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A STUDY ON THE EXISTENCE OF SOLUTIONS OF GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS

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FEBRUARY 2024

CERTIFICATE

I hereby certify that the thesis entitled "A STUDY ON THE EXISTENCE OF SOLUTIONS OF GENERALIZED FRACTIONAL DIFFERENTIAL EQUATIONS" is a bonafide work carried out by **Smt. Shabna. M. S.,** under my guidance for the award of Degree of Ph.D. in Mathematics of M.E.S Mampad College (Autonomous) and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

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ABSTRACT

Fractional calculus is a branch of Mathematics that studies the derivatives and integrals of non-integer orders. Studying generalized fractional differential equations is significant as it allows broader exploration of mathematical models, incorporating various kernels in the $\psi-$ Caputo and $\psi-$ Hilfer fractional differential equations. This versatility leads to the formulation of diverse fractional differential equations involving classical operators, offering a more comprehensive understanding of complex phenomena in diverse fields. We investigated the existence and uniqueness of neutral fractional differential equation, Impulsive fractional neutral functional differential equation and k-system of fractional neutral differential equation involving $\psi-$Caputo fractional operator, the existence of Hybrid fractional differential equations with both initial and boundary conditions involving $\psi-$Hilfer fractional derivative. Also investigated the existence, uniqueness, Ulam Hyers, generalized Ulam Hyers, Ulam Hyers Rassias and generalized Ulam Hyers stabilities for $\psi-$ Caputo neutral functional differential equation and $\psi-$ Hilfer fractional neutral functional differential equations. Examples illustrating the results and graphs are given.

സംഗ്രഹം

പൂർണ്ണസംഖ്യയല്ലാത്ത ഓർഡറുകളുള്ള അവകലജങ്ങളെ (ഡെറിവേറ്റിവുകളെ) കുറിച്ചും, സമാകലങ്ങളെ (ഇന്റഗ്രലുകളെ) കുറിച്ചും പഠിക്കുന്ന ഗണിതശാസ്ത്ര ശാഖയാണ് ഫ്രാക്ഷണൽ കാൽക്കുലസ്. വൈവിധ്യമാർന്ന മേഖലകളിലെ സങ്കീർണ്ണ പ്രതിഭാസങ്ങളെക്കുറിച്ച് കൂടുതൽ സമഗ്രമായ ധാരണ വാഗ്ദാനം ചെയ്യുന്ന ക്ലാസിക്കൽ ഏകദങ്ങൾ ഉൾപ്പെടുന്ന വൈവിധ്യമാർന്ന ഫ്രാക്ഷണൽ അവകലന സമവാക്യങ്ങളുടെ രൂപീകരണത്തിലേക്ക് നയിക്കുന്നതിനാൽ , സാമാന്യവൽക്കരിച്ച ഫ്രാക്ഷണൽ അവകലന സമവാക്യങ്ങളെ കുറിച്ചുള്ള പഠനം പ്രാധാന്യമർഹിക്കുന്നു. പ്സൈ-ക്യാപൃൂട്ടോ, പ്സൈ-ഹിൽഫർ ഫ്രാക്ഷണൽ അവകലന സമവാക്യങ്ങളിൽ വിവിധ കേർണലുകൾ ഉൾപ്പെടുത്തിയിട്ടുള്ളതിനാൽ വ്യത്യസ്ത ക്ലാസിക്കൽ ഏകദങ്ങൾ ഉൾപ്പെടുന്ന വൈവിധ്യമാർന്ന ഫ്രാക്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യങ്ങളുടെ രൂപീകരണത്തിലേക്ക് നയിക്കുന്നു. ഇത് വൈവിധ്യമാർന്ന മേഖലകളിലെ സങ്കീർണ്ണ പ്രതിഭാസങ്ങളെക്കുറിച്ച് കൂടുതൽ സമഗ്രമായ ധാരണ നൽകുകയും ഗണിതശാസ്ത്ര മോഡലുകളുടെ വിശാലമായ പര്യവേക്ഷണം അനുവദിക്കുകയും ചെയ്യുന്നു .

ന്യൂട്രൽ ഫ്രാക്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യം, ഇംപൾസീവ് ഫ്രാക്ഷണൽ ന്യൂട്രൽ ഫംഗ്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യം, പ്സൈ- ക്യാപ്യൂട്ടോ ഫ്രാക്ഷണൽ ഓപ്പറേറ്റർ ഉൾപ്പെടുന്ന ഫ്രാക്ഷണൽ ന്യൂട്രൽ ഡിഫറൻഷ്യൽ സമവാക്യത്തിന്റെ കെ-സിസ്റ്റം, പ്സൈ-ഹിൽഫർ ഫ്രാക്ഷണൽ ഡെറിവേറ്റീവ് ഉൾപ്പെടുന്ന പ്രാരംഭ, അതിർത്തി വ്യവസ്ഥകളുള്ള ഹൈബ്രിഡ് ഫ്രാക്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യങ്ങളുടെ അസ്തിത്വം എന്നിവ ഞങ്ങൾ പഠനവിധേയമാക്കി. പ്സൈ-ക്യാപ്യൂട്ടോ ന്യൂട്രൽ ഫംഗ്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യത്തിനും പ്സൈ-ഹിൽഫർ ഫ്രാക്ഷണൽ ന്യൂട്രൽ ഫംഗ്ഷണൽ ഡിഫറൻഷ്യൽ സമവാക്യങ്ങൾക്കുമുള്ള അസ്തിത്വം, അദ്വിതീയത, ഉലം ഹൈയേഴ്സ്, സാമാന്യവൽക്കരിച്ച ഉലം ഹൈയേഴ്സ്, ഉലം ഹൈയേഴ്സ് റഷ്യാസ്, സാമാന്യവൽക്കരിച്ച ഉലം ഹൈയേഴ്സ് റഷ്യാസ് എന്നിവയും പഠനവിധേയമാക്കി. സിദ്ധാന്തങ്ങളെ വ്യക്തമാക്കുന്ന ഉദാഹരണങ്ങളും ഗ്രാഫുകളും നൽകിയിട്ടുണ്ട്.

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List of symbols

$\psi(\cdot)$	Continuous function from $[a,b],(-\infty \leq a < b \leq \infty)$ to $\mathbb{R}$ which is increasing and $\psi'(t) \neq 0, \forall t \in [a,b]$.
$E_\alpha(\cdot)$	Mittag-Leffler function.
$\Gamma(\cdot)$	The Gamma function.
$D^{\alpha}_{a+}(\cdot)$	The Riemann-Liouville fractional derivative of order α with lower limit a.
${}^{C}D^{\alpha;\psi}_{a+}(\cdot)$	The $\psi-$Caputo fractional derivative of order α with lower limit a.
${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}(\cdot)$	$\psi-$Hilfer fractional derivative of order α and type β with lower limit a.
$I^{\alpha}_{a+}(\cdot)$	The Riemann-Liouville fractional integral of order α with lower limit a.
$I^{\alpha;\psi}_{a+}f(\cdot)$	The fractional integral of a function f of order α with respect to another function ψ with lower limit a.
$f^{[n]}_{\psi}(t)$	$\left[\frac{1}{\psi'(t)}\frac{d}{dt}\right]^n f(t)$.
$C([a,b],\mathbb{R}^n)$	Banach space of continuous functions from $[a,b] \subset \mathbb{R} \to \mathbb{R}^n$.
$C_{1-\gamma;\psi}[a,b]$	The weighted space of continuous functions f on $(a,b]$ such that $\{f:(a,b] \to \mathbb{R}; (\psi(t)-\psi(a))^{1-\gamma}f(t) \in C([a,b])\}$.
$C^{n}_{\gamma;\psi}[a,b]$	The weighted space of continuous functions f on $(a,b]$ such that $\{f:(a,b] \to \mathbb{R}; f(t) \in C^{n-1}[a,b]; f^{(n)}(t) \in C_{\gamma;\psi}[a,b]\}$.

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Introduction

Literature review

Fractional calculus is a branch of mathematics that studies the derivatives and integrals of non-integer orders. The concept of fractional derivative appeared for the first time in a famous correspondence between G.A de L'Hospital and G.W. Leibniz, in 1695 [1]. It has its roots in the work of Leibniz and Euler in the 17^{th} and 18^{th} centuries, but it was not until the late 20^{th} century that it gained widespread recognition as a useful tool for modelling various phenomena in fields such as engineering, physics, and biology [2, 3]. Fractional calculus has several advantages over classical calculus. One of the main advantages is its ability to model phenomena that involve memory or history dependence, such as the viscoelastic behaviour of materials and the spread of diseases [4]. It also has applications in the study of anomalous diffusion processes and the dynamics of complex systems [5]. Using fractional derivatives, Niels Abel gave the first application of fractional calculus in 1823, by solving the Tautochrone problem - the problem of determining the shape of the curve such that the time of descent of an object sliding down that curve under uniform gravity is independent of the object's starting point [6, 7]. Furthermore, between 1832 and 1837, Joseph Liouville published a series of papers in which he presented fractional operators. Riemann independently constructed a theory of fractional operators. Both Liouville and Riemann defined fractional derivatives using an integral approach that is now known as Riemann-Liouville fractional derivative.

Furthermore, Grunwald and Letnikov defined a fractional derivative using the concept of differ-integral. This approach is a generalization of the integer-order derivative definition as the limit of the difference quotient. The Grunwald and Letnikov interpretation is significant since it is algorithmic and thus particularly useful in computations involving fractional derivatives. Michele Caputo presented an alternative formulation of fractional derivative from the perspective of practical applications in 1967. The initial value problems (IVPs) involving Riemann-Liouville (R-L) fractional derivative require initial conditions in terms of fractional derivatives. Since the physical interpretation is unclear, the usage of R-L derivatives in practical applications is limited. In contrast, Caputo derivative needs initial conditions in terms of ordinary derivatives, making them more useful for practical applications. Hence, Caputo derivative has received serious attention in recent past.

For more than two centuries this subject was relevant only in pure Mathematics and L. Euler, J. P. J. Fourier, N. H. Abel, J. Liouville, B. Riemann, J. Hadamard, among others, have studied fractional operators, by presenting new definitions and studying their most important properties [8, 9, 10]. There are several different approaches to fractional calculus, each with its own benefits and limitations. The most commonly used approaches are the Riemann-Liouville and Caputo fractional derivatives, which are defined in terms of the classical derivatives of integer order [11]. However, there are also other approaches, such as the Grünwald-Letnikov and Weyl fractional derivatives, which have their own unique properties [1]. Viscoelasticity, electrical circuits, electro magnetism, sound propagation, fluid mechanics, edge detection, lateral and longitudinal control, cardiac tissue electrode interface, earth system dynamics are some of them [12, 13, 14].

Motivation

Although the concept of fractional calculus originated as a purely mathematical one, it has recently become widely used in a variety of other scientific disciplines, including physics, mechanics, and biotechnology [15, 16]. Various phenomina of relaxation vibrations, viscoelasticity, electrochemistry, diffusion procedures, etc., are successfully described by fractional differential equations [17, 18, 19, 20, 21]. Because of the large number of definitions that exist for fractional derivatives, one simple way to deal with such a variety is to combine those concepts to a single one by considering fractional derivatives of function with respect to another function [11].

Motivated by the paper of Ricardo Almeida [22, 23, 24] and [25], we started studying about fractional differential equation of a function with respect to another function, $\psi-$Caputo fractional differential equations. Ricardo Almeida has many works on the topic fractional derivative of a function with respect to another function, which will be very useful in applied mathematics. One who interested in the study of Fractional calculus may confuse to select operators. Using general definition, from which different operators can be derived will be a solution to some extend. Recently Sousa and Oliveira [26] proposed a new general fractional derivative, which is named as

$\psi-$Hilfer fractional derivative. They derived around 22 types of existing fractional derivatives and integrals from $\psi-$Hilfer operator. Many works have done on the fractional equations involving $\psi-$Hilfer fractional operator [26, 27, 28, 29, 30, 31]. For the last few decades, many researchers are attracted towards the study of fractional calculus motivated by it's wide application both in pure and applied mathematics [1, 5, 6, 8, 9], [12]-[21], [32].

Special types of FDEs

This thesis updates the following special cases of fractional differential equations.

Impulsive fractional Differential equations

The study of impulsive differential equations is relevant to their applicability in modelling processes and phenomena that experience short-term perturbations during their evolution. The perturbations are carried out in a discrete manner, and their duration is insignificant in relation to the whole duration of the processes and occurrences. In recent years, impulsive differential equations have been a popular research topic due to their extensive applicability to issues in mechanics, medicine, biology, ecology, electrical engineering, and other fields of science.

Numerous systems in physics and biology exhibit impulsive dynamical behaviour as a result of rapid jumps at certain instants in the evolution process. Impulsive differential problems are appropriate models for describing process which at certain moments change their state rapidly and which cannot be described using the classical differential problems. Many physical phenomena in evolution processes are modelled as impulsive fractional differential equations and existence results for such equations [33]. Furthermore, the state of many phenomena and processes considered in biology, biotechnologies, optimal control theory, etc. are frequently subject to instantaneous perturbation and sudden changes (impulses) at specific moments of time. The duration of these variations is relatively short and insignificant in comparison to the whole duration of the action under consideration. Such processes and phenomena with short-term external impacts can be modelled using an impulsive differential equation. The impulsive differential equations based on real-world situations can be used to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Many researchers have been established the solvability of impulsive differential equations due to their importance [3, 5, 34, 35, 36, 37, 38, 39].

Neutral differential equation

The study of fractional neutral functional differential equations is a branch of mathematical analysis that combines the concepts of fractional calculus, functional differential equations, and neutral equations. These equations model dynamic systems where the evolution of a quantity depends not only on its current state but also on its past values, incorporating delays and fractional order derivatives. An example of a fractional neutral functional differential equation is given by

$D^{\alpha}x(t) = f(t, x_t, x'(t), x'_t, x''(t), \cdots)$, where D^{α} represents the fractional derivative of order α, $x(t)$ is the state function and f is a given function involving current and past values of x and it's derivatives.

This type of equation finds applications in various fields such as biology, control systems, and population dynamics, where delays and memory effects play a significant role in the system behavior. Many authors have discussed about neutral differential equations, as it have importance in many areas in applied mathematics [25, 34, 42, 43, 44].

Hybrid differential equations

For studying about the dynamical systems described by non-linear differential and integral equations, the perturbation techniques are very useful. The perturbed differential equations are categorized into various types. Quadratic perturbations of nonlinear fractional differential equations, which is an important type of these perturbations (hybrid differential equations) have achieved a great deal of interest and attention of several researchers. Dhange and Lakshmikantham [45, 46] and Dhange and Jadhav [47] have discussed the existence and uniqueness theorems of the solution to the ordinary first order hybrid differential equations with perturbation of first and second kind respectively. Much work has done in this theory and we refer the readers to the articles [40, 48, 49, 50, 51, 52]

Tools and Methods

Mittag - Leffler function [53]

The Mittag - Leffler function is a generalization of the exponential function e^z. It takes two forms;

a one parameter form defined by,

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

and a two parameter form defined by,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

We know that e^z is important in the solution of integer-order differential equations. By extension, the Mittag-Leffler function takes a similar role in fractional calculus. In particular, the second form is very useful in fractional calculus in the various transform methods and also in numerical methods.

Fixed point method is used to prove the existence and uniqueness theorems for different fractional differential systems. The Krasnoselskii's fixed point theorem and the Banach fixed point theorem are used to investigate the existence and uniqueness of solutions of different $\psi-$Caputo fractional differential equations. Also, Dhange's fixed point theorems are used to establish a sufficient condition for the existence of solutions of hybrid fractional differential equations. Monch's fixed point theorem is used to prove the existence of solutions of various types of fractional differential equations. The non-compactness measure is also can be used to analyse the existence of solution. Generalization of Leibnitz rule is applied to find the series solution. Generalized Gronwall's inequality is used for stability analysis. For plotting the graphs, Python programming is used.

Thesis Outline

In chapter I, we provide some basic definitions and lemmas regarding the $\psi-$Caputo and $\psi-$Hilfer fractional operators.

In chapter II, we investigate the existence and uniqueness of solutions for $\psi-$Caputo fractional impulsive neutral functional differential equation. Under certain assumptions, the existence is proved using Krasnoselskii's fixed point theorem and uniqueness is proved using Banach fixed point theorem.

In chapter III, we study the existence and uniqueness of the $k-$dimensional system of $\psi-$ Caputo fractional neutral functional differential equations. Under the given assumptions the existence of the solution is proved by Krasnoselskii's fixed point theorem and uniqueness is proved using Banach fixed point theorem. An example is given to justify the results.

In chapter IV, we investigate the existence, uniqueness and stabilities of the $\psi-$ Caputo fractional neutral differential equation. The existence and uniqueness of the solution is obtained using Krasnoselskii's fixed point theorem and Banach fixed point theorem respectively. We prove the two types of stabilities of the solutions using generalized Gronwall's identity.

Chapter V is devoted to the study of the existence of $\psi-$Hilfer hybrid fractional differential equations of first and second types. The existence results are proved by Dhange's fixed point theorem. Some examples with graphical representations are given.

In chapter VI, the $\psi-$Hilfer hybrid fractional differential equations with boundary conditions is studied. Under certain assumptions, and using Dhange's fixed point theorem, we prove the existence of the solution.

Chapter VII refers to the study of non-linear $\psi-$Hilfer fractional differential equations. To prove the existence and uniqueness, measure of non-compactness and Banach contraction principle are used respectively.

Chapter 1

Preliminaries

1.1 Basic definitions

This chapter will go through the basic definitions and principles of fractional differential and integral operators, specifically the classical operators: Riemann- Liouville fractional operators, Caputo fractional operators and the generalised operators: $\psi-$Caputo fractional operators, $\psi-$Hilfer fractional operators. Fixed point theorems and results which are used to investigate the existence, uniqueness and stability of solutions of different fractional differential equations are also included.

1.1.1 Riemann-Liouville fractional operator

Definition 1.1.1. *[9] Let $\Delta = [a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. For a function f, the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{R}$ with lower limit a and upper limit b is defined as following.*

Left-sided fractional integral is given by:

$$(I_{a+}^{\alpha} f)(\eta) = \frac{1}{\Gamma(\alpha)} \int_{a}^{\eta} \frac{f(s)}{(\eta - s)^{1-\alpha}} ds, \quad \eta > a, \ \alpha > 0.$$

and right-sided fractional integral is given by:

$$(I_{b-}^{\alpha} f)(\eta) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{b} \frac{f(s)}{(\eta - s)^{1-\alpha}} ds, \quad \eta < b, \ \alpha > 0.$$

Definition 1.1.2. *[9, 25] Let $\Delta = [a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$, for the function $f : [a,b] \longrightarrow \mathbb{R}$, Riemann-Liouville derivatives of order α with lower limit a and upper limit b for a function f are defined by:*

$$(D_{a+}^{\alpha} f)(\eta) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\eta^n} \int_{a}^{\eta} \frac{f(s)}{(\eta - s)^{\alpha+1-n}} ds, \quad \eta > a, \ n-1 < \alpha < n.$$

$$(D_{b-}^{\alpha} f)(\eta) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{d\eta^n} \int_{\eta}^{b} \frac{f(s)}{(s - \eta)^{\alpha+1-n}} ds, \quad \eta < b, \ n-1 < \alpha < n.$$

1.1.2 Caputo fractional operator

Definition 1.1.3. *[9, 25] Let $\Delta = [a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$, for the functionf $: [a,b] \longrightarrow \mathbb{R}$, Caputo fractional derivatives of order α with lower limit a and upper limit b for a function are defined by:*

$$({}^{C}D_{a+}^{\alpha} f)(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{\eta} \frac{f^{(n)}(s)}{(\eta - s)^{\alpha+1-n}} ds = (I_{a+}^{n-\alpha} f^{(n)})(\eta), \eta > a, \ n-1 < \alpha < n.$$

$$({}^{C}D_{b-}^{\alpha} f)(\eta) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_{\eta}^{b} \frac{f^{(n)}(s)}{(s - \eta)^{\alpha+1-n}} ds = (-1)^n (I_{b-}^{n-\alpha} f^{(n)})(\eta),$$
$$\eta < b, n-1 < \alpha < n.$$

Obviously, Caputo derivative of a constant is equal to zero.

Remark 1.1.4. *[9, 25]*

Relation between Riemann-Liouville and Caputo fractional derivative of order α is given by the following equations

$$({}^{C}D_{a+}^{\alpha}f)(\eta) = \left(D_{a+}^{\alpha}\left[f(\eta) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(\eta-a)^{k}\right]\right)(\eta).$$

$$({}^{C}D_{b-}^{\alpha}f)(\eta) = \left(D_{b-}^{\alpha}\left[f(\eta) - \sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k+1)}(b-\eta)^{k}\right]\right)(\eta).$$

1.1.3 Fractional derivatives and fractional integrals with respect to another function

Definition 1.1.5. *[9, 22] Let $\Delta = [a,b]$, $(-\infty < a < b < \infty)$ be a finite interval on the real axis, f an integrable function defined on Δ and $\psi \in C^{1}(I)$ an increasing function such that $\psi'(\eta) \neq 0, \forall \eta \in \Delta$. Fractional integrals of a function f with respect to another function ψ are defined as:*

$$(I_{a+}^{\alpha;\psi}f)(\eta) = \frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}\psi'(s)\left[\psi(\eta)-\psi(s)\right]^{\alpha-1}f(s)ds,$$

and

$$(I_{b-}^{\alpha;\psi}f)(\eta) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}\psi'(s)\left[\psi(s)-\psi(\eta)\right]^{\alpha-1}f(s)ds.$$

Fractional derivatives of a function f with respect to another function ψ are defined as:

$$\begin{aligned}(D_{a+}^{\alpha;\psi}f)(\eta) &= \left[\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^{n} I_{a+}^{n-\alpha;\psi}f(\eta)\\ &= \frac{1}{\Gamma(n-\alpha)}\left[\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^{n}\int_{a}^{\eta}\psi'(s)\left[\psi(\eta)-\psi(s)\right]^{n-\alpha-1}f(s)ds,\end{aligned}$$

and

$$(D_{b-}^{\alpha;\psi})f(\eta) = \left[-\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n I_{b-}^{n-\alpha;\psi} f(\eta)$$
$$= \frac{1}{\Gamma(n-\alpha)}\left[-\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n \int_\eta^b \psi'(s)\left[\psi(s)-\psi(\eta)\right]^{n-\alpha-1} f(s)ds,$$

where $n = [\alpha]+1$.

1.1.4 $\psi-$Caputo fractional differential operator

Definition 1.1.6. *[22] Let* $\Delta = [a,b], (-\infty < a < b < \infty)$ *be a finite interval on the real axis,* $f, \psi \in C^n(\Delta)$*, two functions such that* ψ *is an increasing function such that* $\psi'(\eta) \neq 0, \forall \eta \in \Delta$*. The left* $\psi-$*Caputo fractional derivative of* f *of order* α *is given by*

$${}^C D_{a+}^{\alpha;\psi} f(\eta) = I_{a+}^{n-\alpha;\psi}\left[\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n f(\eta),$$

and the right fractional derivative of f *by*

$${}^C D_{b-}^{\alpha;\psi} f(\eta) = I_{a+}^{n-\alpha;\psi}\left[-\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n f(\eta),$$

where $n = [\alpha]+1$ *for* $\alpha \notin \mathbb{N}$ *and* $n = \alpha$ *for* $\alpha \in \mathbb{N}$.

To simplify the notation, we are using the abbreviated symbol

$$f_\psi^{[n]} f(\eta) = \left[\frac{1}{\psi'(\eta)}\frac{d}{d\eta}\right]^n f(\eta).$$

From the definition it is clear that, given $\alpha = k \in \mathbb{N}$, ${}^C D_a^{\alpha;\psi} f(\eta) = f_\psi^{[k]}(\eta)$.

and if $\alpha \notin \mathbb{N}$, then

$${}^C D_{a+}^{\alpha;\psi} f(\eta) = \frac{1}{\Gamma(n-\alpha)}\int_a^\eta \psi'(s)\left[\psi(\eta)-\psi(s)\right]^{n-\alpha-1} f_\psi^{[n]}(s)ds,$$

and

$$ {}^{C}D_{b-}^{\alpha;\psi} f(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_{\eta}^{b} \psi'(s) \left[\psi(\eta) - \psi(s)\right]^{n-\alpha-1} (-1)^n f_{\psi}^{[n]}(s) ds. $$

In particular, if $0 < \alpha < 1$

$$ {}^{C}D_{a+}^{\alpha;\psi} f(\eta) = \frac{1}{\Gamma(1-\alpha)} \int_{a}^{\eta} \left[\psi(\eta) - \psi(s)\right]^{-\alpha} f'(s) ds, $$

and

$$ {}^{C}D_{b-}^{\alpha;\psi} f(\eta) = -\frac{1}{\Gamma(1-\alpha)} \int_{\eta}^{b} \left[\psi(s) - \psi(\eta)\right]^{-\alpha} f'(s) ds. $$

Theorem 1.1.7. *[22, 23, 24] (Relation between $\psi-$Caputo fractional derivative and fractional derivative with respect to another function.)*

Given a function $f \in C^n[a,b]$ and $\alpha > 0$, we have

$$ {}^{C}D_{a}^{\alpha;\psi} f(\eta) = D_{a+}^{\alpha;\psi} \left[f(\eta) - \sum_{m=0}^{n-1} \frac{f_{\psi}^{[m]}(a)}{m!} \left[\psi(\eta) - \psi(a)\right]^m \right] $$

and

$$ {}^{C}D_{b-}^{\alpha;\psi} f(\eta) = D_{b-}^{\alpha;\psi} \left[f(\eta) - \sum_{m=0}^{n-1} \frac{(-1)^m f_{\psi}^{[m]}(a)}{m!} \left[\psi(b) - \psi(\eta)\right]^m \right] $$

where $n = [\alpha] + 1$.

$$ I_{a+}^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_{\psi}^{[k]}(a)}{k!} \left[\psi(x) - \psi(a)\right]^k. $$

In particular, if $0 < \alpha < 1$, we have

$$ I_{a+}^{\alpha,\psi} \quad {}^{C}D_{a+}^{\alpha,\psi} f(t) = f(t) - f(a). $$

1.1.5 $\psi-$Hilfer fractional differential operator

Definition 1.1.8. *[26] Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, $\Delta = [a,b]$ is the interval such that $-\infty \leq a < b \leq \infty$ and f, $\psi \in C^n([a,b],\mathbb{R})$ be two functions such that ψ is increasing* *and* $\psi'(x) \neq 0, \forall x \in \Delta$*. The* $\psi-$*Hilfer fractional derivative (left-sided and right-sided)* ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}$ *of a function of order* α *and type* $0 \leq \beta \leq 1$*, are defined by:*

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(\eta) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(\eta)} \frac{d}{d\eta} \right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(\eta) $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(\eta) = I_{a+}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(\eta)} \frac{d}{d\eta} \right)^{n} I_{b-}^{(1-\beta)(n-\alpha);\psi} f(\eta). $$

Lemma 1.1.9. *[26] If* $f \in C^{n}[a,b], 0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$*, then*

$$ I_{a+}^{\alpha;\psi}\, {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\phi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a). \tag{1.1} $$

Lemma 1.1.10. *[26] If* $f \in C^{1}[a,b], 0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$*, then*

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi} f(t) = f(t). \tag{1.2} $$

Lemma 1.1.11. *[22] Let* $\alpha > 0$ *and* $\delta > 0$*. If* $f(t) = (\psi(t) - \psi(a))^{\delta-1}$*, then*

$$ I_{a+}^{\alpha;\psi} f(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t) - \psi(a))^{\alpha+\delta-1}. \tag{1.3} $$

Lemma 1.1.12. *[26] Let* $\psi \in C^{1}([a,b],\mathbb{R})$ *be a function such that* ψ *is increasing and* $\psi'(t) \neq 0, \forall t \in [a,b]$*. If* $\gamma = \alpha + \beta(1-\alpha)$*, where* $0 < \alpha < 1$ *and* $0 \leq \beta \leq 1$*, then* $\psi-$*Riemann-Liouville fractional integral operator* $I_{a+}^{\alpha;\psi}$ *is bounded from* $C_{1-\gamma;\psi}[a,b]$ *to* $C_{1-\gamma;\psi}[a,b]$*.*

$$ \|I_{a+}^{\alpha;\psi} h\|_{C_{1-\gamma;\psi}[a,b]} \leq M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(t) - \psi(a))^{\alpha}, $$

where M *is the bound of a function* $(\psi(t) - \psi(a))^{1-\gamma} h(t)$*.*

1.1.6 Fixed point theorems

Theorem 1.1.13. *(Kranoselskii's fixed point theorem) [62]*

Let X *be a Banach space,* E *be a bounded closed convex subset of* X *and* S*,* U *be maps* *of E in to X such that $S\zeta + U\vartheta \in E$ for every pair $\zeta, \eta \in E$. If S is a contraction and U is completely continuous, Then the equation $S\zeta + U\zeta = \zeta$ has a solution on E.*

Theorem 1.1.14. ***(Banach contraction principle)*** *[31]*

For a non-empty complete metric space (E,d), let $0 \leq K < 1$ and let the mapping $A : E \to F$ satisfy the inequality

$$d(Ax,Ay) \leq Kd(x,y) \quad \textit{for every} \quad x,y \in E.$$

Then, A has a uniquely determined fixed point x^*. Furthermore, for any $x_0 \in E$, the sequence $(A^j(x_0))_{j=1}^{\infty}$ converges to this fixed point x^*.

Theorem 1.1.15. *(Dhange's fixed point theorem) [45]*

Let S be a non empty, closed, convex and bounded subset of the Banach algebra X, and let $A : X \to X$ and $B : S \to X$ be two operators such that,

(a) *A is Lipschitzian with a Lipschitz constant α*

(b) *B is completely continuous*

(c) $x = AxBy \implies x \in S$ *for all* $y \in S$

(d) $M\zeta(r) < r$, *where* $M = \|B(S)\| = sup\{\|B(x)\| : x \in S\}$.

Then, the operator equation $AxBx = x$ has a solution in S.

Theorem 1.1.16. *(Dhange's fixed point theorem) [45]*

Let S be a closed convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that,

(a) A is a nonlinear contraction

(b) B is continuous and compact

(c) $x = Ax + By$ *for all* $y \in S \implies x \in S$

Then, the operator equation $Ax + By = x$ *has a solution in S.*

Theorem 1.1.17. *(Arzelá-Ascoli theorem) [54]*

Let X *be a compact metric space and* $\mathscr{F}$ *is a compact subspace of* $C(X)$ *if and only if* $\mathscr{F}$ *is closed, uniformly bounded and equicontinuous.*

Theorem 1.1.18. *(Lebesgue dominated convergence theorem) [54]*

Let $\|f_n\}$ *be a sequence of measurable functions on* E*. Suppose there is a function g that is integrable over E and dominates* $\{f_n\}$ *on E in the sense that* $|f_n| \leq g$ *on E for all n. If* $\{f_n\} \to f$ *pointwise a.e on E, then f is integrable over E and* $\lim_{n\to\infty} \int_E f_n = \int_E f$*.*

Chapter 2

$\psi-$Caputo Fractional Impulsive Neutral Functional Differential Equation

2.1 Introduction

In this chapter, the initial value problem is discussed for a class of fractional impulsive neutral functional differential equation of a function with respect to another function. The criteria on existence and uniqueness are obtained using Krasnoselskii's fixed point theorem and Banach fixed point theorem respectively.

Consider the fractional impulsive neutral functional differential equation involving the Caputo fractional derivative of a function x with respect to another function ψ.

$$\begin{cases} {}^{C}D_{t_0}^{\alpha,\psi}(x(t)-g(t,x_t)) = f(t,x_t), \quad t\in(t_0,\infty), t_0\geq 0, t\neq t_k \\ \Delta x(t_k) = I_k(x(t_k-)) \\ x_{t_0} = \phi \end{cases} \tag{2.1}$$

where ${}^{C}D^{\alpha,\psi}$ is the $\psi-$Caputo fractional derivative of order $0<\alpha<1$ with respect to another function ψ. Consider $f, g : ([t_0, \infty) \times C([-r, 0], \mathbb{R}^n)) \to \mathbb{R}^n$ are given functions satisfying certain assumptions, which will be specified later. $a > 0$ and $\phi \in C([-r, 0], \mathbb{R}^n)$. If $x \in C([t_0 - r, t_0 + a], \mathbb{R}^n)$, and for $t \in [t_0, t_0 + a]$, define x_t by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r, 0]$. Let $\psi \in C^1[t_0, \infty)$ be a continuous increasing function such that $\psi'(x) \neq 0, \forall x \in [t_0, \infty)$. Let $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = lim_{h\to 0^+} x(t_k + h), x(t_k^-) = lim_{h\to 0^-} x(t_k - h)$, $k = 1, 2, \cdots, m$ for $t_0 < t_1 < t_2 < \cdots < t_m$.

Through this chapter we are discussing the initial value problem for a class of $\psi-$Caputo fractional neutral functional differential equations with bounded delay. In the preliminary section we present essential results. Then based on many assumptions and Krasnoselskii's fixed point theorem, we prove that IVP (2.1) has at least one solution, by deducing IVP (2.1) into equivalent volterra integral equation. Moreover we prove the uniqueness of the solution.

2.2 Preliminaries

Note that

$$\begin{aligned} x(t) =& x_0 - \frac{1}{\Gamma(\alpha)} \int_0^a \psi'(s)(\psi(a) - \psi(s))^{\alpha-1} h(s)ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)\left(\psi(t) - \psi(s)\right)^{\alpha-1} h(s)ds \end{aligned}$$

solves the Cauchy Problems:

$$\begin{cases} {}^C D^{\alpha,\psi} x(t) = h(t), \quad t \in J = [0, T], T \geq 0 \\ x(0) = x_0 - \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^a \psi'(s)(\psi(a) - \psi(s))^{\alpha-1} h(s)ds \end{cases} \tag{2.2}$$

One can obtain the following result immediately.

Lemma 2.2.1. *Let $\alpha \in (0,1)$ and $h : J \to \mathbb{R}$ be continuous. A function $x \in C(J,\mathbb{R})$ is a solution of the fractional integral equation*

$$x(t) = x_0 - \frac{1}{\Gamma(\alpha)}\int_0^a \psi'(s)(\psi(a)-\psi(s))^{\alpha-1}h(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s)ds$$

if and only if x is a solution of the following fractional Cauchy problems,

$$\begin{cases} {}^{C}D^{\alpha,\psi}x(t) & = h(t), \quad t \in J \\ x(a) & = x_0, \quad a > 0. \end{cases} \tag{2.3}$$

2.3 Existence results

Let $I_0 = [t_0, t_0+\delta]$,

$$A(\delta,\gamma) = \{x \in C([t_0-r, t_0+\delta],\mathbb{R}^n) | x_{t_0} = \phi, \sup_{t_0 \le t \le t_0+\delta} |x(t)-\phi(0)| \le \gamma\},$$

where δ, γ are positive constants. Before starting and proving the main results, we introduce the following hypotheses.

(H1) $f(t,\phi)$ is measurable with respect to t on I_0.

(H2) $f(t,\phi)$ is continuous with respect to ϕ on $C([-r,0],\mathbb{R}^n)$.

(H3) There exist $\alpha_1 \in (0,\alpha)$ and a real valued function $m(t) \in L^{\frac{1}{\alpha_1}}(I_0)$ such that for any $x \in A(\delta,\gamma), |f(t,x_t)| \le m(t), M = ||m||_{L^{\frac{1}{\alpha_1}}(I_0)}$ for $t \in I_0$.

(H4) For any $x \in A(\delta,\gamma)$, $g(t,x_t) = g_1(t,x_t) + g_2(t,x_t)$.

(H5) g_1 is continuous and for any $x', x'' \in A(\delta,\gamma)$, $t \in I_0$,

$|g_1(t,x'_t) - g_1(t,x''_t)| \le l||x'-x''||$, where $l \in (0,1)$.

(H6) g_2 is completely continuous and for any bounded set Λ in $A(\delta,\gamma)$, the set $\{t \to g_2(t,x_t) : x \in \Lambda\}$ is equicontinuous in $C(I_0,\mathbb{R}^n)$.

(H7) The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are continuous and there exist a constant $0 < L < 1$ such that $||I_k(x) - I_k(y)|| \leq \frac{L}{m}||x-y||, x,y \in \mathbb{R}^n$, $m > 0$, $k = 1,2,\cdots,m$.

Lemma 2.3.1. *If there exist* $\delta \in (0,a)$ *and* $\gamma \in (0,\infty)$ *such that* $(H1)-(H3)$ *are satisfied, then for* $t \in (t_0,t_0+\delta]$*, IVP* (2.1) *is equivalent to the following equation.*

$$x(t) = \begin{cases} \phi(0) - g(t_0,\phi) + g(t,x_t) + I_{t_0}^{\alpha,\psi} f(t,x_t), & for \quad t \in [t_0,t_1] \\ \phi(0) - g(t_0,\phi) + g(t,x_t) + I_1(x(t_1-)) + I_{t_0}^{\alpha,\psi} f(t,x_t), & for \quad t \in (t_1,t_2] \\ \phi(0) - g(t_0,\phi) + g(t,x_t) + I_1(x(t_1-)) + I_2(x(t_2-)) + I_{t_0}^{\alpha,\psi} f(t,x_t), & for \quad t \in (t_2,t_3] \\ \vdots & \\ \phi(0) - g(t_0,\phi) + g(t,x_t) + \sum_{t_0<t<t_k} I_i(x(t_i-)) + I_{t_0}^{\alpha,\psi} f(t,x_t), \ for & t \in (t_m,t_0+\delta] \end{cases} \tag{2.4}$$

$x_{t_0} = \phi$

Proof. Assume x satisfies IVP (2.1).
If $t \in [t_0,t_1]$, then ${}^C D_{t_0}^{\alpha,\psi}(x(t) - g(t,x_t)) = f(t,x_t), \quad t \in (t_0,t_1]$.
Integrating from t_0 to t, by virtue of Definition (1.1.5) and Theorem (1.1.7) we can obtain:

$$x(t) = \phi(0) - g(t_0,\phi) + g(t,x_t) + I_{t_0}^{\alpha,\psi} f(t,x_t).$$

If $t \in (t_1,t_2]$,
then ${}^C D_{t_0}^{\alpha,\psi}(x(t) - g(t,x_t)) = f(t,x_t)$, with $x(t_1+) = x(t_1-) + I_1(u(t_1-))$.

By Lemma (2.2.1), one obtain

$$\begin{aligned} x(t) &= x(t_1+) - g(t_1, x_{t_1}) + g(t, x_t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} f(s, x_s) ds \\ &\quad + I_{t_0}^{\alpha,\psi} f(t, x_t) \\ &= x(t_1-) + I_1(x(t_1-)) - g(t_1, x_{t_1}) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \psi'(s)(\psi(t_1) - \psi(s))^{\alpha-1} f(s, x_s) ds \\ &\quad + I_{t_0}^{\alpha,\psi} f(t, x_t). \end{aligned}$$

$$x(t) = \phi(0) - g(t_0, \phi) + g(t, x_t) + I_1(x(t_1-) + I_{t_0}^{\alpha,\psi} f(t, x_t).$$

If $t \in (t_2, t_3]$, then

$$\begin{aligned} x(t) &= x(t_2+) - g(t_2, x_{t_2}) + g(t, x_t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} f(s, x_s) ds \\ &\quad + I_{t_0}^{\alpha,\psi} f(t, x_t) \\ &= x(t_2-) + I_2(x(t_2-)) - g(t_2, x_{t_2}) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_2} \psi'(s)(\psi(t_2) - \psi(s))^{\alpha-1} f(s, x_s) ds \\ &\quad + I_{t_0}^{\alpha,\psi} f(t, x_t). \end{aligned}$$

Which implies,

$$x(t) = \phi(0) - g(t_0, \phi) + g(t, x_t) + I_1(x(t_1-) + I_2(x(t_2-) + I_{t_0}^{\alpha,\psi} f(t, x_t).$$

If $t \in (t_m, t_0 + \delta]$, then again by Lemma (2.2.1), we get

$$x(t) = \phi(0) - g(t_0, \phi) + g(t, x_t) + \Sigma_{i=1}^{m} I_i(x(t_i-) + I_{t_0}^{\alpha,\psi} f(t, x_t).$$

Conversely, assume that x satisfies equation (2.4).

If $t \in (t_0, t_1]$, we get ${}^{C}D_{t_0}^{\alpha,\psi}(x(t) - g(t, x_t) = f(t, x_t)$.

If $t \in (t_k, t_{k+1}],\ k = 1, 2, \cdots, m$ and using the fact of $\psi-$ Caputo fractional derivative of a constant is equal to zero, we obtain:

${}^{C}D_{t_0}^{\alpha,\psi}(x(t) - g(t, x_t)) = f(t, x_t),\ t \in (t_{k-1}, t_k]$ and

$u(t_k+) - u(t_k-) = I_k(u(t_k-)),\ k = 1,2,\cdots,m.$

This completes the proof. □

Theorem 2.3.2. *Assume that there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1)-(H7)$ are satisfied, then IVP(2.1) has at least one solution*

$$\begin{cases} x(t) = \phi(0) - g(t_0,\phi) + g(t,x_t) & + \sum_{t_0<t_k<t} I_k(x(t_k-)) \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} f(s,x_s)ds \\ x_{t_0} = \phi \end{cases} \tag{2.5}$$

on $[t_0, t_0+\eta]$ for some positive number η.

Proof. According to $(H4)$, equation (2.4) is equivalent to the following equation

$$\begin{aligned} x(t) =& \phi(0) - g_1(t_0,\phi) - g_2(t_0,\phi) + g_1(t,x_t) + g_2(t,x_t) + \Sigma_{t_0<t_k<t} I_k(x(t_k-)) \\ & + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)[\psi(t)-\psi(s)]^{\alpha-1} f(s,x_s)ds, \quad t \in I_0 \end{aligned}$$

$$x_{t_0} = \phi.$$

Let $\tilde{\phi} \in A(\delta,\gamma)$ be defined as

$$\tilde{\phi}_{t_0} = \phi, \quad \tilde{\phi}(t_0+t) = \phi(0) \quad \forall t \in [0,\delta].$$

Let x be a solution of the IVP(2.1) and $x(t_0+t) = \tilde{\phi}(t_0+t) + y(t)$, $t \in [-r,\delta]$.

Then we have, $\quad x_{t_0+t} = \tilde{\phi}_{t_0+t} + y_t$, $t \in [0,\delta]$. Thus

$$\begin{aligned} y(t) =& \\ & - g_1(t_0,\phi) - g_2(t_0,\phi) + g_1(t_0+t, y_t + \tilde{\phi}_{t_0+t}) + g_2(t_0+t, y_t + \tilde{\phi}_{t_0+t}) + \Sigma_{t_0<t_k<t} I_k(x(t_k-)) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha-1} f(t_0+s, y_s + \tilde{\phi}_{t_0+s})ds,\ t \in [0,\delta]. \end{aligned} \tag{2.6}$$

Since g_1, g_2 are continuous and x_t is continuous in t, there exist $\delta' > 0$ such that for $0 < t < \delta'$,

$$|g_1(t_0+t, y_t+\tilde{\phi}_{t_0+t}) - g_1(t_0,\phi)| < \frac{\gamma}{4}, \tag{2.7}$$

$$|g_2(t_0+t, y_t+\tilde{\phi}_{t_0+t}) - g_2(t_0,\phi)| < \frac{\gamma}{4}, \tag{2.8}$$

choose

$$\eta = min\left\{\delta, \delta', \left(\frac{\gamma\Gamma(\alpha+1)}{4M}\right)^{1/\alpha}\right\}. \tag{2.9}$$

Define $E(\eta,\gamma)$ as follows

$$E(\eta,\gamma) = \left\{y \in PC([-r,\eta],\mathbb{R}^n)/y(s) = 0 \; for \; s \in [-r,0] \; and \; ||y|| \leq r\right\}.$$

Then $E(\eta,\gamma)$ is a closed bounded and convex subset of $PC([-r,\eta],\mathbb{R}^n)$.

On $E(\eta,\gamma)$, we define the operators S and U as follows:

$$Sy(t) = \begin{cases} 0 & \text{if } t \in [-r,0] \\ -g_1(t_0,\phi) + g_1(t_0+t, y_t+\tilde{\phi}_{t_0+t}) & \text{if } t \in [0,\eta] \end{cases}$$

$$Uy(t) = \begin{cases} 0 & \text{if } t \in [-r,0] \\ -g_2(t_0,\phi) + g_2(t_0+t, y_t+\tilde{\phi}_{t_0+t}) + \sum_{t_0<t_k<t} I_k(x(t_k-)) & \\ +\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s+t_0)\left(\psi(t+t_0)-\psi(s+t_0)\right)^{\alpha-1} f(t_0+s, y_s+\tilde{\phi}_{t_0+s})ds & \\ & \text{if } t \in [0,\eta]. \end{cases} \tag{2.10}$$

It is easy to see that the operator equation

$$y = Sy + Uy \tag{2.11}$$

has a solution $y \in E(\eta,\gamma)$ if and only if y is a solution of (2.6).

Thus $x(t_0+t) = y(t) + \tilde{\phi}(t_0+t)$ is a solution of (2.1) on $[0,\eta]$. Therefore the existence of a solution of the IVP (2.1) is equivalent that equation (2.11) has a fixed point in $E(\eta,\gamma)$.

Now we show that $S+U$ has a fixed point in $E(\eta,\gamma)$. The proof is divided in to three steps.

Step I: $Sz+Uy \in E(\eta,\gamma)$ for every pair z, $y \in E(\eta,\gamma)$.

In fact, for every pair $z,y \in E(\eta,\gamma)$, $Sz+Uy \in PC([-r,\eta],\mathbb{R}^n)$. Also it is obvious that $(Sz+Uy)(t)=0, t \in [-r,0]$.

Moreover, for $t \in [0,\eta]$, by $(2.7),(2.8),(2.9)$ and the condition $(H3)$ and if $(\psi(t+t_0)-\psi(t_0)) < \delta$, we have:

$$
\begin{aligned}
|Sz(t)-Uy(t)| \leq & |-g_1(t_0,\phi)+g_1(t_0+t,z_t+\tilde{\phi}_{t_0+t})| \\
& +|-g_2(t_0,\phi)+g_2(t_0+t,y_t+\tilde{\phi}_{t_0+t})|+\sum_{t_0<t_k<t}|I_k(x(t_k-))| \\
& +\frac{1}{\Gamma(\alpha)}\int_0^t |\psi'(s+t_0)\left[\psi(t+t_0)-\psi(s+t_0)\right]^{\alpha-1} f(t_0+s,y_s+\tilde{\phi}_{t_0+s})|ds \\
\leq & \frac{\gamma}{2}+\frac{M}{\Gamma(\alpha+1)}(\psi(t+t_0)-\psi(t_0))^{\alpha}+L\gamma \\
\leq & \frac{\gamma}{2}+\frac{M}{\Gamma(\alpha+1)}\eta^{\alpha}+L\gamma \\
\leq & \gamma.
\end{aligned}
$$

Therefore, $||Sz+Uy|| = \sup_{t\in[0,\eta]}|(Sz)(t)+(Uy)(t)| \leq \gamma$, which means that $Sz+Uy \in E(\eta,\gamma)$, for any $z,y \in E(\eta,\gamma)$.

Step II: S is a contraction on $E(\eta,\gamma)$.

For any $y',y'' \in E(\eta,\gamma)$, $y'_t+\tilde{\phi}_{t_0+t}$, $y''_t+\tilde{\phi}_{t_0+t} \in A(\delta,\gamma)$. Also by $(H5)$, we get that

$$|Sy'(t)-Sy''(t)| = |g_1(t_0+t,y'_t+\tilde{\phi}_{t_0+t})-g_1(t_0+t,y''_t+\tilde{\phi}_{t_0+t})| \leq l||y'-y''||.$$

Which implies that

$$||Sy' - Sy''|| \leq l||y' - y''||.$$

In view of $0 < l < 1$, S is a contraction on $E(\eta, \gamma)$.

Step III: Now we show that U is a completely continuous operator.

Consider,

$$U_1 y(t) = \begin{cases} 0 & t \in [-r, 0], \\ -g_2(t_0, \phi) + g_2(t_0 + t, y_t + \tilde{\phi}_{t_0+t}) & t \in [0, \eta]. \end{cases}$$

And

$$U_2 y(t) = \begin{cases} 0 & t \in [-r, 0] \\ \sum_{t_0 < t_k < t} I_k(x(t_k-)) \\ + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha - 1} f(t_0 + s, y_s + \tilde{\phi}_{t_0+s}) ds \\ & t \in [0, \eta]. \end{cases}$$

Clearly $U = U_1 + U_2$.

Now we have g_2 is completely continuous, U_1 is continuous and $\{U_1 y : y \in E(\eta, \gamma)\}$ is uniformly bounded. Hence from the condition that the set $\{t \to g_2(t, x_t) : x \in \Lambda\}$ be equicontinuous for any bounded set Λ in $A(\delta, \gamma)$, we can conclude that U_1 is a completely continuous operator.

On the other hand for any $t \in [0, \eta]$, we have:

$$||U_2 y(t)|| \leq \sum_{t_0 < t_k < t} |I_i(x(t_i-))| \\ + \frac{1}{\Gamma(\alpha)} \int_0^t |\psi'(s + t_0) \left[\psi(t + t_0) - \psi(s + t_0)\right]^{\alpha - 1} f(t_0 + s, y_s + \tilde{\phi}_{t_0+s})| ds$$

$$\leq L\gamma + (\psi(t+t_0) - \psi(t_0)^{\alpha} \frac{M}{\Gamma(\alpha+1)}$$
$$\leq L\gamma + \eta^{\alpha} \frac{M}{\Gamma(\alpha)}.$$

Hence $\{U_2(y) : y \in E(\eta,\gamma)\}$ is uniformly bounded.

Now we will prove that $\{U_2 y : y \in E(\eta,\gamma)\}$ is equicontinuous.

For any $0 \leq t_1 < t_2 \leq \eta$ and $y \in E(\eta,\gamma)$,

and if $|\psi(x) - \psi(y)| \leq N||x-y||$, $0 < N < 1$, $\forall x,y \in [0,\eta]$ we get that:

$$\begin{aligned}
&|U_2 y(t_2) - U_2 y(t_1)| \leq \\
&\quad \frac{1}{\Gamma(\alpha)} \int_0^{t_2} |\psi'(s+t_0) \left[(\psi(t_2+t_0) - \psi(s+t_0))^{\alpha-1} \right] f(t_0+s, y_s + \tilde{\phi}_{t_0+s})| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^{t_1} |\psi'(s+t_0) [\psi(t_1+t_0) - \psi(s+t_0)]^{\alpha-1} f(t_0+s, y_s + \tilde{\phi}_{t_0+s})| ds \\
&\qquad + \sum_{t_0 < t_k < t_2 - t_1} I_k(x(t_k-)) \\
&\leq \frac{M}{\Gamma(\alpha+1)} [(\psi(t_2+t_0) - \psi(t_0))^{\alpha} - (\psi(t_1+t_0) - \psi(t_0))^{\alpha}] + \sum_{t_0 < t_k < t_2 - t_1} I_k(x(t_k-))
\end{aligned}$$

which converges to zero as $t_1 \to t_2$.

Hence $\{U_2 y : y \in E(\eta,\gamma)\}$ is equicontinuous. Moreover, it is clear that U_2 is continuous. So, U_2 is completely continuous operator. Then $U = U_1 + U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\eta,\gamma)$ and hence the IVP (2.1) has a solution $x(t) = \phi(0) + y(t-t_0)$ for all $t \in [t_0, t_0+\eta]$. This completes the proof. □

In the case where $g_1 \equiv 0$, we get the following result.

Corollary 2.3.3. *[25] Assume that there exist* $\delta \in (0,a)$ *and* $\gamma \in (0,\infty)$ *such that*

$(H1)-(H3)$ *hold and* $(H5)'$ *g is continuous and for any* $x', x'' \in A(\delta,\gamma)$, $t \in I_0$

$$|g(t,x'_t)-g(t,x''_t))| \leq l||x'-x''||,\, l \in (0,1).$$

Then IVP (2.1) *has at least one solution on* $[t_0,\, t_0+\eta]$ *for some positive number* η.

In the case where $g_2 \equiv 0$, we have the following result.

Corollary 2.3.4. *[25] Assume that there exist* $\delta \in (0,\, a)$ *and* $\gamma \in (0,\, \infty)$ *such that* $(H1)-(H3)$ *hold and* $(H6)'$ *g is completely continuous and for any bounded set* Λ *in* $A(\delta,\, \gamma)$*, the set* $\{t \to g(t,\, x_t) : x \in \Lambda\}$ *is equicontinuous on* $C(I_0,\, \mathbb{R}^n)$*. Then IVP* (2.1) *has at least one solution on* $[t_0,\, t_0+\eta]$ *for some positive number* η.

2.4 Uniqueness result

Theorem 2.4.1. *Assume that the function* f *is Lipschitz continuous with respect to the second variable. i.e, there exist a positive constant* L_1 *such that* $||f(t,x_{1_t})-f(t,x_{2_t})|| \leq L_1||x_1-x_2||\forall t \in [a,b], x_1,x_2 \in C([t_0-r,t_0+a))$ *with H5 and H7, then there is a constant* $h \in \mathbb{R}^+$ *such that there exist a unique solution to the IVP(1) on the interval* $[t_0,t_0+h] \subseteq [a,b]$ *if* $(\frac{L_1}{\Gamma(\alpha+1)}(\psi(t_0+h)-\psi(t_0))^{\alpha}+L+l) < 1$.

Proof. Define the function $F(x,t)$ by

$$F(x,t) = \phi(0)-g(t_0,\phi)+g(t,x_t)+\sum_{t_0<t<t_k} I_i(x(t_i-))+I_{t_0}^{\alpha,\psi}f(t,x_t).$$

Let $U = \{x \in C([t_0-r,t_0+a],\mathbb{R}^n) : {}^{C}D_{t_0}^{\alpha,\psi}x(t)$ exists and is continuous in $[t_0,t_0+h]\}$. It is enough to prove that $F : U \to U$ is a contraction. Let us see that F is well defined. i.e., $F(U) \subseteq U$.

Given the function $x \in U$, we see that ${}^{D}{}_{t_0}^{\alpha,\psi}(F(x)(t) - g(t,x_t)) = f(t,x_t)$ is continuous and

$$F(x)(t) = \phi(0) + g(t_0,\phi) + g(t,x_t) + \sum_{t_0<t<t_k} I_i(x(t_i-)) + I_{t_0}^{\alpha,\psi} f(t,x_t),$$

which satisfies the required conditions.

Now let $x_1, x_2 \in U$ be arbitrary, then by assumptions, we have:

$$\begin{aligned}
&||F(x_1,t) - F(x_2,t)|| \\
&\quad \leq ||I_{t_0}^{\alpha,\psi}(f(t,x_{1_t}) - f(t,x_{2_t}))|| + \sum_{t_0<t<t_k} ||(I_i(x_1(t_i-)) - I_i(x_1(t_i-)))|| \\
&\qquad\qquad + ||g(t,x_{1_t}) - g(t,x_{2_t})|| \\
&\quad \leq \left[\frac{L_1}{(\Gamma(\alpha+1))}(\psi(t_0+h) - \psi(t_0))^{\alpha} + L + l\right] ||x_1 - x_2||.
\end{aligned}$$

which proves that F is a contraction. Using the Banach fixed point theorem, $IVP(2.1)$ has a unique solution. □

2.5 Conclusion

The generalised fractional impulsive neutral functional differential equation involving $\psi-$Caputo fractional derivative is considered. Under certain assumptions the criteria on existence and uniqueness of the solution are obtained using Krasnoselskii's fixed point theorem and Banach fixed point theorem respectively.

Chapter 3

A k- Dimensional System of Fractional Neutral Functional Differential Equations Involving $\psi-$Caputo Fractional Derivative.

3.1 Introduction

This Chapter is devoted to the study of the initial value problem for a class of $k-$ dimensional system of fractional neutral functional differential equations involving generalised fractional derivative namely Caputo-type fractional derivative with respect to another function. Existence and uniqueness results for the problem are established by means of Krasnoselskii's and Banach fixed point theorems respectively.

The aim of this chapter is to investigate the existence of solutions for a class of $k-$

dimensional system of fractional neutral functional differential equation with bounded delay involving the Caputo-type fractional derivative of a function x with respect to another function ψ.

$$\begin{cases} {}^{C}D_{t_0}^{\alpha_1,\psi}(x_1(t)-g_1(t,x_t)) & = f_1(t,x_t) \\ {}^{C}D_{t_0}^{\alpha_2,\psi}(x_2(t)-g_2(t,x_t)) & = f_2(t,x_t) \\ \vdots & \\ {}^{C}D_{t_0}^{\alpha_k,\psi}(x_k(t)-g_k(t,x_t)) & = f_k(t,x_t) \end{cases} \tag{3.1}$$

$$x_{1_{t_0}} = \phi_1, \quad x_{2_{t_0}} = \phi_2 \quad \cdots \quad x_{k_{t_0}} = \phi_k$$

where $t_0 \geq 0$, $a > 0$ and $r > 0$ are constants $t \in (t_0,\infty)$, $0 < \alpha_i < 1$, for $i = 1,2\cdots k$. ${}^{C}D_{t_0}^{\alpha_i,\psi}$ is the Caputo-type fractional derivative of a function x_i with respect to another function ψ. $f_i, g_i : ([t_0,\infty) \times C([-r,0],\mathbb{R}^n) \times C([-r,0],\mathbb{R}^n) \times \cdots \times C([-r,0],\mathbb{R}^n)) \to \mathbb{R}^n$; $i = 1,2\cdots k$ are given functions satisfying certain assumptions, which will be specified later. $a > 0$, $x_{it} = (x_{1_t}, x_{2_t}, \cdots x_{k_t})$ and $\phi_i \in C([-r,0],\mathbb{R}^n)$ for $i = 1,2\cdots k$. If $x_i \in C([t_0-r, t_0+a],\mathbb{R}^n)$, then for any $t \in [t_0, t_0+a]$, define x_{it} by $x_{it}(\theta) = x_i(t+\theta)$ for $\theta \in [-r,0]$. Let $\psi \in C^n[t_0,\infty)$ be a continuous increasing function such that $\psi'(x) \neq 0$, $\forall x \in [t_0,\infty)$.

Through this chapter, we are discussing the initial value problem for a class of $k-$ dimensional system of fractional neutral functional differential equations with bounded delay of a function with respect to another function. In the second section we present essential definitions and results and in the third section, we prove the existence and uniqueness of the IVP (3.1). Using Krasnoselskii's fixed point theorem, we prove that IVP (3.1) has at least one solution, by deducing in to equivalent integral equation. To prove uniqueness, we adopt Banach fixed point theorem.

Let I be an interval in $\mathbb{R}$ and $X = C(I,\mathbb{R}^n)$ with the norm $||x|| = sup_{t\in I}|x(t)|$ where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}^n$.

Consider the product Banach space $(X^k = \underbrace{X\times X\times\cdots\times X}_{k}, ||\cdot||_*)$ with the norm $||(x_1,x_2\cdots x_k)||_* = max\{||x_1||,||x_2||,\cdots||x_k||\}$.

3.2 Existence results

Consider the Initial Value Problem (3.1). Let δ and γ be the positive constants, $I_0 = [t_0,t_0+\delta]$ and define:

$$A(\delta,\gamma) = \{(x_1,x_2\cdots x_k) : x_{i_{t_0}} = \phi_i, \sup_{t_0\le t\le t_0+\delta} |x_i(t)-\phi_i(0)| \le \gamma \quad \forall i = 1,2\cdots k\} \tag{3.2}$$

where $x_i \in C([t_0-r,t_0+\delta],\mathbb{R}^n)$.

Before starting and proving the main results, we introduce the following hypotheses:

(H1) $f_i(t,\phi_1,\phi_2\cdots\phi_k)$ is measurable with respect to t on I_0, $\forall i = 1,2,3\cdots k$.

(H2) $f_i(t,\phi_1,\phi_2\cdots\phi_k)$ is continuous with respect to ϕ_j on $C([-r,0],\mathbb{R}^n)$, $\forall i, j = 1,2\cdots k$.

(H3) There exist $\alpha_{i_1} \in (0,\alpha_i)$ and a real valued function $m_i(t) \in L^{\frac{1}{\alpha_{i_1}}}(I_0)$ such that for any $(x_1,x_2\cdots x_k) \in A(\delta,\gamma)$, $i = 1,2\cdots k$

$$|f_i(t,x_t)| \le m_i(t), \quad t \in I_0. \tag{3.3}$$

(H4) For any $(x_1,x_2\cdots x_k) \in A(\delta,\gamma)$, $g_i(t,x_t) = g_{i_1}(t,x_t)+g_{i_2}(t,x_t)$.

(H5) g_{i_1} is continuous and

$$|g_{i_1}(t,x_t)-g_{i_1}(t,y_t)| \le l_i||x-y||_* \tag{3.4}$$

where $l_i \in (0,1)$, $\forall x = (x_1, x_2 \cdots x_k)$, $y = (y_1, y_2 \cdots y_k) \in A(\delta, \gamma)$, $t \in I_0$

$f ori = 1, 2 \cdots k.$

(H6) g_{i_2} is completely continuous and for any bounded set Λ in $A(\delta, \gamma)$,

the set $\{t \to g_{i_2}(t, x_t) : (x_1, x_2 \cdots x_k) \in \Lambda\}$ is equicontinuous on

$\underbrace{C(I_0, \mathbb{R}^n) \times C(I_0, \mathbb{R}^n) \times \cdots \times C(I_0, \mathbb{R}^n)}_{k}$, $\forall i = 1, 2 \cdots k$.

(H7) $\psi \in C^1([t_0, \infty])$ is a continuous increasing function with

$|\psi(t) - \psi(s)| \leq N|t - s|$, $N \in (0,1)$ and $|\psi\prime(s)| < K$, K be any positive integer.

Lemma 3.2.1. *If there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1)-(H3)$ are satisfied, then for $t \in (t_0, t_0 + \delta]$, IVP (3.1) is equivalent to the following equation.*

$$\begin{cases} x_i(t) = & \phi_i(0) - g_i(t_0, \phi_1, \phi_2 \cdots \phi_k) + g_i(t, x_t) \\ & + \frac{1}{\Gamma(\alpha_i)} \int_{t_0}^{t} \psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha_i - 1} f_i(s, x_s) ds, \ t \in I_0 \\ x_{i_{t_0}} = \phi_i \end{cases} \tag{3.5}$$

for $i = 1, 2 \cdots k$ and $t \in I_0$.

Proof. From the conditions $(H1)$ and $(H2)$, it is obvious that $f_i(t, x_t)$ is Lebesgue measurable on I_0. A direct calculation using $(H7)$ gives that

$$\left(\psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha_i - 1}\right) \in L^{\frac{1}{1-\alpha_{i_1}}}([t_0, t]) \, t \in I_0.$$

In the light of Holder's inequality and $(H3)$,
we obtain that $\left(\psi\prime(s) \left[\psi(t) - \psi(s)\right]^{\alpha - 1}\right) f_i(s, x_s)$ is Lebesgue integrable with respect to $s \in [t_0, t]$, $\forall t \in I_0, i = 1, 2 \cdots, k$ and $(x_1, x_2 \cdots x_k) \in A(\delta, \gamma)$

and

$$\int_{t_0}^{t}\left(\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}\right)f_i(s,x_s)ds \leq$$
$$\|\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}\|_{L^{\frac{1}{1-\alpha_{i_1}}}(I_0)}\|m_i\|_{L^{\frac{1}{\alpha_{i_1}}}(I_0)} \tag{3.6}$$

where

$$\|F\|_{L^p(J)}=\left(\int_J|f(t)|^p dt\right)^{\frac{1}{p}},$$

for any p integrable function $F: J\to\mathbb{R}$.

According to the definition of fractional integral of function with respect to another function ψ and $\psi-$Caputo derivative of order α_i, it is easy to see that if x_i is a solution of the IVP (3.1), then x_i is a solution of equation (3.5).

On the other hand, if equation (3.5) is satisfied then $\forall t\in(t_0,t_0+\delta]$, we have:

$$^{C}D_{t_0}^{\alpha_i,\psi}\left(x_i(t)-g_i(t,x_t)\right)=$$
$$^{C}D_{t_0}^{\alpha_i,\psi}\left(\phi_i(0)-g_i(t_0,\phi_1,\phi_2\cdots\phi_k)+\frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t}\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}f_i(s,x_s)ds\right)$$
$$={}^{C}D_{t_0}^{\alpha_i,\psi}\left(\frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t}\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1}f_i(s,x_s)ds\right)$$

$$\begin{aligned}
^{C}D_{t_0}^{\alpha_i,\psi}\left(x_i(t)-g_i(t,x_t)\right) &= {}^{C}D_{t_0}^{\alpha_i,\psi}I_{t_0}^{\alpha_i,\psi}f_i(t,x_t)\\
&= D_{t_0}^{\alpha_i,\psi}\left[I_{t_0}^{\alpha_i,\psi}f_i(t,x_t)-\sum_{k=0}^{n-1}\frac{\left[I^{\alpha_i,\psi}f_i(t,x_t)\right]^{[k]}(t_0)}{k!}\left(\psi(t)-\psi(t_0)\right)^k\right]\\
&= D_{t_0}^{\alpha_i,\psi}\,I_{t_0}^{\alpha_i,\psi}f_i(t,x_t)-\sum_{k=0}^{n-1}\frac{\left[I^{\alpha_i,\psi}f_i(t,x_t)\right]^{[k]}(t_0)}{\Gamma(k-\alpha_i+1)}\left(\psi(t)-\psi(t_0)\right)^{k-\alpha_i}\\
&= f_i(t,x_t)-\left[I^{\alpha_i,\psi}f_i(t,x_t)\right]_{t=t_0}\frac{\left(\psi(t)-\psi(t_0)\right)^{-\alpha_i}}{\Gamma(1-\alpha_i)}=f_i(t,x_t),
\end{aligned}$$

since $\left[I^{\alpha_i,\psi}f_i(t,x_t)\right]_{t=t_0}=0$.

Hence, we get $^{C}D_{t_0}^{\alpha_i,\psi}\left(x_i(t)-g_i(t,x_t)\right)=f_i(t,x_t),\quad t\in(t_0,t_0+\delta]$.

And this completes the proof. □

Theorem 3.2.2. *Assume that there exist* $\delta \in (0,a)$ *and* $\gamma \in (0,\infty)$ *such that* $(H1)-(H7)$ *are satisfied, then IVP(3.1) has at least one solution on* $[t_0, t_0+\eta]$ *for some positive number* η.

Proof:

According to $(H4)$, equation (3.5) is equivalent to the following equation

$$\begin{cases} x_i(t) & = \phi_i(0) - g_{i_1}(t_0,\phi_1,\phi_2\cdots\phi_k) - g_{i_2}(t_0,\phi_1,\phi_2\cdots\phi_k) + g_{i_1}(t,x_t) + g_{i_2}(t,x_t) \\ & \quad + \frac{1}{\Gamma(\alpha_i)}\int_{t_0}^{t}\psi\prime(s)\left[\psi(t)-\psi(s)\right]^{\alpha_i-1} f_i(s,x_s)ds, \quad t\in I_0 \\ x_{i_{t_0}} & = \phi_i \quad i=1,2\cdots k \end{cases}$$

Let $(\tilde{\phi}_1,\tilde{\phi}_2\cdots\tilde{\phi}_k) \in A(\delta,\gamma)$ be defined as

$$\tilde{\phi}_{i_{t_0}} = \phi_i, \quad \tilde{\phi}_i(t_0+t) = \phi_i(0) \quad \forall t\in[0,\delta], i=1,2\cdots k.$$

If $x=(x_1,x_2\cdots x_k)$ is a solution of the IVP (3.1), let $x_i(t_0+t) = \tilde{\phi}_i(t_0+t)+y_i(t)$, $t\in[-r,\delta]$, $i=1,2\cdots k$.

Then we have, $x_{i_{t_0+t}} = \tilde{\phi}_{i_{t_0+t}} + y_{i_t}$, $t\in[0,\delta]$, $i=1,2\cdots k$.

Thus,

$$\begin{aligned} y_i(t) &= -g_{i_1}(t_0,\phi_1,\phi_2\cdots\phi_k) - g_{i_2}(t_0,\phi_1,\phi_2\cdots\phi_k) \\ &+ g_{i_1}(t_0+t, y_{1_t}+\tilde{\phi}_{1_{t_0+t}}, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}) \\ &+ g_{i_2}(t_0+t, y_{1_t}+\tilde{\phi}_{1_{t_0+t}}, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}) \\ &+ \frac{1}{\Gamma(\alpha_i)}\int_0^t \psi'(s+t_0)\left[\psi(t+t_0)-\psi(s+t_0)\right]^{\alpha_i-1} \\ &\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s}}, y_{2_s}+\tilde{\phi}_{2_{t_0+s}},\cdots y_{k_s}+\tilde{\phi}_{k_{t_0+s}})ds, \end{aligned} \tag{3.7}$$

$t\in[0,\delta]$, $i=1,2\cdots k$.

Since, g_{i_1}, g_{i_2} are continuous and x_{i_t} is continuous in t for all $i = 1, 2\cdots k$, there exist $\delta' > 0$ such that

$$|g_{i_1}(t_0+t,\, y_{1_t}+\tilde{\phi}_{1_{t_0+t}},\, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}) - g_{i_1}(t_0,\phi_1,\phi_2,\cdots\phi_k)| < \frac{\gamma}{3} \quad (3.8)$$

$$|g_{i_2}(t_0+t, y_{1_t}+\tilde{\phi}_{1_{t_0+t}}, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}) - g_{i_2}(t_0,\phi_1,\phi_2,\cdots\phi_k)| < \frac{\gamma}{3} \quad (3.9)$$

for $0 < t < \delta'$ and $i = 1, 2\cdots k$.

Choose

$$\eta = min\left\{\delta,\, \delta',\, \left(\frac{\gamma\Gamma(\alpha_i)(1+\beta_i)^{(1-\alpha_{i_1})}}{3M_iNK}\right)^{\frac{1}{(1+\beta_i)(1-\alpha_{i_1})}}\right\}, \quad (3.10)$$

where $\beta_i = \frac{\alpha_i-1}{1-\alpha_{i_1}} \in (-1,0)$ and $M_i = ||m_i||_{L^{\frac{1}{\alpha_{i_1}}}(I_0)}$, $i = 1, 2\cdots k$.

Define $E(\eta,\gamma)$ as follows:

$E(\eta,\gamma) =$

$$\Big\{(y_1,y_2\cdots y_k) : y_i \in C([-r,\eta],\mathbb{R}^n)|y_i(s) = 0 \,\text{for}\, s \in [-r,0]\; and\; ||y_i|| \leq r,\, i = 1,2\cdots k\Big\}.$$

Then $E(\eta,\gamma)$ is a closed, bounded and convex subset of $C([-r,\eta],\mathbb{R}^n)\times C([-r,\eta],\mathbb{R}^n)\times \cdots \times C([-r,\eta],\mathbb{R}^n)$.

On $E(\eta,\gamma)$, we define the operators S and U by:

$$S(y_1,y_2\cdots y_k)(t) = \begin{pmatrix} S_1(y_1,y_2\cdots y_k)(t) \\ S_2(y_1,y_2\cdots y_k)(t) \\ \vdots \\ S_k(y_1,y_2\cdots y_k)(t) \end{pmatrix},$$

and

$$U(y_1,y_2\cdots y_k)(t) = \begin{pmatrix} U_1(y_1,y_2\cdots y_k)(t) \\ U_2(y_1,y_2\cdots y_k)(t) \\ \vdots \\ U_k(y_1,y_2\cdots y_k)(t) \end{pmatrix}.$$

$$S_i(y_1,y_2\cdots y_k)(t)=\begin{cases}0, & t\in[-r,0]\\ -g_{i_1}(t_0,\phi_1,\phi_2\cdots\phi_k) & \\ +g_{i_1}(t_0+t,\, y_{1_t}+\tilde{\phi}_{1_{t_0+t}},\, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}), & t\in[0,\eta]\end{cases}.$$

and

$$U_i(y_1,y_2\cdots y_k)(t)=\begin{cases}0, & t\in[-r,0]\\ -g_{i_2}(t_0,\phi_1,\phi_2\cdots\phi_k) & \\ +g_{i_2}(t_0+t,\, y_{1_t}+\tilde{\phi}_{1_{t_0+t}},\, y_{2_t}+\tilde{\phi}_{2_{t_0+t}}\cdots y_{k_t}+\tilde{\phi}_{k_{t_0+t}}) & \\ +\frac{1}{\Gamma(\alpha_i)}\int_0^t \psi'(s+t_0)\left(\psi(t+t_0)-\psi(s+t_0)\right)^{\alpha_i-1} & \\ f_i(t_0+s,\, y_{1_s}+\tilde{\phi}_{1_{t_0+s}},\, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}\cdots y_{k_s}+\tilde{\phi}_{k_{t_0+s}})ds, & t\in[0,\eta]\end{cases}$$

for $i=1,2\cdots k$.

It is easy to see that the operator equation $y=Sy+Uy$ has a solution $y=(y_1,y_2\cdots y_k)\in E(\eta,\gamma)$ if and only if y_i is a solution of (3.7), $\forall i=1,2\cdots k$.

Thus, $x_i(t_0+t)=y_i(t)+\tilde{\phi}_i(t_0+t)$ is a solution of equation (3.1) on $[0,\eta]$. Therefore the existence of a solution of the IVP (3.1) is equivalent to the existence of a fixed point for the operator $S+U$ on $E(\eta,\gamma)$. Hence, it is sufficient to show that $S+U$ has a fixed point in $E(\eta,\gamma)$.

The proof is divided into three steps.

Step I: $Sz+Uy\in E(\eta,\gamma)$, for every pair $z=(z_1,\, z_2\cdots z_k)$, $y=(y_1,\, y_2\cdots y_k)\in E(\eta,\gamma)$.

In fact, for every pair $z,\, y\in E(\eta,\,\gamma)$, $S_iz+U_iy\in C([-r,\,\eta],\mathbb{R}^n)$, $i=1,2\cdots k$. Also it is obvious that $(Sz+Uy)(t)=0$, $\forall t\in[-r,\,0]$.

Also, we have:

$$
\begin{aligned}
&|S_i z(t) - U_i y(t)| \leq \\
&|-g_{i_1}(t_0, \phi_1, \phi_2 \cdots \phi_k) + g_{i_1}(t_0+t, z_{1_t} + \tilde{\phi}_{1_{t_0+t}}, z_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, z_{k_t} + \tilde{\phi}_{k_{t_0+t}})| \\
&+ |-g_{i_2}(t_0, \phi_1, \phi_2 \cdots \phi_k) + g_{i_2}(t_0+t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}})| \\
&+ \frac{1}{\Gamma(\alpha_i)} \int_0^t |\psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha_i - 1} \times \\
&\qquad\qquad f_i(t_0+s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s})|ds} \\
&\leq \frac{2\gamma}{3} + \frac{1}{\Gamma(\alpha_i)} \left(\int_0^t |\psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha_i - 1}|^{\frac{1}{1-\alpha_{i_1}}} ds \right)^{1-\alpha_{i_1}} \\
&\qquad\qquad \left(\int_{t_0}^{t_0+t} (m_i(s))^{\frac{1}{\alpha_{i_1}}} ds \right)^{\alpha_{i_1}} \\
&\leq \frac{2\gamma}{3} + \frac{M_i K N^{\alpha_i - 1}}{\Gamma(\alpha_i)} \frac{\eta^{(1+\beta_i)(1-\alpha_{i_1})}}{(1+\beta_i)^{1-\alpha_{i_1}}} \\
&\leq \gamma,\ \forall t \in [0,\eta] \text{ and } i = 1, 2, \cdots k.
\end{aligned}
$$

Therefore

$$||S_i z + U_i y|| = \sup_{t\in[0,\eta]} |(S_i z)(t) + (U_i y)(t)| \leq \gamma,\ \forall i = 1, 2, \cdots k.$$

Which means that $Sz + Uy \in E(\eta\,, \gamma)$, for any $z, y \in E(\eta, \gamma)$.

Step II: S *is a contraction on* $E(\eta, \gamma)$.

Let $y' = (y'_1, y'_2, \cdots, y'_k)$, $y'' = (y''_1, y''_2, \cdots, y''_k) \in E(\eta, \gamma)$.Then,

$$((y'_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y'_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y'_{k_t} + \tilde{\phi}_{k_{t_0+t}}), (y''_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y''_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y''_{k_t} + \tilde{\phi}_{k_{t_0+t}}))$$

is an element of $A(\delta, \gamma)$. Also by $(H5)$, we get that

$$
\begin{aligned}
|S_i y'(t) - S_i y''(t)| = &|g_{i_1}(t_0+t, y'_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y'_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y'_{k_t} + \tilde{\phi}_{k_{t_0+t}}) \\
&- g_{i_1}(t_0+t, y''_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y''_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y''_{k_t} + \tilde{\phi}_{k_{t_0+t}})| \leq l_i ||y' - y''||_*.
\end{aligned}
$$

Which implies that $||Sy' - Sy''||_* \leq l||y' - y''||_*$ where $l = max\{l_1, l_2, \cdots, l_k\}$.

In view of $0 < l < 1$, S is a contraction on $E(\eta, \gamma)$.

Step III: Now, we show that U is a completely continuous operator.

$$U_{i_1}(y_1, y_2, \cdots, y_k)(t) =$$
$$\begin{cases} 0, & t \in [-r, 0], \\ -g_{i_2}(t_0, \phi_1, \phi_2, \cdot, s\phi_k) & \\ +g_{i_2}(t_0 + t, y_{1_t} + \tilde{\phi}_{1_{t_0+t}}, y_{2_t} + \tilde{\phi}_{2_{t_0+t}}, \cdots, y_{k_t} + \tilde{\phi}_{k_{t_0+t}}), & t \in [0, \eta]. \end{cases}$$

and

$$U_{i_2}(y_1, y_2, \cdots, y_k)(t) =$$
$$\begin{cases} 0, & t \in [-r, 0] \\ \dfrac{1}{\Gamma(\alpha_i)} \displaystyle\int_0^t \psi'(s + t_0)\,[\psi(t + t_0) - \psi(s + t_0)]^{\alpha_i - 1} & \\ f_i(t_0 + s, y_{1_s} + \tilde{\phi}_{1_{t_0+s}}, y_{2_s} + \tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s} + \tilde{\phi}_{k_{t_0+s}})ds, & t \in [0, \eta] \end{cases}$$

for $i = 1, 2, \cdots, k$.

Clearly U= $\begin{pmatrix} U_{11} + U_{12} \\ U_{21} + U_{22} \\ \vdots \\ U_{k1} + U_{k2} \end{pmatrix}$.

Since g_{i_2} is completely continuous for all $i = 1, 2, \cdots, k$; U_{i1} is continuous and also $\{U_{i1}(y) : y \in E(\eta, \gamma)\}$ is uniformly bounded. By using the condition (H6), it is easy to check that $\{U_{i1}(y) : y \in E(\eta, \gamma)\}$ is a completely continuous operator.

On the other hand

$$
\begin{aligned}
|U_{i_2}y(t)| &\le \frac{1}{\Gamma(\alpha_i)}\int_0^t |\psi'(s+t_0)\,[\psi(t+t_0)-\psi(s+t_0)]^{\alpha_i-1} \\
&\qquad\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s}}, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}, \cdots, y_{k_s}+\tilde{\phi}_{k_{t_0+s}})|ds \\
&\le \frac{1}{\Gamma(\alpha_i)}\left(\int_0^t |\psi'(s+t_0)\,[\psi(t+t_0)-\psi(s+t_0)]^{\alpha_i-1}|^{\frac{1}{1-\alpha_{i_1}}}\right)^{1-\alpha_{i_1}} \\
&\qquad\qquad \left(\int_0^t (m_i(s))^{\frac{1}{\alpha_{i_1}}}\,ds\right)^{\alpha_{i_1}} \\
&\le \frac{1}{\Gamma(\alpha_i)}\frac{\eta^{(1+\beta_i)(1-\alpha_{i_1})}}{(1+\beta_i)^{1-\alpha_{i_1}}}\frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}, \quad \forall t\in[0,\eta], \quad i=1,2,\cdots,k.
\end{aligned}
$$

Hence $\{U_{i_2}(y): y\in E(\eta,\gamma)\}$ is uniformly bounded.

Now we will prove that $\{U_{i_2}y: y\in E(\eta,\gamma)\}$ is equicontinuous.

For any $0\le t_1<t_2\le\eta$ and $y\in E(\eta,\gamma)$, we get

$$
\begin{aligned}
&|U_{i_2}y(t_2)-U_{i_2}y(t_1)| \\
&\le \frac{1}{\Gamma(\alpha_i)}\int_0^{t_1} |\psi'(s+t_0)\left[(\psi(t_2+t_0)-\psi(s+t_0))^{\alpha_i-1}-(\psi(t_1+t_0)-\psi(s+t_0))^{\alpha_i-1}\right] \\
&\qquad\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s)}},\, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}, \cdots,\, y_{k_s}+\tilde{\phi}_{k_{t_0+s)}}|ds \\
&+\frac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2} |\psi'(s+t_0)\,[\psi(t_2+t_0)-\psi(s+t_0)]^{\alpha_i-1} \\
&\qquad\qquad f_i(t_0+s, y_{1_s}+\tilde{\phi}_{1_{t_0+s}},\, y_{2_s}+\tilde{\phi}_{2_{t_0+s}}, \cdots,\, y_{k_s}+\tilde{\phi}_{k_{t_0+s}})|ds \\
&\le \frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}\left[\int_0^{t_1}(t_1-s)^{\beta_i}-(t_2-s)^{\beta_i}ds\right]^{\alpha_i-1}+\frac{M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)}\left[\int_{t_1}^{t_2}(t_2-s)^{\beta_i}ds\right]^{1-\alpha_{i_1}} \\
&\le \frac{2M_iKN^{\alpha_i-1}}{\Gamma(\alpha_i)(\beta_i+1)^{1-\alpha_{i_1}}}(t_2-t_1)^{(1+\beta_i)(1-\alpha_{i_1})},
\end{aligned}
$$

which means that $\{U_{i2}y: y\in E(\eta,\gamma)\}$ is equicontinuous. Moreover, it is clear that U_2 is continuous. So U_2 is completely continuous operator. Then $U=U_1+U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\eta,\gamma)$ and hence the IVP (3.1) has a solution $x=(x_1, x_2,\cdots, x_k)$ where $x_i(t) = \phi_i(0)+y_i(t-t_0)$ for all $t\in[t_0,t_0+\eta]$, $i=1,2,\cdots,k$.

This completes the proof.

In the case where $g_{i_1}\equiv 0$, $\forall i=1, 2, \cdots, k$, we get the following result.

Corollary 3.2.3. *Assume that there exist $\delta\in(0,a)$ and $\gamma\in(0,\infty)$ such that $(H1)-(H3)$ hold, g_{i1} is continuous for all $i=1, 2, \cdots, k$ and*

$$|g_i(t,x_t)-g_i(t,y_t))|\leq l_i||x-y||_*, \forall x=(x_1, x_2,\cdots, x_k), y=(y_1, y_2, \cdots, y_k)\in A(\delta,\gamma)$$

and $t\in I_0$ where $l_i\in(0,1)$ is a constant for all $i=1, 2,\cdots, k$. Then IVP (3.1) has at least one solution on $[t_0,t_0+\eta]$ for some positive number η.

In the case where $g_{i_2}\equiv 0$, we have the following result.

Corollary 3.2.4. *Assume that there exist $\delta\in(0, a)$ and $\gamma\in(0, \infty)$ such that $(H1)-(H3)$ hold, g_i is completely continuous for all $i=1, 2, \cdots, k$ and the family $\{t\to g_i(t,x_t):(x_1,x_2,\cdots x_k)\in\Lambda\}$ is equicontinuous on $C(I_0,\mathbb{R}^n)\times C(I_0,\mathbb{R}^n)\times\cdots\times C(I_0,\mathbb{R}^n)$ for all bounded set Λ in $A(\delta,\gamma)$. Then IVP (3.1) has at least one solution on $(t_0, t_0+\eta)$ for some positive number η.*

3.3 Uniqueness result

Theorem 3.3.1. *Assume that the functions f and g are Lipschitz continuous with respect to the second variable, that is, there exist positive constants L_{i1} and L_{i2} such that*

$$\|f_i(t,x_{it})-f_i(t,x_{i2t})\|\leq L_{i1} \text{ and } \|g_i(t,x_{it})-g_i(t,x_{i2t})\|\leq L_{i2}.$$

Then there is a constant $h \in \mathbb{R}^+$ such that there exists a unique solution to the IVP (3.1) on the interval $[t_0, t_0+h] \subseteq [a,b]$ if $\left(\frac{L_{i1}}{\Gamma(\alpha_i+1)}(\psi(t_0+h)-\psi(t_0))_i^{\alpha}+L_{i2}\right) < 1$.

Proof. For $t \in I_0$, define the function F by:

$$F_i(x,t) = \phi_i(0) - g_{i1}(t_0,\, \phi_1,\, \phi_2, \ldots,\, \phi_k) - g_{i2}(t_0,\, \phi_1,\, \phi_2, \ldots,\, \phi_k) + g_{i1}(t,x_t) + g_{i2}(t,x_t) + \frac{1}{\Gamma(\alpha_i)} \int_{t_0}^{t} \psi'(s) \left(\psi(t)-\psi(s)\right)^{\alpha_i-1} f_i(s,\, x_s) ds.$$

Let $U = \{x_i \in C([t_0-r, t_0+a], \mathbb{R}^n) : {}^C D_{t_0}^{\alpha_i,\psi} x_i(t)$ exists and is continuous in $[t_0, t_0+h]\}$. It is enough to prove that $F_i : U \to U$ is a contraction.

Let us see that F_i is well defined, i.e., $F_i(U) \subseteq U$.

Given the function $x_i \in U$, we can see that ${}^C D_{t_0}^{\alpha_i,\psi}(F_i(x_i)(t) - g_i(x_{it})) = f_i(t, x_{it})$ is continuous and $F_i(x_i)(t) = I_{t_0}^{\alpha_i,\psi} f_i(t,x_{it}) + g_i(t,x_{it})$.

Now let $x_{i1}, x_{i2} \in U$ be arbitrary, then by assumptions (H_1), (H_2), we have

$$\begin{aligned}\|F_i(x_{i1}) - F_i(x_{i2})\| &\leq \|I_{t_0}^{\alpha_i,\psi}(f_i(t,x_{i1t}) - f_i(t,x_{i2t}))\| + \|g_i(t,x_{i1t}) - g_i(t,x_{i2t})\| \\ &\leq \left[\frac{L_{i1}}{\Gamma(\alpha_i+1)}\left(\psi(t_0+h)-\psi(t_0)\right)_i^{\alpha} + L_{i2}\right] \|x_{i1} - x_{i2}\|,\end{aligned}$$

which proves that F_i is a contraction. By the Banach fixed point theorem, we get the result of the theorem. □

3.4 Example

Here, we give an example to demonstrate our results. Consider the 3-dimensional system of $\psi-$Caputo neutral fractional differential equations,

$$\begin{cases} D^{\frac{1}{2},x} & \left(x_1(t) - \frac{e^{-3t}}{12\sqrt{6400+t^4}}(\sin x_1(t) + \cos x_2(t) + \sin x_3(t))\right) \\ & = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}} \frac{\sin^4(x_1(t))}{1+|(x_2(t))|} \times \frac{(x_3(t))^2}{1+|x_3(t)|^3} \\ D^{\frac{1}{4},x} & \left(x_2(t) - \frac{1}{12\sqrt{3600+t^2}}\left(\cos x_1(t) + \frac{|x_2(t)|}{2+|x_2(t)|} + \frac{|x_3(t)|}{4+|x_3(t)|}\right)\right) \\ & = \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}} \frac{\cos^2(x_1(t))}{1+\sin^4(x_3(t))+(x_2(t))^2} \\ D^{\frac{1}{3},x} & (x_3(t) - \left(\frac{e^t}{18} + \frac{\cos^2 x_1(t)}{9} + \frac{25|x_2(t)|}{10+|x_2(t)|}\right) \\ & = \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}} \frac{|x_1(t))|}{1+(x_1(t))^2+6|x_2(t)|^5} \\ & \text{for } t \in (0,1) \end{cases}$$

$$x_{i_0} = t,\ i = 1,2,3,\ t \in [-1,0].$$

Define the maps

$$\begin{aligned} f_1(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}} \frac{\sin^4(x_1(t-1))}{1+|(x_2(t-1))|} \times \frac{(x_3(t-1))^2}{1+|x_3(t-1)|^3} \\ f_2(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}} \frac{\cos^2(x_1(t-1))}{1+\sin^4(x_3(t-1))+(x_2(t-1))^2}) \\ f_3(t,x_1,x_2,x_3) &= \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}} \frac{|x_1(t))|}{1+(x_1(t))^2+6|x_2(t)|^5} \\ g_1(t,x_1,x_2,x_3) &= \frac{e^{-3t}}{12\sqrt{6400+t^4}}(\sin x_1(t) + \cos x_2(t) + \sin x_3(t)) \\ g_2(t,x_1,x_2,x_3) &= \frac{1}{12\sqrt{3600+t^2}}\left(\cos x_1(t) + \frac{|x_2(t)|}{2+|x_2(t)|} + \frac{|x_3(t)|}{4+|x_3(t)|}\right) \\ g_3(t,x_1,x_2,x_3) &= \frac{e^t}{18} + \frac{\cos^2 x_1(t)}{9} + \frac{25|x_2(t)|}{10+|x_2(t)|} \end{aligned}$$

with $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{3}$, $\psi(x) = x$.

For, $m_1(t) = \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}(t+3)^{\frac{-3}{4}}$, $m_2(t) = \frac{\Gamma(\frac{3}{8})}{\Gamma(\frac{1}{8})}(t+\frac{3}{2})^{\frac{-7}{8}}$, $m_3(t) = \frac{\Gamma(\frac{5}{6})}{\sqrt{\pi}}(t+1)^{\frac{-1}{2}}$, it is easy to check that $|f_1(t,x_{t_1}, x_{t_2}, x_{t_3})(t)| \leq m_1(t)$, $|f_2(t, x_{t_1}, x_{t_2}, x_{t_3})(t)| \leq m_2(t)$, and $|f_3(t, x_{t_1}, x_{t_2}, x_{t_3})(t)| \leq m_3(t)$.

Also $g_1(t, x_{t_1}, x_{t_2}, x_{t_3})$, $g_2(t, x_{t_1}, x_{t_2}, x_{t_3})(t)$ and $g_3(t, x_{t_1}, x_{t_2}, x_{t_3})(t)$ satisfy Lipschitz condition with $l_1 = \frac{1}{320}$, $l_2 = \frac{1}{240}$ and $l_3 = \frac{1}{18}$ respectively.

Thus, all conditions of Theorem (3.2.2) hold and so this system of $\psi-$ Caputo fractional functional differential equation has a solution.

3.5 Conclusion

Different phenomena can be interpreted with the help of systems of equations more effectively than with single equation. In this chapter we have worked on generalized fractional differential operators in $k-$systems and proved the existence and uniqueness of solutions of a $k-$systems of $\psi-$Caputo fractional neutral functional differential equations under the specified conditions using Krasnoselskii's fixed point theorem and Banach's fixed point theorem respectively. Also, we have given an example to illustrate our results.

Chapter 4

A Study on the Solutions of $\psi-$Caputo Fractional Neutral Functional Differential Equation

4.1 Introduction

This chapter is devoted to the study of the existence and uniqueness of solutions for fractional neutral functional differential equation involving the ψ-Caputo type fractional derivative of a function with respect to another function. Further we prove two different types of Ulam stability results of solution for the given initial value problem. Krasnoselskii's and Banach fixed point theorems and generalized Gronwall inequality are used to obtain the results.

Let us consider the following fractional neutral functional differential equation involving the Caputo fractional derivative of a function ζ with respect to another function ψ.

$$\begin{cases} {}^{C}D_{t_0}^{\alpha,\psi}(\zeta(t)-v(t,\zeta_t)) & = u(t,\zeta_t), \quad t\in(t_0,\infty)\, t_0\geq 0 \\ \zeta_{t_0}=\phi \end{cases} \tag{4.1}$$

where ${}^{C}D^{\alpha,\psi}$ is the standard Caputo fractional derivative of order $0<\alpha<1$ with respect to another function ψ. Let $u, v:[t_0,\infty)\times C([-r,0],\mathbb{R}^n)\to\mathbb{R}^n$ be given functions satisfying certain assumptions, which will be specified later. Let $r\in\mathbb{R}^+$, $\eta>0$ and $\phi\in C([-r,0],\mathbb{R}^n)$. If $\zeta\in C([t_0-r,t_0+\tau],\mathbb{R}^n)$, then for any $t\in[t_0,t_0+\eta]$, define ζ_t by $\zeta_t(\theta)=\zeta(t+\theta)$ for $\theta\in[-r,0]$. Let $\psi\in C^1[t_0,\infty)$ be a continuous increasing function such that $\psi'(\zeta)\neq 0, \forall\zeta\in[t_0,\infty)$.

Through this chapter, we are discussing the initial value problem for a class of fractional neutral functional differential equations with bounded delay of a function with respect to another function. Based on many assumptions and Krasnoselskii's fixed point theorem, we prove that IVP (4.1) has at least one solution. To prove the uniqueness, we used Banach fixed point theorem and generalized Gronwall inequality for proving Ulam Hyers and generalized Ulam Hyers stabilities.

4.2 Preliminaries

Lemma 4.2.1. *(Generalized Gronwall's Inequality)[63]*

Let ζ,ϑ be two integrable functions and σ continuous with domain $[a,\infty)$. Let $\psi\in C^1([a,\infty),\mathbb{R})$ increasing function such that $\psi'(t)\neq 0, \forall t\in[a,\infty)$. Assume that ζ,ϑ are non-negative and σ is non-negative and non-decreasing. If

$$\zeta(t)\leq\vartheta(t)+\sigma(t)\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\zeta(s)ds,$$

then, for all $t \in [t_0, \infty)$, we have

$$\zeta(t) \leq \vartheta(t) + \int_0^t \sum_{k=1}^{\infty} \frac{[\sigma(t)\Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(s)(\psi(t) - \psi(s))^{\alpha k - 1} \vartheta(s) ds.$$

Remark 4.2.2. *Consider ϑ, a nondecreasing function on $[a, \infty)$, and under the assumptions of Lemma* (4.2.1)*, then*

$$\zeta(t) \leq \vartheta(t) E_\alpha(\sigma(t)\Gamma(\alpha)(\psi(t) - \psi(a))^\alpha),$$

where $E_\alpha(\cdot)$ is the Mttag-Leffler function defined by

$$E_\alpha(\cdot) = \sum_{k=0}^{\infty} \frac{\sigma^k}{\Gamma(k\alpha + 1)}.$$

4.3 Existence results

Let $I_0 = [t_0, t_0 + \delta]$

$$A(\delta, \gamma) = \{\zeta \in C([t_0 - r, t_0 + \delta], \mathbb{R}^n) / \zeta_{t_0} = \phi, \sup_{t_0 \leq t \leq t_0 + \delta} |\zeta(t) - \phi(0)| \leq \gamma\},$$

where δ, γ are positive constants. Before proving the main results, we introduce the following hypotheses.

(H1) $u(t, \phi)$ is measurable with respect to t on I_0.

(H2) $u(t, \phi)$ is continuous with respect to ϕ on $C([-r, 0], \mathbb{R}^n)$ and there exists $T \in (0, \infty)$ such that $\|u(t, \zeta_t') - u(t, \zeta_t'')\| \leq T \|\zeta_t' - \zeta_t''\|$.

(H3) There exist $\alpha_1 \in (0, \alpha)$ and a real valued function $m(t) \in L^{\frac{1}{\alpha_1}}(I_0)$ such that for any $\zeta \in A(\eta, \gamma), |u(t, \zeta_t)| \leq m(t)$ for $t \in I_0$.

(H4) For any $\zeta \in A(\eta, \gamma), v(t, \zeta_t) = v_1(t, \zeta_t) + v_2(t, \zeta_t)$.

(H5) v_1 is continuous and for any $\zeta', \zeta'' \in A(\eta, \gamma), t \in I_0$,

$\|v_1(t, \zeta_t') - v_1(t, \zeta_t'')\| \leq l\|\zeta' - \zeta''\|$, where $l \in (0,1)$,

$\|v(t, \zeta_t') - v(t, \zeta_t'')\| \leq L\|\zeta_t' - \zeta_t''\|, L \in (0, \infty)$.

(H6) v_2 is completely continuous and for any bounded set Λ in $A(\eta, \gamma)$,

the set $\{t \to v_2(t, \zeta_t) : \zeta \in \Lambda\}$ is equicontinuous in $C(I_0, \mathbb{R}^n)$.

(H7) $\psi \in C^1([t_0, \infty], \mathbb{R})$ is a continuous increasing function with

$|\psi(t) - \psi(s)| \leq N|t - s|, N \in (0,1)$ and $|\psi\prime(s)| < K$, K be any positive integer.

Lemma 4.3.1. *If there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1) - (H3)$ are satisfied, then for $t \in (t_0, t_0 + \delta]$, IVP (4.1) is equivalent to the following equation.*

$$\begin{cases} \zeta(t) = \phi(0) - v(t_0, \phi) + v(t, \zeta_t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1} u(s, \zeta_s) ds, \ t \in I_0 \\ \zeta_{t_0} = \phi. \end{cases} \tag{4.2}$$

Proof. From the conditions $(H1)$ and $(H2)$, it is obvious that $f(t, \zeta_t)$ is Lebesgue measurable on I_0. A direct calculation using $(H7)$ gives that

$$\left(\psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1}\right) \in L^{\frac{1}{1-\alpha_1}}([t_0, t]) \quad t \in I_0.$$

Holder's inequality and $(H3)$ implies that $\left(\psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1}\right) f(s, \zeta_s)$ is Lebesgue integrable with respect to $s \in [t_0, t], \ \forall t \in I_0, \ \zeta \in A(\delta, \gamma)$ and

$$\int_{t_0}^{t} \left(\psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1}\right) u(s, \zeta_s) ds \leq \|\psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1}\|_{L^{\frac{1}{1-\alpha_1}}(I_0)} \|m\|_{L^{\frac{1}{\alpha_1}}(I_0)}, \tag{4.3}$$

where

$$\|F\|_{L^p(J)} = \left(\int_J |\zeta(t)|^p dt\right)^{\frac{1}{p}},$$

for any p integrable function $F : J \to \mathbb{R}$.

According to the definition of fractional integral of function with respect to another function ψ and Caputo derivative of order α, it is easy to see that if ζ is a solution of the IVP (4.1), then ζ is a solution of equation (4.2).

On the other hand, if equation (4.2) is satisfied then $\forall t \in (t_0, t_0 + \delta]$, we have

$$\begin{aligned}
&{}^{C}D_{t_0}^{\alpha,\psi}\left(\zeta(t) - v(t,\zeta_t)\right)\\
&\quad = {}^{C}D_{t_0}^{\alpha,\psi}\left(\phi(0) - v(t_0,\phi) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(s)\left[\psi(t) - \psi(s)\right]^{\alpha-1} u(s,\zeta_s)ds\right)\\
&\quad = {}^{C}D_{t_0}^{\alpha,\psi}\left(\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(s)\left[\psi(t) - \psi(s)\right]^{\alpha-1} u(s,\zeta_s)ds\right).
\end{aligned}$$

Also

$$\begin{aligned}
{}^{C}D_{t_0}^{\alpha,\psi}\left(\zeta(t) - v(t,\zeta_t)\right) &= {}^{C}D_{t_0}^{\alpha,\psi} I_{t_0}^{\alpha,\psi} u(t,\zeta_t)\\
&= {}^{C}D_{t_0}^{\alpha,\psi}\left[I_{t_0}^{\alpha,\psi} u(t,\zeta_t) - \sum_{k=0}^{n-1} \frac{\left[I^{\alpha,\psi} u(t,\zeta_t)\right]^{[k]}(t_0)}{k!}\left(\psi(t) - \psi(t_0)\right)^{k}\right]\\
&= {}^{C}D_{t_0}^{\alpha,\psi} I_{t_0}^{\alpha,\psi} u(t,\zeta_t) - \sum_{k=0}^{n-1} \frac{\left[I^{\alpha,\psi} u(t,\zeta_t)\right]^{[k]}(t_0)}{\Gamma(k-\alpha+1)}\left(\psi(t) - \psi(t_0)\right)^{k-\alpha}\\
&= u(t,\zeta_t) - \left[I^{\alpha,\psi} u(t,\zeta_t)\right]_{t=t_0} \frac{\left(\psi(t) - \psi(t_0)\right)^{-\alpha}}{\Gamma(1-\alpha)} = u(t,\zeta_t).
\end{aligned}$$

Since $[I^{\alpha,\psi} u(t,\zeta_t)]_{t=t_0} = 0$, we get ${}^{C}D_{t_0}^{\alpha,\psi}\left(\zeta(t) - v(t,\zeta_t)\right) = u(t,\zeta_t)$, $t \in (t_0, t_0 + \eta]$. And this completes the proof. □

Theorem 4.3.2. *If there exist $\delta \in (0,a)$ and $\gamma \in (0,\infty)$ such that $(H1) - (H7)$ are satisfied, then IVP (4.1) has at least one solution on $[t_0, t_0 + \eta]$ for some positive number η.*

Proof. According to $(H4)$, equation (4.2) is equivalent to the following equation

$$\begin{cases} \zeta(t) & = \phi(0) - v_1(t_0,\phi) - v_2(t_0,\phi) + v_1(t,\zeta_t) + v_2(t,\zeta_t) \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s) \left[\psi(t) - \psi(s)\right]^{\alpha-1} u(s,\zeta_s) ds, \quad t \in I_0 \\ \zeta_{t_0} & = \phi. \end{cases}$$

Let $\tilde{\phi} \in A(\delta,\gamma)$ be defined as

$$\tilde{\phi}_{t_0} = \phi, \quad \tilde{\phi}(t_0+t) = \phi(0), \quad \forall t \in [0,\delta].$$

If ζ is a solution of the IVP(4.1), let $\zeta(t_0+t) = \tilde{\phi}(t_0+t) + \vartheta(t), t \in [-r,\eta]$.

Then we have, $\zeta_{t_0+t} = \tilde{\phi}_{t_0+t} + \vartheta_t, \quad t \in [0,\eta]$. Thus

$$\begin{aligned} \vartheta(t) &= -v_1(t_0,\phi) - v_2(t_0,\phi) + v_1(t_0+t,\vartheta_t + \tilde{\phi}_{t_0+t}) + v_2(t_0+t,\vartheta_t + \tilde{\phi}_{t_0+t}) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s+t_0) \left[\psi(t+t_0) - \psi(s+t_0)\right]^{\alpha-1} u(t_0+s,\zeta_s + \tilde{\phi}_{t_0+s}) ds, \\ &\qquad t \in [0,\eta]. \end{aligned} \tag{4.4}$$

Since v_1, v_2 are continuous and ζ_t is continuous in t, when $0 < t < \eta'$, there exist $\eta' > 0$ such that

$$|v_1(t_0+t,\vartheta_t + \tilde{\phi}_{t_0+t}) - v_1(t_0,\phi)| < \frac{\gamma}{3}. \tag{4.5}$$

$$|v_2(t_0+t,\vartheta_t + \tilde{\phi}_{t_0+t}) - v_2(t_0,\phi)| < \frac{\gamma}{3}. \tag{4.6}$$

$$\text{Choose } \tau = min\left\{ \eta, \eta', \left(\frac{\gamma\Gamma(\alpha)(1+\beta)^{(1-\alpha_1)}}{3MN^{\alpha-1}K} \right)^{\frac{1}{(1+\beta)(1-\alpha_1)}} \right\}, \tag{4.7}$$

where $\beta = \frac{\alpha-1}{1-\alpha_1} \in (-1,0)$ and $M = ||m||_{L^{\frac{1}{\alpha_1}}(I_0)}$.

Define $E(\tau,\gamma)$ as follows

$$E(\tau,\gamma) = \left\{ \vartheta \in C([-r,\tau],\mathbb{R}^n) / \vartheta(s) = 0 \; for \; s \in [-r,0] \; and \; ||\vartheta|| \le \gamma \right\}.$$

Then $E(\tau,\gamma)$ is a closed, bounded and convex subset of $C([-r,\eta],\mathbb{R}^n)$.

On $E(\tau,\gamma)$, we define the operators S and U as follows:

$$S\vartheta(t) = \begin{cases} 0 & \text{if } t \in [-r, 0] \\ -v_1(t_0, \phi) + v_1(t_0 + t, \vartheta_t + \tilde{\phi}_{t_0+t}) & \text{if } t \in [0, \tau]. \end{cases}$$

$$U\vartheta(t) =$$
$$\begin{cases} 0, & \text{if } t \in [-r, 0] \\ -v_2(t_0, \phi) + v_2(t_0 + t, \vartheta_t + \tilde{\phi}_{t_0+t}) + & \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s + t_0) \left(\psi(t + t_0) - \psi(s + t_0)\right)^{\alpha-1} & u(t_0 + s, \vartheta_s + \tilde{\phi}_{t_0+s}) ds, \\ & \text{if } t \in [0, \tau]. \end{cases}$$

It is easy to see that the operator equation

$$\vartheta = S\vartheta + U\vartheta \tag{4.8}$$

has a solution $\vartheta \in E(\tau, \gamma)$ if and only if ϑ is a solution of (4.4).

Thus $\zeta(t_0 + t) = \vartheta(t) + \tilde{\phi}(t_0 + t)$ is a solution of (4.1) on $[0, \tau]$. Therefore the existence of a solution of the IVP (4.1) is equivalent that equation (4.8) has a fixed point in $E(\tau, \gamma)$.

Now we show that $S + U$ has a fixed point in $E(\tau, \gamma)$.

The proof is divided in to three steps.

Step I: $Sz + Uy \in E(\tau, \gamma)$ for every pair $z, y \in E(\tau, \gamma)$.

In fact, for every pair $z, y \in E(\tau, \gamma)$, $Sz + Uy \in C([-r, \tau], \mathbb{R}^n)$.

Also it is obvious that $(Sz + Uy)(t) = 0$, $t \in [-r, 0]$.

Moreover, for $t \in [0, \tau]$, by (4.5), (4.6), (4.7) and the condition $(H3)$ and $(H7)$, we have:

$$
\begin{aligned}
|Sz(t)+U\vartheta(t)| &\leq |-v_1(t_0,\phi)+v_1(t_0+t,z_t+\tilde{\phi}_{t_0+t})| \\
&\quad +|-v_2(t_0,\phi)+v_2(t_0+t,\vartheta_t+\tilde{\phi}_{t_0+t})| \\
&\quad +\frac{1}{\Gamma(\alpha)}\int_0^t|\psi'(s+t_0)\,[\psi(t+t_0)-\psi(s+t_0)]^{\alpha-1}\,u(t_0+s,\vartheta_s+\tilde{\phi}_{t_0+s})|ds \\
&\leq \frac{2\gamma}{3}++\frac{1}{\Gamma(\alpha)}\left(\int_0^t|\psi'(s+t_0)\,[\psi(t+t_0)-\psi(s+t_0)]^{\alpha-1}|^{\frac{1}{1-\alpha_1}}ds\right)^{1-\alpha_1} \\
&\qquad\qquad \times\left(\int_{t_0}^{t_0+t}(m(s))^{\frac{1}{\alpha_1}}ds\right)^{\alpha_1} \\
&\leq \frac{2\gamma}{3}+\frac{MKN^{\alpha-1}}{\Gamma(\alpha)}\frac{\tau^{(1+\beta)(1-\alpha_1)}}{(1+\beta)^{1-\alpha_1}}\leq\gamma.
\end{aligned}
$$

Therefore

$$
||Sz+U\vartheta||=\sup_{t\in[0,\tau]}|(Sz)(t)+(U\vartheta)(t)|\leq\gamma,
$$

which means that $Sz+U\vartheta\in E(\tau,\gamma)$, for any $z,\vartheta\in E(\tau,\gamma)$.

Step II: S is a contraction on $E(\tau,\gamma)$.

For any $\vartheta',\vartheta''\in E(\eta,\gamma),\ \vartheta'_t+\tilde{\phi}_{t_0+t},\ \vartheta''_t+\tilde{\phi}_{t_0+t}\in A(\eta,\gamma)$.

Also by $(H5)$, we get that

$$
|S\vartheta'(t)-S\vartheta''(t)|=|v_1(t_0+t,\vartheta'_t+\tilde{\phi}_{t_0+t})-v_1(t_0+t,\vartheta''_t+\tilde{\phi}_{t_0+t})|\leq l||\vartheta'-\vartheta''||.
$$

Which implies that

$$
||S\vartheta'-S\vartheta''||\leq l||\vartheta'-\vartheta''||.
$$

In view of $0<l<1$, S is a contraction on $E(\tau,\gamma)$.

Step III: Now we show that U is a completely continuous operator.

$$
U_1\vartheta(t)=\begin{cases}0 & \text{if } t\in[-r,0],\\ -v_2(t_0,\,\phi)+v_2(t_0+t,\,\vartheta_t+\tilde{\phi}_{t_0+t}) & \text{if } t\in[0,\,\tau].\end{cases}
$$

And $U_2\vartheta(t) =$

$$\begin{cases} 0 & \text{if } t \in [-r, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha-1} & u(t_0+s,\, \vartheta_s + \tilde{\phi}_{t_0+s})ds \\ & \text{if } t \in [0, \tau]. \end{cases}$$

Clearly $U = U_1 + U_2$.

Since v_2 is completely continuous, U_1 is continuous and $\{U_1\vartheta : \vartheta \in E(\tau,\gamma)\}$ is uniformly bounded. From the condition that the set $\{t \to v_2(t,\zeta_t) : \zeta \in \Lambda\}$ be equicontinuous for any bounded set Λ in $A(\eta,\gamma)$, we can conclude that U_1 is a completely continuous operator.

On the other hand for any $t \in [0,\tau]$, we have:

$$\begin{aligned} &|U_2\vartheta(t)| \\ &\quad \leq \frac{1}{\Gamma(\alpha)} \int_0^t |\psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha-1} u(t_0+s, \vartheta_s + \tilde{\phi}_{t_0+s})|ds \\ &\quad \leq \frac{1}{\Gamma(\alpha)} \left(\int_0^t |\psi'(s+t_0)\,[\psi(t+t_0) - \psi(s+t_0)]^{\alpha-1}|^{\frac{1}{1-\alpha_1}} \right)^{1-\alpha_1} \\ &\qquad\qquad\qquad\qquad \times \left(\int_{t_0}^{t+t_0} (m(s))^{\frac{1}{\alpha_1}} ds \right)^{\alpha_1} \\ &\quad \leq \frac{\tau^{(1+\beta)(1-\alpha_1)}}{(1+\beta)^{1-\alpha_1}} \frac{MKN^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

Hence $\{U_2(\vartheta) : \vartheta \in E(\tau,\gamma)\}$ is uniformly bounded.

Now we will prove that $\{U_2\vartheta : \vartheta \in E(\tau,\gamma)\}$ is equicontinuous.

For any $0 \leq t_1 < t_2 \leq \tau$ and $\vartheta \in E(\tau,\gamma)$, we get that:

$$\begin{aligned} &|U_2\vartheta(t_2) - U_2\vartheta(t_1)| \leq \\ &\frac{1}{\Gamma(\alpha)} \int_0^{t_1} |\psi'(s+t_0) u(t_0+s, \vartheta_s + \tilde{\phi}_{t_0+s)}| \\ &\times \left[(\psi(t_2+t_0) - \psi(s+t_0))^{\alpha-1} - (\psi(t_1+t_0) - \psi(s+t_0))^{\alpha-1} \right] ds \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}|\psi'(s+t_0)\,[\psi(t_2+t_0)-\psi(s+t_0)]^{\alpha-1}u(t_0+s,\vartheta_s+\tilde{\phi}_{t_0+s})|ds$$

$$\leq \frac{MKN^{\alpha-1}}{\Gamma(\alpha)}\left(\left[\int_0^{t_1}(t_1-s)^{\beta}-(t_2-s)^{\beta}ds\right]^{\alpha-1}+\left[\int_{t_1}^{t_2}(t_2-s)^{\beta}ds\right]^{1-\alpha_1}\right)$$

$$\leq \frac{2MKN^{\alpha-1}}{\Gamma(\alpha)(\beta+1)^{1-\alpha_1}}(t_2-t_1)^{(1+\beta)(1-\alpha_1)}.$$

Hence we can conclude that, $\{U_2\vartheta : \vartheta \in E(\tau,\gamma)\}$ is equicontinuous. Moreover, it is clear that U_2 is continuous. So U_2 is completely continuous operator. Then $U = U_1 + U_2$ is a completely continuous operator.

Therefore, Krasnoselskii's fixed point theorem shows that $S+U$ has a fixed point on $E(\tau,\gamma)$ and hence the IVP (4.1) has a solution $\zeta(t) = \phi(0)+\vartheta(t-t_0)$ for all $t \in [t_0,t_0+\tau]$.

This completes the proof. □

In the case where $v_1 \equiv 0$, we get the following result:

Corollary 4.3.3. *Assume that there exist* $\eta \in (0,a)$ *and* $\gamma \in (0,\infty)$ *such that* $(H1)-(H3)$ *and* $(H7)$ *hold and* $(H5)'$: *v is continuous and for any* $\zeta',\zeta'' \in A(\eta,\gamma)$, $t \in I_0$, $|v(t,\zeta_t')-v(t,\zeta_t''))| \leq l||\zeta'-\zeta''||$, $l \in (0,1)$. *Then IVP* (4.1) *has at least one solution on* $[t_0,\, t_0+\tau]$ *for some positive number* τ.

In the case where $v_2 \equiv 0$, we have the following result:

Corollary 4.3.4. *Assume that there exist* $\eta \in (0,\, a)$ *and* $\gamma \in (0,\, \infty)$ *such that* $(H1)-(H3),(H7)$ *hold and* $(H6)'$: *v is completely continuous and for any bounded set* Λ *in* $A(\eta,\,\gamma)$, *the set* $\{t \to v(t,\,\zeta_t) : \zeta \in \Lambda\}$ *is equicontinuous on* $C(I_0,\,\mathbb{R}^n)$. *Then IVP* (4.1) *has at least one solution on* $[t_0,\, t_0+\tau]$ *for some positive number* τ.

4.4 Uniqueness result

Theorem 4.4.1. *Assume that the assumptions* (H_2) *and* (H_5) *are satisfied and if* $\tau < \left(\psi^{-1}\left[\psi(t_0)+\left(\frac{\Gamma(\alpha+1)}{T+L}\right)^{1/\alpha}\right]-t_0\right)$, *then the IVP*$(4.1)$ *has unique solution on the subinterval* $[t_0, t_0+\tau]$.

Proof. For $t \in I_0$, define the operator F as follows:

$$(F\zeta)(t) = \phi(0) - v(t_0,\phi) + v(t,\zeta_t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} u(s,\zeta_s)ds. \tag{4.9}$$

For each $t \in [t_0, t_0+\eta]$; $\zeta, \zeta' \in C([t_0-r, t_0+\tau], \mathbb{R}^n)$, we have:

$$\begin{aligned}
&\|F\zeta(t) - F\zeta'(t)\| \\
&\leq \|v(t,\zeta_t) - v(t,\zeta'_t)\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|u(s,\zeta_s)-u(s,\zeta'_s)\|ds \\
&\leq T\|\zeta_t - \zeta'_t\| + \frac{L}{\Gamma(\alpha)}\int_{t_0}^{t} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|\zeta_t-\zeta'_t\|ds \\
&\leq (T+L)\|\zeta_t-\zeta'_t\|\frac{(\psi(t)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}$$

Hence,

$$\|F\zeta(t) - F\zeta'(t)\| \leq (T+L)\|\zeta_t-\zeta'_t\|\frac{(\psi(t_0+\tau)-\psi(t_0))^{\alpha}}{\Gamma(\alpha+1)}.$$

Also, since $\tau < \left(\psi^{-1}\left[\psi(t_0)+\left(\frac{\Gamma(\alpha+1)}{T+L}\right)^{1/\alpha}\right]-t_0\right)$, we have

$$\|F\zeta(t) - F\zeta'(t)\| \leq \sigma\|\zeta_t-\zeta'_t\|, 0 \leq \sigma < 1,$$

where, $\sigma = \frac{(T+L)}{\Gamma(\alpha+1)}[\psi(t_0+\tau)-\psi(t_0)]^{\alpha}$.

This shows that F is a contraction mapping. Hence, by using Banach fixed point theorem, there exist a local unique solution $\zeta \in C([t_0-r, t_0+\tau], \mathbb{R}^n)$. □

4.5 Stability analysis

Definition 4.5.1. *[60] The problem* (4.1) *is UH stable if there exist a real number* $\lambda_u > 0$ *such that for every* $\varepsilon > 0$ *and for each solution* $\tilde{\zeta} \in C([I_0, \mathbb{R}^n])$ *of the following inequality*

$$\|{}^C D_{t_0+}^{\alpha,\psi}(\tilde{\zeta}(t) - v(t, \tilde{\zeta}_t)) - u(t, \tilde{\zeta}_t)\| \leq \varepsilon, \tag{4.10}$$

there exists a solution $\zeta \in C([I_0, \mathbb{R}^n])$ *satisfying the IVP (4.1) with*

$$\|\tilde{\zeta}(t) - \zeta(t)\| \leq \lambda_u \varepsilon, t \in I_0.$$

Definition 4.5.2. *[60] The problem* (4.1) *GUH stable if there exists* $\tau_u \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\tau_u(t_0) = 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\tilde{\zeta} \in C([I_0, \mathbb{R}^n])$ *of the inequality* (4.10) *there exists a solution* $\zeta \in C([I_0, \mathbb{R}^n])$ *of the problem* (4.1) *with*

$$\|\tilde{\zeta}(t) - \zeta(t)\| \leq \tau_u(\varepsilon), t \in I_0.$$

Definition 4.5.3. *[60] The problem* (4.1) *is UHR stable with respect to* $\tau \in C(I_0, \mathbb{R})$*, if there exists a real number* $\sigma_{u,\tau} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\tilde{\zeta} \in C([I_0, \mathbb{R}^n])$ *of the inequality*

$$\|{}^C D_{t_0+}^{\alpha,\psi}(\tilde{\zeta}(t) - v(t, \tilde{\zeta}_t)) - u(t, \tilde{\zeta}_t)\| \leq \varepsilon \tau(t), t \in I_0, \tag{4.11}$$

there exists a solution $\zeta \in C([I_0, \mathbb{R}^n])$ *of the problem* (4.1) *such that*

$$\|\tilde{\zeta}(t) - \zeta(t)\| \leq \sigma_{u,\tau} \varepsilon \tau(t), t \in I_0.$$

Definition 4.5.4. *[60] The problem* (4.1) *is GUHR stable with respect to* $\tau \in C(I_0, \mathbb{R})$*, if there exists a real number* $\sigma_{u,\tau} > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\tilde{\zeta} \in C([I_0, \mathbb{R}^n])$ *of the inequality*

$$\|{}^C D_{t_0+}^{\alpha,\psi}(\tilde{\zeta}(t) - v(t, \tilde{\zeta}_t)) - u(t, \tilde{\zeta}_t)\| \leq \tau(t), t \in I_0, \tag{4.12}$$

there exists a solution $\zeta \in C([I_0,\mathbb{R}^n])$ *of the problem* (4.1) *such that*

$$\|\tilde{\zeta}(t)-\zeta(t)\| \leq \sigma_{u,\tau}\tau(t),\, t \in I_0.$$

Remark 4.5.5. *[60] A function* $\tilde{\zeta} \in C([I_0,\mathbb{R}^n])$ *is a solution of the inequality (4.10) if and only if there exist a function* $w \in C([I_0,\mathbb{R}^n])$*, where* w *depends on the solution of* $\tilde{\zeta}$ *such that:*

(i) $\|w(t)\| \leq \varepsilon, \forall t \in I_0,$

(ii) ${}^{C}D^{\alpha,\psi}_{t_0+}(\tilde{\zeta}(t)-v(t,\tilde{\zeta}_t)) = u(t,\tilde{\zeta}_t)+w(t).$

Remark 4.5.6. *[60] A function* $\tilde{\zeta} \in C([I_0,\mathbb{R}^n])$ *is a solution of the inequality (4.11) if and only if there exist two functions* $w \in C([I_0,\mathbb{R}^n])$ *and* $\tau \in C([I_0,\mathbb{R}])$*,* w *depends on solution* $\tilde{\zeta}$ *such that*

(i) $\|w(t)\| \leq \varepsilon\tau(t), \forall t \in I_0,$

(ii) ${}^{C}D^{\alpha,\psi}_{t_0+}(\tilde{\zeta}(t)-v(t,\tilde{\zeta}_t)) = u(t,\tilde{\zeta}_t)+w(t).$

Theorem 4.5.7. *Under the assumptions* $(H_2)-(H_5)$ *and if the inequality* (4.10) *is satisfied, then the problem (4.1) is UH stable in* $C(I_0,\mathbb{R}^n)$*.*

Proof. Let $\varepsilon > 0$ and let $\tilde{\zeta} \in C([I_0,\mathbb{R}^n])$ be a function which satisfies the inequality and let $\zeta \in C([I_0,\mathbb{R}^n])$ be a local unique solution of the problem (4.1).
In view of the Lemma (4.2), we have

$$\begin{aligned}
&\|\tilde{\zeta}(t)-\phi(0)-v(t_0,\phi)-v(t,\tilde{\zeta}_t)-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s,\tilde{\zeta}_s)ds\| \\
&\qquad = \|\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}w(s)ds\| \\
&\qquad \leq \frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{\Gamma(\alpha+1)}\varepsilon.
\end{aligned}$$

From our assumptions, we obtain:

$$
\begin{aligned}
&\|\tilde{\zeta}(t)-\zeta(t)\| \\
=&\|\tilde{\zeta}(t)-\phi(0)-v(t_0,\phi)-v(t,\zeta_t)-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s,\zeta_s)ds\| \\
\leq&\|\tilde{\zeta}(t)-\phi(0)-v(t_0,\phi)-v(t,\tilde{\zeta}_t)-\int_{t_0}^{t}\frac{1}{\Gamma(\alpha)}\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}u(s,\tilde{\zeta}_s)ds\| \\
+&\|v(t,\tilde{\zeta}_t)-v(t,\zeta_t)\|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}\|u(s,\tilde{\zeta}_s)-u(s,\zeta_s)\|ds \\
\leq&\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{\Gamma(\alpha+1)}\varepsilon+L\|\tilde{\zeta}_t-\zeta_t\| \\
&\qquad+\frac{M}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}\|\tilde{\zeta}_s-\zeta_s\|ds.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&\|\tilde{\zeta}_t-\zeta_t\|\leq \\
&\quad\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon+\frac{M}{(1-L)\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)[\psi(t)-\psi(s)]^{\alpha-1}\|\tilde{\zeta}_s-\zeta_s\|ds.
\end{aligned}
$$

Using Gronwall's inequality,

$$
\begin{aligned}
\|\tilde{\zeta}_t-\zeta_t\|\leq&\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon+ \\
&\int_{t_0}^{t}\sum_{k=1}^{\infty}\frac{M^k}{(1-L)^k}\psi'(s)[\psi(t)-\psi(s)]^{\alpha k-1}\frac{1}{\Gamma(\alpha k)}\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon\,ds \\
\leq&\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon\times \\
&\qquad\left[1+\int_{t_0}^{t}\sum_{k=1}^{\infty}\frac{M^k}{\Gamma(\alpha k)(1-L)^k}\psi'(s)(\psi(t)-\psi(s))^{\alpha k-1}ds\right] \\
\leq&\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon\left[1+\sum_{k=1}^{\infty}\frac{M^k}{(1-L)^k}\frac{[\psi(t_0+\delta)-\psi(t_0)]^{\alpha k}}{\Gamma(\alpha k+1)}\right] \\
\leq&\frac{(\psi(t_0+\delta)-\psi(t_0))^{\alpha}}{(1-L)\Gamma(\alpha+1)}\varepsilon E_{\alpha}\left[\frac{M}{1-L}(\psi(t_0+\delta)-\psi(t_0))^{\alpha}\right].
\end{aligned}
$$

For

$$
\lambda_u=\frac{(\psi(t_0+\delta)-\psi(t_0))^{\alpha}}{(1-L)\Gamma(\alpha+1)}E_{\alpha}\left[\frac{M}{1-L}(\psi(t_0+\delta)-\psi(t_0))^{\alpha}\right],\ t\in I_0,
$$

we get $\|\tilde{\zeta}_t - \zeta_t\| \leq \varepsilon$.

Hence the problem is UH stable. □

We consider the following hypothesis:

(H8) There exist an increasing function $\tau \in C(I_0, \mathbb{R})$ and there exist $\lambda_\tau > 0$ such that $\forall t \in I_0, I_{t_0+}^{\alpha;\psi} \tau(t) \leq \lambda_\tau \tau(t)$.

Lemma 4.5.8. *Let $\tilde{\zeta} \in C(I_0, \mathbb{R}^n)$ is a solution of the inequality (4.11), the $\tilde{\zeta}$ is a solution of the following integral inequality*

$$\|\tilde{\zeta}(t) - \phi(0) - v(t_0, \phi) - v(t, \tilde{\zeta}_t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s, \tilde{\zeta}_s) ds\| \leq \varepsilon \lambda_\tau \tau(t).$$

Proof. By Remark (4.5.6), we have that

$${}^{C}\mathbb{D}_{t_0+}^{\alpha,\psi}(\tilde{\zeta}(t) - v(t, \tilde{\zeta}_t)) = u(t, \tilde{\zeta}_t) + w(t), t \in I_0.$$

In view of Theorem (4.4.1) and using Remark (4.5.6), we have

$$\begin{aligned}\tilde{\zeta}(t) = \phi(0) + v(t_0, \phi) + v(t, \tilde{\zeta}_t) &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s, \tilde{\zeta}_s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} w(s) ds.\end{aligned}$$

$$\begin{aligned}\|\tilde{\zeta}(t) - \phi(0) - v(t_0, \phi) - v(t, \tilde{\zeta}_t) - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s, \tilde{\zeta}_s) ds\| \\ \leq \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \|w(s)\| ds \\ \leq \frac{\varepsilon}{\Gamma(\alpha)} \int_{t_0}^{t} \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \tau(s) ds \\ \leq \varepsilon \lambda_\tau \tau(t).\end{aligned}$$

□

Theorem 4.5.9. *Assume that $(H_2)-(H_5)$ hold, if (4.11) satisfies and $L+T\lambda_\tau \neq 1$, then the problem (4.1) is UHR and UGHR stable.*

Proof. Let $\varepsilon>0$ and if $\tilde{\zeta}\in C(I_0,\mathbb{R}^n)$ is a function which satisfies inequality (4.11). Also consider $\zeta\in C(I_0,\mathbb{R}^n)$ is a local unique solution of the problem (4.1). Using Theorem (4.4.1), we have

$$\zeta(t)=\phi(0)+v(t_0,\phi)+v(t,\zeta_t)+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s,\zeta_s)ds.$$

$$\begin{aligned}
&\|\tilde{\zeta}(t)-\zeta(t)\|\\
&\le \|\tilde{\zeta}(t)-\phi(0)-v(t_0,\phi)-v(t,\zeta_t)-\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}u(s,\zeta_s)ds\|\\
&\quad+\|v(t,\tilde{\zeta}_t)-v(t,\zeta_t)\|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|u(s,\tilde{\zeta}_s)-u(s,\zeta_s)\|ds\\
&\quad\le \varepsilon\lambda_\tau\tau(t)+L\|\tilde{\zeta}_t-\zeta_t\|+\frac{T}{\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|\tilde{\zeta}_s-\zeta_s\|ds.
\end{aligned}$$

Hence

$$\|\tilde{\zeta}(t)-\zeta(t)\|\le\frac{\varepsilon}{1-L}\lambda_\tau\tau(t)+\frac{T}{(1-L)\Gamma(\alpha)}\int_{t_0}^{t}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|\tilde{\zeta}_s-\zeta_s\|ds.$$

By Lemma (4.2.1), Remark (4.2.2) and (H_8), we obtain

$$\begin{aligned}
&\|\tilde{\zeta}(t)-\zeta(t)\|\\
&\le\frac{\varepsilon}{1-L}\lambda_\tau\tau(t)\\
&\quad+\int_{t_0}^{t}\sum_{k=1}^{\infty}\frac{T^k}{(1-L)^k}\frac{1}{\Gamma(\alpha k)}\psi'(s)(\psi(t)-\psi(s))^{\alpha k-1}\frac{\varepsilon}{1-L}\lambda_\tau\tau(s)ds\\
&\le\frac{\varepsilon}{1-L}\lambda_\tau\tau(t)+\left[\frac{\varepsilon\lambda_\tau}{1-L}\int_{t_0}^{t}\frac{T}{1-L}\frac{1}{\Gamma(\alpha)}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\tau(s)ds\right]\\
&\quad+\left[\frac{\varepsilon\lambda_\tau}{1-L}\int_{t_0}^{t}\frac{T^2}{(1-L)^2}\frac{1}{\Gamma(2\alpha)}\psi'(s)(\psi(t)-\psi(s))^{2\alpha-1}\tau(s)ds+\cdots\right]\\
&\le\frac{\varepsilon}{1-L}\lambda_\tau\tau(t)+\frac{\varepsilon}{1-L}\lambda_\tau\left[\frac{T}{1-L}I_{t_0}^{\alpha;\psi}\tau(t)+\left(\frac{T}{1-L}\right)^2 I_{t_0+}^{2\alpha;\psi}\tau(t)+\cdots\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\varepsilon}{1-L}\lambda_\tau \tau(t) + \frac{\varepsilon}{1-L}\lambda_\tau \left[\frac{T}{1-L}\lambda_\tau \tau(t) + \left(\frac{T}{1-L}\lambda_\tau\right)^2 \tau(t) + \cdots\right] \\
&\le \frac{\varepsilon}{1-L}\lambda_\tau \tau(t) \sum_{k=0}^{\infty} \left(\frac{T}{1-L}\lambda_\tau\right)^k \\
&\le \frac{\lambda\tau}{1-L-T\lambda_\tau}\varepsilon\tau(t).
\end{aligned}
$$

Then, for $\sigma_{u,\tau} = \frac{\lambda_\tau}{1-L-T\lambda_\tau}$, we conclude that

$$\|\tilde{\zeta}(t) - \zeta(t)\| \le \sigma_{u,\tau}\varepsilon\tau(t), t \in I_0.$$

Hence the problem (4.1) is stable.

Further by putting $\varepsilon = 1$, we can easily find that

$$\|\tilde{\zeta}(t) - \zeta(t)\| \le \sigma_{u,\tau}\tau(t), t \in I_0.$$

□

4.6 Example

Here we give an example which establishes our results.

Example 4.6.1. *Consider for $t \in [1, \infty)$,*

$$ {}^{C}D_{1+}^{\frac{1}{3},\ln t}\left(\zeta(t) - \frac{\Gamma(5/6)}{\sqrt{\pi}}\ln t|\zeta(t+3/2)|\right) = \frac{1}{81}\ln t^{1/3}\cos^2(\zeta(t+3/2))\zeta(t+3/2).$$

$$\zeta_1 = 0,$$

$$\alpha = \frac{1}{3}, \psi(t) = \ln t,$$

$$v(t,\zeta_t) = \frac{\Gamma(5/6)}{\sqrt{\pi}}\ln t|\zeta(t+3/2)|,$$

$$u(t,\zeta_t) = \frac{1}{81}\ln t^{1/3}\cos^2(\zeta(t+3/2))\zeta(t+3/2).$$

.

$$T = \frac{1}{243}.$$

$$\|v(t,\zeta_t') - v(t,\zeta_t'')\| \leq \frac{\Gamma(5/6)}{2\sqrt{\pi}}.$$

$$i.e., L = \frac{\Gamma(5/6)}{2\sqrt{\pi}}.$$

$$\sigma = \frac{\frac{1}{243} + \frac{\Gamma(5/6)}{2\sqrt{\pi}}}{\Gamma(4/3)} \ln(1+e)^{1/3} \approx 0.4 < 1.$$

Hence by the Theorem (4.4.1), the problem has a unique solution with $\tau = e$.

Further, as shown in the Theorem (4.5.7), for $\varepsilon > 0$ and if $\tilde{\zeta} \in C([1,e],\mathbb{R}^n)$ satisfies $\|^C D_{1+}^{1/3,\ln t}(\tilde{\zeta}(t) - v(t,\tilde{\zeta}_t) - u(t,\tilde{\zeta}_t))\| \leq \frac{1}{3}$ and there exist a unique solution ζ such that $\|\tilde{\zeta}(t) - \zeta(t)\| \leq \frac{1}{3}\lambda_u$, where

$$\lambda_u = \frac{\ln(1+e)^{1/3}}{\left(1 - \frac{\Gamma(5/6)}{2\sqrt{\pi}}\right)\Gamma(4/3)} E_{1/3}\left[\frac{2\sqrt{\pi}}{243(2\sqrt{\pi}) - \Gamma(5/6)}\right] \approx 1.4 > 0.$$

Hence the problem is UH stable.

Finally we consider $\tau(t) = (\ln t)^{1/3}$,

$$\begin{aligned} I_{1+}^{1/3,\ln t}(\ln t)^{1/3} &= \frac{1}{\Gamma(1/3)} \int_1^t \left(\ln\left(\frac{t}{s}\right)\right)^{-2/3} (\ln s)^{1/3} \frac{ds}{s} \\ &\leq \frac{3}{\Gamma(1/3)} \ln(1+e)^{1/3} \tau(t). \end{aligned}$$

Thus the hypothesis (H_8) is satisfied with

$$\lambda_\tau = \frac{3}{\Gamma(1/3)} \ln(1+e)^{1/3} \approx 1.23 > 0.$$

Also $\sigma_{u,\tau} = \dfrac{\lambda_\tau}{1 - L - T\lambda_\tau} = \dfrac{\frac{3}{\Gamma(1/3)} \ln(1+e)^{1/3}}{1 - \frac{\Gamma(5/6)}{2\sqrt{\pi}} - \frac{1}{243}\frac{3}{\Gamma(1/3)} \ln(1+e)^{1/3}} \approx 1.8 > 0.$

4.7 Conclusion

In this chapter we have studied the $\psi-$Caputo fractional neutral functional differential equation. Using some of the hypothesis, have we proved the existence and uniqueness of the solution by Krasnoselskii's fixed point theorem and Banach fixed point theorem respectively. We also proved the Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability, generalized Ulam-Hyers-Rassias stability of solutions.

Chapter 5

On Existence of $\psi-$Hilfer Hybrid Fractional Differential Equation

5.1 Introduction

In this chapter, we derive existence results for the solutions to the hybrid fractional differential equations with perturbation of first kind and second kind involving $\psi-$Hilfer fractional derivative using different fixed point theorems. The result is illustrated with an example. We consider the existences of hybrid fractional differential equations of first and second types involving $\psi-$Hilfer fractional derivative, which are given by

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}\frac{x(t)}{f(t,x(t))} = g(t,x(t)), \quad a.e \quad t\in I=[0,T] \\ I_{0+}^{1-\gamma,\psi}\frac{x(0)}{f(0,x(0))} = x_0 \end{cases} \tag{5.1}$$

and

$$\begin{cases} {}^{H}\mathbb{D}_{t_0+}^{\alpha,\beta;\psi}[x(t)-f(t,x(t))] &= g(t,x(t)),\ a.e.\ t\in J=[t_0,t_0+a] \\ I_{t_0}^{1-\gamma;\psi}[x(t_0)-f(t_0,x(t_0))] &= \sigma\in\mathbb{R} \end{cases} \tag{5.2}$$

where ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$. $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$. $J = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$. $I = [t_0, t_0 + a]$, with $t_0 = 0$ and $a = T$.

This chapter is organized as follows: In section 5.2 we give the existence result for $\psi-$Hilfer fractional hybrid differential equation of first type based on Dhange's fixed point theorem. In section 5.3 we give the existence result for $\psi-$Hilfer fractional hybrid differential equation of second type based on fixed point theorem is given. We also gave an example to illustrate the results and plotted graphs for different functions.

5.2 $\psi-$Hilfer fractional hybrid differential equation of the first type.

We take $X = C_{1-\gamma;\psi}([0,T])$, $T > 0$ throughout this section. We have the following lemma.

Lemma 5.2.1. *Any function satisfies IVP (5.1) will also satisfy the integral equation*

$$\begin{aligned}
x(t) &= f(t,x(t))\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{x(0)}{f(0,x(0))}\right] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds, \\
&= f(t,x(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds\right],
\end{aligned} \tag{5.3}$$

where $t \in [0, T]$.

In addition, if the function $x \to \frac{x}{f(0,x)}$ is injective, and $I_{0+}^{\alpha;\psi} g(t, x(t))$ is an abso-

lutely continuous function, then the converse is true.

Proof. From Lemma (1.1.9), the proof is clear [26, 27]. □

Theorem 5.2.2. *Assume the following.*

(H_1) *The function $x \to \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$, $\forall t \in I$.*

(H_2) *There exists a constant $L_f > 0$ such that $|f(t,x) - f(t,y)| \leq L_f|x-y|$, for all $t \in I$ and $x, y \in \mathbb{R}$.*

(H_3) $K(\psi(T) - \psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| < 1$, then the $\psi-$Hilfer hybrid fractional differential equation has a solution defined on I, where K is the bound of a bounded function $(\psi(t) - \psi(a))^{1-\gamma}g(t)$.

Then IVP (5.1) *has a mild solution on I.*

Proof. We define a subset S of X by $S = \{x \in X : \|x\| \leq N\}$,

where,

$$N = \frac{F_0\left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)x_0 + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)\right]}{1 - L_f\left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right) + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)}. \tag{5.4}$$

and $F_0 = \sup_{t\in I}\|f(t,0)\|_{C_{1-\gamma;\psi}[0,T]}$.

It is clear that S satisfies the hypotheses of Dhange's fixed point theorem (1.1.15).

Also IVP (5.1) is equivalent to the $\psi-$Hilfer hybrid volterra integral equation:

$$x(t) =$$
$$f(t, x(t))\left[\frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}g(s,x(s))ds\right],$$
$$t \in [0,T]. \tag{5.5}$$

Define two operators $A : X \to X$ and $B : S \to X$ by:

$$Ax(t) = f(t, x(t)),\ t \in I. \tag{5.6}$$

and

$$Bx(t) = \frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s, x(s)) ds. \tag{5.7}$$

Then equation (5.5) is transformed into the operator equation as

$$x(t) = Ax(t)\, Bx(t),\ t \in I. \tag{5.8}$$

We shall show that the operators A and B satisfy all the conditions of Dhange's fixed point theorem (1.1.15).

Claim I: A is Lipschitzian with Lipschitz constant L_f.

Let $x, y \in X$, then by Hypothesis (H_2)

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq L_f |x(t) - y(t)| \leq L_f \|x - y\|,\ \forall t \in I.$$

Taking supremum over t, we obtain:

$$\|Ax - Ay\|_{C_{1-\gamma;\psi}[I\times\mathbb{R},\mathbb{R}|\{0\}]} \leq L_f \|x - y\|\, x, y \in X.$$

Claim II: B is continuous in S.

Let x_n be a sequence in S converging to a point $x \in S$. Then by Lebesgue dominated convergence theorem (1.1.18),

$$\begin{aligned} &\lim_{n\to\infty} Bx_n(t) \\ &\quad = \lim_{n\to\infty} \left[\frac{(\psi(t) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s, x_n(s)) ds \right], \end{aligned}$$

$$= \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\lim_{n\to\infty} g(s,x_n(s))ds.$$

Hence

$$\lim_{n\to\infty} Bx_n(t) = Bx(t),\ \forall\, t\in I.$$

Claim III: B is a Compact operator on S.

First, we show that $B(S)$ is a uniformly bounded set in X. Let $x\in S$, then by hypothesis (H_3), $\forall\, t\in I$.

$$\begin{aligned}
|Bx(t)| &\le \left|\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds\\
&\le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(t)-\psi(0))^{\alpha}\Gamma(1-\alpha)\frac{(\psi(t)-\psi(0))^{\alpha}}{\alpha\Gamma(\alpha)}\\
&\le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

$\implies \|Bx\| \le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)},\ \forall x\in X.$

This shows that B is uniformly bounded on S.

Next, we show that $B(S)$ is an equicontinuous set on X. Let $t_1,t_2\in I$, then for any $x\in S$,

$$|Bx(t_1)-Bx(t_2)| = \frac{1}{\Gamma(\alpha)}\times$$

$$\left|\int_0^{t_1}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds - \int_0^{t_2}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds\right|$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds\\
&\le K\frac{1}{\Gamma(\alpha)}\frac{(\psi(t_1)-\psi(t_2))^{\alpha}}{\alpha}.
\end{aligned}$$

Hence for $\varepsilon>0$, there exist a $\delta>0$ such that whenever $|t_1-t_2|<\delta$, then $|B(x(t_1))-B(x(t_2))|<\varepsilon$, $\forall\, t_1,t_2\in I$ and $\forall\, x\in X$. This shows that $B(S)$ is an equicontinuous set in X. Then by the Arzelá-Ascoli theorem (1.1.17), B is a continuous and compact operator on S.

Claim IV: $x = AxBy \implies x \in S, \forall y \in S.$

The hypothesis (c) of Dhange's fixed point theorem (1.1.15).

Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$, then:

$$\begin{aligned}
|x(t)| &= |Ax(t)||By(t)| \\
&\leq |f(t,x(t))| \left| \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right| \\
&\leq [f(t,x(t)) - f(t,0)] + \\
&\qquad |f(t,0)| \left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right] \\
&\leq (L_f|x(t)| + F_0) \left[\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right].
\end{aligned}$$

Thus

$$|x(t)| \leq \frac{F_0 \left[\left(\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \right) x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right]}{1 - L_f \left(\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 \right) + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}}. \tag{5.9}$$

Taking supremum over $t \in I$,

$$||x|| \leq \frac{F_0 \left[\left(\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \right) x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right]}{1 - L_f \left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 \right) + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}} = N. \tag{5.10}$$

Thus, $x \in S$ and hypothesis (c) of Dhange's fixed point theorem (1.1.15) is satisfied.

Finally, we have

$$M = ||B(S)|| = \sup\{||Bx|| : x \in S\}$$

and so

$$\alpha M \leq K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| < 1.$$

Thus, all conditions of Dhange's fixed point theorem (1.1.15) are satisfied. Hence the operator equation $AxBx = x$ has a solution in S. As a result, equation (5.1) has a solution defined on I.

This completes the proof. □

5.3 $\psi-$Hilfer fractional hybrid differential equation of second type.

Consider the $\psi-$Hilfer fractional hybrid differential equation of the form:

$$\begin{cases} {}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{t_0+}[x(t)-f(t,x(t))] & = g(t,x(t)),\ a.e.\ t\in J=[t_0,t_0+a], \\ I^{1-\gamma;\psi}_{t_0}[x(t_0)-f(t_0,x(t_0))] & = \sigma\in\mathbb{R}. \end{cases} \tag{5.11}$$

Lemma 5.3.1. *Any function satisfies IVP (5.11) will also satisfy the integral equation*

$$x(t) = f(t,x(t)) + \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} I^{1-\gamma}_{t_0+}(x(t_0)-f(t_0,x(t_0))) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds, \tag{5.12}$$

$$= f(t,x(t)) + \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds, \tag{5.13}$$

$t \in [t_0, t_0 + a].$

In addition if the function $x \to x - f(0,x)$ *is injective, and* $I_{0+}^{\alpha;\psi} g(t,x(t))$ *is an absolutely continuous function, then the converse is true.*

Proof. From Lemma (1.1.9), the proof is clear [26, 27]. □

Theorem 5.3.2. *Consider the following assumptions*

($\mathscr{H}1$) *There exists constants* $M_f \geq L_f > 0$ *such that*
$|f(t,x) - f(t,y)| \leq \frac{L_f|x-y|}{(M_f+|x-y|)}$, $\forall t \in J$ *and* $x, y \in \mathbb{R}$.

($\mathscr{H}2$) *There exists* $r > 0$ *such that*
$r > L_f + F_0 + \left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a) - \psi(t_0))^{2\alpha}$,
where $F_0 = \sup_{t\in J}|f(t,0)|$

($\mathscr{H}3$) K *is the bound of a bounded function* $(\psi(t) - \psi(a))^{1-\gamma} g(t)$.

Then equation (5.11) has a mild solution on J.

Proof. Let $X = C_{1-\gamma;\psi}([t_0, t_0 + a]), T > 0$ and define the set $S \subset X$ by
$S = \{x \in X : ||x|| \leq r\}$.

We prove the existence of a mild solution to problem (5.11) by discussing the solution to the integral equation (5.13) which is equivalent to the operator equation,

$$Ax(t) + Bx(t) = x(t),\ t \in J. \tag{5.14}$$

where,

$Ax(t) = f(t, x(t)).$

$$Bx(t) = \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds.$$

Now we prove our existence of the solution by proving that the conditions of Dhange's fixed point theorem (1.1.16) are satisfied.

Step I: A is a nonlinear contraction.

Using the hypothesis $(\mathscr{H}1)$ we get:

$$\begin{aligned}|Ax(t)-Ay(t)| =& |f(t,x(t))-f(t,y(t))|\\ \leq& \frac{L_f|x(t)-y(t)|}{M_f+|x(t)-y(t)|}\\ \leq& \frac{L_f||x-y||}{M_f+||x-y||}.\end{aligned}$$

Thus, the operator A is a nonlinear contraction with the function ϕ defined by $\phi(r) = \frac{L_f r}{M_f+r}$.

Step II: B is continuous and compact.

We show that B is continuous in S. Let x_n be a sequence in S converging to a point $x \in S$. Then by Lebesgue Dominated convergence theorem (1.1.18),

$$\begin{aligned}&\lim_{n\to\infty} Bx_n(t)\\ &\quad= \lim_{n\to\infty}\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x_n(s))ds\right],\end{aligned}$$

$$= \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\lim_{n\to\infty} g(s,x_n(s))ds.$$

Hence

$$\lim_{n\to\infty} Bx_n(t) = Bx(t),\ \forall\, t\in I.$$

Now, to prove B is a Compact operator on S, first, we show that $B(S)$ is a uniformly bounded set in X.

Let $x \in S$, then by hypothesis $(\mathscr{H}_3)$, $\forall\, t \in I$.

$$\begin{aligned}
&|Bx(t)| \\
&\quad \leq \left|\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds \\
&\quad \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\sigma| + K(\psi(t)-\psi(0))^{\alpha}\Gamma(1-\alpha)\frac{(\psi(t)-\psi(0))^{\alpha}}{\alpha\Gamma(\alpha)} \\
&\quad \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\sigma| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

$\implies \|Bx\| \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|\sigma| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}$, $\forall x \in X$.

This shows that B is uniformly bounded on S.

Next, we show that $B(S)$ is an equicontinuous set on X. Let $t_1, t_2 \in I$, then for any $x \in S$,

$$\begin{aligned}
&|Bx(t_1) - Bx(t_2)| = \frac{1}{\Gamma(\alpha)}\times \\
&\left|\int_0^{t_1}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds - \int_0^{t_2}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds\right| \\
&\qquad\qquad \leq \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds \\
&\qquad\qquad \leq K\frac{1}{\Gamma(\alpha)}\frac{(\psi(t_1)-\psi(t_2))^{\alpha}}{\alpha}.
\end{aligned}$$

Hence for $\varepsilon > 0$, there exist a $\delta > 0$ such that whenever $|t_1 - t_2| < \delta$, then $|B(x(t_1)) - B(x(t_2))| < \varepsilon$, $\forall\, t_1, t_2 \in I$ and $\forall\, x \in X$. This shows that $B(S)$ is an equicontinuous set in X. Then by the Arzelá-Ascoli theorem (1.1.17), B is a continuous and compact operator on S.

Step III:

Let $x \in X$ be fixed and $y \in S$ be arbitrary such that $x = Ax + By$, then we have:

$$
\begin{aligned}
&|x(t)| \\
&\quad \leq |Ax(t)| + |By(t)|, \\
&\quad \leq |f(t,x(t))| + \left| \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \sigma \right| \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |g(s,x(s))| ds, \\
&\quad \leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \left| \frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \sigma \right| \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |g(s,x(s))| ds, \\
&\quad \leq L_f + F_0 + \left| \frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \sigma \right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} (\psi(t_0+a)-\psi(t_0))^{2\alpha}, \\
&\quad \leq r.
\end{aligned}
$$

which proves that $||x|| \leq r$. Therefore $x \in S$.

Thus the conditions of Dhange's fixed point theorem (1.1.16) are satisfied. Hence the operator equation $Ax(t) + Bx(t) = x(t)$ has a solution in S which proves the existence of a mild solution to problem (5.11) in J. □

5.4 Example

Consider the example

Example 5.4.1. *Consider the $\psi-$Hilfer fractional Hybrid differential equation*

$$
\begin{aligned}
&{}^{H}\mathbb{D}_{0+}^{0.5,0;x} \left(x(t) - \frac{sin(t)|x(t)|}{2+|x(t)|} \right) = \frac{tx(t)}{1+|x(t)|}, \\
&I_{0+}^{0.5;x}(x(0) - f(0,x(0))) = 1,\ t \in [0,\pi].
\end{aligned}
$$

We have,

$$|f(t,x(t)) - f(t,y(t))| \leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|+|y(t)|}, \\ \leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|}.$$

Also $|g(t,x(t))| \leq t$.

It is clear that all hypotheses of Theorem (5.3.2) are satisfied with

$$L_f = 1, M_f = 2, T = \pi, F_0 = 0.$$

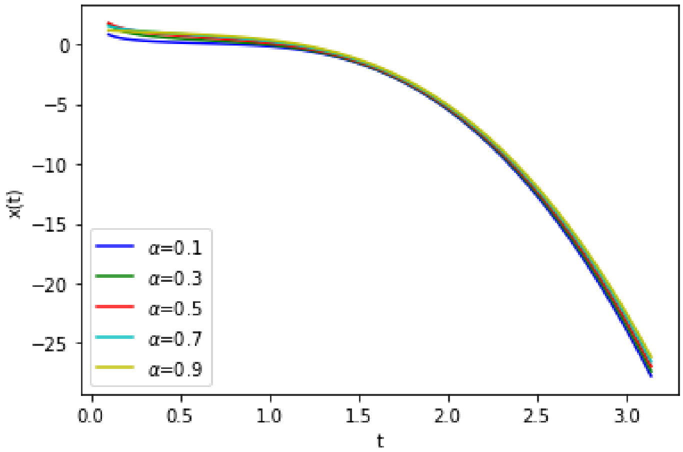
We conclude that

$$L_f + F_0 + \left| \frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \sigma \right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} (\psi(t_0+a)-\psi(t_0))^{2\alpha} =$$

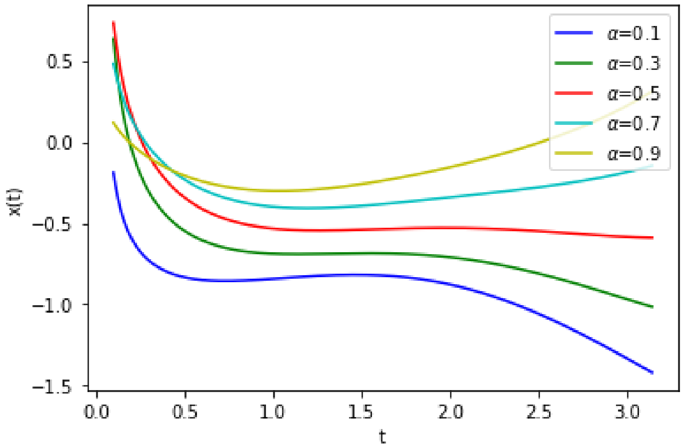
$$1 + \frac{1}{\pi} + \pi^2\sqrt{\pi} < 19.$$

For different $x(t)$ and α the solutions corresponding to the problem are plotted below. In each case red colour indicated the solution of the above problem.

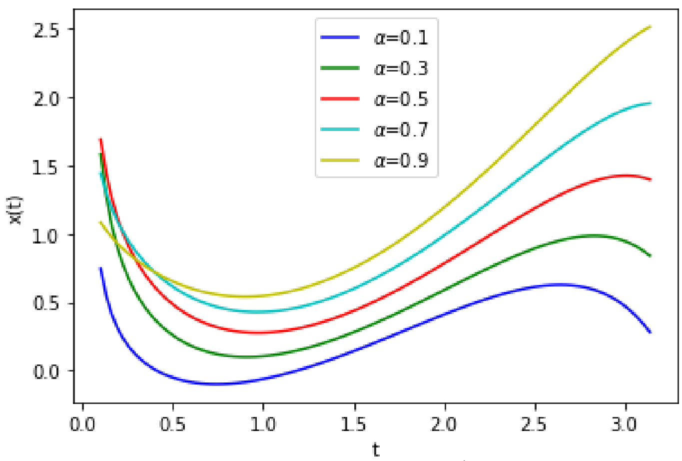
Figure 5.1: $\psi-$Hilfer hybrid fractional differential equation with $\psi(t) = t$.



(a) $x(t) = t^3$

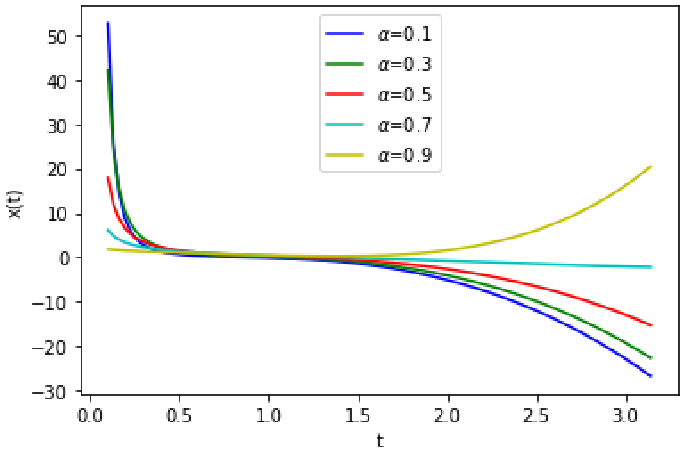


(b) $x(t) = 1 + t$

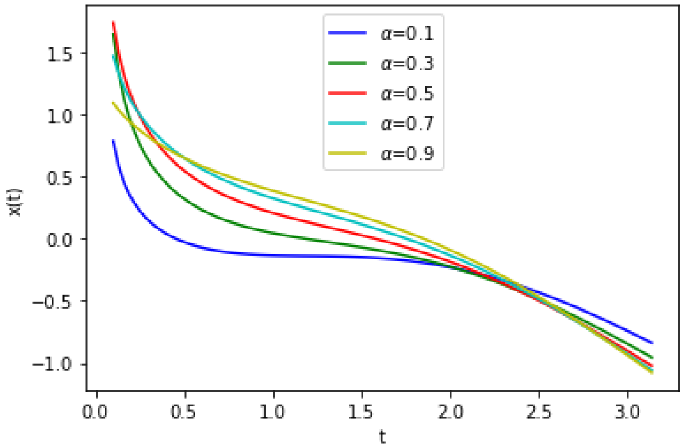


(c) $x(t) = sint$

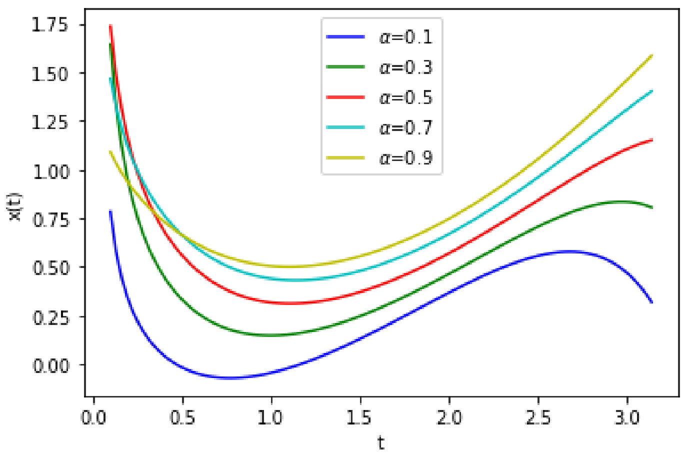
Figure 5.2: $\psi-$Hilfer hybrid fractional differential equation.



(a) $x(t) = t^3, \psi(t) = t^3$



(b) $x(t) = t, \psi(t) = ln(t+1)$



(c) $x(t) = sint, \psi(t) = ln(t+1)$

5.5 Conclusion

In this chapter, using Dhange's fixed point theorems we have proved the existences of solutions of $\psi-$Hilfer hybrid fractional differential equations with perturbations of first and second kind. We have given and example to illustrate our result. We have concluded the chapter with an example. Graphical representation for different values for the functions ψ is also given. For different kernel function, we plotted solutions of different fractional differential equations.

Chapter 6

On Existence of $\psi-$ Hilfer Hybrid Fractional Differential Equations with Boundary Condition

6.1 Introduction

In this chapter, we discuss the existences of $\psi-$Hilfer hybrid fractional differential equations of first type with boundary condition, which is given by

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi} \frac{u(t)}{f(t,u(t))} = g(t,u(t)), \quad a.e \quad t \in J = [0,T] \\ a I_{0+}^{1-\gamma,\psi} \frac{u(0)}{f(0,u(0))} + b \frac{u(T)}{f(T,u(T))} = c \end{cases} \tag{6.1}$$

where ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$ and $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R}|\{0\})$, $g \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real numbers with $b(\psi(T) - \psi(0)) \neq 0$.

By a solution of (6.1), we mean a function $u \in C(J, \mathbb{R})$ such that

(i) The function $t \to \frac{u}{f(t,u)}$ is continuous for each $t \in \mathbb{R}$

(ii) u satisfies the equation in (6.1).

6.2 Preliminaries

For the weighed function $C_{1-\gamma;\psi}[a,b]$,

(i) The map $t \to g(t,u)$ is measurable for each $u \in \mathbb{R}$, and

(ii) The map $u \to g(t,u)$ is continuous for each $t \in [a,b]$;

(iii) For each $g \in C_{1-\gamma;\psi}[a,b]$, $g(t,u(t))$ is $\psi-$integrable.

6.3 Existence results

We take $X = C_{1-\gamma;\psi}([0,T])$, $T > 0$ throughout this section. In this section, we prove the existence results for the boundary value problems for generalised Hilfer hybrid differential equation with $0 < \alpha < 1$, $0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$ under mixed Lipschitz and Carath*é*odory conditions on the nonlinearities involved in it.
We define the multiplication in X by

$$(uv)(t) = u(t)\, v(t), \quad u,\, v \in X.$$

We make the following assumptions:

(H_0) The function $t \to \frac{t}{f(t,u)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(H_1) There exists a constant $L > 0$ such that

$$|f(t,u) - f(t,v)| \leq L|u - v|$$

for all $t \in J$ and $u, v \in \mathbb{R}$.

(H_2) There exist a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t,u)| \leq h(t) \qquad a.e\ t \in J.$$

In the following lemma we are finding the volterra integral equation equivalent to the given $\psi-$Hilfer hybrid differential equation with boundary condition.

Lemma 6.3.1. *Assume that hypothesis (H_0) holds and a, b, c are real constants with $b(\psi(T) - \psi(0)) \neq 0$. Then for any $h \in L^1(J, \mathbb{R})$, the function $u \in C(J, \mathbb{R})$ is a solution of the boundary value problem*

$$\begin{cases} {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \frac{u(t)}{f(t,ut))} = h(t), \quad a.e \quad t \in J = [0,T] \\ aI_{0+}^{1-\gamma,\psi} \frac{u(0)}{f(0,u(0))} + b\frac{u(T)}{f(T,u(T))} = c \end{cases} \tag{6.2}$$

if and only if u satisfies the hybrid integral equation

$$\begin{aligned} u(t) =& f(t,u(t)) \left[\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} h(s)) ds \right] \\ & - f(t,u(t)) \left[\frac{(\psi(t) - \psi(0))^{\gamma-1}}{b(\psi(T) - \psi(0))^{\gamma-1}} \left(\frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T) - \psi(s))^{\alpha-1} h(s) ds - c \right) \right]. \end{aligned} \tag{6.3}$$

Proof. Assume that u is a solution of problem (6.3). By definition $\frac{u(t)}{f(t,u(t))}$ is continuous. Applying the $\psi-$Hilfer fractional operator, we obtain the first equation in (6.2). Again substituting $t = 0$ and $t = T$ in (6.3), we have

$$b\frac{u(T)}{f(T,u(T))} + aI_{0+}^{1-\gamma;\psi} \frac{u(0)}{f(0,u(0))} =$$

$$
\begin{aligned}
&\left[\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}h(s))ds\right] \\
-&\left[\frac{(\psi(T)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\left(\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}h(s)ds-c\right)\right] \\
&= c.
\end{aligned}
$$

Conversely, by applying Lemma (1.1.9) for equation (6.2), we have

$$
\begin{aligned}
\frac{u(t)}{f(t,u(t))} = &\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{u(0)}{f(0,u(0))}\right] \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(0))^{\alpha-1}h(s)ds.
\end{aligned} \tag{6.4}
$$

Now,

$$
\begin{aligned}
b\frac{u(T)}{f(T,u(T))} = &\\
&b\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{u(0)}{f(0,u(0))}\right] \\
&+b\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(0))^{\alpha-1}h(s)ds.
\end{aligned} \tag{6.5}
$$

Thus,

$$
\begin{aligned}
aI_{0+}^{1-\gamma;\psi}\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(T,u(T))} = &\\
&b\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{u(0)}{f(0,u(0))}\right] \\
&+b\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(0))^{\alpha-1}h(s)ds \qquad = c.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
I_{0+}^{1-\gamma;\psi}\frac{u(0)}{f(0,u(0))} = &\\
&\frac{\Gamma(\gamma)}{b(\psi(T)-\psi(0))^{\gamma-1}}\left[c-\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(0))^{\alpha-1}h(s)ds\right].
\end{aligned}
$$

Consequently,

$$
u(t) = f(t,u(t))\left[\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}h(s))ds\right]
$$

$$-f(t,u(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\left(\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}h(s)ds-c\right)\right]. \tag{6.6}$$

□

Lemma 6.3.2. *By Lemma (6.3.1) $u \in C(J,\mathbb{R})$ is a solution of the boundary value problem*

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi}\frac{u(t)}{f(t,u(t)))} = g(t,u(t)), \quad a.e \quad t \in J = [0,T] \\ aI_{0+}^{1-\gamma,\psi}\frac{u(0)}{f(0,u(0))} + b\frac{u(T)}{f(T,u(T))} = c \end{cases} \tag{6.7}$$

if and only if u satisfies the hybrid integral equation

$$\begin{aligned} u(t) =& f(t,u(t))\left[\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,u(s)))ds\right] \\ &- f(t,u(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\left(\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}g(su(s))ds-c\right)\right]. \end{aligned} \tag{6.8}$$

Theorem 6.3.3. *Assume that hypotheses $(H_0)-(H_2)$ hold and a,b,c are real constants with $b(\psi(T)-\psi(0)) \neq 0$.*
Further, if $\left(\frac{M\Gamma(\gamma)}{|b|\Gamma(\gamma+\alpha)}(\psi(T)-\psi(0))^{\alpha}(1+|b|)+\frac{|c|}{|b|}\right) < 1$, then the $\psi-$Hilfer Hybrid differential equation (6.7) has a solution defined on J.

Proof. We define a subset S of X by $S = \{u \in X : \|u\| \leq N\}$

where,

$$N = \frac{\left(F_0\frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|}(\psi(T)-\psi(0))^{\alpha}\left[|b|+\left(1+|bc|\frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)}\right)\right]\right)}{\left(1-L\frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|}(\psi(T)-\psi(0))^{\alpha}\left[|b|+\left(1+|bc|\frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)}\right)\right]\right)} \tag{6.9}$$

and $F_0 = \sup_{t\in I}\|f(t,0)\|_{C_{1-\gamma;\psi}[0,T]}$.
It is clear that S satisfies the hypotheses of Dhange's fixed point theorem (1.1.15).

Also, we have the $\psi-$Hilfer hybrid volterra integral equation:

$$u(t) = f(t,u(t)\left[\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,u(s))ds\right] - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}} f(t,u(t)\left[\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}g(s,u(s))ds - c\right] t\in[0,T]. \tag{6.10}$$

Define two operators $A : X \to X$ and $B : S \to X$ by

$$Au(t) = f(t,u(t)),\ t \in I. \tag{6.11}$$

and

$$Bu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,u(s))ds - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}g(s,u(s))ds - c \tag{6.12}$$

Then equation 6.10 is transformed into the operator equation as

$$u(t) = Au(t)\,Bu(t),\ t \in I. \tag{6.13}$$

We shall show that the operators A and B satisfy all the conditions of Dhange's fixed point theorem (1.1.15).

Claim I: A is Lipschitzian with Lipschitz constant L.

Let $u, v \in X$, then by hypothesis (H_2)

$$|Au(t) - Av(t)| = |f(t,u(t)) - f(t,v(t))| \leq L|u(t)-v(t)| \leq L\|u-v\|,\ \forall t \in I$$

Taking supremum over t, we obtain

$$\|Au - Av\|_{C_{1-\gamma;\psi}[I\times\mathbb{R},\mathbb{R}]} \leq L\|u-v\|,$$

where $u, v \in X$.

Claim II: B is continuous in S.

Let u_n be a sequence in S converging to a point $u \in S$. Then by Lebesgue dominated convergence theorem (1.1.18),

$$\begin{aligned}\lim_{n\to\infty} Bu_n(t) = \\ &\lim_{n\to\infty}\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,u_n(s))ds \\ &-\lim_{n\to\infty}\frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\frac{1}{\Gamma(\alpha)}\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}g(s,u_n(s))ds-c.\end{aligned}$$

Hence

$$\lim_{n\to\infty} Bu_n(t) = Bu(t),\ \forall t \in I.$$

Claim III: B is a Compact operator on S.

First, we show that $B(S)$ is a uniformly bounded set in X. Let $u \in S$, then by hypothesis (H_3), $\forall t \in I$,

$$\begin{aligned}|Bu(t)| \leq \\ &\frac{1}{\Gamma(\alpha)}\int_0^t |\psi'(s)|(\psi(t)-\psi(s))^{\alpha-1}|g(s,u(s))|ds \\ &+\frac{(\psi(t)-\psi(0))^{\gamma-1}}{b(\psi(T)-\psi(0))^{\gamma-1}}\frac{1}{\Gamma(\alpha)}\int_0^T |\psi'(s)|(\psi(T)-\psi(s))^{\alpha-1}|g(s,u(s))|ds \\ &+\left[\frac{(\psi(t)-\psi(0))}{(\psi(T)-\psi(0))}\right]^{\gamma-1}\left|\frac{c}{b}\right| \\ &\leq \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|}(\psi(t)-\psi(0))^{\alpha}\left[|b|+\left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)}\right)^{\gamma-\alpha-1}\left(1+|bc|\frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)}\right)\right] \\ &t \in [0,T].\end{aligned}$$

Thus

$$\|Bu\| \leq \frac{M\Gamma(\gamma)}{|b|\Gamma(\gamma+\alpha)}(\psi(T)-\psi(0))^{\alpha}(1+|b|)+\frac{|c|}{|b|}\ \forall u \in X.$$

This shows that B is uniformly bounded on S.

Next, we show that $B(S)$ is an equicontinuous set on X.

Let $t_1, t_2 \in I$, then for any $u \in S$,

$$\begin{aligned}
&|Bu(t_1) - Bu(t_2)| \leq \\
&\frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds - \int_0^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right| + \\
&\frac{1}{|b|\Gamma(\alpha)} \left[\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} |g(s,u(s))| ds \right] \left[(\psi(t_1)-\psi(0))^{\gamma-1} - (\psi(t_2)-\psi(0))^{\gamma-1} \right] \\
&\leq \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |g(s,x(s))| ds + \\
&\frac{1}{|b|\Gamma(\alpha)} \left[\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} |g(s,u(s))| ds \right] \left[(\psi(t_1)-\psi(0))^{\gamma-1} - (\psi(t_2)-\psi(0))^{\gamma-1} \right] \\
&\leq M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} \left[(\psi(t_2)-\psi(t_1))^{\alpha} + \frac{1}{b\Gamma(\alpha)} \left((\psi(t_2)-\psi(0))^{\gamma-1} - (\psi(t_1)-\psi(0))^{\gamma-1} \right) \right].
\end{aligned}$$

Hence, for $\varepsilon > 0$, there exist a $\delta > 0$ such that:

whenever $|t_1 - t_2| < \delta$, then $|B(u(t_1)) - B(u(t_2))| < \varepsilon$, $\forall\, t_1, t_2 \in I$ and $\forall\, u \in X$. This shows that $B(S)$ is an equicontinuous set in X. Then by the Arzel*á*-Ascoli theorem, B is a continuous and compact operator on S.

Claim IV: $u = AuBv \implies u \in S, \forall v \in S$

The hypothesis (c) of Dhange's fixed point theorem is satisfied.

Let $u \in X$ and $v \in S$ be arbitrary such that $u = AuBv$, then

$$\begin{aligned}
|u(t)| &= |Au(t)||Bv(t)| \\
&\leq |f(t,u(t))| \times \\
&\quad \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |g(s,u(s))| ds \right. \\
&\quad \left. - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{|b|(\psi(T)-\psi(0))^{\gamma-1}} \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} |g(s,u(s))| ds + |c| \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq ([f(t,x(t)) - f(t,0)] + |f(t,0)|) \times \\
&\quad \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} |g(s,u(s))| ds \right. \\
&\quad \left. - \frac{(\psi(t)-\psi(0))^{\gamma-1}}{|b|(\psi(T)-\psi(0))^{\gamma-1}} \frac{1}{\Gamma(\alpha)} \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1} |g(s,u(s))| ds + |c| \right\} \\
&\leq (L|u(t)| + F_0) \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(t)-\psi(0))^{\alpha} \left[|b| + \left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)} \right)^{\gamma-\alpha-1} \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right].
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&|u(t)| \left(1 - L \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(t)-\psi(0))^{\alpha} \left[|b| + \left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)} \right)^{\gamma-\alpha-1} \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right) \\
&\quad \leq \left(F_0 \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(t)-\psi(0))^{\alpha} \left[|b| + \left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)} \right)^{\gamma-\alpha-1} \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right).
\end{aligned}
$$

Thus,

$$
|u(t)| \leq \frac{\left(F_0 \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(t)-\psi(0))^{\alpha} \left[|b| + \left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)} \right)^{\gamma-\alpha-1} \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right)}{\left(1 - L \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(t)-\psi(0))^{\alpha} \left[|b| + \left(\frac{\psi(t)-\psi(0)}{\psi(T)-\psi(0)} \right)^{\gamma-\alpha-1} \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right)}. \tag{6.14}
$$

Taking supremum over $t \in I$,

$$
||u|| \leq \frac{\left(F_0 \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(T)-\psi(0))^{\alpha} \left[|b| + \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right)}{\left(1 - L \frac{M\Gamma(\gamma)}{\Gamma(\gamma+\alpha)|b|} (\psi(T)-\psi(0))^{\alpha} \left[|b| + \left(1 + |bc| \frac{\Gamma(\gamma+\alpha)}{M\Gamma(\gamma)} \right) \right] \right)} = N. \tag{6.15}
$$

Thus $u \in S$ and hypothesis (c) of Dhange's fixed point theorem is satisfied.

Finally, we have

$$
M = ||B(S)|| = \sup\{||Bu|| : u \in S\}
$$

$$\leq \frac{M\Gamma(\gamma)}{|b|\Gamma(\gamma+\alpha)}(\psi(T)-\psi(0))^{\alpha}(1+|b|)+\frac{|c|}{|b|}.$$

And so

$$\alpha M \leq \frac{M\Gamma(\gamma)}{|b|\Gamma(\gamma+\alpha)}(\psi(T)-\psi(0))^{\alpha}(1+|b|)+\frac{|c|}{|b|} < 1.$$

Thus all conditions of Dhange's fixed point theorem are satisfied and hence the operator equation $Au\,Bu = u$ has a solution in S. As a result, equation (6.1) has a solution defined on I.

This completes the proof. □

6.4 Conclusion

In this chapter we have proved the existences of Hybrid fractional differential equation involving $\psi-$Hilfer fractional derivative with boundary conditions using Dhange's fixed point theorem.

Chapter 7

Existence, Uniqueness and Stability Results on ψ-Hilfer Fractional Differential Equations

7.1 Introduction

This chapter addresses the fractional-order system in terms of ψ-Hilfer fractional differential equations. The Banach contraction principle and the non-compactness measure are used to analyse the existence and uniqueness of the mild solution.

In our study, we consider,

$$\begin{cases} {}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi} p(u) = q(u,\, p(u)), \quad a.e \quad u \in J = [a_0,\, T], \\ p(a_0) = \phi_0 \end{cases} \tag{7.1}$$

where ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1, 0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$ and $q \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$. $I = [a_0, a_0 + a]$ is a bounded interval in $\mathbb{R}$ for some a_0.

Non-compactness measure and Banach contraction principle have been used to check the existence and uniqueness of solution respectively. This chapter is set up as follows: section 7.2 presents several fundamental lemmas and definitions. Additionally, it contains a few findings that are necessary to support our main findings. In section 7.3 we analyse the existence and uniqueness of the mild solution for the given $\psi-$ Hilfer fractional differential equation (7.1), with initial condition.

7.2 Preliminaries

Definition 7.2.1. *[27] For the class of all bounded subsets $\mathscr{M}_U$ of the metric space U, a map $\mu : \mathscr{M}_U \to [0,\infty)$ is called a measure of non-compactness on U, if the following properties are satisfied for $M_1, M_2 \in \mathscr{M}_U$.*

$\mathscr{C}_1$ *: $\mu(M) = 0$ if and only if M is pre compact.*

$\mathscr{C}_2$ *:$\mu(M) = \mu(\overline{M})$.*

$\mathscr{C}_3$ *:$\mu(M_1 \cup M_2) = \max\{\mu(M_1), \mu(M_2)\}$.*

Lemma 7.2.2. *[31] Let $(a,b]$ with $-\infty \leq a < b \leq \infty$ an interval in real line, $\alpha > 0$ and $\psi(u)$ a monotone growing and positive function in $[a,b]$, whose derivative is continuous in (a,b). Thus,*

$$I_{a+}^{\alpha;\psi} p(u) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} p^{(k)}(u) \frac{(\psi(u) - \psi(a))^{\alpha+k}}{\Gamma(\alpha+k+1)},$$

and the left fractional integral of a two-function product is given by:

$$I_{a+}^{\alpha;\psi}(pf)(u) = \sum_{k=0}^{\infty} \binom{-\alpha}{k} p^{(k)}(u) I_{a+}^{\alpha+k;\psi}$$

where $p^{(k)}(u)$ is the $k-$ th derivative of the entire order and $u > a$.

Theorem 7.2.3. ***(Banach contraction principle)*** *[31]*

For a non-empty complete metric space (E,d), let $0 \leq K < 1$ and let the mapping $A : E \to F$ satisfy the inequality

$$d(Ax,Ay) \leq Kd(x,y) \quad \textit{for every} \quad x,y \in E.$$

Then, A has a uniquely determined fixed point x^*. Furthermore, for any $x_0 \in E$, the sequence $(A^j(x_0))_{j=1}^{\infty}$ converges to this fixed point x^*.

Theorem 7.2.4. ***(Monch's fixed point theorem)****[31]*

Let Y be a bounded, closed and convex subset of a Banach space X such that which contains 0 and let A be a continuous mapping of Y into itself. If the implication

$$E = conv\mathrm{A}(\mathrm{E}) \; or \; \mathrm{E} = \mathrm{A}(\mathrm{E}) \cup \{0\} \implies \mathrm{E} \textit{is compact}$$

holds for every subset E of Y, then A has a fixed point.

7.3 Existence and Uniqueness results

Lemma 7.3.1. *Any function satisfies the IVP given by*

$$\begin{cases} {}^H\mathbb{D}_{a_0+}^{\alpha,\beta;\psi} p(u) = q(u,p(u)), \quad a.e \quad u \in J = [a_0,T], \\ p(u_0) = \phi_0 \end{cases}$$

is equivalent to the equation

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} q(s,p(s))ds. \tag{7.2}$$

Proof. Let us assume to have the $\psi-$Hilfer fractional differential equation (7.1) and prove that $p(u) \in C([a_0,T],E)$ satisfies the volterra integral equation (7.2). By properties of $\psi-$Hilfer fractional derivative, we have

$$^{H}\mathbb{D}^{\alpha,\beta;\psi}p(u) = I^{\beta(n-\alpha);\psi}D^{\gamma,\psi}p(u) = I^{\gamma-\alpha;\psi}D^{\gamma;\psi}p(u) \tag{7.3}$$

where $\gamma = \alpha + \beta(n-\alpha)$. Using equation (7.3), we have

$$^{H}\mathbb{D}^{\alpha,\beta;\psi}p(u) = q(u,p(u)).$$

which implies

$$I^{\gamma-\alpha;\psi}D^{\gamma;\psi}p(u) = q(u,p(u)).$$

Applying $I^{\alpha;\psi}$ on both sides of the above equation

$$I^{\alpha;\psi}I^{\gamma-\alpha;\psi}D^{\gamma;\psi}p(u) = I^{\alpha;\psi}q(u,p(u)).$$

$$\implies I^{\gamma;\psi}D^{\gamma;\psi}p(u) = I^{\alpha;\psi}q(u,p(u)).$$

Hence

$$p(u) - p(u_0) = \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)\left(\psi(u)-\psi(s)\right)^{\alpha-1}q(s,p(s))ds$$
$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)\left(\psi(u)-\psi(s)\right)^{\alpha-1}q(s,p(s))ds.$$

Now let us assume that $p(u) \in C([a_0,T],E)$ satisfies the volterra integral equation (7.2) and prove that the $\psi-$ Hilfer fractional differential equation holds. By applying $^{H}\mathbb{D}^{\alpha,\beta;\psi}$ to both sides of the equation (7.2), we can obtain

$$\begin{aligned}
^{H}\mathbb{D}^{\alpha,\beta;\psi}p(u) &= {}^{H}\mathbb{D}^{\alpha,\beta;\psi}\phi_0 + {}^{H}\mathbb{D}^{\alpha,\beta;\psi}\left[\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)\left(\psi(u)-\psi(s)\right)^{\alpha-1}q(s,p(s))ds\right] \\
&= {}^{H}\mathbb{D}^{\alpha,\beta;\psi}\phi_0 + I^{\gamma-\alpha;\psi}D^{\gamma,\psi}\left[\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)\left(\psi(u)-\psi(s)\right)^{\alpha-1}q(s,p(s))ds\right] \\
&= I^{\gamma-\alpha;\psi}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^{n}I^{(1-\beta)(n-\alpha);\psi}\left[\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)\left(\psi(u)-\psi(s)\right)^{\alpha-1}q(s,p(s))ds\right]
\end{aligned}$$

$$= I^{\gamma-\alpha;\psi}\left(\frac{1}{\psi'(u)}\frac{d}{du}\right)^{n} I^{n-\beta n+\alpha\beta;\psi} q(u,p(u)).$$

Hence, we obtain

$$^{H}\mathbb{D}^{\alpha,\beta;\psi} p(u) = I^{\gamma-\alpha;\psi} D^{\gamma-\alpha;\psi} q(u,p(u)) = q(u,p(u)).$$

□

Definition 7.3.2. *A function $p(u) \in C([a_0,T],E)$ is said to be a mild solution of the system*

$$\begin{cases} ^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a_0+} p(u) = q(u,p(u)), \quad a.e \quad u \in J = [a_0,T], \\ p(u_0) = \phi_0 \end{cases}$$

if it satisfies the equation

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1} q(s,x(s))ds,$$
$$u \in [a_0,T].$$

Theorem 7.3.3. *Under the assumptions:*

(H_1) *For the Banach space* X *(real or complex), $p : I \times X \to X$, the function $u \to q(u,p(u))$ is measurable on I, $\forall p \in X$ and the function $p \to q(u,p(u))$ is continuous on X for a.e $u \in [a_0,T]$.*

(H_2) *There exist $K > 0$ such that*

$$\|q(u,p_1) - q(u,p_2)\| < K\|p_1 - p_2\|, \ \forall u \in [a_0,T], \ \forall p_1,p_2 \in \mathrm{X},$$

then the IVP (7.1) *has a unique mild solution on $[a_0,T]$, provided $K \leq \frac{\Gamma(\alpha)}{\psi(T)^{1-\alpha}}$.*

Proof. Consider the operator $T : C_{1-\alpha}(J,\mathrm{X}) \to C_{1-\alpha}(J,\mathrm{X})$ on $[a_0,T]$ by

$$(Fp)(u) = \psi(u)^{\alpha-1} p(u_0) + I^{\alpha,\psi} q(u,p(u))du.$$

Choose $M_1 = \sup_{u\in[a_0,T]} \|q(u_0, p(u_0))\|$ and $\psi(a_0) = 0$.

Let $\mathscr{B}_r = \{p \in C_{1-\alpha}(J,\mathrm{X}),\ \|p\|_\alpha \leq r\}$ where, $r \geq 2\left[|\phi_0| + \dfrac{M_1\psi(t)}{\Gamma(\alpha+1)}\right]$.

Then we first need to show that $F(\mathscr{B}_r) \subset \mathscr{B}_r$.

$$\begin{aligned}
\psi(u)^{1-\alpha}\|(Fp)(u)\| &\leq |\phi_0| + \frac{\psi(u)^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p(s))\|ds\\
&\leq |\phi_0| + \frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\left[\|q(s,p(s))-q(s,\phi_0)\| + \|q(s,\phi_0)\|\right]ds\\
&\leq |\phi_0| + \frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\|q(s,p(s))-q(s,\phi_0)\|_\alpha + \frac{M_1\psi(T)}{\Gamma(\alpha+1)}\\
&\leq |\phi_0| + \frac{M_1\psi(T)}{\Gamma(\alpha+1)} + \frac{\psi(T)^{1-\alpha}}{\Gamma(\alpha)}\mathscr{B}_{(\alpha,\alpha)}r \leq r.
\end{aligned}$$

which proves that $F(\mathscr{B}_r) \subset \mathscr{B}_r$.

Now take $p_1, p_2 \in C_{1-\alpha(J,X)}$, then we obtain

$$\|(Fp_1)(u)-(Fp_2)(u)\| = \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p_1(s))-q(s,p_2(s))\|ds.$$

Now applying $\psi(u)^{1-\alpha}$ on both sides,

$$\begin{aligned}
&\psi(u)^{1-\alpha}\|(Fp_1)(u)-(Fp_2)(u)\|\\
&\quad=\psi(u)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}\|q(s,p_1(s))-q(s,p_2(s))\|ds\\
&\quad\leq\frac{K}{\Gamma(\alpha)}\psi(T)^{1-\alpha}\|p_1-p_2\|_\alpha.
\end{aligned}$$

Which implies F is a contraction mapping. Hence by Banach fixed point theorem, F has a unique fixed point, which is given by the mild solution to the IVP (7.1). □

Theorem 7.3.4. *The IVP* (7.1) *has at least one mild solution defined on I, under the assumptions*

($\mathscr{A}_1$) *For the Banach space* X *(real or complex),* $p : J \times X \to X$, *the function* $u \to q(u, p(u))$ *is measurable on* $J, \forall p \in X$ *and the function* $p \to q(u, p(u))$ *is continuous on* X *for a.e* $u \in [a_0, T]$.

($\mathscr{A}_2$) *There exist a function* $l \in L^\infty(I)$ *such that*

$\|q(u, p(u))\| \leq l(u)(1 + \|u\|); \quad \forall u \in J; \quad$ *and* $\quad p \in X.$

($\mathscr{A}_3$) $\mu(p(u, B)) \leq m(u)\mu(B); \quad \forall B \subset X.$

($\mathscr{A}_4$) *The function* $\psi(\eta)$ *is uniformly continuous and* $\psi(a_0) = 0$

hold and $k = T\frac{\psi(T)^\alpha}{\Gamma(\alpha+1)}m^* < 1$

Proof. Let us transform the IVP (7.1) into a fixed point theorem.

Consider the operator $F : BC \to BC$ on $[a_0, T]$ defined by,

$$(Fu)(\eta) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} q(s, u(s))ds.$$

Let $\tau > 0$ such that $\tau \geq \frac{\Gamma(\alpha+1)\|\phi_0\| + Tl\psi(T)^\alpha}{1 - l\psi(T)^\alpha}$.

Consider the ball $\mathscr{B}_\tau = \mathscr{B}(a_0, \tau) = \{u \in BC : \|u - a_0\|_{BC} \leq \tau\}$,

for any $\mathscr{B}_\tau$ and $t \in J$

$$\begin{aligned}
\|(Fu)(\eta)\| &\leq \|\phi_0\| + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} \|q(s, u(s))\| ds \\
&\leq \|\phi_0\| + \frac{Tl^*(1+\tau)}{\Gamma(\alpha)} \int_{a_0}^{\eta} \psi'(s)[\psi(\eta) - \psi(s)]^{\alpha-1} ds \\
&\leq \|\phi_0\| + \frac{Tl(1+\tau)\psi(T)^\alpha}{\Gamma(\alpha+1)} \leq \tau.
\end{aligned}$$

Thus, $\|(Fu)(\eta)\|_{BC} \leq \tau$, which proves that F transforms the ball $\mathscr{B}_\tau$ into itself.

Now we will demonstrate that the operator $F : \mathscr{B}_\tau \to \mathscr{B}_\tau$ complies with the Monch's theorem's assumptions. So, the succeeding three steps need to be proven.

Step (1) : $F : \mathscr{B}_\tau \to \mathscr{B}_\tau$ is continuous.

Let $\{\zeta_m\}_{m\in\mathbb{N}}$ be a sequence such that $\zeta_m \to \zeta$ as $m \to \infty$ in $\mathscr{B}_\tau$.
Then $\forall \eta \in J$, we'll obtain

$$\|(F\zeta_m)(\eta)-(F\zeta)(\eta)\| = \frac{1}{\Gamma(\alpha)}\int_{a_0}^{\eta} \psi'(s)[\psi(\eta)-\psi(s)]^{\alpha-1}\|q(s,\zeta_m(s))-q(s,\zeta(s))\|ds.$$

Since $\zeta_m \to \zeta$ as $m \to \infty$ in $\mathscr{B}_\tau$ and q is continuous, then by the Lebesgue dominated convergence theorem (1.1.18), $\|(F\zeta_m)(\eta)-(F\zeta)(\eta)\|_{BC} \to 0$ as $m \to \infty$.
Then, $F : \mathscr{B}_\tau \to \mathscr{B}_\tau$ is continuous.

Step (2) : $F(\mathscr{B}_\tau)$ is bounded and equicontinuous.

$F(\mathscr{B}_\tau)$ bounded and $F(\mathscr{B}_\tau) \subset \mathscr{B}_\tau$, implies that $F(\mathscr{B}_\tau)$ is bounded.
Let $\eta, x \in I$ and let $x < \eta$, $\zeta \in \mathscr{B}_\tau$. Then,

$$\begin{aligned}
&\psi(\eta)^{1-\alpha}\|(F\zeta)(\eta)-(F\zeta)(x)\| \\
&= [\psi(\eta)-\psi(x)]^{1-\alpha}\|\phi_0\| + \frac{[\psi(\eta)-\psi(x)]^{1-\alpha}}{\Gamma(\alpha)}\int_{a_0}^{x}\psi'(s)[\psi(\eta)-\psi(s)]^{\alpha-1}\|q(s,\zeta(s))\|ds \\
&\leq [\psi(\eta)-\psi(x)]^{1-\alpha}\|\phi_0\| + \frac{[\psi(\eta)-\psi(x)]^{1-\alpha}}{\Gamma(\alpha+1)}l^*(1+\tau)[\psi(\eta)-\psi(x)]^{\alpha} \\
&\leq [\psi(\eta)-\psi(x)]^{1-\alpha}\|\phi_0\| + \frac{[\psi(\eta)-\psi(x)]}{\Gamma(\alpha+1)}l^*(1+\tau).
\end{aligned}$$

Since, $\psi(\eta)$ is uniformly continuous implies that the right hand side of the above inequality converges to zero, as $\eta \to x$.
Similarly, the case $\eta, x \in I, \eta < x$ can be proved. Which implies that $F(\mathscr{B}_\tau)$ is equicontinuous.

Step (3) : The implication $E = conv\,\mathrm{F(E)}\ or\ \mathrm{E} = \mathrm{F(E)} \cup \{0\} \implies$ E is compact holds.

Let E be a subset of B_τ such that E $\subset$ F(E) $\cup \{0\}$. From step (2), it is clear that E is bounded and equicontinuous, hence $u \to e(u) = \mu(E(u))$ is continuous on J.

Using the properties of measure μ, for each $u \in J$, we obtain:

$$\begin{aligned} e(u) &\leq \mu((FE)(u) \cup \{0\}) \leq \mu((FE)(u)) \\ &\leq \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1} m(s)\mu(E(s))ds \\ &\leq \frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* \int_{a_0}^{u} e(s)ds \\ &\leq T\frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* \|e\|_\infty. \end{aligned}$$

$$\implies e(u) \leq k\|e\|_\infty,$$

where, $k = T\frac{\psi(T)^\alpha}{\Gamma(\alpha+1)} m^* < 1$.

Now, $\|e\|_\infty = 0 \to e(s) = \mu(E(u)) = 0, \forall u \in J$.

Hence, $E(s)$ is relatively compact X and E is relatively compact in $\mathscr{B}_\tau$, by Ascoli-Arzela theorem, E is relatively compact in $\mathscr{B}_\tau$.

Hence in view of Monch's fixed point theorem, we conclude that F has a fixed point which is a mild solution of our $IVP(7.1)$. □

7.4 Stability analysis

In this section, we study $E_\alpha-$Ulam Hyers, generalized $E_\alpha-$Ulam Hyers, $E_\alpha-$ Ulam Hyers Rassias, and generalized $E_\alpha-$Ulam Hyers Rassias stability for the solutions of $\psi-$Hilfer fractional differential equations.

Definition 7.4.1. *[61] IVP (7.1) is $E_\alpha-$Ulam Hyers stable if there exists a real number*

$\lambda > 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\tilde{p} \in C(J,\mathbb{R})$ *of the inequality*

$$|{}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}\tilde{p}(u) - q(u,\tilde{p}(u))| \leq \varepsilon,\, u \in J, \tag{7.4}$$

there exists a solution $p \in C(J,\mathbb{R})$ *of (7.1) with*

$$|\tilde{p}(u) - p(u)| \leq \lambda \varepsilon E_{\alpha}(L_p(\psi(u) - \psi(a_0))^{\alpha}),\, u \in J \tag{7.5}$$

Definition 7.4.2. *[61] IVP (7.1) is generalized* $E_{\alpha}-$*Ulam Hyers stable if there exists* $f \in C(\mathbb{R}+,\mathbb{R}+)$ *with* $f(0) = 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\tilde{p} \in C(J,\mathbb{R})$ *of the inequality (7.4), there exists a solution* $p \in C(J,\mathbb{R})$ *of (7.1) with*

$$|\tilde{p}(u) - p(u)| \leq f(\varepsilon) E_{\alpha}(L_p(\psi(u) - \psi(a_0))^{\alpha}),\, u \in J \tag{7.6}$$

Definition 7.4.3. *[61] IVP (7.1) is* $E_{\alpha}-$*Ulam Hyers Rassias stable with respect to* χ *if there exists a real number* $\lambda_{\chi} > 0$ *such that, for each* $\varepsilon > 0$ *and for each solution* $\tilde{p} \in C(J,\mathbb{R})$ *of inequality*

$$|{}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}\tilde{p}(u) - q(u,\tilde{p}(u))| \leq \varepsilon\chi(u),\, u \in J, \tag{7.7}$$

there exists a solution $p \in C(J,\mathbb{R})$ *of (7.1) with*

$$|\tilde{p}(u) - p(u)| \leq \lambda_{\chi}\varepsilon\chi(u) E_{\alpha}(L_p(\psi(u) - \psi(a_0))^{\alpha}),\, u \in J. \tag{7.8}$$

Definition 7.4.4. *[61] IVP (7.1) is generalized* $E_{\alpha}-$*Ulam Hyers Rassias stable with respect to* χ *if there exists a real number* $\lambda_{\chi} > 0$ *such that, for each solution* $\tilde{p} \in C(J,\mathbb{R})$ *of inequality*

$$|{}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}\tilde{p}(u) - q(u,\tilde{p}(u))| \leq \chi(u),\, u \in J, \tag{7.9}$$

there exists a solution $p \in C(J,\mathbb{R})$ *of (7.1) with*

$$|\tilde{p}(u) - p(u)| \leq \lambda_{\chi}\chi(u) E_{\alpha}(L_p(\psi(u) - \psi(a_0))^{\alpha}),\, u \in J. \tag{7.10}$$

Remark 7.4.5. *[61] A function $\tilde{p} \in C(J,\mathbb{R})$ is a solution of the inequality (7.4) if and only if there exists a function $r \in C(J,\mathbb{R})$ such that*

(i) $|r(u)| \leq \varepsilon$, $u \in J$.

(ii) ${}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}\tilde{p}(u) = q(u,\tilde{p}(u)) + r(u)$, $u \in J$.

Remark 7.4.6. *[61] A function $\tilde{p} \in C(J,\mathbb{R})$ is a solution of the inequality (7.7) if and only if there exists a function $r \in C(J,\mathbb{R})$ such that*

(i) $|r(u)| \leq \varepsilon\chi(u)$, $u \in J$.

(ii) ${}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}\tilde{p}(u) = q(u,\tilde{p}(u)) + r(u)$, $u \in J$.

Theorem 7.4.7. *Assume that (H_1) and (H_2) hold. Then the IVP (7.1) is $E_\alpha-$Ulam Hyers stable on J and as a result IVP (7.1) is generalized $E_\alpha-$Ulam Hyers stable.*

Proof. Let $\tilde{p} \in C(J,\mathbb{R})$ be a function and $\varepsilon > 0$, which satisfies the inequality (7.4) and let $p \in C(J,\mathbb{R})$ the unique solution of the following problem

$$\begin{cases} {}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi}p(u) = q(u,\, p(u)), \quad a.e \quad u \in J = [a_0,\, T], \\ p(u_0) = \phi_0 \end{cases} \tag{7.11}$$

By Lemma (7.3.1) we have

$$\tilde{p}(u) = \phi_0 + \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}q(s,\tilde{p}(s)) + r(s))ds. \tag{7.12}$$

On the other hand, we have, for each $u \in J$,

$$\begin{aligned} |\tilde{p}(u) - p(u)| \leq & \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}|r(s)|ds \\ & + \frac{1}{\Gamma(\alpha)}\int_{a_0}^{u}\psi'(s)(\psi(u)-\psi(s))^{\alpha-1}|q(s,\tilde{p}(s)) - q(s,p(s))|ds \end{aligned}$$

$$\leq \frac{\varepsilon}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1} ds$$
$$+ \frac{K}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1} |\tilde{p}(s)-p(s)| ds$$
$$\leq \frac{(\psi(u)-\psi(a_0))^{\alpha}}{\Gamma(\alpha+1)} \varepsilon + \frac{K}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u)-\psi(s))^{\alpha-1} |\tilde{p}(s)-p(s)| ds.$$

Applying the Gronwall inequality to the above inequality with

$$x(t) = |\tilde{p}(u)-p(u)|,$$
$$y(t) = \frac{(\psi(u)-\psi(a_0))^{\alpha}}{\Gamma(\alpha+1)} \varepsilon,$$
$$z(t) = \frac{K}{\Gamma(\alpha)}.$$

$$|\tilde{p}(u)-p(u)| \leq \frac{(\psi(T)-\psi(a_0))^{\alpha}}{\Gamma(\alpha+1)} \varepsilon E_{\alpha}(K((\psi(u)-\psi(a_0))^{\alpha}))\, u \in J. \tag{7.13}$$

Choosing $\lambda = \frac{(\psi(T)-\psi(a_0))^{\alpha}}{\Gamma(\alpha+1)}$, We have

$$|\tilde{p}(u)-p(u)| \leq \lambda \varepsilon E_{\alpha}(K((\psi(u)-\psi(a_0))^{\alpha})),\, u \in J.$$

Hence the problem (7.1) is $E_{\alpha}-$Ulam Hyers stable.

Further, if we set $f(\varepsilon) = \lambda \varepsilon$, $f(0) = 0$, then the problem is generalized $E_{\alpha}-$Ulam Hyers stable.

This completes the proof. □

Theorem 7.4.8. *Assume that the hypothesis (H_1) and H_2 hold and if there exists an increasing function $\chi \in C(J, \mathbb{R}+)$ and there exists $\tau_{\chi} > 0$ such that for any $t \in J$*

$$I_{a_0+}^{\alpha;\psi} \chi(t) \leq \tau_{\chi} \chi(t).$$

Then, the IVP (7.1) is $E_{\alpha}-$Ulam Hyers Rassias stable and hence generalized $E_{\alpha}-$Ulam Hyers Rassias stable.

Proof. Let $\tilde{p} \in C(J,\mathbb{R})$ be a function and $\varepsilon > 0$, which satisfies the inequality (7.7) and let $p \in C(J,\mathbb{R})$ the unique solution of the following problem

$$\begin{cases} {}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi} p(u) = q(u, p(u)), \quad a.e \quad u \in J = [a_0, T], \\ p(u_0) = \phi_0 \end{cases} \tag{7.14}$$

By Lemma (7.3.1) we have

$$p(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} q(s, p(s))ds. \tag{7.15}$$

Also by Remark (7.4.6)

$$\begin{cases} {}^{H}\mathbb{D}_{a_0+}^{\alpha,\beta;\psi} \tilde{p}(u) = q(u, \tilde{p}(u)) + r(u), \quad a.e \quad u \in J = [a_0, T], \\ \tilde{p}(u_0) = \phi_0 \end{cases} \tag{7.16}$$

Again by Lemma (7.3.1) we have,

$$\tilde{p}(u) = \phi_0 + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} q(s, \tilde{p}(s)) + r(s))ds. \tag{7.17}$$

On the other hand, for each $u \in J$

$$\begin{aligned} |\tilde{p}(u) - p(u)| \leq & \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} |r(s)| ds \\ & + \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} |q(s, \tilde{p}(s)) - q(s, p(s))| ds \end{aligned}$$

Hence by using Remark (7.4.6) and (H_2), we can obtain

$$\begin{aligned} |\tilde{p}(u) - p(u)| \leq & \frac{\varepsilon}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} \chi(s) ds \\ & + \frac{K}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} |\tilde{p}(s) - p(s)| ds \\ \leq & \varepsilon \tau_\chi \chi(u) + \frac{K}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} |\tilde{p}(s) - p(s)| ds. \end{aligned}$$

Applying the Gronwall inequality to the above inequality with

$$x(t) = |\tilde{p}(u) - p(u)|,$$

$$y(t) = \varepsilon\tau_\chi\chi(u),$$
$$z(t) = \frac{K}{\Gamma(\alpha)}.$$

$$|\tilde{p}(u) - p(u)| \leq \varepsilon\tau_\chi\chi(u)E_\alpha(K(\psi(u) - \psi(a_0))^\alpha)\, u \in J. \tag{7.18}$$

Now by choosing

$$\lambda_\chi = \tau_\chi,$$

then equation (7.18) becomes

$$|\tilde{p}(u) - p(u)| \leq \varepsilon\lambda_\chi\chi(u)E_\alpha(K(\psi(u) - \psi(a_0))^\alpha)\, u \in J. \tag{7.19}$$

Then the IVP (7.1) is $E_\alpha-$Ulam Hyers Rassias stable.

Further if we set $\varepsilon = 1$, then the IVP (7.1) is generalized $E_\alpha-$Ulam Hyers Rassias stable.

This completes the proof. □

7.5 Examples

In this section, we give some examples Consider the following fractional differential equation:

Example 7.5.1.

$$\begin{cases} {}^H\mathbb{D}^{\alpha,\beta;\psi}_{0+} & p(u) = e + \frac{cosp(u)}{e^u+e^{-u}} \quad u \in [0,1] \\ & p(a_0) = 0. \end{cases},$$

where $q : J \times \mathbb{R} \to \mathbb{R}$ is given by

$$q(u, p(u)) = e + \frac{cosp(u)}{e^u + e^{-u}} \quad u \in J,\, p(u) \in C(J, \mathbb{R}).$$

It is clear that q is continuous and which satisfies

$$|q(u,p_1) - q(u,p_2)| \leq \frac{1}{1+e}|p_1 - p_2|,$$

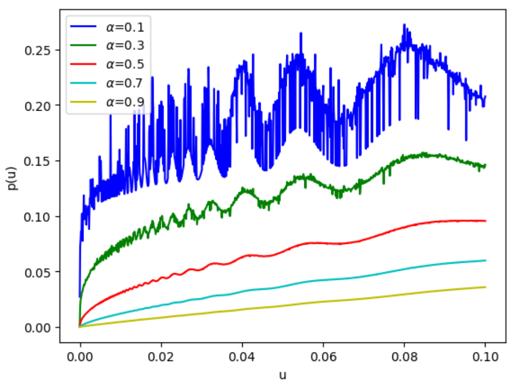
Hence the condition (H_2) *holds for* $K = \frac{1}{1+e}$.

The exact solution to this equation is

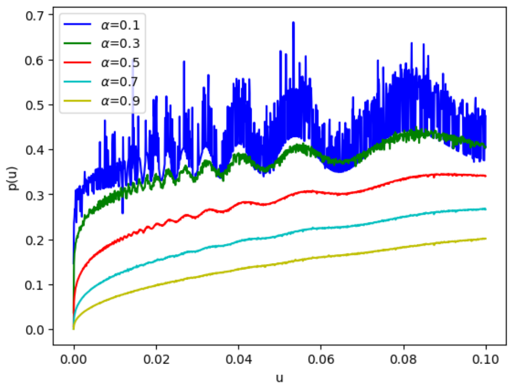
$$p(u) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} \left(e + \frac{cosp(s)}{e^s + e^{-s}} \right) ds.$$

For different $\psi(u)$ *and* $p(u)$*, the solutions are plotted below*

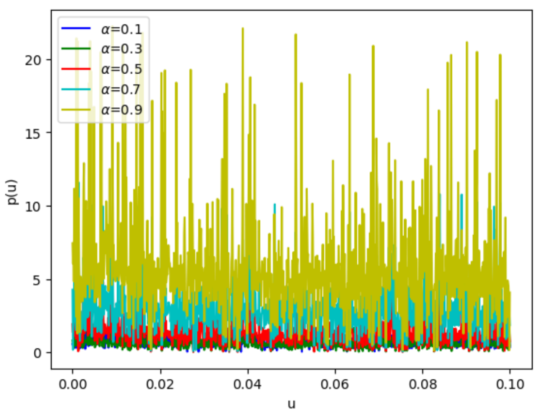
Figure 7.1: $\psi-$Hilfer differential equation with $q(u,p) = \left(e + \frac{cosp(s)}{e^u + u^{-s}}\right)$, $p(u) = \frac{1}{|u|}$.



(a) $\psi(u) = u$, $p(u) = \frac{1}{|u|}$, $a_0 = 0$

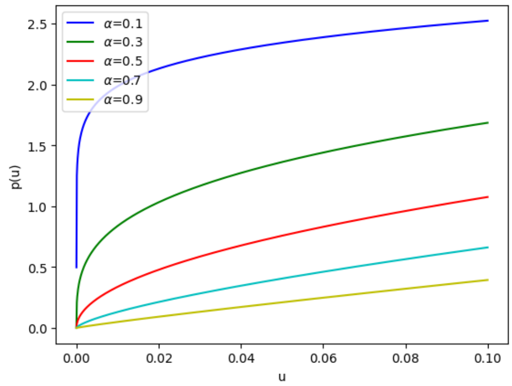


(b) $\psi(u) = u^{0.5}$, $p(u) = \frac{1}{|u|}$, $a_0 = 0$

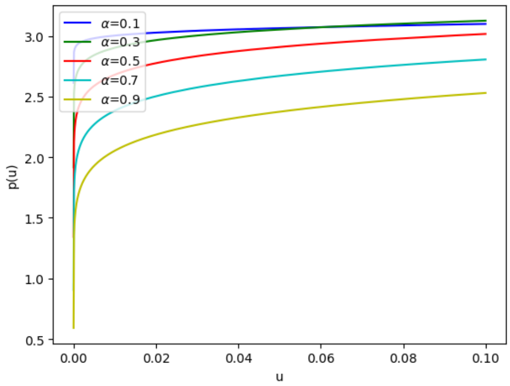


(c) $\psi(u) = \ln u$, $p(u) = \frac{1}{|u|}, a_0 = 1$

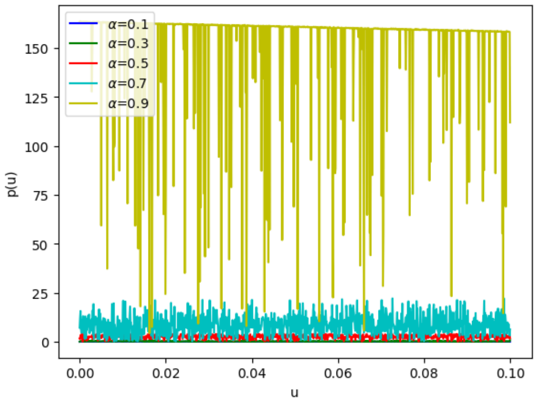
Figure 7.2: $\psi-$Hilfer fractional differential equation with $q(u,p) = \left(e + \frac{cos p(s)}{e^{u}+u^{-s}}\right)$, $p(u) = e^{-u}$.



(a) $\psi(u) = u$, $p(u) = e^{-u}, a_0 = 0$



(b) $\psi(u) = u^{0.5}$, $p(u) = e^{-u}, a_0 = 0$



(c) $\psi(u) = \ln u$, $p(u) = e^{-u}, a_0 = 1$

Example 7.5.2.

$$\begin{cases} {}^{H}\mathbb{D}_{0+}^{\alpha,\beta;\psi} & p(u) = \frac{5}{8}e^{-4u} - \frac{5}{8} + \frac{5}{2}u + \frac{\sin p(u)}{10} \quad u \in [0,1] \\ & p(0) = 0. \end{cases},$$

where $q : J \times \mathbb{R} \to \mathbb{R}$ is given by

$$q(u, p(u)) = \frac{5}{8}e^{-4u} - \frac{5}{8} + \frac{5}{2}u + \frac{\sin p(u)}{10} \quad u \in J,\ p(u) \in C(J, \mathbb{R}).$$

It is clear that q is continuous and which satisfies

$$|q(u,p_1) - q(u,p_2)| \leq \frac{1}{10}|p_1 - p_2|,$$

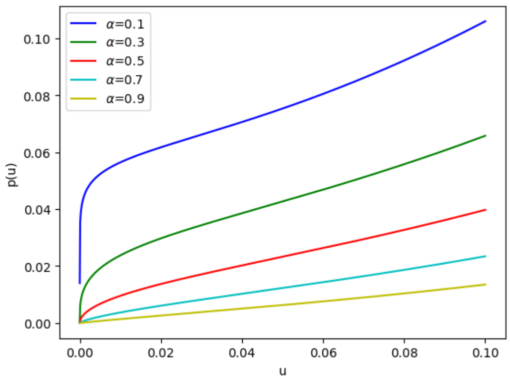
Hence the condition (H_2) holds for $K = \frac{1}{10}$.

The exact solution to this equation is

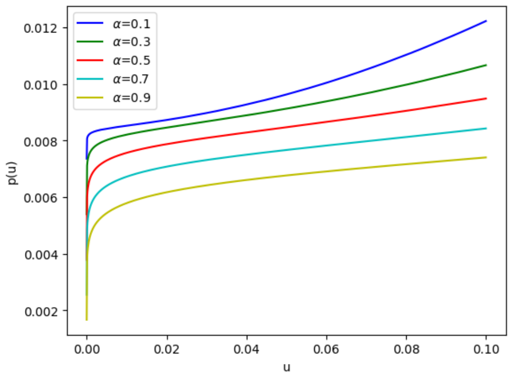
$$p(u) = \frac{1}{\Gamma(\alpha)} \int_{a_0}^{u} \psi'(s)(\psi(u) - \psi(s))^{\alpha-1} \left(\frac{5}{8}e^{-4s} - \frac{5}{8} + \frac{5}{2}s + \frac{\sin p(s)}{10} \right) ds.$$

For different $\psi(u)$ and $p(u)$, the solutions are plotted below

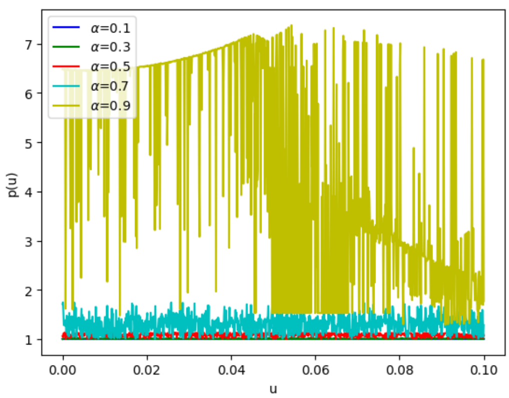
Figure 7.3: $\psi-$Hilfer differential equation with $q(u,p(u)) = \frac{5}{8}e^{-4u} - \frac{5}{8} + \frac{5}{2}u + \frac{\sin p(u)}{10}$



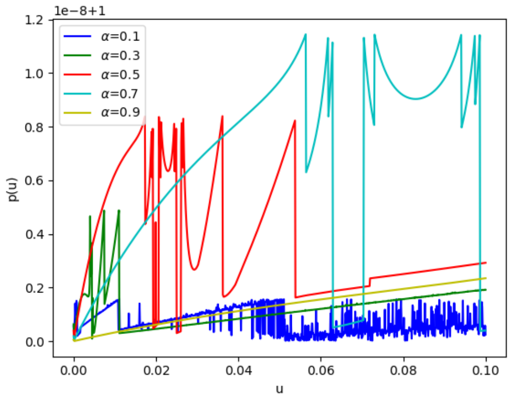
(a) $\psi(u) = u$, $p(u) = 1+u$



(b) $\psi(u) = u^{0.5}$, $p(u) = 1+u$



(c) $\psi(u) = \ln u$, $p(u) = 1+u$, $u \in [1,2]$



(d) $\psi(u) = \ln u$, $p(u) = u$, $u \in [1,2]$

7.6 Conclusion

Using Monch's fixed point theorem and Banach contraction principle, we have investigated the existence and uniqueness of the mild solution for the $\psi-$ Hilfer fractional differential equation. $E_\alpha-$Ulam Hyers, generalized $E_\alpha-$Ulam Hyers, $E_\alpha-$ Ulam Hyers Rassias, and generalized $E_\alpha-$Ulam Hyers Rassias stability for the solutions of $\psi-$Hilfer fractional differential equations were studied. Examples illustrating the results was given.

Chapter 8

Conclusion

Applications for generalised fractional differential equations (GFDEs) can be found in a wide range of scientific and technical fields. GFDEs help represent viscoelastic materials in materials science and mechanics by capturing the complex interaction between strain and deformation history. GFDEs provide an appropriate description of electrochemical processes, which are essential in energy systems such as batteries and fuel cells. They take into account memory in dynamic responses as well as non-local effects. The versatility of GFDEs in characterising system dynamics with memory is advantageous to control systems, particularly in robotics and aerospace, since it improves the accuracy of predictive modelling and control techniques.

GFDEs are used in signal processing, which extends beyond physics and engineering. They help with non-stationary signal analysis and make it easier to create filters with memory. These formulas are useful in biomedical engineering modelling physiological processes, including medication distribution in pharmacokinetics, where fractional derivatives precisely capture the intricate dynamics. Furthermore, long-range relationships in time series data are captured by GFDEs, which is useful for economic and financial modelling and leads to better forecasts and risk assessments. Whether used in the study of chaos in dynamical systems, fluid mechanics, environmental science, or heat transport in materials, GFDEs are an effective mathematical tool that offer a more nuanced understanding of complicated processes by including fractional derivatives.

The $\psi-$Caputo and $\psi-$Hilfer fractional derivatives are substitute methods for characterising non-local and memory-dependent behaviour in mathematical models. These fractional operators have certain disadvantages in addition to their benefits. Compared to the classical Riemann-Liouville fractional derivatives, the $\psi-$Caputo and $\psi-$Hilfer fractional derivatives are less well-studied, have fewer known features, and less analytical tools. Because of this, solving analytically and doing mathematical analyses for equations incorporating these derivatives can be difficult.

Compared to the Riemann-Liouville fractional derivatives, the $\psi-$Caputo and $\psi-$Hilfer fractional derivative formulas are more complicated. This complexity could make it harder to understand and apply, especially for users who are more accustomed to using the conventional fractional calculus operators.

It can be computationally hard to solve equations with $\psi-$Caputo and $\psi-$ Hilfer derivatives numerically. The absence of well-proven numerical techniques and algorithms designed especially for these derivatives could lead to higher computing expenses and difficulties in finding precise solutions. In the scientific and technical fields, the $\psi-$Caputo and $\psi-$ Hilfer derivatives are not as extensively used as the standard fractional derivatives. Researchers and practitioners working with these derivatives may find themselves lacking in literature, resources, and expertise as a result of this limited acceptance. Consistency and compatibility problems between results achieved using $\psi-$Caputo and $\psi-$ Hilfer derivatives and accepted theories and applications could arise. Because of this, it could be challenging to smoothly incorporate these variants into current frameworks and models.

On the following topics, the author developed some significant inferences:

1. Existence and uniqueness results for the $\psi-$Caputo fractional impulsive neutral functional differential equation.

2. Existence and uniqueness results for the $k-$dimensional system of $\psi-$Caputo fractional impulsive neutral functional differential equation.

3. Existence and uniqueness and stability results for the $\psi-$Caputo fractional neutral functional differential equation.

4. Existence result for the $\psi-$Hilfer hybrid fractional differential equations of first and second kind with initial conditions.

5. Existence result for the $\psi-$Hilfer hybrid fractional differential equations with boundary conditions.

6. Existence, uniqueness and stabiity results for $\psi-$Hilfer fractional differential equation.

Chapter 9

Recommendations

Future research on the following topics is suggested in light of the study's findings

- In our work we have studied the existence of solutions of generalized fractional differential equations for $0 < \alpha < 1$. Future research could extend study for other values of α.
- Future research could further examine existence, uniqueness and stability results for generalized stochastic, nonlocal differential and integro differential equations. Also can be extended to the equations involving resolvant, sectorial operators.
- Study could further extended for equations involving (k, ψ)Hilfer fractional operator or $\psi-$Hilfer generalized propotional fractional operators.
- A generalized fractional operator could have been developed, from which the fractional operators like Atangana Baleanu, Caputo Fabrizio and Conformal differential operators and the study can be further extended.

- A numerical method could be developed for generalized fractional differential equations.
- Generalized operators and studies could be used to model real-world scenarios for greater acceptability.

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APPENDIX I

List of publications

1. Shabna. M. S., & Ranjini. M, C. (2019). Fractional impulsive neutral functional differential equations involving $\psi-$Caputo fractional derivative. Malaya Journal of Matematik (MJM), (1), 493-499.https://doi.org/10.26637/MJM0S01/0089.

2. Shabna. M. S., & Ranjini. M. C. (2020). On existence of $\psi-$Hilfer hybrid fractional differential equations. South East Asian Journal of Mathematics & Mathematical Sciences, 16(2).

3. Shabna. M. S., & Ranjini. M. C. (2020). A k-dimensional systems of fractional neutral functional differential equations involving $\psi-$ Caputo fractional derivative. series Mathematics (85).

4. Shabna. M. S., & Ranjini. M. C. (2024). On the study of $\psi-$Caputo fractional neutral functional differential equation. The Journal of the Indian Mathematical Society, 91(3-4), 457–469. https://doi.org/10.18311/jims/2024/32507.

5. Ranjini. M. C., Shabna. M. S., Shameema. V. Existence, uniqueness and numerical results on $\psi-$Hilfer fractional differential equation. (Communicated).

6. Shabna. M. S., & Ranjini. M. C. "Existence and stability of $\psi-$Hilfer hybrid fractional differential equation with boundary conditions". (Communicated)