$$\pi_j = \frac{E(N(j))}{E(N)} \quad , \ j = a, a+1, \ldots, K-1. \qquad (3.2.3)$$

where N(j) = number of served customers in a busy period, leaving j customers in the system.

Now,

$E(N(j))$ = Expected up crossings of the process $\{X(t), t \geq 0\}$ to state j + 1 during $[0,T_1]$

$$= \lambda\, E(T(j)). \qquad (3.2.4)$$

Note that,

$$\begin{aligned} E(N) &= \sum_{j=a}^{K-1} E(N(j)) \\ &= \sum_{j=a}^{K-1} \lambda E(T(j)) \\ &= \lambda\,[E(T) - E(T(K)]. \end{aligned}$$

Using (3.2.2),

$$p_K = \frac{E(T(K))}{E(T)} \Rightarrow E(T(K)) = E(T).p_K\,.$$

$$E(N) = \lambda[E(T) - E(T).p_K)]$$

$$= \lambda[(1 - p_K)E(T)]. \qquad (3.2.5)$$

From (3.2.3),

$$E(N) = \frac{E(N(j))}{\pi_j} .$$

There fore, $\pi_j = \dfrac{p_j}{1 - p_K}$; $\qquad j = a, a + 1, a + 2, \ldots, K - 1.$

3.3 Expected Busy period

For the well-known M/G/1 queuing system with infinite waiting room Takacs (1967) and Cohen (1969) obtained the distribution of the maximum number of customers present simultaneously during a busy period. Cohen (1971) obtained an expression for the distribution of the busy period for the M/G/1 system with finite system capacity, ie, the busy period in the M/G/1/K system with the standard (0,1) - policy.

A random variable X in the standard model will be denoted in the (a, b) - policy with finite capacity K as $X_{a,b}^K$ and this notation will be followed in the sequel.

Let $B_{a,b}^K$ be the random variable denoting the busy period in the M/G/1/K model with server operating under (a, b) – policy. Put $\alpha_{a,b}^K = E(B_{a,b}^K)$ *and* $\alpha_K = \alpha_{0,1}^K$. It is evident that, the random variables $B_{a,b}^{K}$ and $B_{0,b-a}^{K-a}$ have the same distribution. Also one can view $B_{0,b-a}^{K-a}$ as the sum of independent random variables as given below.

$$B_{0,b-a}^{K-a} = \sum_{i=0}^{b-a-1} B_{0,1}^{K-a-i} .$$

$$\text{Hence } \alpha_{a,b}^{k} = \sum_{n=K-b+1}^{K-a} \alpha_n .$$

Let p(j; n, 1) = Prob. {j customers enter the M/G/1/n system during a service beginning with one customer in the system}.

Then,

$\alpha_1 = E(S)$ and

$$\alpha_n = E(S) + \sum_{j=1}^{n-1} p(j;n,1) \sum_{i=0}^{j-1} \alpha_{n-i} \quad , \quad n > 1$$

$$= E(S) + \sum_{i=0}^{n-2} \alpha_{n-i} (1 - x_0 - x_1 - \ldots - x_i), \quad n > 1$$

where $x_i = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^i}{i!} dS(t)$, $i \geq 0$ and S(t) = cumulative distribution function of service time.

This may be written as follows.

$\alpha_1 = E(S)$ and

$$x_0 \alpha_n = E(S) + \sum_{i=1}^{n-2} (1 - x_0 - \ldots - x_i) \alpha_{n-i} \; ; \; n \geq 2.$$

This may be re-written as $\begin{bmatrix} x_0 & 0 & 0 & \cdots & 0 \\ -y_1 & x_0 & 0 & \cdots & 0 \\ -y_2 & -y_1 & x_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -y_{K-a-2} & -y_{K-a-3} & -y_{K-a-4} & \cdots & x_0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{K-a} \end{bmatrix} = \begin{bmatrix} E(S) \\ E(S) \\ E(S) \\ \vdots \\ E(S) \end{bmatrix}$

where $y_i = 1 - x_0 - x_1 - \ldots - x_i$ for $i = 1, 2, \ldots, K - a - 2$.

Solving this linear system we get, α_n ; $n = 2, 3, \ldots, K - a$.

3.4 Calculation of Stationary Probabilities.

The stationary probabilities π_j's can be computed as follows:

We have $\pi_j = \dfrac{E(N_{a,b}^K(j))}{E(N_{a,b}^K)}$,

$$\alpha_{a,b}^K = \sum_{n=K-b+1}^{K-a} \alpha_n$$

and $E(N_{a,b}^K).E(S) = \alpha_{a,b}^K$.

We shall first of all find an expression for $E(N_{a,b}^K(j))$; $j = a, a+1, \ldots, K-1$.

Clearly $E(N_{a,b}^K(j)) = E(N_{0,b-a}^{K-a}(j-a))$. (3.4.1)

For convenience we shall put $N_{0,b-a}^{K-a}(j-a) = N_{b'}^{K'}(j')$.

Let $\eta_{b'}^{K'}(j') = E(N_{b'}^{K'}(j'))$; $j' = 0, 1, \ldots, K' - 1$. (3.4.2)

Obviously $\eta_{b'}^{K'}(0) = 1$. Also $\alpha_1 = E(S)$

$$\therefore \eta_{b'}^{K'}(0).E(S) = \alpha_1 .$$

We shall prove that

$$\eta_{b'}^{K'}(j').E(S) = \alpha_{j'+1} \quad ; \; 0 \leq j' < b'.$$

For $j' = 1, 2, \ldots, K' - 1$, we have,

$$\eta_{b'}^{K'}(j') = \sum_{i=0}^{\min(j', b'-1)} \eta_1^{K'-i}(j' - i) \tag{3.4.3}$$

$$\text{and } \eta_1^{K'}(j') = p(j'; K', 1) + \sum_{l=1}^{K'-1} p(l; K', 1)\eta_l^{K'}(j') \cdot \tag{3.4.4}$$

Using (3.4.3) and (3.4.4) we get

$$x_0\eta_1^{K'}(j') = (1 - x_0 - \ldots - x_{j'-1})[1 + \eta_1^{K'-j'-1}(1)](1 - \delta_{j',1}) + \sum_{l=1}^{j'-2}(1 - x_0 - \ldots - x_l)\eta_1^{K'-l}(j' - l)$$

for $1 \leq j' \leq K' - 1$ (3.4.5)

where $\delta_{j',1}$ is Kronckers Delta

$$\Rightarrow \eta_1^{K'}(1).E(S) = \frac{1 - x_0}{x_0} E(S).$$

Now using (3.3.1), the above implies

$$\eta_1^{K'}(1).E(S) = \alpha_2 - \alpha_1 . \tag{3.4.6}$$

Also from (3.3.1) we have the recursion relation for $\alpha_{j'+1} - \alpha_{j'}$, for $j' \geq 2$,

$$x_0(\alpha_{j'+1} - \alpha_{j'}) = (1 - x_0 - \ldots - x_{j'-1})\alpha_2 + \sum_{i=1}^{j'-2}(1 - x_0 - \ldots - x_i)(\alpha_{j'-i+1} - \alpha_{j'-i}) . \tag{3.4.7}$$

Since $\alpha_1 = E(S)$, using (3.4.5) and (3.4.6) it follows that

$$\alpha_2 = 1 + \eta_1^{K'-j'-1}(1).$$

So we conclude that for $b' = 1$,

$$\eta_{b'}^{K'}(j').E(S) = \alpha_{j'+1} \quad ; \quad 0 < j' < b'.$$

Consider the case $b' > 1$.

We have $\eta_{b'}^{K'}(j') = \sum_{i=0}^{\min(j', b'-1)} \eta_1^{K'-i}(j' - i)$.

For $1 \le j' < b'$,

$$\begin{aligned} \eta_{b'}^{K'}(j').E(S) &= \sum_{i=0}^{j'} \eta_1^{K'-1}(j' - i).E(S) \\ &= \alpha_1 + \sum_{i=o}^{j'-1}(\alpha_{j'-i+1} - \alpha_{j'-i}) \\ &= \alpha_{j'+1}. \end{aligned}$$

For $b' \le j' \le K' - 1$,

$$\begin{aligned} \eta_{b'}^{K'}(j').E(S) &= \sum_{i=0}^{b'-1} \eta_1^{K'-i}(j' - i).E(S) \\ &= \sum_{i=0}^{b'-1}(\alpha_{j'-i+1} - \alpha_{j'-i}) \\ &= \alpha_{j'+1} - \alpha_{j'-b'+1}. \end{aligned}$$

Hence we get,

$$\eta_{b'}^{K'}(j').E(S) = \begin{cases} \alpha_{j'+1} \quad ; & 0 \le j' < b' \\ \alpha_{j'+1} - \alpha_{j'-b'+1} ; & b' \le j' \le K' - 1. \end{cases} \tag{3.4.8}$$

This gives

$$E(N_{a,b}^{K}(j)).E(S) = \begin{cases} \alpha_{j-a+1} \quad ; & a \le j < b \\ \alpha_{j-a+1} - \alpha_{j-b+1}; & b \le j \le K - 1. \end{cases} \tag{3.4.9}$$

That is, expected number of served customers in a Busy period leaving ' j ' customers in the system for M/G/1/K under (a,b) - policy is equal to the expected increment of the served customers in a busy period for the model with (0,1) - policy, when the capacity is increased from max (0 , j - b + 1) to j - a + 1

Hence we get,

$$\pi_j = \begin{cases} \dfrac{\alpha_{j-a+1}}{\sum_{n=K-b+1}^{K-a} \alpha_n}; & a \le j < b \\ \dfrac{\alpha_{j-a+1} - \alpha_{j-b+1}}{\sum_{n=K-b+1}^{K-a} \alpha_n}; & b \le j \le K-1 \end{cases} \tag{3.4.10}$$

Using equation (3.2.5) we have,

$$E(N_{a,b}^K) = \lambda[1 - p_{a,b}^K(K)]E(T_{a,b}^K). \tag{3.4.11}$$

$$\text{Now, } E(T_{a,b}^K) = \frac{b-a}{\lambda} + \alpha_{a,b}^K$$

$$and\ E(N_{a,b}^K)E(S) = \alpha_{a,b}^K = \sum_{n=K-b+1}^{K-a} \alpha_n .$$

So from equation (3.4.11) we get,

$$1 - p_{a,b}^K(K) = \frac{E(N_{a,b}^K)}{\lambda\, E(T_{a,b}^K)}$$

$$= \frac{\lambda \sum_{l=K-b+1}^{K-a} \alpha_l}{E(S)\left[(b-a) + \lambda \sum_{l=K-b+1}^{K-a} \alpha_l\right]} . \tag{3.4.12}$$

Note:- The steady state probabilities can be computed in terms of the moments of busy period as described above.

Remark 3.1 Similar argument can be used to arrive at the result for the infinite capacity case by assuming that $\rho < 1$.

3.5 NUMERICAL EXAMPLE

Let S follows an exponential distribution with parameter μ.

Then we have

$$x_n = \frac{\lambda^n \mu}{(\lambda + \mu)^{n+1}} \quad ; \quad n = 0, 1, 2, \ldots$$

and

$$\alpha_n = \frac{1}{\mu} \cdot \frac{1 - \rho^n}{1 - \rho} \quad ; \quad n = 1, 2, \ldots \quad \text{where} \quad \rho = \frac{\lambda}{\mu}.$$

Following the method suggested in the last section we compute the stationary probability distributions $\{\pi_j\}$ and $\{p_j\}$. When S follows an exponential distribution with parameter μ, then the stationary probabilities $\{\pi j\}$ and $\{p_j\}$ for different values of λ and μ (For the same a, b and K) are given as follows:

1) $\lambda = 0.5$, $\mu = 0.8$, $a = 5$, $b = 10$ and $K = 25$.

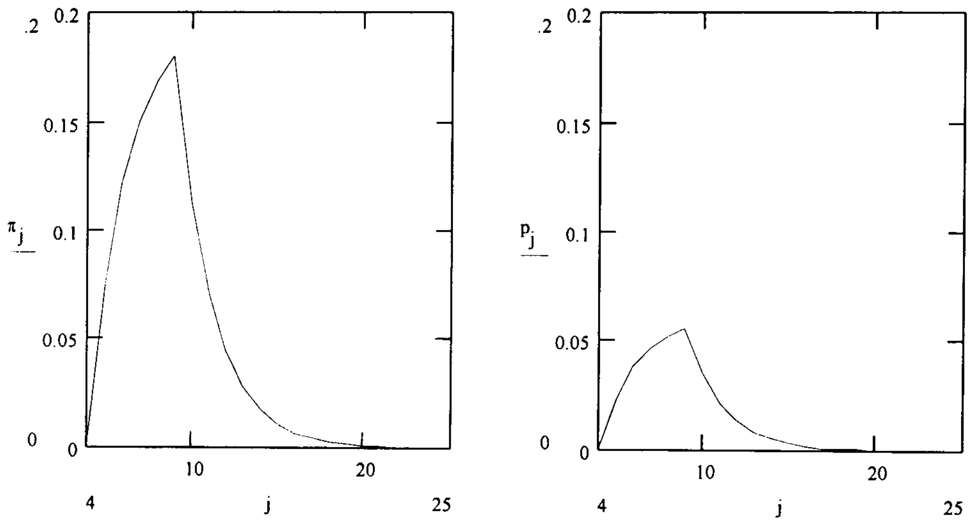
j	π_j	p_j
1	0	0
2	0	0
3	0	0
4	0	0
5	0.07502	0.02308
6	0.12191	0.03751
7	0.15121	0.04652
8	0.16953	0.05216
9	0.18098	0.05568
10	0.11311	0.03480
11	0.07069	0.02175
12	0.04418	0.01359
13	0.02761	0.00850
14	0.01726	0.00531
15	0.01079	0.00332
16	0.00674	0.00207
17	0.00421	0.00130
18	0.00263	0.00081
19	0.00165	0.00051
20	0.00103	0.00032
21	0.00064	0.00020
22	0.00040	0.00012
23	0.00025	0.00008
24	0.00016	0.00005
25	-	0.69232

Table 3.1

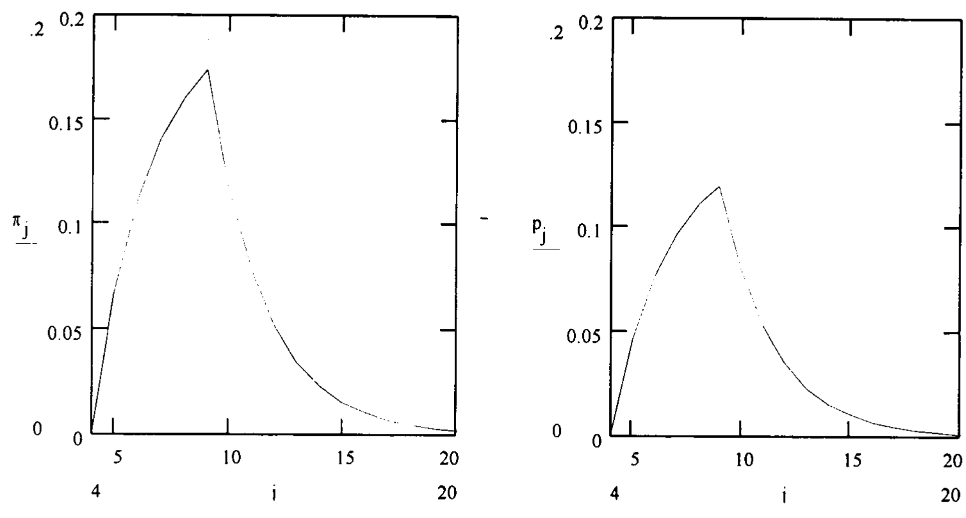
2) $\lambda = 0.8$, $\mu = 1.2$, $a = 5$, $b = 10$ and $K = 25$

j	π_j	p_j
1	0	0
2	0	0
3	0	0
4	0	0
5	0.06672	0.04574
6	0.11120	0.07623
7	0.14085	0.09656
8	0.16062	0.11012
9	0.1738	0.11915
10	0.11587	0.07943
11	0.07724	0.05296
12	0.05150	0.03530
13	0.03433	0.02354
14	0.02289	0.01569
15	0.01526	0.01046
16	0.01017	0.00697
17	0.00678	0.00465
18	0.00452	0.00310
19	0.00301	0.00207
20	0.00201	0.00138
21	0.00134	0.00092
22	0.00089	0.00061
23	0.00060	0.00041
24	0.00040	0.00027
25	-	0.31444

Table 3.2



Graph of Table 3.1



Graph of Table 3.2

CHAPTER 4

SINGLE AND BATCH SERVICE M/M/1 QUEUE WITH A SECONDARY LIMIT ON BATCH SIZE

4.1 Introduction

We consider a Markovian queueing system with single and batch service. Customers arrive according to a Poisson Process with parameter λ and are served by a single server. The server serves the customers either one at a time or in batches according to the control limit policy - (a, c) for the batch size. Specifically after a service completion epoch, if the system size (n) is less than or equal to the control limit 'c', then the server serves the single customer according to FCFS rule and the service time is exponentially distributed with parameter μ_1 and if $n > c$, then he serves all the units in a batch where the service time is exponentially distributed with parameter μ_2 independent of batch size. But once the batch service (machine mode) has been initiated it is changed to single service (manual mode) not when the batch size is 'c' but at a lower level 'a' below 'c'. This will avoid perpetual transfer from one type of service to the other.

Several authors considered Poisson arrival queues with single as well as batch service. For example see Neuts (1967), Cohen (1969), Chaudhry and Templeton (1984) etc. There are many real life queueing situations in which service is rendered with control limit policies. For example it may be possible to process jobs manually or by machine. When the number of jobs to be processed is not more than a fixed number it will be profitable to do them manually and if the number of jobs exceeds that fixed number, processing by machine is turns out to be cheaper.

A single and batch service with single control limit 'c' on the batch service are found to be important in many real life situations, but when the control limit 'c' is very large the model suffers from the risk of perpetual change-over from batch service (machine) to single service (manual). This problem may be overcome by the introduction of a secondary limit 'a' ($< c$) so that when a batch service has been on, the server may continue the batch service even with a size less than 'c' but up to the limit $a + 1$ and when the number of customers drops below $a + 1$ manual service commences.

Baburaj and Manoharan (1997a) have considered a Markovian queueing system with single and batch services and obtained expressions for the steady state probabilities of the system size. Jacob et. al. (1997) dealt with a single server queueing model with Poisson arrivals and a new bulk service rule for which steady state probabilities and waiting time distributions are obtained. Dshalalow and Dikong (1999) studied M/G/1 type queues with modulated bulk input, state dependent batch service and (r, N) – hysteretic discipline of idle and busy periods and obtained explicit formulas for the stationary distribution of the queueing process and other characteristics using semi regenerative and first excess level techniques. Dikong and Dshalalow (1999) dealt with a bulk input-batch service queueing system and (r, N) – hysteretic control with state dependent service and analyzed the model using first excess level technique. Baburaj (2000) have given a transient distribution of a single and batch service queueing system with accessibility to the batches.

In this chapter we consider a single and batch service M/M/1 model with a secondary limit on batch size and obtain the transient state and steady state probabilities, expected queue length, busy period distribution and hence expected busy period and variance of busy period. A numerical illustration is given in the last section.

4.2 Analysis of the Model

Let X(t) denote the number of customers in the waiting line at time t and Y(t) denote the type of service given by the server at time t. Specifically Y(t) = 0, 1, 2 according as the server is idle, busy with a single service or busy with a batch. Then {Y(t), X(t)} constitutes a Markov chain with state space $S = S_1 \cup S_2 \cup S_3$ where $S_1 = \{(0, 0)\}$,

$S_2 = \{(1, n); n = 0, 1, 2, \ldots, c - 1\}$ and $S_3 = \{(2, n); n \geq 0\}$.

Let us denote $P(i, j, t) = P\{Y(t) = i \text{ and } X(t) = j\}$. The transient distribution of the system $P(i, j, t)$, then satisfy the following system of difference differential equations.

$$P'(0, 0, t) = -\lambda P(0, 0, t) + \mu_1 P(1, 0, t) + \mu_2 P(2, 0, t) \quad (4.2.1)$$

$$P'(1, 0, t) = -(\lambda + \mu_1)P(1, 0, t) + \lambda P(0, 0, t) + \mu_1 P(1, 1, t) + \mu_2 P(2, 1, t) \quad (4.2.2)$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n - 1, t) + \mu_1 P(1, n + 1, t) + \mu_2 P(2, n + 1, t);$$
$$1 \leq n \leq a - 1 \quad (4.2.3)$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n - 1, t) + \mu_1 P(1, n + 1, t); \; a \leq n \leq c-2$$
$$(4.2.4)$$

$$P'(1, c - 1, t) = -(\lambda + \mu_1)P(1, c - 1, t) + \lambda P(1, c - 2, t) \quad (4.2.5)$$

$$P'(2, 0, t) = -(\lambda + \mu_2)P(2, 0, t) + \lambda P(1, c - 1, t) + \mu_2 \sum_{n=a+1}^{\infty} P(2, n, t) \quad (4.2.6)$$

$$P'(2, n, t) = -(\lambda + \mu_2)P(2, n, t) + \lambda P(2, n - 1, t) \; ; n > 0. \quad (4.2.7)$$

Let $P^*(i, j, s)$ denote the Laplace transform of $P(i, j, t)$. We shall assume that $P(0,0,0) = 1$. So,

$$(s + \lambda)P^*(0, 0, s) - 1 = \mu_1 P^*(1, 0, s) + \mu_2 P^*(2, 0, s) \quad (4.2.8)$$

$$(s + \lambda + \mu_1)P^*(1, 0, s) = \lambda P^*(0, 0, s) + \mu_1 P^*(1, 1, s) + \mu_2 P^*(2, 1, s) \quad (4.2.9)$$

$$(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n - 1, s) + \mu_1 P^*(1, n + 1, s) + \mu_2 P^*(2, n + 1, s);$$
$$1 \leq n \leq a - 1 \quad (4.2.10)$$

$$(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n - 1, s) + \mu_1 P^*(1, n + 1, s); \; a \leq n \leq c - 2 \quad (4.2.11)$$

$$(s + \lambda + \mu_1)P^*(1, c - 1, s) = \lambda P^*(1, c - 2, s) \tag{4.2.12}$$

$$(s + \lambda + \mu_2)P^*(2, 0, s) = \lambda P^*(1, c - 1, s) + \mu_2 \sum_{n=a+1}^{\infty} P(2, n, s) \tag{4.2.13}$$

$$(s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n - 1, s); n > 0 . \tag{4.2.14}$$

Solving (4.2.14) as a difference equation in $P^*(2, n, s)$ we get,

$$P^*(2, n, s) = A(s)R^n; n \geq 0 \tag{4.2.15}$$

where $A(s) = P^*(2, 0, s)$ and

$$R = \frac{\lambda}{s + \lambda + \mu_2} .$$

From (4.2.13),

$$P^*(1, c - 1, s) = \frac{A(s)}{\lambda}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1 - R}\right) . \tag{4.2.16}$$

From (4.2.11),

$$P^*(1, n, s) = P^*(1, a, s)\, \theta^{n-a}; a \leq n \leq c - 2 \tag{4.2.17}$$

where θ is the positive root less than unity of the equation

$$\mu_1 z^2 - (s + \lambda + \mu_1)z + \lambda = 0 .$$

So using (4.2.12) we get,

$$P^*(1, a, s) = \frac{A(s)}{\lambda^2 \theta^{c-2-a}}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1 - R}\right)$$

$$= A(s).K_1(s) . \qquad \text{(Say)} \tag{4.2.18}$$

From (4.2.10),

$$P^*(1, n, s) = P^*(1, 0, s)\theta^n - \mu_2 \frac{A(s)R^{n+1}}{K(R)} \quad ; \qquad 1 \le n \le a-1 \qquad (4.2.19)$$

where $K(z) = \mu_1 z^2 - (s + \lambda + \mu_1)z + \lambda$.

So,

$$P^*(1, 0, s) = A(s)\frac{1}{\theta^{a-1}}\left(K_1(s)\left(\frac{s+\lambda+\mu_1}{\lambda} - \mu_1\theta\right) - \frac{\mu_2 R^a}{K(R)}\right)$$

$$= A(s).K_2(s). \quad \text{(Say)} \qquad (4.2.20)$$

$$P^*(0, 0, s) = A(s)\frac{1}{s+\lambda}\left(\mu_1 K_2(s) + \mu_2\right) + \frac{1}{s+\lambda}. \qquad (4.2.21)$$

Now using the condition,

$$P^*(0, 0, s) + \sum_{n=0}^{c-1} P^*(1, n, s) + \sum_{n=0}^{\infty} P^*(2, n, s) = \frac{1}{s}$$

we get,

$$A(s) = \frac{\lambda}{s(s+\lambda)}\begin{bmatrix} K_2(s)\left[\frac{\mu_1}{s+\lambda} + \frac{1-\theta^a}{1-\theta}\right] + K_1(s)\left[\frac{1-\theta^{c-a-1}}{1-\theta}\right] + \\ \mu_2\left[\frac{1}{s+\lambda} - \frac{R^2(1-R^{a-1})}{K(R)(1-R)} - \frac{R^{a+1}}{\lambda(1-R)}\right] + \\ \frac{s+\lambda+\mu_2}{\lambda} + \frac{1}{1-R} \end{bmatrix}^{-1}. \qquad (4.2.22)$$

4.3 Steady State Distribution

Using final value theorem on Laplace transforms the steady state distribution of the queue size can be obtained as

$$P(i, j) = \lim_{t \to \infty} P(i, j, t) = \lim_{s \to 0} sP^*(i, j, s).$$

Hence from (4.2.15) to (4.2.22) the steady state distribution can be obtained as

$$P(0, 0) = B.\frac{1}{\lambda}[\mu_1 L_2 + \mu_2] \tag{4.3.1}$$

$$P(1, 0) = B.L_2 \tag{4.3.2}$$

$$P(1, n) = B\left[L_2\theta_1^{\,n} - \mu_2 \frac{r^{n+1}}{K^{'}(r)}\right] \quad ; \quad 1 \le n \le a - 1 \tag{4.3.3}$$

$$P(1, n) = B.L_1\theta_1^{\,n-a} \quad ; \quad a \le n \le c - 2 \tag{4.3.4}$$

$$P(1, c-1) = B.\frac{1}{\lambda}\left[\lambda + \mu_2 - \mu_2 \frac{r^{a+1}}{1-r}\right] \tag{4.3.5}$$

$$P(2, n) = B.r^n \quad ; \quad n \ge 0 \tag{4.3.6}$$

where

$$L_1 = \lim_{s \to 0} K_1(s)$$

$$= \frac{1}{\theta_1^{\,c-2-a}\lambda^2}\left[\lambda + \mu_2 - \mu_2 \frac{r^{a+1}}{1-r}\right]$$

$$L_2 = \lim_{s\to 0} K_2(s)$$
$$= \frac{1}{\theta_1^{a-1}}\left[L_1\left(\frac{\lambda+\mu_1}{\lambda} - \mu_1\theta_1\right) - \mu_2 \frac{r^a}{K'(r)}\right]$$

$$B = \lim_{s\to 0} s.A(s)$$
$$= \begin{bmatrix} L_2\left[\frac{\mu_1}{\lambda} + \frac{1-\theta_1^{a}}{1-\theta_1}\right] + L_1\frac{1-\theta_1^{c-a-1}}{1-\theta_1} + \mu_2\left[\frac{1}{\lambda} - \frac{r^2(1-r^{a-1})}{K'(r)(1-r)} - \frac{r^{a+1}}{\lambda(1-r)}\right] + \\ \frac{\lambda+\mu_2}{\lambda} + \frac{1}{1-r} \end{bmatrix}^{-1}$$

where $K'(r) = \mu_1 r^2 - (\lambda + \mu_1)r + \lambda$,

$$\theta_1 = \frac{\lambda}{\mu_1} \quad and \quad r = \lim_{s\to 0} R = \frac{\lambda}{\lambda+\mu_2}.$$

4.4 Expected Queue Length

The expected queue length L_q is given by

$$L_q = \sum_{n=1}^{c-1} n.P(1,n) + \sum_{n=1}^{\infty} n.P(2,n)$$

$$= \sum_{n=1}^{a-1} n.B\left[L_2\theta_1^{n} - \mu_2\frac{r^{n+1}}{K'(r)}\right] + \sum_{n=a}^{c-2} n.B.L_1\theta_1^{n-a} + (c-1)B\frac{1}{\lambda}\left[\lambda+\mu_2 - \mu_2\frac{r^{a+1}}{1-r}\right]$$
$$+ \sum_{n=1}^{\infty} nBr^{n}$$

$$= B \left\{ \begin{array}{l} L_2\theta_1\left[(1-\theta_1)^{-2}\left(1-\theta_1^{\,a-1}\right)-(a-1)\theta_1^{\,a-1}(1-\theta_1)^{-1}\right]+ \\ \dfrac{L_1}{\theta_1^{\,a-1}}\left[(1-\theta_1)^{-2}\left(\theta_1^{\,a-1}-\theta_1^{\,c-2}\right)+(1-\theta_1)^{-1}\left[(a-1)\theta_1^{\,a-1}-(c-2)\theta_1^{\,c-2}\right]\right]- \\ \dfrac{\mu_2}{K'(r)}r^2\left[(1-r)^{-2}\left(1-r^{a-1}\right)-(a-1)r^{a-1}(1-r)^{-1}\right]+ \\ \dfrac{c-1}{\lambda}\left[\lambda+\mu_2-\mu_2\dfrac{r^{a+1}}{1-r}\right]+r(1-r)^{-2} \end{array} \right\}. \quad (4.4.1)$$

4.5 Busy Period Distribution

In this model the server is idle only when the system is empty. Thus the busy period T commences with the arrival of a unit to the empty system and lasts till the system is empty again.

Let $b(t) = P\{t \leq T < t + dt, X(t + dt) = 0\}$.

Then $b(t) = \dfrac{d}{dt}P(0, 0, t)$.

Hence the Laplace transform $b^*(s)$ is given by

$b^*(s) = s.P^*(0, 0, s)$.

In order to find the distribution of the busy period we consider the system avoiding the zero state. Here we assume that $P(1, 0, 0) = 1$.

Hence we get the following difference differential equations.

$$P'(0, 0, t) = \mu_1 P(1, 0, t) + \mu_2 P(2, 0, t) \qquad (4.5.1)$$

$$P'(1, 0, t) = -\mu_1 P(1, 0, t) + \mu_1 P(1, 1, t) + \mu_2 P(2, 1, t) \qquad (4.5.2)$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) + \mu_2 P(2, n+1, t);$$

$$1 \le n \le a-1 \qquad (4.5.3)$$

$$P'(1, n, t) = -(\lambda + \mu_1)P(1, n, t) + \lambda P(1, n-1, t) + \mu_1 P(1, n+1, t) \ ; \ a \le n \le c-2$$

$$(4.5.4)$$

$$P'(1, c-1, t) = -(\lambda + \mu_1)P(1, c-1, t) + \lambda P(1, c-2, t) \qquad (4.5.5)$$

$$P'(2, 0, t) = -(\lambda + \mu_2)P(2, 0, t) + \lambda P(1, c-1, t) + \mu_2 \sum_{n=a+1}^{\infty} P(2, n, t) \qquad (4.5.6)$$

$$P'(2, n, t) = -(\lambda + \mu_2)P(2, n, t) + \lambda P(2, n-1, t) \ ; n > 0 . \qquad (4.5.7)$$

Taking Laplace Transform on both sides of (4.5.1) - (4.5.7) we have the following set of equations.

$$sP^*(0, 0, s) = \mu_1 P^*(1, 0, s) + \mu_2 P^*(2, 0, s) \qquad (4.5.8)$$

$$(s + \mu_1)P^*(1, 0, s) - 1 = \mu_1 P^*(1, 1, s) + \mu_2 P^*(2, 1, s) \qquad (4.5.9)$$

$$(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s) + \mu_2 P^*(2, n+1, s) \ ;$$

$$1 \le n \le a-1 \qquad (4.5.10)$$

$$(s + \lambda + \mu_1)P^*(1, n, s) = \lambda P^*(1, n-1, s) + \mu_1 P^*(1, n+1, s); \ a \le n \le c-2 \qquad (4.5.11)$$

$$(s + \lambda + \mu_1)P^*(1, c-1, s) = \lambda P^*(1, c-2, s) \qquad (4.5.12)$$

$$(s + \lambda + \mu_2)P^*(2, 0, s) = \lambda P^*(1, c-1, s) + \mu_2 \sum_{n=a+1}^{\infty} P(2, n, s) \qquad (4.5.13)$$

$$(s + \lambda + \mu_2)P^*(2, n, s) = \lambda P^*(2, n - 1, s);\ n \geq 0. \tag{4.5.14}$$

Solving (4.5.14) as a difference equation in $P^*(2, n, s)$ we get,

$$P^*(2, n, s) = A_1(s)\, R^n\ ;\ n \geq 0 \tag{4.5.15}$$

where $A_1(s) = P^*(2, 0, s)$ and

$$R = \frac{\lambda}{s + \lambda + \mu_2}.$$

From (4.5.13),

$$P^*(1, c - 1, s) = \frac{A_1(s)}{\lambda}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1 - R}\right). \tag{4.5.16}$$

From (4.5.11),

$$P^*(1, n, s) = P^*(1, a, s)\, \theta^{n-a};\ a \leq n \leq c - 2 \tag{4.5.17}$$

where θ is the positive root less than unity of the equation

$$\mu_1 z^2 - (s + \lambda + \mu_1)z + \lambda = 0.$$

So using (4.5.12) we get,

$$P^*(1, a, s) = \frac{A_1(s)}{\lambda^2 \theta^{c-2-a}}\left(s + \lambda + \mu_2 - \mu_2 \frac{R^{a+1}}{1 - R}\right)$$

$$= A_1(s).K_1(s). \qquad \text{(Say)} \tag{4.5.18}$$

From (4.5.10),

$$P^*(1,n,s) = P^*(1,0,s)\,\theta^n - \mu_2 \frac{A_1(s)R^{n+1}}{K(R)} \quad ; \; 1 \le n \le a-1 \tag{4.5.19}$$

where $K(z) = \mu_1 z^2 - (s + \lambda + \mu_1)z + \lambda$.

So,

$$P^*(1,0,s) = A_1(s)\frac{1}{\theta^{a-1}}\left(K_1\,(s)(\frac{s+\lambda+\mu_1}{\lambda} - \mu_1\theta) - \frac{\mu_2 R^a}{K(R)} \right)$$

$$= A_1(s).K_2(s) . \quad \text{(Say)} \tag{4.5.20}$$

From (4.5.9),

$$(s+\mu_1)A_1(s)K_2(s) - 1 = -\mu_1 A_1(s)K_2(s)\theta + \mu_2 A_1(s)\frac{R^2}{K(R)} + \mu_2 A_1(s)R$$

$$A_1(s) = \left[(s+\mu_1)K_2(s) + \mu_1 K_2(s)\theta - \mu_2 \frac{R^2}{K(R)} - \mu_2 R \right]^{-1}$$

$$= \left[K_2(s)(s+\mu_1-\mu_1\theta) - \mu_2 R(1+\frac{R}{K(R)}) \right]^{-1} . \tag{4.5.21}$$

From (4.5.8),

$$sP^*(0,0,s) = \mu_1 A_1(s)K_2(s) + \mu_2 A_1(s)$$

$$= A_1(s)[\mu_1 K_2(s) + \mu_2] .$$

Hence the Laplace transform $b^*(s)$ of $b(t)$ is given by

$$b^*(s) = A_1(s).[\mu_1 K_2(s) + \mu_2] . \tag{4.5.22}$$

From the above expression one may compute the expectation and variance of the busy period using the following formula.

$$E(T) = \lim_{s \to 0}\left[-\frac{d}{ds}\left[b^*(s)\right]\right]$$

$$V(T) = \lim_{s \to 0}\left[\frac{d^2}{ds^2}\left[b^*(s)\right]\right] - [E(T)]^2 .$$

Remark 4.1. Expected busy period in the steady state is given as follows:

In this model the busy periods T and idle periods I alternate and form a busy cycle. From the theory of renewal process we have

$$P(0, 0) = \lim_{t \to 0} P\left[X(t) = 0, Y(t) = 0\right]$$

$$= \frac{E(I)}{E(I) + E(T)} \cdot$$

$$\therefore E(T) = \frac{\left[1 - P(0, 0)\right]E(I)}{P(0, 0)}$$

$$= \frac{\lambda - B\left[\mu_1 L_2 + \mu_2\right]}{\lambda B\left[\mu_1 L_2 + \mu_2\right]} \cdot \qquad (4.5.23)$$

4.4 Numerical Example

1. Consider the model with $\lambda = 0.5$, $\mu_1 = 0.8$, $\mu_2 = 0.4$, $a = 5$ and $c = 10$. We use equations (4.3.1) - (4.3.6) for the determination of the steady state probabilities. The steady state probabilities $\{P(i, n), i = 0, 1, 2 \text{ and } n = 0, 1, 2, \ldots\}$ are calculated using the equations(4.3.1) – (4.3.6) on a computer, the results are given below.

$P(0, 0) = 0.38625$

N	P(1, n)	P(2, n)
0	0.24078	0.00126
1	0.14419	0.00070
2	0.09056	0.00039
3	0.05684	0.00021
4	0.03566	0.00012
5	0.01801	0.00007
6	0.01126	0.00004
7	0.00703	0.00002
8	0.00440	0.00001
9	0.00220	0
10	-	0
≥11	-	0

Table 4.1

The total probability is verified to be equal to one.

2. We consider some models with different values of a and c. The values of expected busy periods in steady state using (4.5.23) are given in the following table. Here also $\lambda = 0.5$, $\mu_1 = 0.8$ and $\mu_2 = 0.4$.

a↓c→	10	11	12	14	16	18	20
2	3.01483	3.01810	3.02009	3.02207	3.02283	3.02312	3.02324
3	3.08974	3.10886	3.12054	3.13220	3.13670	3.13845	3.13913
4	3.14266	3.16925	3.18556	3.20189	3.20820	3.21066	3.21162
5	3.17803	3.20838	3.22704	3.24577	3.25302	3.25584	3.25694
6	3.20118	3.23345	3.25334	3.27333	3.28108	3.28410	3.28527
7	3.21617	3.24943	3.26966	3.29062	3.29864	3.30177	3.30299
8	3.22583	3.25958	3.28044	3.30147	3.30963	3.31282	3.31406

Table 4.2

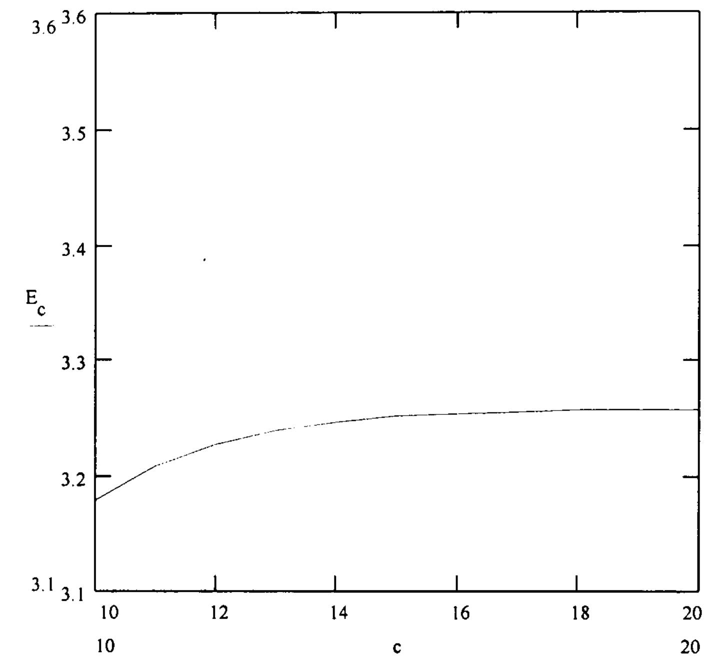
Expected busy period is monotonically increasing, but the rate of increase is very small for larger values of a and c.

2. For the same model as in example 2, we computed the expected queue length using (4.4.1) and the results are given in the following table. Here also $\lambda = 0.5$, $\mu_1 = 0.8$ and $\mu_2 = 0.4$.

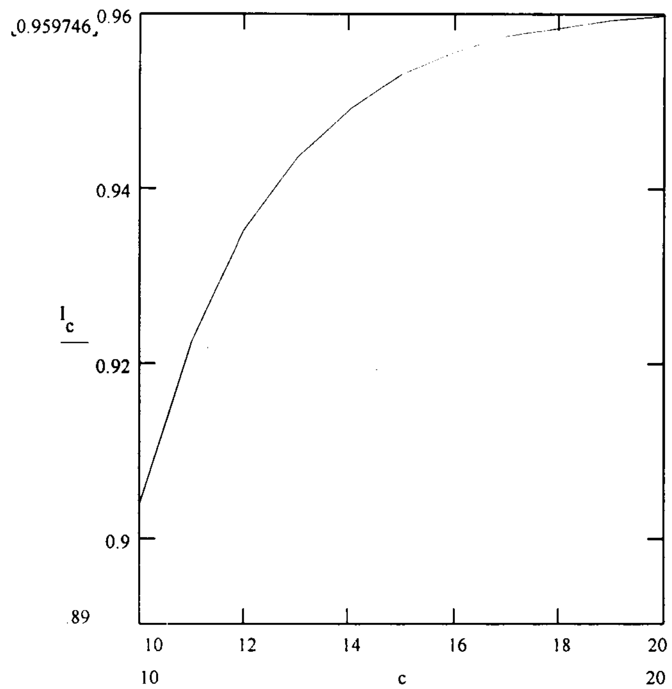
a↓c→	10	11	12	14	16	18	20
2	0.84968	0.85795	0.86419	0.87210	0.87611	0.87803	0.87893
3	0.86337	0.87620	0.88520	0.89580	0.90082	0.90314	0.90418
4	0.88361	0.89993	0.91103	0.92372	0.92954	0.93216	0.93332
5	0.90379	0.92253	0.93512	0.94929	0.95566	0.95850	0.95975
6	0.92125	0.94160	0.95518	0.97035	0.97711	0.98010	0.98140
7	0.93530	0.95668	0.97090	0.98672	0.99373	0.99682	0.99816
8	0.94611	0.96812	0.98274	0.99897	1.00615	1.00930	1.01066

Table 4.3

Expected queue length also monotonically increasing, but the rate of increase is very small for larger values of a and c.



Graph of Table 4.2 for a = 5.



Graph of Table 4.3 for a = 5.

CHAPTER 5

A GENERAL PRODUCTION INVENTORY (S – 1, S) POLICY

5.1 Introduction

Inventory systems of (S - 1, S) type had been studied quite extensively in the past. The one for one (S - 1, S) policy which is also known as " sell one buy one policy " when demands are unit sized, is a special case of the well known (s, S) inventory control policy with s = S - 1. Thus when demand is discrete the (S - 1, S) policy calls for the placement of a replenishment order after each demand equal in magnitude to the size of the demand. This policy has often been advocated for controlling inventory of expensive and slow moving items.

Higa et al (1975) studied the customer waiting in an (S - 1, S) production / inventory system with geometric Poisson demand arrivals and exponentially distributed replacement times. Smith (1977) obtained an appropriate formula for the cost minimizing S in an (S - 1, S) pure inventory system with Poisson demand arrivals and an arbitrary replenishment time distribution. Examples of more recent studies of multi-echelon repairable systems employing the (S - 1, S) policy include Muckstadt and Thomas (1980), Shanker (1981), Hausman and Scudder (1982), Graves (1985) and Sherbrooke (1986) etc.

Schults (1990) established that (S - 1, S) policy is optimal when the renewal function of demand sizes is concave. Further he has shown that if the ratio of the re-order cost to the expected time between demands is smaller than a specified function of the lead time demand distribution and the holding and penalty costs, then it is optimal to order after each demand. Parthasarathy and Vijayalakshmi (1996) carried out a numerical analysis of an (S – 1, S) inventory model with a constant rate of decay and random lead time having an exponential distribution. Cheng, Ki Ling (1996) analyzed an (S – 1, S) inventory model with compound Poisson demands and derived expressions of performance measures such as the steady state distribution and the expectation of the number of backlogged units. Smith and Dekker (1997) discussed the (S – 1, S) stock model where demand follows a renewal process and the lead time is deterministic.

In this chapter we are considering a production inventory (S - 1, S) model in which there is a production facility and a finished product warehouse for a single commodity. The production continues until the on-hand stock reaches ' S ', that is when there are S units in the system, ready for sale, S - 1 will be in the warehouse and one will be at the production facility. In the finished product warehouse there are S spaces to allocate the finished products. If all the S spaces are full then the production facility will not produce further units. In a similar manner, if all the S spaces in the warehouse are vacant (that is free of products) then no demands will be entertained. That is the demands will be rejected at that time.

In this inventory model, the time between successive demands is assumed to follow a PH-distribution F(.) which has the irreducible representation (α, T) of order m and is given by

$$F(x) = 1 - \alpha \exp(Tx)\, e_m \; ; \; x \geq 0. \qquad (5.1.1)$$

The row vector α and the matrix T are of dimension m. e_m is a column vector of order m with all its components equal to one. The vector T^0 is defined by

$$T^0 = - T\, e_m.$$

The production times are mutually independent and independent of the demand process and have common probability distribution G(.) of phase type with the irreducible representation (β, U) of order n and is given by

$$G(x) = 1 - \beta \exp(U x)\, e_n \; ; \; x \geq 0. \qquad (5.1.2)$$

The row vector β and the matrix U are of dimension n. e_n is a column vector of order n with all its components equal to one. The vector U^0 is defined by

$$U^0 = - U e_n .$$

Thus the stock level can be described by a stochastic process $\{I(t), t \geq 0\}$ with $I(0) = S$. So I(t) denotes the on-hand inventory level at arbitrary time t. The principal quantity of interest is the probability mass function of the inventory level at any arbitrary time t on the time axis.

That is,

$$P\{I(t) = n\}, n = 0, 1, 2, \ldots, S.$$

Here T and U are generators of Markov process describing the generation of demands and production. The PH - distribution has a probabilistic interpretation by its definition so that it can be considered as the distribution of time between demands and the distribution of production time in two networks where the sojourn time at each node of the network has an exponential distribution. So F(x) and G(x) could be interpreted as the distribution of time spent in their networks.

The above production inventory system can be studied by a direct analysis of the behavior of the physical inventory level and the status of the processor. The reason why the PH - distributions are chosen in this work is that many researchers have been investigating these distributions and developing procedures to fit these distributions to given data sets (See for example Bobbio and Telek (1994) and Asmussen et al (1996)). Also various algorithms have been already developed for problems in applied probability where PH - distributions are involved (see Neuts (1981)). This way production facility, which stops not when the warehouse is full, will achieve a higher level of productivity than the case where it stops when the warehouse is full.

The next section introduces the notations used and some preliminaries. In section 5.3, the system is analyzed in detail to obtain the stationary distribution of inventory level.

5.2 Notations and Preliminaries

Let I_m denotes the identity matrix of order m, I_n denotes the identity matrix of order n and A⊗B denotes the Kronecker (tenser) product of the matrices A and B. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are rectangular matrices of dimensions $m_1 \times n_1$ and $m_2 \times n_2$, then their Kronecker product A⊗B is the matrix of dimension $m_1m_2 \times n_1n_2$ which can be written in block partitioned form as follows:

$$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n_1}B \\ a_{21}B & a_{22}B & \cdots & a_{2n_1}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m_11}B & a_{m_12}B & \cdots & a_{m_1n_1}B \end{bmatrix}.$$

Some useful properties of the Kronecker product that are repeatedly used in this chapter are given as follows: (Refer Bellman (1974))

I. $A \otimes (B + C) = A \otimes B + A \otimes C$

II. $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$

III. $(A \otimes B)(C \otimes D) = AC \otimes BD$.

5.3 Stationary Distribution of the Inventory level

Let X(t) denote the state of the inventory at time t. Then $\{X(t),\ t \geq 0\}$ may be considered as homogeneous Markov process on the state space

$\Omega = \{(0, j)/j = 1, 2, \ldots, m\} \cup \{(i, j, k)/\ i = 1, 2, \ldots, S;\ j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n\}$

where (0, j) represents that the system is empty and the demand is at phase j, while (i, j, k) represents that there are ' i ' units in the system and the arriving demand and being produced unit are at phases ' j ' and ' k ' respectively.

The states are labeled in the lexicographic order. The infinitesimal generator of the process in block-partitioned form is given by

$$Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & \cdots & S-1 & S \end{array} \\ \left[\begin{array}{cccccccc} T & A' & 0 & 0 & \cdots & 0 & 0 \\ B' & C & A & 0 & \cdots & 0 & 0 \\ 0 & B & C & A & \cdots & 0 & 0 \\ 0 & 0 & B & C & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C & A \\ 0 & 0 & 0 & 0 & \cdots & B & D \end{array}\right] \end{array}$$

where $A = T^0 \alpha \otimes I_n$

$B = I_m \otimes U_0 \beta$

$C = T \otimes I_n + I_m \otimes U$

$$D = (T + T^0\alpha) \otimes I_n + I_m \otimes U$$

$$A' = T^0 \alpha \otimes \beta$$

$$B' = T \otimes U^0.$$

Let π_ω denote the stationary probability of the state $\omega \in \Omega$.

Define $\pi_0 = (\pi_{01}, \pi_{02}, \ldots, \pi_{0m})$

$$\pi_i = (\pi_{i,1,1}, \pi_{i,1,2}, \ldots, \pi_{i,1,m}, \pi_{i,2,1}, \ldots, \pi_{i,m,n}); \; i = 1, 2, \ldots, S.$$

Under our assumption, the stationary distribution π_ω; $\omega \in \Omega$ is the unique solution of the steady state equilibrium equation (See Neuts (1981))

$$\pi Q = 0, \; \pi e = 1 \text{ where } \pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_S).$$

That is,

$$\pi_0 T + \pi_1 B' = 0 \tag{6.3.1}$$

$$\pi_0 A' + \pi_1 C + \pi_2 B = 0 \tag{6.3.2}$$

$$\pi_{i-1} A + \pi_i C + \pi_{i+1} B = 0 \; ; \; i = 2, 3, \ldots, S - 1 \tag{6.3.3}$$

$$\pi_{S-1} A + \pi_S D = 0 \tag{6.3.4}$$

$$\pi_0 e_m + \sum_{i=1}^{S} \pi_i e_{mn} = 1 . \tag{6.3.5}$$

Multiplying on the right of both sides of (6.3.2), (6.3.3) and (6.3.4) by the matrix $I_m \otimes e_n$ and using the properties of Kronecker product, we get

$$\pi_0 T^0 \alpha + \pi_1 (T \otimes e_n - I_m \otimes U^0) + \pi_2 (I_m \otimes U^0) = 0 \tag{6.3.6}$$

$$\pi_{i-1} (T^0 \alpha \otimes e_n) + \pi_i (T \otimes e_n - I_m \otimes U^0) + \pi_{i+1} (I_m \otimes U^0) = 0;$$

$$i = 2, 3, \ldots, S - 1 \tag{6.3.7}$$

$$\pi_{S-1}(T^0 \alpha \otimes e_n) + \pi_S[(T + T^0\alpha) \otimes e_n - I_m \otimes U^0)] = 0 \tag{6.3.8}$$

Adding equations (6.3.1), (6.3.6), (6.3.7) and (6.3.8) we get

$$\pi_0 (T + T^0 \alpha) + \pi_1[(T + T^0 \alpha) \otimes e_n] + \pi_2 [(T + T^0 \alpha) \otimes e_n]$$
$$+ \dots + \pi_{S-1} [(T + T^0 \alpha) \otimes e_n] + \pi_S [(T + T^0 \alpha) \otimes e_n] = 0 \cdot$$

Since $A \otimes e_n = (I_m \otimes e_n) A$, we can write the above equation as

$$\left[\pi_0 + \sum_{i=1}^{S} \pi_i (I_{nm} \otimes e_n)\right] \left(T + T^0 \alpha\right) = 0 \cdot \qquad (6.3.9)$$

Now multiply on the right both sides of (6.3.1), (6.3.6), (6.3.7) and (6.3.8) by e_m, we get,

$$-\pi_0 T^0 + \pi_1 (e_m \otimes U^0) = 0$$

$$\pi_0 T^0 - \pi_1 [(T^0 \otimes e_n + e_m \otimes U^0)] + \pi_2 (e_m \otimes U^0) = 0$$

$$\pi_{i-1} (T^0 \otimes e_n) - \pi_i [(T^0 \otimes e_n) + (e_m \otimes U^0)] + \pi_{i+1} (e_m \otimes U^0) = 0 ; i = 2, 3, \dots S - 1.$$

and $\pi_{S-1}(T^0 \otimes e_n) - \pi_S(e_m \otimes U^0) = 0.$

From the above equations, we have

$$\pi_1 (e_m \otimes U^0) = \pi_0 T^0 \text{ and}$$

$$\pi_{i+1} (e_m \otimes U^0) = \pi_i (T^0 \otimes e_n) ; i = 1, 2, 3, \dots, S - 1. \qquad (6.3.10)$$

Again multiplying on the right both sides of the equations (6.3.2) and (6.3.3) by the matrix $I_m \otimes (e_n\beta - I_n)$ we have,

$$\pi_1[T \otimes I_n + I_m \otimes U] [I_m \otimes (e_n \beta - I_n) = 0$$

and $\pi_{i-1}[T^0 \alpha \otimes (e_n \beta - I_n) + \pi_i[T \otimes I_n + I_m \otimes U] [I_m \otimes (e_n \beta - I_n) = 0$;

$i = 2, 3, \dots, S - 1$

The first equation reduces to

$\pi_1\{T \otimes (e_n \beta - I_n) + I_m \otimes [U \ (e_n \beta - I_n)]\} = 0$

That is, $\pi_1[T \otimes (e_n \beta - I_n) - I_m \otimes U] = \pi_1(I_m \otimes U^0\beta)$

$$= \pi_1 [(I_m \otimes U^0)(I_m \otimes \beta)]$$

$$= - \pi_0[T \otimes \beta] \quad \text{(Using (6.3.1))}$$

and the second equation becomes,

$$\pi_i[T \otimes (e_n \beta - I_n) - I_m \otimes U] = \pi_i (I_m \otimes U^0\beta) + \pi_{i-1}[T^0 \alpha \otimes I_n] - \pi_{i-1}[T^0 \alpha \otimes e_n \beta]$$

$$= \pi_i (I_m \otimes U^0\beta) + \pi_{i-1}[T^0 \alpha \otimes I_n] - \pi_{i-1}[T^0 \otimes e_n][\alpha \otimes \beta]$$

$$= \pi_i (I_m \otimes U^0\beta) + \pi_{i-1}[T^0 \alpha \otimes I_n] - \pi_i[e_m \otimes U^0][\alpha \otimes \beta]$$

(Using (6.3.10))

$$= \pi_i (I_m \otimes U^0\beta) + \pi_{i-1}[T^0 \alpha \otimes I_n] - \pi_i[e_m\alpha \otimes U^0\beta] ;$$

$i = 2, 3, \ldots, S - 1.$

Putting $U^* = T \otimes (e_n \beta - I_n) - I_m \otimes U$, the above two equations becomes

$$\pi_1 U^* = - \pi_0[T \otimes \beta] \qquad (6.3.11)$$

$$\pi_i U^* = \pi_i (I_m \otimes U^0\beta) + \pi_{i-1}[T^0 \alpha \otimes I_n] - \pi_i[e_m\alpha \otimes U^0\beta] ; \ i = 2, 3, \ldots S - 1. \qquad (6.3.12)$$

Similarly after multiplying on the right both of (6.3.2) and (6.3.3) by the matrix $(e_m \alpha - I_m) \otimes I_n$, and putting $T^* = (e_m \alpha - I_m) \otimes U - T \otimes I_n$, we get

$$\pi_i T^* = \pi_i (T^0 \alpha \otimes I) + \pi_{i+1}[(I_m - e_m\alpha) \otimes U^0\beta]; \quad i = 1, 2, 3, \ldots S - 1. \qquad (6.3.13)$$

Now combining (6.3.11) - (6.3.13) and (6.3.4), we have

$$\pi_1 U^* = - \pi_0 (T \otimes \beta) \qquad (6.3.14)$$

$$\pi_i U^* = - \pi_{i-1} T^* ; \ i = 2, 3, \ldots, S - 1 \qquad (6.3.15)$$

and $\pi_S[(T + T^0 \alpha) \otimes I_n + I_m \otimes U] = - \pi_{S-1}(T^0 \alpha \otimes I_n)$. (6.3.16)

Following Neuts (1981) we see that the irreducible matrix $Q^* = T + T^0 \alpha$ is the generator of a homogeneous Markov process and it is sub-stable (A matrix is stable if all its eigen values have their real parts are strictly less than zero. It is sub-stable if their real parts are less than or equal to zero.). The eigen values of $Q^* \otimes I_n + I_m \otimes U$ are the sum of the eigen values of the sub-stable matrix Q^* and the stable matrix U (See Bellman (1974)). Hence $Q^* \otimes I + I \otimes U$ is stable and invertible.

Next we prove that T^* and U^* are invertible.

Consider $T^* = -(T \otimes I_n + (I_m - e_m \alpha) \otimes U)$. By our assumption T and U are stable matrices. T^* can be rearranged as

$T^* = -(T \otimes I_n)[I \otimes U^{-1} + T^{-1}(I_m - e_m \alpha) \otimes I](I_m \otimes U)$. (6.3.17)

First of all we prove that $T^{-1}(I_m - e_m \alpha)$ is sub-stable. Let λ be an eigen value of $T^{-1}(I_m - e_m \alpha)$ and ν be the corresponding eigen vector. Assume that $Re(\lambda) > 0$ where $Re(\lambda)$ = real part of λ. We have $\lambda\nu = T^{-1}\nu - (\alpha\nu) T^{-1} e_m$.

Since T and T^{-1} are stable, λ^{-1} cannot be an eigen value of T and λ cannot be an eigen value of T^{-1}.

Hence $\alpha\nu \neq 0$.

Further $\nu = \left(\frac{\alpha\nu}{\lambda}\right)\left(\frac{1}{\lambda} I_m - T\right)^{-1} e_m$.

$$\therefore \alpha\nu = \left(\frac{\alpha\nu}{\lambda}\right)\alpha\left(\frac{1}{\lambda}I_m - T\right)^{-1} e_m$$

$$\Rightarrow \frac{1}{\lambda}\alpha\left(\frac{1}{\lambda}I_m - T\right)^{-1} e_m = 1 . \qquad (6.3.18)$$

The Laplace Stieltjes transform of F(x) is given by

$F^*(s) = 1 - s\,\alpha\,(s\,I_m - T)^{-1} e_m$

$$\therefore F^*\left(\frac{1}{\lambda}\right) = 1 - \frac{1}{\lambda}\alpha\left(\frac{1}{\lambda}I_m - T\right)^{-1} e_m = 0 \text{ by (6.3.18)}.$$

But the Laplace Stieltjes transform of a non-negative random variable is strictly positive in the right plane. Hence $T^{-1}(I_m - e_m\,\alpha)$ is sub-stable.

Now, since U^{-1} is stable and $T^{-1}(I_m - e_m\,\alpha)$ is sub-stable, $I \otimes U^{-1} + T^{-1}(I_m - e_m\,\alpha) \otimes I_n$ is stable and invertible. More over $(T \otimes I_n)$ and $(I_m \otimes U)$ are invertible. Hence from (6.3.17) we conclude that T^* is invertible.

By the same argument U^* is invertible.

Now the stationary distribution π_i; i = 1, 2, …, S is given by

$\pi_1 = -\,\pi_0\,(T \otimes \beta)U^{*-1}$

$\pi_i = -\,\pi_{i-1}T^*\,U^{*-1}$; i = 2, 3, …, S - 1

and $\pi_S = -\,\pi_{S-1}(T^0\alpha \otimes I_n)[(T + T^0\alpha) \otimes I_n + I_m \otimes U]^{-1}$.

By defining the matrices,

$$M_0 = -(T \otimes \beta)\, U^{*-1} \tag{6.3.19}$$

$$M = T^{*}\, U^{*-1} \tag{6.3.20}$$

and $M_{S-1} = -(T^0 \alpha \otimes I_n)[(T + T^0 \alpha) \otimes I_n + I_m \otimes U]^{-1}$ (6.3.21)

which are respectively of dimensions m × mn, mn × mn and mn × mn. It is clear that

π_i; i = 1, 2, …, S

can be obtained by the following formula

$$\pi_i = \begin{cases} \pi_0 M_0 M^{i-1}; & i = 1,2,\ldots S-1 \\ \pi_0 M_0 M^{S-2} M_{S-1}; & i = S \end{cases} \tag{6.3.22}$$

In order to find π_0 we substitute the above values in (6.3.9) to get

$$\pi_0 \left[I_m + \left(\sum_{i=0}^{S-2} M_0 M^i + M_0 M^{S-2} M_{S-1} \right) (I_m \otimes e_n) \right] \left[T + T^0 \alpha \right] = 0$$

That is

$$\pi_0 W (T + T^0 \alpha) = 0 \tag{6.3.23}$$

where $W = I_m + \left(\sum_{i=0}^{S-2} M_0 M^i + M_0 M^{S-2} M_{S-1} \right) (I_m \otimes e_n)$ (6.3.24)

Since $T + T^0 \alpha$ is irreducible, the system

$$v(T + T^0 \alpha) = 0; \quad v e_m = 1 \tag{6.3.25}$$

has a unique solution. Therefore from (6.3.23) we get $\pi_0 W = cv$, where c is a constant and v is a vector.

On applying the normalizing condition (6.3.5) we get c = I.

Hence in order to obtain π_0 we need to solve the system

$\pi_0 W = v$,

and the stationary probabilities can be calculated using the formula (6.3.22).

If we denote $p_i = \sum_{j=1}^{m}\sum_{k=1}^{n}\pi_{ijk}$; $\quad i = 1, 2, ..., S$

and $p_0 = \sum_{j=1}^{m}\pi_{0j}$.

Then $\{p_i\ ;\ i = 0, 1, 2, \ldots, S\}$ represents the stationary distribution of the inventory level.

Calculation of the stationary probability distribution of the inventory level:-

Based on the forgoing analysis the following algorithmic procedure is suggested for the computation of $\{p_i; i = 0, 1, 2, \ldots, S\}$.

Inputs:

Integers m, n and S

Row vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$

$\beta = (\beta_1, \beta_2, \ldots, \beta_n)$

Matrices $T = (T_{ij})_{m\times m}$

$U = (U_{ij})_{n\times n}$.

Compute:

$U^0 = -Te_m$

$U^* = T \otimes (e_n\beta - I_n) - I_m \otimes U$

$T^* = (e_m\alpha - I) \otimes U - T \otimes I_n$

$$M_0 = -(T \otimes \beta)U^{*^{-1}}$$

$$M = T^* U^{*^{-1}}$$

$$M_{S-1} = -(T^0\alpha \otimes I)\,[(T + T^0\alpha) \otimes I_n + I_m \otimes U]^{-1}$$

$$W = I_m + \left(\sum_{i=0}^{S-2} M_0 M^i + M_0 M^{S-2} M_{S-1}\right)(I_m \otimes e_n)\cdot$$

Solve for $v = (v_1, v_2, \ldots, v_m)$ from the system $v(T + T^0\alpha) = 0$, $ve_m = 1$.

Solve for $\pi_0 = (\pi_{01}, \pi_{02}, \ldots, \pi_{0m})$ from the system $\pi_0 W = v$.

Calculate:

$$\pi_i = \begin{cases} \pi_0 M_0 M^{i-1} & \text{for } i = 1, 2, \ldots, S-1 \\ \pi_0 M_0 M^{S-2} M_{S-1} & \text{for } i = S\,. \end{cases}$$

The stationary probabilities are:

$$p_i = \pi_i e_{mn}\ ;\ i = 1, 2, \ldots, S$$

$$p_0 = \pi_0 e_m.$$

□

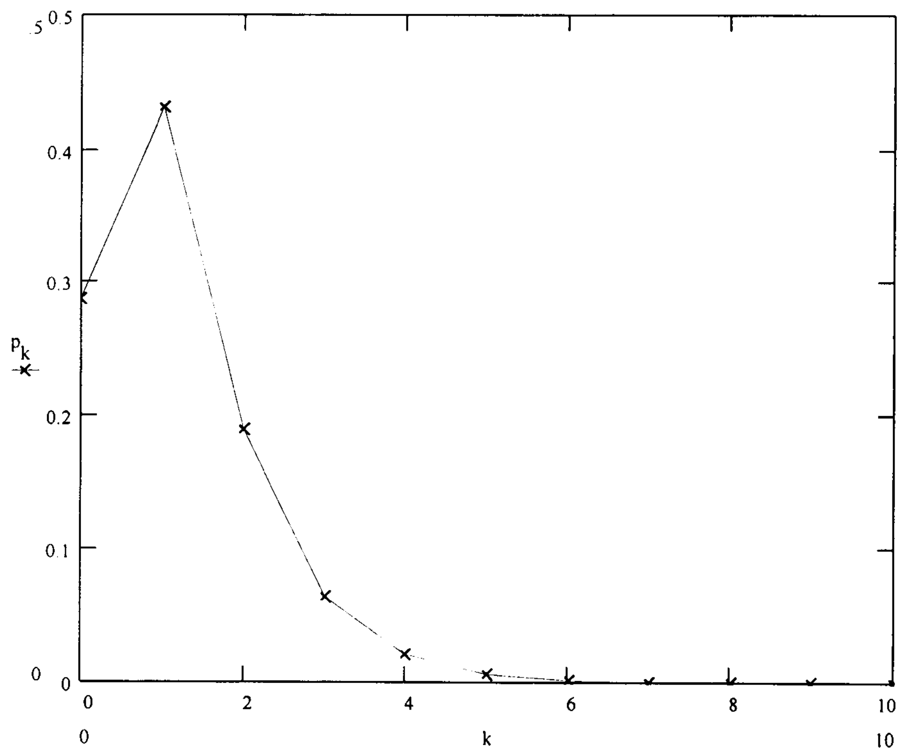
Numerical Illustration:-

Suppose that in a production – inventory system with S = 10 the inter-demand times follow a CPH $(\alpha, T)_m$ where m = 3, $\alpha = (1, 0, 0)$ and $T = \begin{bmatrix} -2.5 & 2.5 & 0 \\ 0 & -2.5 & 2.5 \\ 0 & 0 & -2.5 \end{bmatrix}$.

The production times follow a CPH $(\beta, U)_n$ where n = 4, $\beta = (1, 0, 0, 0)$ and

$$U = \begin{bmatrix} -3.5 & 3.5 & 0 & 0 \\ 0 & -3.5 & 3.5 & 0 \\ 0 & 0 & -3.5 & 3.5 \\ 0 & 0 & 0 & -3.5 \end{bmatrix}.$$

Using the above procedure the stationary distribution $\{p_i \; ; \; i = 0, 1, 2, \ldots, 10\}$ is computed and its graph is given below:



Graph of $\{p_i\}$

CHAPTER 6

INTERMITTENTLY USED COMPLEX N-UNIT SYSTEM WITH STOCHASTIC DEMAND

6.1 Introduction

A system in general can be classified as one of two types according to its usage. One is when a system is constantly in use and second is one, which is intermittently used. In this intermittently used systems continuous failure free performance may not be necessary. For example telephone devices computer etc. Gaver (1963) first discussed a one unit intermittently used system and introduced "Disappointment time". Srinivasan (1966) extended Gaver's results to two-unit redundant system with arbitrary failure and exponential repair time distributions to obtain the first passage time to disappointment and mean time to disappointment using supplementary variable technique. Natarajan and Shahul Hameed (1988) studied an intermittently used k out of n: F system with the assumption that the failures will be detected only during the usage period and obtained expression for the distribution of time to the first disappointment etc by identifying suitable regeneration points. Sharafali et. al. (1988) considered a complex two-unit system with stochastic demand and obtained the various reliability measures. We extend this work to an intermittently used n-unit systems. In most of the intermittently used systems one or more units may be required for the satisfactory performance of the system. In such situations the number of units needed at any time may be described by a stochastic process called the "need process". The next section treats the model, in which all the underlying distributions are assumed to be exponential. For this model apart from obtaining expressions for time to the first disappointment and mean number of disappointments, an attempt is made to derive the first order product density of a disappointment. The last section considers the sojourn times.

6.2 The Model

The system consists of n units subject to failure and a single repair facility. The need process is governed by a Markov process. The assumptions for the model are given below.

1.There are n identical units, whose lifetime distributions are negative exponential with mean $\frac{1}{a}$.

2.There is only one repair facility and repairs are taken in FIFO order.

3.Each unit is new after repair.

4.The repair time distribution is negative exponential with mean $\frac{1}{\mu}$.

5.The need process is governed by a Markov Process $\{Y(t), t \geq 0\}$ on the state space $\{0, 1, 2, \ldots, n\}$. Here Y(t) represents the number of units required for the satisfactory performance of the system at time t. The infinitesimal generator of this Markov Process $\{Y(t), t \geq 0\}$ is assumed to be

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & n \end{array} \\ \begin{bmatrix} -\lambda_0 & \lambda_{01} & \lambda_{02} & \cdots & \lambda_{0n} \\ \lambda_{10} & -\lambda_1 & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{20} & \lambda_{21} & -\lambda_2 & \cdots & \lambda_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{n0} & \lambda_{n1} & \lambda_{n2} & \cdots & -\lambda_n \end{bmatrix} \end{array}.$$

6.If at any time the number of operable units is less than the number of units required for the satisfactory performance of the system, then the system enters the down state. If more no of units are in the operable system than required, then the operable units not in use will behave like cold standbys.

7.During the system down, the need will last for a duration governed by the respective exponential distribution and will not wait for a system recovery

6.3 State of the System

Let X(t) represent the number of units in the repair facility and Y(t), the state of the need process at time t. Clearly {X(t), Y(t)} is a Markov Process on the state space

$E = E_0 \cup E_1 \cup \ldots \cup E_n$, where $E_i = \{(i,0), (i,1), \ldots, (i,n)\}$, $i = 0,1, \ldots, n$. The infinitesimal generator of this process $\{(X(t), Y(t))\}$, $t \geq 0\}$ is easily seen to be Q given as follows.

$$
Q = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_{n-2} \\ E_{n-1} \\ E_n \end{array}
\begin{array}{c}
\begin{array}{ccccccc} E_0 & E_1 & E_2 & \cdots & E_{n-2} & E_{n-1} & E_n \end{array} \\
\begin{bmatrix}
A - a\Delta e_n & a\Delta e_n & 0 & \cdots & 0 & 0 & 0 \\
\mu I & A - \mu I - a\Delta e_{n-1} & a\Delta e_{n-1} & \cdots & 0 & 0 & 0 \\
0 & \mu I & A - \mu I - a\Delta e_{n-2} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A - \mu I - a\Delta e_2 & a\Delta e_2 & 0 \\
0 & 0 & 0 & \cdots & \mu I & A - \mu I - a\Delta e_1 & a\Delta e_1 \\
0 & 0 & 0 & \cdots & 0 & \mu I & A - \mu I
\end{bmatrix}
\end{array}
$$

where

$$\Delta(e_n) = \text{diag}(0, 1, 2, \ldots, n)$$

$$\Delta(e_{n-1}) = \text{diag}(0, 1, 2, \ldots, n-1, 0)$$

$$\Delta(e_{n-2}) = \text{diag}(0, 1, 2, \ldots, n-2, 0, 0)$$

$$\vdots \quad = \quad \vdots$$

$$\Delta(e_2) = \text{diag}(0, 1, 2, 0, \ldots, 0)$$

$$\Delta(e_1) = \text{diag}(0, 1, 0, \ldots, 0).$$

Let $P_{ij}(t) = P\,[X(t) = i, Y(t) = j]$, $i, j = 0, 1, 2, \ldots, n$ and $P(t) = [P_{00}(t), P_{01}(t), \ldots, P_{nn}(t)]^T$.
Then by Chapman-Kolmogrov equation

$$P'(t) = P(t)\,Q$$

$$\text{or } P(t) = \exp\{Q(t)\}P(0)$$

where P(0) is the initial state probability.

6.4. Stationary Distribution

Let $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ with $\pi_k = (\pi_{k0}, \pi_{k1}, \ldots, \pi_{kn})$

where k = 0, 1, 2, ..., n be the stationary distribution. This is the solution of the equations $\pi Q = 0$ with $\sum_{k=0}^{n} \pi_k e = 1$.

$\pi Q = 0$ gives

$$\left.\begin{array}{ll} \pi_0[A - \Delta e_n] + \mu\pi_1 & = 0 \\ a\pi_0 \Delta e_n + \pi_1(A - \mu I - a\Delta e_{n-1}) + \mu\pi_2 & = 0 \\ a\pi_1 \Delta e_{n-1} + \pi_2(A - \mu I - a\Delta e_{n-2}) + \mu\pi_3 & = 0 \\ \vdots & = \vdots \\ a\pi_{n-2} \Delta e_2 + \pi_{n-1}(A - \mu I - a\Delta e_1) + \mu\pi_n & = 0 \\ a\pi_{n-1} \Delta e_1 + \pi_n(A - \mu I) & = 0 \end{array}\right\}. \qquad (6.4.1)$$

Adding these equations we get

$$(\pi_0 + \pi_1 + \ldots + \pi_n)A = 0.$$

This implies that $\pi_0 + \pi_1 + \ldots + \pi_n$ must be invariant measure for the Markov Process with the generator A. Assume that A possesses the invariant measure and let it be η. Here

$$\pi_0 + \pi_1 + \ldots + \pi_n = \eta. \qquad (6.4.2)$$

By solving (6.4.1) and (6.4.2),

we get the solution for

$$\pi = (\pi_0, \pi_1, \ldots, \pi_n).$$

6.5. Distribution of the time to the First Disappointment

A disappointment is said to occur whenever the number of operable units at any time becomes less than Y(t) at that instant of time. Thus the set of states of disappointment is D = {(1,n), (2,n-1), (2,n), (3,n-2),(3,n-1),(3,n), …, (n,1), (n,2), …, (n,n)} (Totally $\frac{n(n+1)}{2}$ in number). So the infinitesimal generator of $\{(X(t), Y(t)), t \geq 0\}$ can be rearranged corresponding to the set of up-states U and set of disappointment states D as follows.

$$Q = \begin{array}{c} \\ U \\ D \end{array}\begin{array}{c} \begin{array}{cc} U & D \end{array} \\ \begin{bmatrix} Q_U & B_D \\ B_U & Q_D \end{bmatrix} \end{array} \qquad (6.5.1)$$

where

$$Q_U = \begin{array}{c} (0,0) \\ \vdots \\ (0,n) \\ (1,0) \\ \vdots \\ (1,n-1) \\ (2,0) \\ \vdots \\ (n-1,1) \\ (n,0) \end{array}
\overset{\begin{array}{cccccccccc} (0,0) & \cdots & (0,n) & (1,0) & \cdots & (1,n-1) & (2,0) & \cdots & (n-1,1) & (n,0) \end{array}}{\begin{bmatrix}
-\lambda_0 & \cdots & \lambda_{0n} & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_{n0} & \cdots & -(\lambda_n + na) & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\mu & \cdots & 0 & \nu_0 & \cdots & \lambda_{0,n-1} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \lambda_{n-1,0} & \cdots & \nu_{n-1} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & \mu & \cdots & 0 & -(\lambda_0 + \mu) & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \nu_1 & 0 \\
0 & \vdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & -(\lambda_0 + \mu)
\end{bmatrix}}$$

with $\nu_n = -(\lambda_n + \mu + na)$,

$$
B_D = \begin{array}{c} \\ (0,0) \\ \vdots \\ (0,n) \\ (1,0) \\ \vdots \\ (1,n-1) \\ (2,0) \\ \vdots \\ (n-1,1) \\ (n,0) \end{array}
\begin{array}{c} \begin{array}{ccccccc} (1,n) & (2,n-1) & (2,n) & \cdots & (n,1) & \cdots & (n,n) \end{array} \\
\left[\begin{array}{ccccccc}
0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
na & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\lambda_{0n} & 0 & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\lambda_{n-1,n} & (n-1)a & 0 & \cdots & 0 & \cdots & 0 \\
0 & \lambda_{0,n-1} & \lambda_{0,n} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & a & \cdots & 0 \\
0 & \cdots & 0 & \cdots & \lambda_{0,1} & \cdots & \lambda_{0,n}
\end{array}\right] \end{array},
$$

$$
B_U = \begin{array}{c} \\ (1,n) \\ (2,n-1) \\ (2,n) \\ \vdots \\ (n,1) \\ \vdots \\ (n,n) \end{array}
\begin{array}{c} \begin{array}{cccccccccc} (0,0) & \cdots & (0,n) & (1,0) & \cdots & (1,n-1) & (2,0) & \cdots & (n-1,1) & (n,0) \end{array} \\
\left[\begin{array}{cccccccccc}
0 & \cdots & \mu & \lambda_{n0} & \cdots & \lambda_{n,n-1} & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & \mu & \lambda_{n-1,0} & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \lambda_{n,0} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \mu & \lambda_{1,0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \lambda_{n,0}
\end{array}\right] \end{array},
$$

and

$$
Q_D = \begin{array}{c} \\ (1,n) \\ (2,n-1) \\ (2,n) \\ \vdots \\ (n,1) \\ \vdots \\ n,n \end{array}
\begin{array}{c} \begin{array}{ccccccc} (1,n) & (2,n-1) & (2,n) & \cdots & (n,1) & \cdots & (n,n) \end{array} \\
\left[\begin{array}{ccccccc}
-(\lambda_n+\mu) & 0 & 0 & \cdots & 0 & \cdots & 0 \\
0 & -(\lambda_{n-1}+\mu) & \lambda_{n-1,n} & \cdots & 0 & \cdots & 0 \\
\mu & \lambda_{n,n-1} & -(\lambda_n+\mu) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -(\lambda_1+\mu) & \cdots & \lambda_{1,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \lambda_{n,1} & \cdots & -(\lambda_n+\mu)
\end{array}\right] \end{array}.
$$

Our problem is to obtain the distribution of random variable D, which represents the time to the first disappointment. Hence we dump together the states of the Markov process $\{(X(t), Y(t)), t \geq 0\}$ into a single absorbing state D. We so obtain the absorbing Markov process with generator

$$\hat{Q} = \begin{bmatrix} Q_U & B_D e \\ 0 & 0 \end{bmatrix}.$$

Let us assume that the process starts in a state U and so let $\hat{P}_U(0)$ be the row vector of initial state probabilities corresponding to the situation. Now the time to the first disappointment is the same as the time to absorption in the Markov process with the generator $\hat{Q}$ given as above. If $G_D(t)$ is the distribution function of the time to the first disappointment then

$$G_D(t) = 1 - \hat{P}_U(0)\exp\{tQ_U\}.$$

One may note that this is phase type distribution representing the distribution of the absorption time in the Markov process described above. Clearly the moments are given by

$$E(D^k) = (-1)^k k!\hat{P}_U(0)Q_U^{-k}e, \quad k = 0, 1, 2, \cdots\cdots$$

6.6 Product Density of a Disappointment

Let $h_D(t)$ be the first order product density of a disappointment at t. Now $h_D(t)$ is the probability that a disappointment occurs in (t, t + dt).

By considering the various possibilities of entering into the states of disappointment we have

$$h_D(t) = a \sum_{i=1}^{n} i p_{n-i,i}(t) + \sum_{j=i}^{n} \sum_{i=0}^{n-j} \sum_{k=1}^{j} \lambda_{i,n-j+k} p_{j,i}(t).$$

The expected number of disappointment in (0, t) is given by

$$E[N(D,t) = \int_0^t h_D(u)du.$$

6.7 Total Sojourn Time

Let $S_{r,i}^{m,j}(t)$ denote the total sojourn up to t in the state (m, j) starting from (r, i) at t = 0.

Let the elements of Q in (6.5.1) can be written as

$$Q = \left(Q_{r,i}^{m,j}\right)$$

where

$$Q_{r,i}^{m,j} < 0 \quad \text{for } (m, j) = (r, i)$$

and $Q_{r,i}^{m,j} \geq 0$ for (m, j) ≠ (r, i).

Define

$$M_{r,j}(z;t) = E\left[\exp\left\{-\sum_{(m,j)} z_{m,j} S_{r,i}^{m,j}(t)\right\}\right]$$

and $M(z; t) = (M_{00}(z; t), M_{01}(z; t), \ldots, M_{nn}(z; t))^T$

with $z = (z_{00}, z_{01}, \ldots, z_{nn})$.

So for (r, i) ∈ E,

$$M_{r,i}(z;t) = \exp\{-(z_{r,i} - Q_{r,i}^{r,i})t\} + \sum_{(m,k)\in E} \int_0^t \exp\{-(z_{r,i} - Q_{r,i}^{r,i})u\} Q_{r,i}^{m,i} M_{m,k}(z; t-u)\, du.$$

Therefore

$$M^*(z;s) = [I - \Delta^*(z;s)]^{-1} \Delta^*(z;s)\, e \qquad (6.7.1)$$

where $M^*(z;s)$ is the Laplace transform of M(z; t) and

$$\Delta^*(z;s) = diag\left(\frac{1}{s + z_{0,0} - Q_{0,0}^{0,0}}, \frac{1}{s + z_{0,1} - Q_{0,1}^{0,1}}, \ldots, \frac{1}{s + z_{n,n} - Q_{n,n}^{n,n}}\right).$$

The Laplace transform of the Moment Generating Function of the total time spent in the state of disappointment up to any time t is obtained by putting

$Z = z\,(0, 0, 0, \ldots, 0, 1, 1, \ldots, 1)^T$ (with $\frac{n(n+1)}{2}$ 1's) in (6.7.1).

Total time spent in each of the up states prior to the occurrence of a disappointment is obtained by taking $S^{m,j}$, (m,j) ∈ U be the total time spent in the state (m,j)

we get,

$$E\left[\exp\left\{-\sum_{(m,j)\in U} z_{m,j} S_{r,i}^{m,j}\right\}\right] = P_U(0)[\Delta(z) - Q_U]^{-1} B_D e \qquad (6.7.2)$$

where $\Delta(z) = diag(z_{0,0}, z_{0,1}, \ldots, z_{n,0})$.

By differentiating (6.7.2), we get the mean vector

$$\mu = \left(E[S^{0,0}], E[S^{0,1}], \cdots, E[S^{n,0}] \right)^T$$

$$= -P_U(0)\, Q_U^{-1} e.$$

Remark 6.1. When the number of units n = 2, then the results of this chapter are in agreement with those of Sharafali et. al. (1988).

CHAPTER 7

SOME GENERALIZED POISSON SHOCK MODELS AND THEIR CLOSURE PROPERTIES

7.1 Introduction

Suppose that a device is subjected to shocks occurring randomly over time according to counting process **N** = {N(t): t ≥ 0}. Let $\bar{P}_k$ be the probability that the device survives k shocks, k = 0, 1, 2, …, where it is reasonable to assume that $1 = \bar{P}_0 \geq \bar{P}_1 \geq \ldots\ldots$. Then it follows that the probability $\bar{H}(t)$ of survival function of the device is given by,

$$\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\} \bar{P}_k \cdot \qquad (7.1.1)$$

Shock models of this kind have been studied by Esary et al (1973) when **N** is homogeneous Poisson process, by A - Hameed and Proschan (1973) when **N** is a non-homogeneous Poisson process or a birth process and by Block and Savits (1978) when

N is still more general counting process. In all these cases the authors prove that $\bar{H}(t)$ is IFR, IFRA, DMRL, NBU or NBUE under suitable conditions on **N** if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ has the corresponding discrete property. Klefsjo (1981) proved that $\bar{H}(t)$ is HNBUE if $\bar{P}_k$ has the discrete HNBUE property and Abouammoh et al (1988) have proved the NBUFR and NBAFR preservation results. In this chapter we are proving the corresponding theorems for the ILR class of life distributions. We also get the dual class DLR.

Let F be a life distribution (F(0-) = 0) and $\bar{F} = 1 - F$ be the corresponding survival function. Also let f be its density function.

Definition 7.1. A life distribution F (or its survival function $\bar{F}$) is said to be ILR or Polya Frequency of order 2 (PF_2) if log f is concave.

That is $\dfrac{f(t+\Delta)}{f(t)}$ is decreasing in ' t ' for any $\Delta > 0$

or equivalently $\dfrac{f'(t)}{f(t)}$ is decreasing in ' t '.

If log f is convex, then F is DLR.

Definition 7.2. A discrete distribution $\{p_k\}$ (or its survival probabilities $\bar{P}_k = \sum_{j=k+1}^{\infty} p_j$, k = 0, 1, 2, ...) is called discrete increasing likelihood ratio (Discrete ILR) or discrete PF_2 if $\frac{p_{k+m}}{p_k}$ is decreasing in k for every m > 0. (The reader may refer to Ross (1983) for the above definitions of ILR and DLR class of distributions)

In section 7.2 and 7.3, we prove the preservation of ILR (DLR) properties when N is a homogeneous and non-homogeneous Poisson processes respectively. Section 7.4 is concerned with a mixed Poisson shock model and similar results are proved under suitable conditions on the probability of shocks and the mean value function of the process.

Throughout this chapter increasing is short for non-decreasing and decreasing is short for non-increasing. We remark that the result of section 7.3 in fact was proved in Esary et al (1973) but the method used here may be of intrinsic interest.

7.2 Shock models and Preservation Results

Let a device be subject to shocks occurring randomly over time according to a homogeneous Poisson process with intensity λ. Then the probability $\bar{H}(t)$ of survival of the device until time ' t ' is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \bar{P}_k \quad , \quad t \geq 0 \cdot \qquad (7.2.1)$$

Survival functions with this form have a number of attractive properties. Some of them are discussed in this chapter. If $\left(\bar{P}_k\right)_{k=0}^{\infty}$ of surviving k shocks in (7.2.1) is assumed to be a deterministic function of k alone and not any random damages, it is reasonable to assume that $1 \geq \bar{P}_0 \geq \bar{P}_1 \geq \ldots\ldots$. The density of H(t) has a jump of magnitude $1 - \bar{P}_0$ at the origin and for t > 0

$$h(t) = \lambda \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} p_{k+1}\,;\; t > 0 \tag{7.2.2}$$

where $p_{k+1} = \bar{P}_k - \bar{P}_{k+1}$, $k = 0, 1, 2, \ldots\ldots$

Moreover the hazard rate

$$\lambda(t) = \lambda . \left\{ 1 - \frac{\displaystyle\sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \bar{P}_{k+1}}{\displaystyle\sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} \bar{P}_k} \right\} \quad , \quad t > 0$$

is seen to be less than or equal to λ.

Also λ(t) = λ for some t > 0 $\Leftrightarrow$ $\bar{P}_k = 0 \;\forall\; k > 0$ in which case λ(t) = λ ∀ t > 0.

An interesting example of $\bar{H}(t)$ as defined in (7.2.1) occurs with $\bar{P}_k = \theta^k$, $0 < \theta < 1$, k = 0, 1, 2, ... In this case $\bar{H}(t)$ = exp (-λ (1 - θ) t) is exponential. It can be seen that $\bar{H}(t)$ is exponential if and only if $\bar{P}_k = \theta^k$ for some $0 \leq \theta < 1$.

7.2.1. Properties of $\bar{H}(t)$ from the properties of $\bar{P}_k$

Certain kinds of properties when imposed on $\bar{P}_k$ in (7.2.1) are reflected as analogous properties of $\bar{H}(t)$. For example, if $\bar{P}_k$ is decreasing in k then $\bar{H}(t)$ is decreasing in t. When $\bar{P}_k$ has geometric distribution then $\bar{H}(t)$ has exponential distribution. Various other properties of the $\bar{P}_k$ of interest in reliability are found to carry over to $\bar{H}(t)$.

The main preservation results are given in the following theorem.

Theorem 7.1. (i) If $\bar{P}_k$ is decreasing in k then $\bar{H}(t)$ is decreasing in t.

(ii) Let $\bar{H}(t)=\sum_{k=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\bar{P}_k$ when $1=\bar{P}_0 \geq \bar{P}_1 \geq$ ...., then $\bar{H}(t)$ has a PF_2 density

if $\frac{p_{k+1}}{p_k}$ is decreasing in k = 1, 2, ... That is if $\{p_k, k \geq 1\}$ is a PF_2 sequence.

(iii) $\bar{H}(t)$ is IFR if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete IFR.

(iv) $\bar{H}(t)$ is IFRA if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete IFRA.

(v) $\bar{H}(t)$ is NBU if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete NBU.

(vi) $\bar{H}(t)$ is NBUE if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete NBUE.

(vii) $\bar{H}(t)$ is DMRL if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete DMRL.

Note: Similar results hold for the dual classes representing beneficial ageing.

Suppose that the process **N** is a mixture of homogeneous Poisson process, then $\bar{H}(t)$ is given by

$$\bar{H}(t) = \int_0^{\infty} \sum_{k=0}^{\infty} \frac{e^{-\lambda t}[\lambda t]^k}{k!} \bar{P}_k dG(\lambda) \tag{7.2.3}$$

where λ is a random variable with distribution function G(.).

Then we have

Theorem.6.2. (Manoharan et. al. (1992)) The PH- distribution is preserved by the transformation (7.2.2) when G(.) is a distribution with finite support.

Specifically, if the shock probabilities $\{p_k\}$ has DPH $(\alpha, T)_m$ G has a support on $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ with mixing density $\{r_1, r_2, \ldots, r_k\}$, then the survival distribution function H(t) has CPH distribution with representation $(\alpha^*, T^*)_{mk}$

where $\alpha^* = (r_1\alpha, r_2\alpha, \ldots, r_k\alpha)$ and

$$T^* = \begin{bmatrix} \lambda_1(T-I) & 0 & \cdots & \cdots \\ 0 & \lambda_2(T-I) & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_k(T-I) \end{bmatrix}$$

Also the mean of the distribution H(t) is

$$\mu'_H = \sum_{i=1}^{k}\left(\frac{r_i}{\lambda_i}\right)\mu'_p$$

where $\mu'_p = \alpha(I-T)^{-1}e$ is the mean of $\{p_k\}$.

In particular, if the shocks arrive according to a homogeneous Poisson process with parameter λ and the shock probabilities $\{p_k\}$ has DPH distribution with mean μ'_p, then H(t) is of CPH with mean $\mu'_H = \left(\frac{1}{\lambda}\right)\mu'_p$.

Now we prove the following preservation result.

Theorem 7.3. The survival function $\bar{H}(t)$ in (7.2.1) has ILR property if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ has the discrete ILR property.

Proof:

Differentiating (7.2.2), we have

$$h'(t) = \lambda^2 \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\left(p_{k+2} - p_{k+1}\right).$$

Let $$g(t) = \frac{h'(t)}{h(t)} = \lambda \frac{\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\left(p_{k+2} - p_{k+1}\right)}{\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} p_{k+1}}$$

For $t_1 < t_2$, let the determinant

$$D = \begin{vmatrix} \sum_{k=0}^{\infty} \frac{(\lambda t_2)^k}{k!}(p_{k+2} - p_{k+1}) & \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k}{k!}(p_{k+2} - p_{k+1}) \\ \sum_{k=0}^{\infty} \frac{(\lambda t_2)^k}{k!} p_{k+1} & \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k}{k!} p_{k+1} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{k=0}^{\infty} \frac{(\lambda t_2)^k}{k!} p_{k+2} & \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k}{k!} p_{k+2} \\ \sum_{k=0}^{\infty} \frac{(\lambda t_2)^k}{k!} p_{k+1} & \sum_{k=0}^{\infty} \frac{(\lambda t_1)^k}{k!} p_{k+1} \end{vmatrix}$$

$$= \sum_k \sum_l \frac{(\lambda t_2)^k}{k!} \frac{(\lambda t_1)^l}{l!} p_{k+2} p_{l+1} - \sum_k \sum_l \frac{(\lambda t_1)^k}{k!} \frac{(\lambda t_2)^l}{l!} p_{k+2} p_{l+1}$$

$$= \sum_k \sum_l \frac{(\lambda t_2)^k}{k!} \frac{(\lambda t_1)^l}{l!} [p_{k+2} p_{l+1} - p_{l+2} p_{k+1}]$$

$$= \left(\sum_{k<l} + \sum_{k>l} \right) \frac{(\lambda t_2)^k}{k!} \frac{(\lambda t_1)^l}{l!} [p_{k+2} p_{l+1} - p_{l+2} p_{k+1}]$$

$$= \sum_{k<l} \left[\frac{(\lambda t_2)^k}{k!} \frac{(\lambda t_1)^l}{l!} - \frac{(\lambda t_2)^l}{l!} \frac{(\lambda t_1)^k}{k!} \right] [p_{k+2} p_{l+1} - p_{l+2} p_{k+1}]$$

$$= \sum_{k<l} \begin{vmatrix} \frac{(\lambda t_2)^k}{k!} & \frac{(\lambda t_2)^l}{l!} \\ \frac{(\lambda t_1)^k}{k!} & \frac{(\lambda t_1)^l}{l!} \end{vmatrix} \begin{vmatrix} p_{k+2} & p_{l+2} \\ p_{k+1} & p_{l+1} \end{vmatrix}.$$

Now for $k < l$ and $t_1 < t_2$, the first determinant is ≤ 0 and since $\{p_k\}$ is discrete ILR, the second determinant is ≥ 0. Hence $D \leq 0$.

$D \leq 0 \Rightarrow g(t_2) - g(t_1) \leq 0$

$\Rightarrow$ g(t) is decreasing in t

$\Rightarrow$ log h(t) is concave

$\Rightarrow$ h(t) is ILR . □

In the DLR case we get the following theorem

Theorem 7.4. The survival function $\bar{H}(t)$ in (7.2.1) is DLR if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ has the discrete DLR property.

7.3 A Non - Homogeneous Poisson Shock Model

If the shocks occur according to a non-homogeneous Poisson process with mean value function $\Lambda(t)$ and event rate $\lambda(t) = \Lambda'(t)$ both defined on the domain $[0, \infty)$, with $\lambda(0)$ as the right hand derivative of $\Lambda(t)$ at $t = 0$ and the device has the probability $\bar{P}_k$ of surviving first k shocks, then its survival function $\bar{H}(t)$ is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} \frac{e^{-\Lambda(t)}\Lambda^k(t)}{k!} \bar{P}_k \ ; \quad t \geq 0 \tag{7.3.1}$$

and density function is

$$h(t) = \lambda(t)\sum_{k=0}^{\infty} \frac{e^{-\Lambda(t)}\Lambda^k(t)}{k!} p_{k+1}, \quad t > 0 \tag{7.3.2}$$

Under the suitable assumptions on $\Lambda(t)$, various geometric probabilities of $\left(\bar{P}_k\right)_{k=0}^{\infty}$ have counter parts as corresponding properties of $\bar{H}(t)$. Similar relationships hold among $\lambda(t)$, p_k, and $h(t)$.

Theorem 7.5. (i) If p_k is decreasing then $\frac{h(t)}{\lambda(t)}$ is decreasing. If in addition $\lambda(t)$ is decreasing then h(t) is decreasing.

(ii) If p_k is decreasing and log concave, and $\lambda(t)$ is increasing then $\frac{h(t)}{\lambda(t)}$ is log concave. If in addition, $\lambda(t)$ is log concave then h(t) is log concave.

(iii) If $\bar{P}_k$ is discrete IFR and $\Lambda(t)$ is convex (that is $\lambda(t)$ is monotonic increasing in t), then $\bar{H}(t)$ is IFR.

(iv) If $\bar{P}_k$ is discrete IFRA and $\Lambda(t)$ is star-shaped, then $\bar{H}(t)$ is IFRA.

(v) If $\bar{P}_k$ is discrete NBU and $\Lambda(t)$ is super-additive, then $\bar{H}(t)$ is NBU.

(vi) If $\bar{P}_k$ is discrete NBUE and $\Lambda(t)$ is star-shaped, then $\bar{H}(t)$ is NBUE.

(vii) If $\bar{P}_k$ is discrete DMRL and $\Lambda(t)$ is convex, then $\bar{H}(t)$ is DMRL.

Note: The same corresponding results hold for the dual classes also.

So we have

Theorem 7.6. If $\left(\bar{P}_k\right)_{k=0}^{\infty}$ has discrete ILR property, p_k is decreasing in k and $\lambda(t)$ is increasing and log concave, then $\bar{H}(t)$ in (7.3.1) is ILR.

Proof:-

Put $$g(t)=\sum_{k=0}^{\infty}\frac{e^{-\Lambda(t)}\Lambda^{k}(t)}{k!}p_{k+1}$$

so that $h(t) = \lambda(t)\, g(t)$. (7.3.3)

First we prove that g(t) is log concave.

Let $$g^{*}(t)=\frac{g'(t)}{g(t)}=\lambda(t)\frac{\sum_{k=0}^{\infty}\frac{\Lambda^{k}(t)}{k!}(p_{k+2}-p_{k+1})}{\sum\frac{\Lambda^{k}(t)}{k!}p_{k+1}}.$$

For $t_1 < t_2$, Consider the determinant

$$D=\begin{vmatrix}\lambda(t_2)\sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_2)}{k!}(p_{k+2}-p_{k+1}) & \lambda(t_1)\sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_1)}{k!}(p_{k+2}-p_{k+1})\\ \sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_2)}{k!}p_{k+1} & \sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_1)}{k!}p_{k+1}\end{vmatrix}$$

$$\le\lambda(t_2)\begin{vmatrix}\sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_2)}{k!}p_{k+2} & \sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_2)}{k!}p_{k+2}\\ \sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_2)}{k!}p_{k+1} & \sum_{k=0}^{\infty}\frac{\Lambda^{k}(t_1)}{k!}p_{k+1}\end{vmatrix}.$$

Since $\lambda(t)$ is increasing and p_k is decreasing. By the same way as done in the proof of theorem (7.2.1) we have

$$D\le\lambda(t_2)\sum_{k<l}\begin{vmatrix}\frac{\Lambda^{k}(t_2)}{k!} & \frac{\Lambda^{l}(t_2)}{l!}\\ \frac{\Lambda^{k}(t_1)}{k!} & \frac{\Lambda^{l}(t_1)}{l!}\end{vmatrix}\begin{vmatrix}p_{k+2} & p_{l+2}\\ p_{k+1} & p_{l+1}\end{vmatrix}.$$

Now since $\Lambda(t_1) \leq \Lambda(t_2)$ and $k < l$, the first determinant is negative and since $\{p_k\}$ is discrete ILR, the second determinant is positive. Hence $D \leq 0$, which implies that $g^*(t)$ is decreasing in t and therefore that log g(t) is concave.

From (6.3.3), since $\lambda(t)$ is concave, we conclude that h(t) is ILR. □

In a similar way we get a dual theorem in the DLR case.

Theorem 7.7. If $\left(\bar{P}_k\right)_{k=0}^{\infty}$ has discrete DLR property, p_k is increasing in k, $\lambda(t)$ is increasing in t and log concave, then $\bar{H}(t)$ in (7.3.1) is DLR.

7.4 Mixed Poisson Shock Model and Closure of ILR Property

If **N** is a mixed Poisson process with mixing distribution given by the random variable λ having distribution G, then the survival function of the device $\bar{H}(t)$ is given by

$$\bar{H}(t) = E\left\{\sum_{k=0}^{\infty} \frac{e^{-\Lambda(t)}\Lambda^k(t)}{k!}\bar{P}_k\right\}$$

where $\Lambda(t) = \lambda A(t)$, λ is a random variable following a distribution G and A(t) is a deterministic differentiable function of t with $A'(t) = a(t)$.

We have

$$\bar{H}(t) = \int_0^{\infty}\sum_{k=0}^{\infty} \frac{e^{-\lambda A(t)}[\lambda A(t)]^k}{k!}\bar{P}_k dG(\lambda). \tag{7.4.1}$$

The density function h(t) is given by

$$h(t) = a(t)\int_0^\infty \lambda e^{-\lambda A(t)} \sum_{k=0}^\infty \frac{[\lambda A(t)]^k}{k!} p_{k+1} dG(\lambda) \quad , \quad t > 0 \cdot \tag{7.4.2}$$

Now we prove

Theorem 7.8. The survival function $\bar{H}(t)$ in (7.4.1) is ILR if $\left(\bar{P}_k\right)_{k=0}^\infty$ is discrete ILR, $\{p_k\}$ is decreasing and a(t) is an increasing log concave function.

Proof:-

h(t) = a(t)g(t) where g(t) = $a(t)\int_0^\infty \lambda e^{-\lambda A(t)} \sum_{k=0}^\infty \frac{[\lambda A(t)]^k}{k!} p_{k+1} dG(\lambda) \cdot$

$$\text{Let } g^*(t) = \frac{g'(t)}{g(t)} = \frac{a(t)\int_0^\infty \lambda^2 e^{-\lambda A(t)} \sum_{k=0}^\infty \frac{[\lambda A(t)]^k}{k!} (p_{k+2} - p_{k+1})\, dG(\lambda)}{\int_0^\infty \lambda e^{-\lambda A(t)} \sum_0^\infty \frac{(\lambda A(t))^k}{k!} p_{k+1}\, dG(\lambda)} \cdot$$

For $t_1 < t_2$, let the determinant

$$D = \begin{vmatrix} a(t_2) D_{11} & a(t_1) D_{12} \\ D_{21} & D_{22} \end{vmatrix}$$

$$\text{where} D_{11} = \int_0^\infty \lambda^2 e^{-\lambda A(t_2)} \sum_{k=0}^\infty \frac{[\lambda A(t_2)]^k}{k!} (p_{k+2} - p_{k+1})\, dG(\lambda)$$

$$D_{12} = \int_0^\infty \lambda^2 e^{-\lambda A(t_1)} \sum_{k=0}^\infty \frac{[\lambda A(t_1)]^k}{k!} (p_{k+2} - p_{k+1})\, dG(\lambda)$$

$$D_{21} = \int_0^\infty \lambda e^{-\lambda A(t_2)} \sum_{k=0}^\infty \frac{[\lambda A(t_2)]^k}{k!} p_{k+1}\, dG(\lambda)$$

$$D_{22} = \int_0^\infty \lambda e^{-\lambda A(t_1)} \sum_{k=0}^\infty \frac{[\lambda A(t_1)]^k}{k!} p_{k+1}\, dG(\lambda) \cdot$$

Since a(t) is increasing and p_k is decreasing,

$D \le a(t_2)[D_{11}D_{22} - D_{12}D_{21}]$.

Now, using the basic composition formula (refer Karlin (1968)) and proceeding in a similar manner as in the proof of theorem (7.2.1) it can be seen that

$$M = \iint_{\lambda_1<\lambda_2} \begin{vmatrix} \lambda_1^2 e^{-\lambda_1 A(t_2)} & \lambda_1^2 e^{-\lambda_1 A(t_1)} \\ \lambda_2 e^{-\lambda_2 A(t_2)} & \lambda_2 e^{-\lambda_2 A(t_1)} \end{vmatrix} \times \sum\sum_{k<l} \begin{vmatrix} \frac{(\lambda_1 A(t_2))^k}{k!} & \frac{(\lambda_1 A(t_1))^k}{k!} \\ \frac{(\lambda_1 A(t_2))^l}{l!} & \frac{(\lambda_1 A(t_1))^l}{l!} \end{vmatrix} \begin{vmatrix} p_{k+2} & p_{l+2} \\ p_{k+1} & p_{l+1} \end{vmatrix} dG(\lambda_1)dG(\lambda_2)$$

where $M = D_{11}D_{22} - D_{12}D_{21}$.

Using the facts that A(t) is increasing in t and $\{p_k\}$ is discrete ILR, it can be verified that the first and third determinants are positive and the second one is negative. Hence $D \le 0$.

Therefore $g^*(t)$ is decreasing in t.

That is log g(t) is concave.

That is h(t) is log concave, since a(t) is log concave. □

Corollary 7.1. The survival function $\bar{H}(t)$ in a homogeneous mixed Poisson model is ILR if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete ILR.

The corresponding dual theorem under mixed Poisson model is the following.

Theorem 7.9. The survival function $\bar{H}(t)$ in (6.4.1) is DLR if $\left(\bar{P}_k\right)_{k=0}^{\infty}$ is discrete DLR, $\{p_k\}$ is increasing and a(t) is decreasing log convex function.

CONCLUDING REMARKS

In this study we have investigated certain stochastic models in queues, inventory and reliability and augmented some interesting results concerning their stationary and time dependent aspects. The unimodality properties of stationary population size distribution of the birth and death process and the characterization result presented in chapter 2 have great importance in queueing networks. This suggests further motivation for studying the stationary characteristics of similar queueing models under general distributional assumption.

In many real life situations we can see that there is a control limit for shutting down and resuming service and we have developed in chapter-3, a simple procedure to obtain the stationary distribution of the appropriate queueing model in such a situation. By introducing a secondary limit to the single and batch service M/M/1 queueing system we can escape from perpetual change over from batch service to single service. We develop such a model and analyzed it in fourth chapter with numerical illustration. Deviating from Markovian assumption to non-Markovian one may develop algorithmic solution using regenerative or semi-regenerative process to compute the steady state and transient state probabilities. In a similar manner if one can relax the exponential assumption and perform the analysis of intermittently used n-unit system, it will be a quite worthwhile work. Since we derive the stationary distribution of (S – 1, S) production/inventory system under a versatile class of PH- distributions, which covers most of the commonly used distribution like exponential, gamma, hyper exponential etc., the results can be considered to be more robust.

There are many interesting problems concerning various stochastic models considered in this thesis. Practical experience over the past has suggested a much more flexible family of models. Current attempts to extend the models have been restricted to models, which have closed form properties. The power of Markov chain Monte Carlo (MCMC) methods may be used to explore new models which do not posses the simple structure necessary to give closed form properties but instead provide a much broader class of models and inference flexibility.

REFERENCES

1. Abouammoh, A. M.; Hendi, M. I. and Ahmed, A. N. (1988) Shock models with NBUFR and NBAFR survivals, *Trabajos De Estadistica*, **3**, 97-113.

2. Abouammoh, A. M.; Hendi, M. I. and Khalique, A. (1995) Shock models with NBRU properties, *Pakistan J. Statist.*, **11**, 35-47.

3. A-Hameed, M. S. and Proschan, F. (1973) Non stationary shock models, *Stoch. Processes Appl.*, **1**, 383-404.

4. Aggoun, L.; Benkherouf, L and Tadj, L. (1997) A hidden Markov model for an inventory system with perishable items, *J. Appl. Math. Stoch. Anal.* **10**, 423-430.

5. Aggoun, L.; Benkherouf, L and Tadj, L. (1999) A stochastic inventory model with perishable and ageing items, *J. Appl. Math. Stoch. Anal.* **12**, 23-29.

6. Altiok, T. and Shiue, G. A. (1995) Single stage, multi-product production/ inventory systems with lost sales, *Naval Res. Logistics*, **42**, 889-914.

7. Arreola-Risa, A. (1996) Integrated multi-item production-inventory systems, *Europ. J. Operat. Res.*, **89**, 326-340.

8. Arrow, K. J.; Karlin, S. and Scarf, H. (1958) *Studies in the Mathematical Theory of Inventory and Production*, Stanford University Press, Stanford.

9. Ashok Kumar and Manju Agarwal, (1980) A review of standby redundant systems, *IEEE Trans. Rel.*, **29**, 290-294.

10. Assmussen, S. (1987) *Applied Probability and Queues*, John Wiley, Chichester.

11. Asmussen, S.; Nerman, O. and Olsson, M. (1996) Fitting of phase-type distribution via the EM algorithm, *Scand. J. Statist.*, **23**, 419 - 441.

12. Atkinson, J.B. (1995) The transient M/G/1/0 queue: some bounds and approximations for light traffic with application to reliability, *J. Appl. Math. Stoch. Anal.* **8**, 347-359.

13. Aviv, Y. and Federgruen, A. (1997) Stochastic Inventory models with limited production capacity and periodically varying parameters, *Prob.Engin. Inf. Sci.*, **11**, 107-135.

14. Aven, T. and Jensen, U. (1999) *Stochastic Models in Reliability*, Springer Verlag, New York.

15. Babu Raj, C. (2000) On the transient distribution of a single and batch service queueing system with accessibility to the batches, *International Journal of Information and Management Sciences*, **11**, 27-36.

16. Baburaj, C. and Manoharan, M. (1997) A bulk service queuing system with removable servers, *International Journal of Information and Management Sciences*, **8**, 29-40.

17. Babu Raj, C. and Manoharan M. (1997a) An M/M/1 queue with single and batch service, *J. Indian Statist. Ass.*, **35**, 39-44.

18. Bailey, N. T. J. (1954) A continuous time treatment of a simple queue using generating function, *J. R. Statist. Soc., B*, **16**, 288-291.

19. Bar – Lev, S. K.; Parlar, M. and Perry, D. (1996) Analysis of a two-sided production policy with inventory-level-dependent production rates, *Appl. Soch. Models Data Anal.*, **12**, 221-237.

20. Barlow, R. E and Proschan, F. (1965) *Mathematical Theory of Reliability*, John Wiley, New York.

21. Barlow, R. E. and Proschan, F. (1975) *Statistical Theory of Reliability and Life testing,* Holt, Rinehart and Winston Inc, New York.

22. Barlow, R. E and Wu, A. S. (1978) Coherent systems with multi state components, *Math. Operat. Res.*, **4**, 275-281.

23. Bell, C.E. (1975) Turning off a server with customers present: is this any way to run a M/M/c queue with removable servers, *Operations Research*, **23**, 571-573.

24. Bellman, R. (1974) *Introduction to Matrix Analysis, 2nd Edition*, Tata McGraw-Hill Publishing Company Ltd., New Delhi.

25. Bellman, R. E and Harris, T. E. (1948) On the theory of age dependent stochastic processes, *Proc. Nat. Acad. Sc.*, *USA*, **34**, 601-604.

26. Berg, M.; Posner, M. J. M., and Zhao, H. (1994) Production/inventory system with unreliable machines, *Operations Research*, **42,** 111-118.

27. Bhat, B. R. (2000) *Stochastic Models - Analysis and Applications*, New Age International Limited, New Delhi.

28. Bhat, U. N. (1968) *A study of the queueing systems M/G/1 and GI/M/1*, Lecture Notes in Operations research and Mathematical Economics, 2, Springer-Verlag, Berline.

29. Birnbaum, Z. W.; Esary, J. D. and Saunders, S. C. (1961) Multi component systems and structures and their reliability, *Technometrics*, **3**, 55-77.

30. Bobbio, A. and Telek, M. (1994) A benchmark for PH estimation Algorithms: Results for Acyclic - PH, *Communications Statistics - Stochastic Models,* **1013,** 661-677.

31. Bogoyavlenskaya, O. I., (1997) The steady state distribution of the queue length for a queue with a bulk arrival and processor sharing discipline, *Probab. Methods in Discrete Math.*, 131-156.

32. Block, H.W. and Savits, T.H. (1978) Shock models with NBUE survival, *J. Appl. Prob.*, **15**, 621 - 628.

33. Brandt, A., Franken, P. and Lisek, B. (1990) *Stationary Stochastic Models*, John Wiley & Sons, New York.

34. Bryson, M. C. and Siddiqui, M. M. (1969) Some criteria for aging, *J. Amer. Statist. Ass.*, **64**, 1472-1483.

35. Chakravarthy, S. R. and Alfa, A. S. (1997) *Matrix-Analytic Methods in Stochastic Models*, Marcel Dekker, Inc., New York.

36. Chaudhry, M. L. and Templeton, J. G. C. (1984) *A First Course in Bulk Queues*, John Wiley and Sons, New York.

37. Chaudhry, M. L. and Gupta, U. C. (1999) Modeling and analysis of $M/G^{a,b}/1/N$ queue – a simple alternative approach, *Queueing Systems Theory Appl.*, **31**, 95-100.

38. Cheng, Ki Ling (1996) On the (S – 1, S) inventory model under compound Pisson demands and i. i. d. unit supply times, *Naval Res. Logistics*, **43**, 563-572.

39. Cinlar, E. (1975) *Introduction to Stochastic Processes*, Prentice Hall, Engle Wood Cliffs, N. J.

40. Cohen, J.W. (1969) *The Single Server Queues,* North Holland, Amsterdam.

41. Cohen, J.W. (1971) On the busy period for the M/G/1 queue with a finite and infinite waiting room, *J. Appl. Prob*, **8**, 821-827.

42. Daley, D. J. (1972) A bivariate Poisson queuing process that is not infinitely divisible, *Proc. Camp. Phil. Soc.* **72**, 449-450.

43. Daley, D. J. (1976) Queueing output processes, *Adv. App. Prob*., **8**, 395-415.

44. Dikong, E. E. and Dshalalow, J. H. (1999) Bulk input queues with hysteretic control, *Queueing Systems Theory Appl.*, **32**, 287-304.

45. Doshy, B. T. (1986) Queueing systems with vacation – a survey, *Queueing Systems*, **1**, 29-66.

46. Doshy, B. T. (1990) Generalizations of the stochastic decomposition results for single server queues with vacation, *Stoch. Models*, **6**, 307-333.

47. Dshalalow, J. H. (1995) *Advances in Queueing: Theory, Methods and Open Problems*, CRC Press, Inc.

48. Dshalalow, J. H. (1997) *Frontiers in Queueing: Models and Applications in Science and Engineering*, CRC Press, Inc.

49. Dshalalow, J. H. and Dikong, E. E. (1999) On generalized hysteretic control queues with modulated input and state dependent service, *Stoch. Anal. Appl.*, **17**, 937-961.

50. Ebrahimi, N. (1999) Stochastic properties of a cumulative damage threshold crossing model, *J. Appl. Prob.*, **36**, 720-732.

51. Epstein, B. and Sobel, M. (1953) Life testing, *J. Amer. Statist. Ass.*, **48**, 486- 502.

52. Esary, J. D.; Marshall,A. W. and Proschan, F.(1973) Shock models and wear processess, *Ann. Prob.*, **1**, 627 - 649.

53. Federgruen, A. and Green, L. (1986) Queueing systems with interruptions, *Operations Research*, **34**, 752-768.

54. Feinberg, E. A. and Kim, Dong J. (1996) Bicriterion optimization of an M/G/1 queue with a removable server, *Prob. Engin. Inf. Sci.*, **10**, 57-73.

55. Feller, W. (1941) On the integral equation of renewal theory, *Ann. Math. Statist.* **12**, 243-267.

56. Frey, A. and Takahashi, Y. (1997) A note on an M/G/1/N queue with vacation time and exhaustive service discipline, *Oper. Res. Lett.*, **21**, 95-100.

57. Frey, A. and Takahashi, Y. (1998) An explicit solution for an M/GI/1/N queue with vacation time and exhaustive service discipline, *J. Operat. Res. Soc. Japan*, **41**, 430-441.

58. Frey, A. and Takahashi, Y. (1999) An M^X/ GI/1/N with close down and vacation times, *J. Appl. Math. Stoch. Anal.*, **12**, 63-83.

59. Gani, J. (1957) Problems in the probability theory of storage systems, *J. Roy. Statist. Soc., B,* **19**, 181-206.

60. Gaver, D. P. Jr., (1963) Time to failure and availability of parallel systems with repair, *IEEE Trans. on Reliab.*, **12**, 30-38

61. Gasemyr, J. and Natwig, B. (1995) Some aspects of reliability analysis in shock models, *Scand. J. Statist.*, **22**, 385-393.

62. Gasemyr, J. and Natwig, B. (1998) The posterior distribution of the parameters of component life times based on autopsy data in a shock model, *Scand. J. Statist.*, **25**, 271-292.

63. Ghare, P. M. and Schrader, G. F. (1963) A model for an exponentially decaying inventory, *Journal of Industrial Engineering*, **14**, 238-243.

64. Glasserman, P. and Tayur, S. (1996) A simple approximation for a multi-stage capacitated production-inventory system, *Naval Res. Logistics*, **43**, 41-58.

65. Gnedenko, B. V. and Kovalenko, I. N. (1968) *Introduction to Queueing Theory* Israel Program for Scientific Translations, Jerusalem.

66. Gnedenko, B.V.and Konig, D. (1983) *Handbook of Queueing Theory: Foundations and Methods, Vol.1*, Akadamic – Verlag, Berlin.

67. Gopalan, M. N. (1999) Busy period analysis of a 1-server 2-unit system with server subject to maintenance, *Bull. Pure and Appl. Sci.*, **18E**, 113-117.

68. Graves, S. C. (1985) A multi-echelon inventory model for a repairable item with one-for-one replenishment, *Mgmt. Sci.*, **31**, 1247-1256.

69. Gross, D. and Harris, C.M. (1998) *Fundamentals of Queueing theory*, John Wiley and Sons, Inc. New York.

70. Gullu, R. (1996) On the value of information in dynamic production/inventory problems under forecast evolution, *Naval Res. Logistics*, **43**, 289-303.

71. Gullu, R. (1998) Base stock policies for production/inventory problems with uncertain capacity levels, *Europ. J. Operat. Res.*, **105**, 43-51.

72. Guttierrez, R. and Valderrama, M.J., (1994) *Selected Topics on Stochastic Modeling*, World Scientific, Singapore.

73. Ha, A. Y. (1997) Stock-rationing policy for a make-to-stock production system with two priority classes and backordering, *Naval Res. Logistics*, **44**, 457-472.

74. Hausman, W. H. and Scudder, G. D. (1982) Priority scheduling rules for repairable inventory systems, *Mgmt. Sci.*, **28**, 1215 - 1232.

75. Higa, I., Feyerherm, A. M. and Machado, A. L. (1975) Waiting time in an (S - 1, S) production/inventory system, *Operat. Res.*, **6**, 693 - 703.

76. Ibragimov, I.A. (1956) On the composition of unimodal distributions, *Theory Prob. App.* **1**, 255 –260.

77. Inderfurth, K. (1997) Simple optimal replenishment and disposal policies for a product recovery system with leadtimes, *OR Spektrum*, **19**, 111-122.

78. Jacob, M.; Mathew, M. and Joseph, M. (1997) The steady state behavior of a single server queue with a new bulk service rule, *J. Decision and Math. Sci.*, **2**, 29-36.

79. Janssen, J.; Skiadas, C. H. and Zopounidis, C. (1995) *Advances in Stochastic Modelling and Data Analysis*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

80. Joseph, K. X. and Manoharan, M. (1997) Analysis of a multi state repairable system, *International Journal of Information and Management Sciences*, **8**, 25-30.

81. Kalashnikov, V. V. (1994) *Mathematical Methods in Queueing Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

82. Kalpakam, S. and Arivarignan, G. (1989) On exhibiting inventory system with Erlangian lifetimes under renewal demand, *Ann. Inst. Stat. Math.*, **41**, 601-616.

83. Kalpakam, S. and Sapna, K. P. (1996) A lost sales (S − 1, S) perishable inventory system with renewal demand, *Naval Res. Logist*ics, **43**, 129-142.

84. Kapoor, K. R. and Kapoor, P. K. (1980) First uptime and disappointment time joint distribution of an intermittently used system, *Micrelectron. Reliab.* **20**, 891-893.

85. Karaesmen, F. and Gupta, S. M. (1997) Control of arrivals in a finite buffered queue with setup costs, *J. Operat. Res. Soc.*, **48**, 1113-1122.

86. Kasahara, S.; Takagi, H.; Takahashi, Y. and Hasegawa, T. (1996) M/G/1/K system with push-out scheme under vacation policy, *J. Appl. Math. Stoch. Anal.*, **39**, 118-131.

87. Karlin, S. (1968) *Total Positivity, Volume 1*, Stanford University Press, Stanford.

88. Ke, J. C. and Wang, K. H. (1999) Cost analysis of the M/M/R machine repair problem with balking, reneging and server break downs, *J. Operat. Res. Soc.*, **50**, 275-282.

89. Keilson, J. (1971) Log-concavity and log-convexity in passage time densities of diffusion and birth-death processes, *J. App. Prob.*, **8**, 391-398.

90. Kendall, D. G. (1953) Stochastic process occurring in the theory of queues and their analysis by the method of imbedded Markov chain, *Ann. Math. Stat.*, **24**, 338-334.

91. Kijima, M. (1997) *Markov Processes for Stochastic Modeling*, Chapman & Hall, London.

92. Kijima, M. and Takimoto, T. (1999) A (T, S) inventory/production system with limited production capacity and uncertain demands, *Oper. Res. Lett.*, **25**, 67-79.

93. Klefsjo, B. (1981) HNBUE survival under some shock models, *Scand. J. Statist.*, **8**, 39 - 47.

94. Krishnamoorthy, A. and Varghese, T. V. (1995) Inventory with disaster, *Optimization*, **35**, 85-93.

95. Krishna Reddy, G. V., Nadarajan, R. and Venkatasubramanian, N.K. (1994) *Stochastic Models, Optimization Techniques and Computer Applications*, Wiley Eastern Limited, New Delhi, India.

96. Kusaka, Y. and Mori, M. (1995) Inventory control policy for repair parts, *J. Operat. Res. Soc. Japan*, **38**, 269-288.

97. Langaris, C. and Katsaros, A. (1997) An M/G/1 queue with finite population and gated service discipline, *J. Operat. Res. Soc. Japan*, **40**, 133-139.

98. Latouche, G. and Taylor, P. (2000) *Advances in Algorithmic Methods for Stochastic Models*, Notable Publications, Inc., New Jersey, USA.

99. Lee, J. Y. and Lee, E. Y. (1995) A random shock model for a linearly deteriorating system, *J. Korean Statist. Soc.*, **24**, 471-479.

100. Li, Wei; Shi, Dinghua and Chao, Xiuli (1997) Reliability analysis of M/G/1 queueing system with server break downs and vacations, *J. Appl. Prob.*, **34**, 546-555.

101. Lukacs, E. (1970), *Characteristic Functions - 2nd edition*, Griffin, London.

102. Manoharan, M. (1998) On a multi state deteriorating system and its optimal replacement policy, *Presented in the International conference in reliability and survival analysis, NIU, Dekalb, USA, May 21-24, 1998.*

103. Manoharan, M.; Harshinder Singh and Neeraj Misra (1992) Preservation of phase type distributions under Poisson shock model, *Adv. App. Prob.*, **24**, 223-225.

104. Manoharan, M. and Krishnamoorthy, A. (1989) Markov renewal theoretic analysis of a perishable inventory system, *Tamkang Journal of Management Science*, **10**, 47-57.

105. Medhi, J. (1984) *Recent developments in Bulk Queues*, Wiley Eastern Ltd., New Delhi.

106. Medhi, J. (1994) Extensions and generalizations of the classical single server queueing system with Poisson input: a survey, *J. Assam Sci. Soc.*, **36**, 211-226.

107. Mitra, M. and Basu, S. K. (1996) Shock models leading to non-monotonic ageing classes of the life distributions, *J. Statist. Planning Infer.*, **55**, 131-138.

108. Miller, L. W. (1964) *Alternating Priorities in Multi-class Queues*, Ph. D. Dissertation, Cornell University, Ithaca.

109. Muckstadt, J. A. and Thomas, L. J. (1980) Are multi-echelon inventory models worth implementing in systems with low-demand rate items?, *Mgmt. Sci.*, **26**, 483-494.

110. Nahmias, S. (1982) Perishable inventory theory: A review, *Operations Research,* **30**, 680-708.

111. Nakagawa, T.; Sawa, Y. and Suzuki, Y. (1976) Reliability analysis of intermittently used systems when failures are detected only during a usage period, *Microelectron. Reliab.*, **15**, 35-38.

112. Natarajan, P. and Shahul Hameed, M. A. (1988) Reliability analysis of intermittently used k-out of n: F system, *J. Math. Phy. Sci.* **2**, 191-208.

113. Natvig, B. (1975) On the input and output processes for a general birth and death queuing model, *Adv. App. Prob.,* **7**, 576-592.

114. Neuts, M. F. (1967) A general class of bulk queues with Poisson input, *Ann. Math. Statist.*, **38**, 759-770.

115. Neuts, M. F. (1975) Probability distributions of phase type, *Liber Amicorum, Prof. Emaritus, H. Florin, Dept. of Math. Univ. Louvain, Belgium*, 173-206.

116. Neuts, M. F. (1981) *Matrix Geometric Solutions in Stochastic Models*, The Johns Hopkins University Press, Baltimore, London.

117. Osaki, S. and Nakagawa, T. (1976) Bibliography for reliability and availability of stochastic systems, *IEEE Trans. Rel.*, **25**, 284-286.

118. Palm, C. (1943) Intensitatats schwankungen in fermsprechverkehr, *Ericsson Technics*, **44**, 1-189.

119. Park, W. J.; Lee, E. Y. and Kim, H. (1996) A model for a continuous state system with (s, S) repair policy, J. Korean Statist. Soc., **25**, 111-122.

120. Parthasarathy, P. R. and Vijayalakshmi, V. (1996) Transient analysis of an (S –1, S) inventory model – a numerical approach, *Internat. J. Comput. Math.*, **59**, 177-185.

121. Pekoz, E. A. (1999) On the number of refusals in a busy period, *Prob. Engin. Inf. Sci.*, **13**, 71-74.

122. Perez-Ocon, R. and Gamiz-Perez, M. L. (1995) On the HNBUE property in a class of correlated cumulative shock models, *Adv. Appl. Prob.*, **27**, 1186- 1188.

123. Perry, D. and Posner, M. J. M. (2000) A correlated M/G/1 – type queue with randomized server repair and maintenance modes, *Oper. Res. Lett.*, **26**, 137-147.

124. Prabhu, N. U. (1965) *Queues and Inventories: A Study of Their Basic Stochastic Processes*, John Wiley, New York.

125. Rhee, H.K. and Sivazlian (1990) Distribution of the busy period in a controllable M/M/2 queue under the triadic. (0, K, N, M) policy, *J. Appl. Prob.*, **27**, 426-432.

126. Roster, U. (1980) Unimodality of first passage times for one dimensional strong Markov processes, *Ann. Prob.*, **8**, 853-859.

127. Ross, S. M. (1983) *Stochastic Processes*, John Wiley and Sons, New York.

128. Saaty, T. L. (1961) *Elements of Queueing Theory with Applications*, Mc Graw Hill, New York.

129. Scarf, H. E.; Gilford, D. M. and Shelly, M. W. (1963) *Multistage Inventory Models and Techniques*, Stanford University Press, Stanford, California.

130. Schultz, C. R. (1990) On the Optimality of the (S - 1, S) Policy, *Naval Res. Logist. Quart.*, **37**, 715 - 723.

131. Shanbhag, D.N. (1973) Characterization for the queuing system M/G/∞, *Proc. Camb. Phil. Soc.*, **74**, 141-143.

132. Shanker, K. (1981) Exact analysis of a two echelon inventory system for recoverable items under batch inspection policy, *Naval Res. Logistic Quart.*, **28**, 579-601.

133. Sharafali, M., Natarajan, R. and Parthasarathy, P. R. (1988) A complex two unit system with stochastic demand, *Microelectron. Reliab.* **28**, 359-361.

134. Sharma, G. C. and Chauhan, M. S. (1996) Preventive maintenance of an $M^X/G/1$ queue-like production system, *Monte Carlo Method Appl.*, **2**, 129-137.

135. Sharma, Y. V. S. and Natarajan, R. (1982) An intermittently used n- unit system, *Microelectron. Reliab.*, **22**, 441-444.

136. Sherbrooke, C. C. (1986) VARI_METRIC: Improved approximations for multi indenture, multi-echelon availability models, *Operations Research*, **34**, 311-319.

137. Sherif, Y. and Smith, M. (1981) Optimal maintenance models for systems subject to failure: A review, *Naval Res. Logist. Quart.*, **28**, 47-74.

138. Sheu, Shey-Huei (1997) Extended block replacement policy of a system subject to shocks, *IEEE Trans. Rel.*, **46**, 375-382.

139. Sheu, Shey-Huei (1998) A generalized age and block replacement of a system subject to shocks, *Europ. J. Operat. Res.*, **108,** *345-362.*

140. Sheu, Shey-Huei (1999) Extended optimal replacement model for deteriorating systems, *Aligarh J. Statist. (1997-98)*, **17/18**, 35-41.

141. Sheu, Shey-Huei and Griffith, W. S. (1996) Optimal number of minimal repairs before replacement of a system subject to shocks, *Nav. Res. Logistics*, **43**, 319-333.

142. Sing, S. K.; Mishra, A. K.; Bhave, R. and Anand, I. (1998) On the effect of shocks on a protective system having two main and one protective unit, *Assam Statist. Rev.*, **12**, 73-84.

143. Skoulakis, G. (2000) A general shock model for a reliability system, *J. Appl. Prob.*, **37**, 925-935.

144. Smith, M. A. J. and Dekker, R. (1997) On the (S – 1, S) stock model for renewal demand processes: Poisson's Poisson, *Prob. Engin. Inf. Sci.*, **11**, 375-386.

145. Smith, S. A. (1977) Optimal inventories for an (S - 1, S) system with back orders, *Mgmt. Sci.*, **23**, 522 - 528.

146. Smith, W. L. (1954) Asymptotic renewal theorems, *Proc. Roy. Soc. Edin.*, **64**, 9-48.

147. Sreehari, M. (2000) A reliability model in a dynamic environment, *Statistical Methods*, **2**, 114-122.

148. Srinivasan, S. K. (1974) *Stochastic Point Processes and its Applications*, Griffin London.

149. Srinivasan, S. K. and Bhasker, D. (1979) Analysis of Intermittently used two dissimilar unit system with single repair facility, *Microelectron. Reliab.*, **19**, 247-252.

150. Srinivasan, V. S. (1966) The effect of standby redundancy in systems failure with repair maintenance, *Oper. Res.*, **14**, 1024-1036.

151. Srivastava, S. S.; Garg, R. C. and Govil. A. K. (1971) Stochastic behavior of an intermittently working system with standby redundancy, *Microelectron. Reliab.* **10**, 375-380.

152. Stidham, S. Jr. (1972) Regenerative process in the theory of queues with the application of alternative priority queue, *Adv. Appl. Prob.*, **4**, 542-577.

153. Stidham, S. Jr. (1974) Stochastic clearing systems, *Stoch. Processes Appl.*, **2**, 85-113.

154. Subramanian, R., Sarma, Y. V. S. and Natarajan, R. (1981) An intermittently used n-unit redundant system, *Proceedings of the Second International Conference on Reliability and Exploitation of Computer systems, Relcomex '81*, 63-67.

155. Takacs, L. (1962) *Introduction to the Theory of queues*, Oxford University Press, New York.

156. Takacs, L. (1967) *Combinatorial Methods in the Theory of Stochastic Processes*, Wiley, New York.

157. Thangaraj, V. and Stanley A. D. J. (1995) Mutual information measure of shock models, *Proc. Third Ramanujam Symp. 1993-94*, 221-234.

158. Thangaraj, V. and Sundararajan, R. (1997) Multivariate optimal replacement policies for a system subject to shocks, *Optimization*, **41**, 173-195.

159. Tijims, H.C. (1994) *Stochastic Models - An Algorithmic Approach*, John Wiley and Sons, New York.

160. Touzi, N. (2000) Optimal insurance demand under marked point processes shocks, *Ann. App. Prob.*, **10**, 283-312.

161. Valdez-Flores, C. and Feldman, R. (1989) A survey of preventive maintenance models for stochastically deteriorating single unit systems, *Naval Res. Logistics*, **36**, 419-446.

162. Van Harn, K. (1978) *Classifying Infinitely Divisible Distributions by Functional Equations*, Mathematisch Centrum, Amsterdam.

163. Van – Houtum, G. J. and Zijm, W. H. M. (1997) Incomplete convolutions in production and inventory models, *OR Spektrum*, **19**, 97-107.

164. Veinott, A.F. Jr. (1966) The status of mathematical inventory theory, *Mgmt. Sci.*, **12**, 745- 777.

165. Von Mises (1936) Leloisde probabilite pourles functions statistiques, *Ann. Inst. H. Poincare*, **6**, 185-209.

166. Wang, K. H. (1997) Optimal control of an $M/E_K/1$ queueing system with removable service station subject to breakdowns, *J. Operat. Res. Soc.,* **48**, 936-942.

167. Wang, Kuo-Hsiung and Huang, Hui-Mei (1995) Optimal control of an $M/ E_K/1$ queueing system with a removable service station, *J. Operat. Res. Soc.*, **46**, 1014-1022.

168. Williams, C. L. and Patuwo, B. E. (1999) A perishable inventory model with positive order lead times, *Europ. J. Operat. Res.*, **116**, 352-373.

169. Wolf, R. W. (1989) *Stochastic Modeling and the Theory of Queues*, Prentice-Hall International, Inc., USA.

170. Yadin, M. and Naor, P. (1963) Queuing systems with removable service station, *Operations Research Quarterly,* **14**, 393-405.