A STUDY ON PROPERTIES OF LOCALES, ACTION OF LOCALES AND THEIR APPLICATIONS

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This is to certify that the thesis entitled ***A Study on Properties of Locales, Action of Locales and Their Applications***, submitted by full-time research scholar Ms.Sabna K.S, Department of Mathematics, K.K.T.M.Government College, Pullut, to the University of Calicut, in partial fulfilment of requirement for the degree of Doctor of Philosophy in Mathematics, is a bonafide record of research work undertaken by her in the Centre for Research in Mathematical Science, St.Joseph's College, Irinjalakuda, under my supervision during the period 2011-2018 and that no part thereof has been presented before for any other degree.

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DECLARATION

I hereby declare that this thesis entitled ***A Study on Properties of Locales, Action of Locales and Their Applications***, is the record of bonafide research I carried out in the Centre for Research in Mathematical Sciences, St.Joseph's College (Autonomous), Irinjalakuda, under the supervision of Dr.Mangalambal N.R, Head, Associate Professor, Department of Mathematics, St.Joseph's College (Autonomous), Irinjalakuda.

I further declare that this thesis, or any part thereof, has not previously formed the basis for the award of any other degree, diploma, associateship, fellowship or any other similar title of recognition.

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Contents

Notations Used

$\sqsubseteq$: partial order relation on the locale L

0_L : the bottom element of the locale L

1_L : the top element of the locale L

f^* : left adjoint of f

f_* : right adjoint of f

a^c : compliment of a in L

$x \ll y$: x is way below y

$\boldsymbol{C}^{op}$: dual or opposite of the category $\boldsymbol{C}$

id_A : identity morphism on the object A

$\Omega(X)$: openset lattice of topological space X

$\downarrow(a)$: principal ideal generated by a

$\uparrow(a)$: principal filter generated by a

$\mathbf{0}$: the least sublocale $\{1\}$ of L

$\mathbf{2}$: two element locale

$o(a)$: open sublocale associated with $a \in L$

$c(a)$: closed sublocale associated with $a \in L$

$\bar{S}$: closure of a sublocale S of a locale L

$a \prec b$: a is rather below b

$a \prec\prec b$: a is completely below b

L/R : quotient frame with respect to the congruence R

Ç : set of core elements of the locale L with respect to the ideal I

Σ_a : the collection of all completely prime filters of the locale L containing $a \in L$

$O(L)$: the collection of all order preserving maps on the locale L

$\langle x \rangle$: subslice generated by x

$(\delta, J/J')$: factor of the L-slice (σ, J) with respect to the subslice (σ, J')

$(\gamma, J/R)$: quotient slice of L-slice (σ, J) with respect to the congruence R

Abbreviations Used

c.p filter : completely prime filter

$supD$: supremum of D

$Sp(L)$: spectrum of the locale L

$Con(J)$: the collection of all relative congruences on the L-slice (σ, J)

$Ann(J)$: annihilator of the L-slice (σ, J)

$kerf$: kernal of f

imf : image of f

Fix_f : the set of fixed points of f

$L - Hom(J, K)$: the collection of all L-slice homomorphisms from J to K

$Span(S)$: span of the set S

$Ob\boldsymbol{C}$: class of objects of the category $\boldsymbol{C}$

$Mor\boldsymbol{C}$: class of morphisms of the category $\boldsymbol{C}$

Frm : Category of frames and frame homomorphisms

Loc : Category of locales and localic maps

Top : Category of topological spaces and continuous maps

L-slice : Category of L-slices and L-slice homomorphisms

TopWMod : Category of topological weak modules and continuous weak module homomorphisms

Fin L-slice : Category of finitely generated L-slices and L-slice homomorphisms

Fin TopWMod : Category of finitely generated topological weak modules and continuous weak module homomorphisms

JSLat : Category of join semilattices with bottom and semilattice homomorphisms

iTopMon : Category of idempotent topological monoids and continuous monoid morphisms

K.E.P : Key Exchange Protocol

Introduction

In 1914, Hausdorff [17] used the concept of open sets in a topological space to study the properties of continuous functions between two topological spaces. Thus from 1914 onwards, a topological space was considered as a non-empty set X together with a lattice $\Omega(X)$ of open subsets of X. The American Mathematician Marshall Stone was the first Mathematician who studied the interrelation between topology and lattice theory in his work on topological representation of Boolean algebras [47] [48] and distributive lattices [49], where two important results have been developed. The first one reveals the importance of ideals in lattice theory by viewing Boolean algebra as a type of Boolean ring. The other result was Stone representation theorem which was a milestone in the development of theory of locales.

Stone's Representation Theorem

Every Boolean algebra is isomorphic to the Boolean algebra of open-closed sets of a totally disconnected compact Hausdorff space.

After Stone, Henry Wallman [51], was the first person who used the lattice theoretic notions to study topological properties. In order to introduce the concept of Wallman compactification of T_1 topological spaces, he used the lattice theoretic ideas. After a few years, American logician McKinsey and a Polish Mathematician Tarski [28] [29], carried out a study on "Algebra of Topology". The book "Grundlagen der

Analytischen Topologie"[33] by Nobeling was the first text book which explain general topology in the view point of lattice theory. Charles Ehresmann and his student Jean Benabou [13] studied topological and differentiable categories and they developed an idea that the lattices with right distributive property should be studied as "generalized topological spaces". They named these "generalized topological spaces" as local lattices. At the same period, Dona and Seymour Papert [35] [36] used similar concepts to study topological spaces.

Isbell [20] [22], in his paper pointed out that generalized spaces have some sort of differences with topological spaces. The main difference is that the product of generalized spaces behaved better than Tychonoff product of topological spaces.

C.H.Dowker [8] suggested the term *Frame* for generalized spaces. Frame theory is point-free topology which views topology $\Omega(X)$ of a topological space $(X, \Omega(X))$ as a lattice of open subsets of X satisfying infinite distributive property. The functor Ω : **Top** → **Frm** from the category **Top** of topological spaces to the category **Frm** of frames, which sends a topological space $(X, \Omega(X))$ to the lattice $\Omega(X)$ of its open sets, is contravariant. Accordingly the category of "generalized topological spaces" must be opposite to the category of frames. The term *Locale* was the contribution of Isbell [20] to the objects in this opposite category. *If we do not refer to the morphisms in the category **Loc** of locales and the category **Frm** of frames, then the objects frames and locales are same.* Dowker, Papert [8] and Isbell [20] had taken up the study of sublocales (quotient frames) and Isbell put forward the term sublocales.

As the properties such as compactness and connectedness are defined in terms of open sets, it is easy to extend these concepts to localic background. But some topological properties such as $T_1 - axiom$ are defined in terms of points. So the extension of such ideas to localic background is not so easy. In such cases, we have to apply alternate definitions.

Localic version of Hausdorff axiom was put forward by Isbell [23], C.H.Dowker and D.Strauss [10], H.Simmons [45], P.T Johnstone and S.H Sun [26] and J.Paseka [37]. Among these most acceptable form was the axiom introduced by Isbell [23]. Isbell defined localic version of Hausdorff axiom in terms of diagonals. But this axiom has a disadvantage. It should not be considered as equivalent to the Hausdorff axiom for topological spaces. Though the definition of regularity in topological spaces involves the points, it has alternate representation in terms of open sets. So extension of regularity to localic background is straight forward. Thus, as locales are extension of topological spaces, most of the topological properties are extended into localic background.

In addition to the above development, among many introductions to topology, a particular view that has arisen in Theoretical Computer Science starts with the theory of domains, as defined by Scott and Strachey [42], to provide a mathematical foundation for semantics of programming languages, establishing that domains could be put into a topological setting. Mike Smyth [46] has developed the idea further. The topology describes an essential computational notion that provide them an independence from the points of topological spaces and this fall into the branch of mathematics, the theory of locales.

Duality between Frames and topological spaces have been utilized to make a connection between syntactical and semantical approach to logic. But the application of Stone duality in modal logic require a duality for Boolean algebras or distributive lattices endowed with additional operations.

The background of theory of locales used as a theory of information is the motivation for the present study in this thesis titled "A study on properties of locales, action of locales and their applications". The above context has inspired the introduction of the concept of "an action of a locale on a join semilattice". Given a locale L and a join semilattice J with bottom element 0_J, we have introduced a new concept called L-slice, denoted by (σ, J), to be an action σ of the locale L on the join semilattice J together with a set of conditions. The L-slice, though algebraic in nature adopts topological properties such as compactness through the action σ. Several different aspects of L-slice for a locale L have been obtained in the present study.

The study in this thesis is begun with an investigation into construction of sublocales from ideals of a locale L. An embedding theorem for a locale L has been derived.

The content in the thesis is described in the following way.

Chapter 1 contains a quick review of the preliminary materials required to read and understand this thesis.

In chapter 2, the following are studied. Sublocales of a locale L are traditionally presented in terms of sublocale homomorphism, frame congruence and nucleus. In [39] Pultr and Picado have shown that there exist a one-one correspondence between sublocales of a locale L and nuclei in L. The work in this chapter discusses a method of construction of sublocales using ideals of a locale L. Given an ideal I of a locale L, a collection $\{I_a; a \in L\}$ of ideals of L with the property $I \subseteq I_a$ for all $a \in L$ has been constructed. The following results are obtained.

- If the ideal I is prime, then the ideals I_a are prime for all $a \in L$.
- If the ideal I is closed under arbitrary join, then there exists a complete join

semilattice homomorphism from the locale L to the complete lattice $M = (\{I_a; a \in L\}, \supseteq)$ and M induces a frame congruence R_I on L. This congruence determines a sublocale of L.

- The topological properties such as subfit, fit, S_2', regularity, normality and compactness of the sublocale S of L thus constructed are obtained using the class of core elements of L with respect to I.

Chapter 3 deals with the following ideas. If X is a topological space and $C(X)$, the ring of continuous real-valued functions on X, then the sets of the form $\{f \in C(X) : f(x) \in V, V \ \ open \ \ in \ \ R\}$, which depends on both points of X and topology of R, forms a subbase for a point-open topology on $C(X)$. Also if L is a locale and Σ_x denotes the set of completely prime filters in L containing $x \in L$, then $Sp(L) = (\{all \ \ completely \ \ prime \ \ filters \ \ of \ \ L\}, \{\Sigma_a; a \in L\})$ is a topological space. In this chapter, as a generalization of above context to localic background, we have proved the following

- For $a, b \in L$, the collection $[a, \Sigma_b] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_b\}$ are ideals in $O(L)$, where $O(L)$ is the locale of order preserving maps on L. Some algebraic properties of these ideals $[a, \Sigma_b]$ have been established.

- For each $a \in L$, ideals of the form $[a, \Sigma_b]$ generates a spatial locale $J_a = \{[a, \Sigma_b], b \in L\}$. Separation properties such as subfit, S_2', regularity and normality pertaining to the locale J_a have been established.

- Using the ideals $[a, \Sigma_b]$, a congruence $\sim_a$ is defined in L and it is proved that $L/\sim_a$ is isomorphic to J_a.

- For each $a \in L$, congruence R_a is defined in $O(L)$ and it is proved that the quotient frame $O(L)/R_a$ of $O(L)$ is isomorphic to the quotient frame $L/\sim_a$ of L.

- Using the congruences on L and $O(L)$, an embedding theorem for locale L is established.

- Setting the coproduct $J = \prod J_a$, the productive properties of J have been proved.

- For $a, b \in L$, the collection $\langle a, \Sigma_b \rangle = \{f \in O(L) : \Sigma_{f(a)} \supseteq \Sigma_b\}$ is a filter in $O(L)$. Sufficient conditions for the filter $\langle a, \Sigma_b \rangle$ to be completely prime is derived.

- For a compact open set Σ_b in spectrum $Sp(L)$ of L, the collection $\{\langle a, \Sigma_b \rangle : a \in L\}, a, b \in L$ determines a compact, connected, T_0 subspace of the spectrum $Sp(O(L))$ of $O(L)$.

In chapter 4, a new concept (σ, J), called L-slice, is introduced as, an action σ of a given locale L, on a join semilattice J with bottom element 0_J. Using the algebraic properties of the L-slice (σ, J), a congruence R on (σ, J) is obtained. We have proved that the pair $(\gamma, J/R)$, of all equivalence classes with respect to the congruence R on (σ, J), is an L-slice, where the action γ is defined in terms of σ. The Factor of L-slice (σ, J) with respect to the subslice (σ, J') of (σ, J), is defined.

Chapter 5 discusses various properties of L-slice homomorphism. We have proved the following

- The collection $L - Hom(J, K)$ of all L-slice homomorphisms from L-slice (σ, J) to the L-slice (μ, K) is an L-slice with respect to the action $\delta : L \times L - Hom(J, K) \to L - Hom(J, K)$ and that every L-slice (σ, J) is isomorphic to a subslice of $(\delta, L - Hom(L, J))$.

- An isomorphism theorem for L-slice is derived. As an application, the notion of finitely generated L-slice of a locale L is introduced and we have shown that every finitely generated L-slice (σ, J) of a locale L with n generators is isomorphic to the quotient slice of the L-slice $(\sqcap, L^n)$.

- For each $a \in L, \sigma_a : (\sigma, J) \to (\sigma, J)$ defined by $\sigma_a(x) = \sigma(a, x)$ is an interior operator on (σ, J).

- The collection $M = \{\sigma_a; a \in L\}$ is a Priestley space and a subslice of $(\delta, L{-}Hom(J, J))$. If the locale L is spatial an isomorphism between the L-slices $(\sqcap, L)$ and (δ, M) has been established.

- Fixed set of $\sigma_a, a \in L$ is a subslice of (σ, J).

- For each $x \in (\sigma, J), \sigma_x : (\sqcap, L) \to (\sigma, J)$ defined by $\sigma_x(a) = \sigma(a, x)$ is an L-slice homomorphism and the collection $P = \{\sigma_x; x \in (\sigma, J)\}$ is an L-subslice of $(\delta, L - Hom(L, J))$.

- The map $x \mapsto \sigma_x$ is an L-slice isomorphism between the L-slices (σ, J) and (δ, P)

- The compactness in L-slice (σ, J) is defined and it is proved that L-slice compactness is stronger than topological compactness and localic compactness.

- A subspace Y of spectrum $Sp(L)$ of L has been constructed using filters $F_x = \{a \in L : \sigma_x(a) = x\}$ for compact elements $x \in (\sigma, J)$ and the compactness of the subspace Y is characterized using the existence of maximal compact element in the L-slice (σ, J).

- It is known that there is a contravariant functor from the category **JSLat** of join semilattice with bottom element, and semilattice homomorphism to the category **iTopMon** of idempotent topological monoids, and continuous monoid homomorphisms. In this context, we have proved that there is a contravarint functor from the category **L-slice** of L-slices of a locale L to the category **TopWMod** of topological weak L-modules.

In chapter 6, as an application of the above study of L-slices, a key exchange protocol has been developed that utilizes the concept of L-slices for the generation of secret and public keys. The L-slice and its properties are utilized to extend the existing Diffie Hellman key exchange protocol that uses groups in algebra, to the background of L-slices of a locale L. A modification is given to the extended Diffie Hellman key exchange protocol using L-slices of a locale L in order to give optimum security to the system.

The thesis is concluded with further scope of study.

Chapter 1

Preliminaries

This chapter includes some preliminary concepts on Order theory, Category theory, Frames and Locales and Cryptography required for the next chapters.

1.1. Order theoretical concepts

Definition 1.1.1. [24] Let L be a set. A partial order on L is a binary relation $\sqsubseteq$ which is

i. reflexive : for all $a \in L$, $a \sqsubseteq a$,

ii. antisymmetric: if $a \sqsubseteq b$ and $b \sqsubseteq a$, then $a = b$, and

iii. transitive: if $a \sqsubseteq b$ and $b \sqsubseteq c$, then $a \sqsubseteq c$.

A partially ordered set (also called poset) is a set equipped with a partial order.

Definition 1.1.2. [5] An element $x \in A \subseteq L$ is called minimal if $a \in A, a \sqsubseteq x$ implies $a = x$. If L has a unique minimal element, then it is called the least element (bottom) of L denoted by 0_L.

Definition 1.1.3. [5] An element $x \in A \subseteq L$ is called maximal if $a \in A, x \sqsubseteq a$ implies $a = x$. If L has a unique maximal element, then it is called the greatest element (top) of L denoted by 1_L.

Definition 1.1.4. [5] An element $x \in L$ is called an upperbound of $A \subseteq L$, if for all $a \in A$, we have $a \sqsubseteq x$. The least element of the set of all upperbounds of A in L, if it exists, is called the least upperbound (supremum)of A. It is denoted by $\bigsqcup A$.

Definition 1.1.5. [5] An element $x \in L$ is called a lowerbound of $A \subseteq L$, if for all $a \in A$, we have $x \sqsubseteq a$. The greatest element of the set of all lowerbounds of A in L, if it exists, is called the greatest lowerbound (infimum) of A. It is denoted by $\bigsqcap A$

Definition 1.1.6. [39] A poset L is called a join-semilattice (resp.meet-semilattice) if there is a supremum $a \sqcup b$ (resp.infimum $a \sqcap b$) for any two $a, b \in L$.

Definition 1.1.7. [24] A partially ordered set L in which for every pair of elements a, b, there exists the supremum $a \sqcup b$ and the infimum $a \sqcap b$ is called a lattice. A partially ordered set L for which every set $A \subseteq L$ has the supremum $\bigsqcup A$ and the infimum $\bigsqcap A$ exist in L is called a complete lattice.

Definition 1.1.8. [24] A lattice L is distributive if $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$ which is equivalent to $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

Definition 1.1.9. [5] A map $f : L \to M$, where L, M are partially ordered sets, is called monotone(order preserving) if $a \sqsubseteq_L b \Rightarrow f(a) \sqsubseteq_M f(b)$ for all $a, b \in L$. If f is bijective and its inverse f^{-1} is also monotone, then it is called an order isomorphism.

Definition 1.1.10. [39] Monotone maps $f : L \to M, g : M \to L$ are Galois adjoint (or are in a Galois connection)-f is a left adjoint to g, and g is a right adjoint of f- if

$$\forall x \in L, \forall y \in M, f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq g(y).$$

Theorem 1.1.11. *[39] Monotone maps $f : L \to M$ and $g : M \to L$ are adjoint (f on the left, g on the right) if and only if there holds $f(g(y)) \sqsubseteq y$ and $x \sqsubseteq g(f(x))$.*

Corollary 1.1.12. *[39] If monotone maps f, g are adjoint, then fgf=f and gfg=g.*

Theorem 1.1.13. *[39] The left Galois adjoint preserves suprema, and the right one preserves infima.*

Theorem 1.1.14. *[39] If L, M are complete lattices, then a monotone map $f : L \to M$ is a left (resp. right) adjoint if and only if it preserves all suprema (resp. infima).*

Remark. The left adjoint of $f : L \to M$ is denoted by f^* and right adjoint by f_*.

Definition 1.1.15. [24] Let L be a distributive lattice with greatest element 1_L and least element 0_L. The complement a^c of an element $a \in L$ is the one satisfying $a \sqcap a^c = 0_L$ and $a \sqcup a^c = 1_L$.

Definition 1.1.16. [24] A Boolean algebra is a distributive lattice with 0_L and 1_L in which every element has a complement.

Definition 1.1.17. [39] An element $p \neq 1$ in a lattice L is said to be meet-irreducible if for any $a, b \in L$, $a \sqcap b \sqsubseteq p$ implies that either $a \sqsubseteq p$ or $b \sqsubseteq p$.

Definition 1.1.18. [39] An element $p \neq 0$ in a lattice L is join-irreducible if for any $a, b \in L, p \sqsubseteq a \sqcup b$ implies that either $p \sqsubseteq a$ or $p \sqsubseteq b$.

Definition 1.1.19. [15] Let L be a poset. We say that x is way below y, $(x \ll y)$ if and only if for all directed subset $D \subseteq L$ for which $supD$ exists, the relation $y \sqsubseteq supD$ always implies the existence of a $d \in D$ with $x \sqsubseteq d$.

Definition 1.1.20. [15] An element satisfying $x \ll x$ is said to be compact or isolated from below.

Definition 1.1.21. [39] A lattice A is said to be a Heyting algebra if for each pair of elements (a, b) in A, there exist an element $a \to b$ such that $c \sqsubseteq (a \to b)$ if and only if $c \sqcap a \sqsubseteq b$.

Definition 1.1.22. [6] Let $(X, \sqsubseteq)$ be a poset. A map $f : X \to X$ is called interior operator if

i. f is order preserving

ii. $f(x) \sqsubseteq x$ for all $x \in X$

iii. $f \circ f = f$.

Definition 1.1.23. Let L be bounded distributive lattice, and let X denote the set of prime filters of L. For each $a \in L$, let $\phi_+(a) = \{x \in X : a \in x\}$. Then (X, τ_+) is a spectral space, where the topology τ_+ on X is generated by $\{\phi_+(a); a \in L\}$. The spectral space (X, τ_+) is called the prime spectrum of L.

The map ϕ_+ is a lattice isomorphism from L onto the lattice of all compact open subsets of (X, τ_+). Similarly, if $\phi_-(a) = \{x \in X : a \notin x\}$ and τ_- denotes the topology generated by $\{\phi_-(a); a \in L\}$, then (X, τ_-) is also spectral space. Let $\subseteq$ be set-theoretic inclusion on the set of prime filters of L and let $\tau = \tau_+ \cup \tau_-$. Then $(X, \tau, \subseteq)$ is a Priestley space.

1.2. Categorical Concepts

Definition 1.2.1. [18] A category $\boldsymbol{C}$ consist of:

i. A class $Ob\boldsymbol{C}$ of objects (notation: $A, B, C....$)

ii. A class $Mor\,\boldsymbol{C}$ of morphisms (notation: $f, g, h...$). Each morphism f has a domain or source A (notation: $dom(f)$) and a codomain or target B (notation: $codom(f)$) which are objects of $\boldsymbol{C}$; this is indicated by writing $f : A \to B$.

iii. A composition law that assign to each pair (f, g) of morphisms satisfying $dom(g) = codom(f)$ a morphism $g \circ f : dom(f) \to codom(g)$, satisfying

(a) $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the compositions are defined.

(b) For each object A of $\boldsymbol{C}$ there is an identiy $id_A : A \to A$ such that $f \circ id_A = f$ and $id_A \circ g = g$ whenever the composition is defined.

Definition 1.2.2. [18] A category $\boldsymbol{B}$ is said to be a subcategory of the category $\boldsymbol{C}$ provided that the following conditions are satisfied.

i. $Ob(\boldsymbol{B}) \subseteq Ob(\boldsymbol{C})$.

ii. $Mor(\boldsymbol{B}) \subseteq Mor(\boldsymbol{C})$.

iii. The domain, codomain and composition functions of $\boldsymbol{B}$ are restriction of the corresponding functions of $\boldsymbol{C}$.

iv. Every $\boldsymbol{B}$-identity is a $\boldsymbol{C}$ -identity.

Theorem 1.2.3. *[18] Every product category of categories is a category.*

Definition 1.2.4. [18] If $\boldsymbol{C}$ is a category we can take the same class of objects and morphisms, and interchange the domains and codomains (which leads to inverted composition). Thus $f : A \to B$ is now $f : B \to A$ and we have a composition $f * g = g \circ f$. Thus obtained category is called the dual or opposite of $\boldsymbol{C}$ and denoted by $\boldsymbol{C}^{op}$.

Definition 1.2.5. [18] A morphism $f : A \to B$ in a category $\boldsymbol{C}$ is said to be section in $\boldsymbol{C}$ provided that there exists some $\boldsymbol{C}$-morphism $g : B \to A$ such that $g \circ f = id_A$.

Definition 1.2.6. [18] A morphism $f : A \to B$ in a category $\boldsymbol{C}$ is said to be a retraction in $\boldsymbol{C}$ provided that there exists some $\boldsymbol{C}$-morphism $g : B \to A$ such that $f \circ g = id_B$.

Definition 1.2.7. [18] A $\boldsymbol{C}$-morphism is said to be an isomorphism in $\boldsymbol{C}$ provided that it is both $\boldsymbol{C}$ -section and $\boldsymbol{C}$ -retraction.

Definition 1.2.8. [18] Let $\boldsymbol{C}$ be a category and $A, B \in Obj(\boldsymbol{C})$. A morphism $f : A \to B$ is epimorphism if $f \circ g = f \circ h$ implies $g = h$ for all morphisms $g, h : B \to C$

Definition 1.2.9. [18] A $\boldsymbol{C}$-morphism $f : A \to B$ is said to be a monomorphism in $\boldsymbol{C}$ provided that for all $\boldsymbol{C}$-morphisms h and k such that $f \circ h = f \circ k$, it follows that $h = k$.

Definition 1.2.10. [18] A $\boldsymbol{C}$-morphism is said to be a bimorphism in $\boldsymbol{C}$ provided that it is both a monomorphism and an epimorphism.

Definition 1.2.11. [18] Let $\boldsymbol{C}$,$\boldsymbol{D}$ be categories. A functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ is a triple ($\boldsymbol{C}$,F,$\boldsymbol{D}$) where F is a function from the class of morphisms of $\boldsymbol{C}$ to the class of morphisms of $\boldsymbol{D}$ (i.e. $F : Mor\boldsymbol{C} \to Mor\boldsymbol{D}$) satisfying the following conditions.

i. F preserves identities: i.e., if e is a $\boldsymbol{C}$-identity, then $F(e)$ is a $\boldsymbol{D}$- identity.

ii. F preserves composition: $F(f \circ g) = F(f) \circ F(g)$;i.e., whenever $dom(f) = codom(g)$, then $dom(F(f)) = codom(F(g))$ and the above equality holds.

Definition 1.2.12. [18] A triple ($\boldsymbol{C}$, F, $\boldsymbol{D}$) is called a contravariant functor from $\boldsymbol{C}$ to $\boldsymbol{D}$ if and only if ($\boldsymbol{C}^{op}, F, \boldsymbol{D}$) is a functor (or, equivalently, if and only if ($\boldsymbol{C}, F, \boldsymbol{D}^{op}$) is a functor).

1.3. Frames and Locales

Definition 1.3.1. [39] A frame is a complete lattice L satisfying the infinite distributivity law $a \sqcap \bigsqcup B = \bigsqcup\{a \sqcap b; b \in B\}$ for all $a \in L$ and $B \subseteq L$.

Definition 1.3.2. [39] A map $f : L \to M$ between frames L, M preserving all finite meets (including the top 1) and all joins (including the bottom 0) is called a frame homomorphism. A bijective frame homomorphism is called a frame isomorphism.

Remark. The category of frames is denoted by **Frm**. The opposite of category **Frm** is the category **Loc** of locales. We can represent the morphism in **Loc** as the infima -preserving $f : L \to M$ such that the corresponding left adjoint $f^* : M \to L$ preserves finite meet. If we do not refer to the morphisms in the category **Loc** of locales and the category **Frm** of frames, then the objects frames and locales are same.

Remark. The category of topological spaces and continuous maps is denoted by **Top**

Definition 1.3.3. [39] The functor $\Omega : \mathbf{Top} \to \mathbf{Frm}$ maps objects and morphisms as follows

i. A topological spaces $(X, \Omega(X))$ is mapped into frame of open sets $\Omega(X)$

ii. Ω sends morphism $f : X \to Y$ in **Top** to the frame homomorphism
$\Omega(f) : \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(V) = f^{-1}(V)$.

Theorem 1.3.4. *[39] The functor Ω : **Top** → **Frm** is a contravariant functor*

Definition 1.3.5. [24] A subset I of a locale L is said to be an ideal if

i. I is a sub-join-semilattice of L; that is $0_L \in I$ and $a \in I, b \in I$ implies $a \sqcup b \in I$;and

ii. I is a lower set; that is $a \in I$ and $b \sqsubseteq a$ imply $b \in I$.

If $a \in L$, the set $\downarrow(a) = \{x \in L; x \sqsubseteq a\}$ is an ideal of L. $\downarrow(a)$ is the smallest ideal containing a and is called the principal ideal generated by a. A proper ideal I is prime if $x \sqcap y \in I$ implies that either $x \in I$ or $y \in I$ [24].

Definition 1.3.6. [39] A subset F of locale L is said to be a filter if

i. F is a sub-meet-semilattice of L; that is $1_L \in F$ and $a \in F$, $b \in F$ imply $a \sqcap b \in F$.

ii. F is an upper set; that is $a \in F$ and $a \sqsubseteq b$ imply $b \in F$.

Definition 1.3.7. [39] A filter F is proper if $F \neq L$, that is if $0_L \notin F$.

A proper filter F in a locale L is prime if $a_1 \sqcup a_2 \in F$ implies that $a_1 \in F$ or $a_2 \in F$.

Definition 1.3.8. [39] A proper filter F in a locale L is a completely prime filter if for any indexing set J and $a_i \in L$, $i \in J$, $\bigsqcup a_i \in F \Rightarrow \exists i \in J$ such that $a_i \in F$. Completely prime filters are denoted by c.p filters.

Example 1.3.9. *[39]* $U(x) = \{V \in \Omega(X); x \in V\}$ *is a completely prime filter in the locale* $\Omega(X)$.

For an element a of a locale L, set $\Sigma_a = \{F \subseteq L; F \neq \phi, F \;\; is \;\; c.p \;\; filters; a \in F\}$. We can easily check that $\Sigma_0 = \phi$, $\Sigma_{\bigsqcup a_i} = \bigcup \Sigma_{a_i}$, $\Sigma_{a \sqcap b} = \Sigma_a \cap \Sigma_b$ and

$\Sigma_1 = \{all \;\; c.p \;\; filters\}$.

The spectrum of a locale is defined as follows.

Sp(L)=($\{all \;\; c.p \;\; filters\}$, $\{\Sigma_a : a \in L\}$). Then $Sp(L)$ is a topological space with the topology $\Omega(Sp(L)) = \{\Sigma_a : a \in L\}$.

Definition 1.3.10. [39] If $f : L \to M$ is a morphism in the category **Loc**, then $Sp(f) : Sp(L) \to Sp(M)$ defined by $Sp(f)(F) = (f^*)^{-1}(F)$ is a morphism in the category **Top**.

Definition 1.3.11. [39] A locale L is said to be spatial if it is isomorphic to $\Omega(X)$ of some topological space X.

We have a frame homomorphism Φ_L : L $\to$ $Lc(Sp(L))$ given by $\Phi_L(a) = \Sigma_a$. Their right Galois adjoint is the localic map $\sigma_L = (\Phi_L)_*$: Lc(Sp(L))$\to$ L.

Proposition 1.3.12. *[39] The following statements on a locale are equivalent.*

i. L is spatial.

ii. σ_L: Lc(Sp(L))$\to$ L is a complete lattice isomorphism.

iii. σ_L^*:L $\to$ Lc(Sp(L)) is a complete lattice isomorphism.

iv. σ_L is onto.

v. σ_L^* is one-one.

Definition 1.3.13. [39] Let L be a frame. An equivalence relation θ on L is said to be a congruence on L if $(a, b) \in \theta \Rightarrow (a \sqcap c, b \sqcap c) \in \theta$ and $(a \sqcup \bigsqcup S, b \sqcup \bigsqcup S) \in \theta$ for all $c \in L, S \subseteq L$.

1.3.1 Products in frame

Definition 1.3.14. [39] If $L_i, i \in J$ are frames, we endow the cartesian product $\prod_{i \in J} L_i$ with the structure of frame coordinatewise (which is same as defining the order by $(x_i)_{i \in J} \sqsubseteq (y_i)_{i \in J}$ iff $x_i \sqsubseteq y_i$). The projections

$$p_j = ((x_i)_{i \in J} \to x_j) : \prod_{i \in J} L_i \to L_j$$

are then frame homomorphisms and we see that for each system $(h_j : M \to L_j)_{j \in J}$ of frame homomorphisms there is precisely one frame homomorphism $h : M \to \prod_{i \in J} L_i$ such that $p_j \circ h = h_j$ for all $j \in J$, namely the one given by $h(x) = (h_j(x))_{j \in J}$.

Remark. [39] Those were the product of non-empty systems. The empty product is the one element frame $\mathbf{1} = \{0_{\mathbf{1}} = 1_{\mathbf{1}}\}$. The constant mapping $L \to \mathbf{1}$ are obviously frame homomorphisms.

Since **Frm** has products, **Loc** has coproducts. Let us present the coproducts injections explicitely.

Definition 1.3.15. [39] For a fixed $j \in J$, define $\alpha_j : L_j \to \prod_{i \in J} L_i$ by setting

$$(\alpha_j(x))_i = \begin{cases} x & \text{if i=j} \\ 1 & \text{otherwise} \end{cases}$$

We immediately see that $p_j((x_i)_{i \in J}) \sqsubseteq x$ iff $(x_i)_{i \in J} \sqsubseteq \alpha_j(x)$.
Thus $(\alpha_j : L_j \to \prod_{i \in J} L_i)_{j \in J}$ constitutes the coproduct in **Loc**.

1.3.2 Subframes and Sublocales

Definition 1.3.16. [39] A subset of a frame L which is closed under the same finite meets and arbitrary joins in the frame is called a subframe. That is a subframe is itself a frame under the induced order of L.

The concept of sublocale is something different, corresponding to quotient frames.

Definition 1.3.17. [39] Let L be a locale. A subset $S \subseteq L$ is a sublocale of L if

i. S is closed under meets, and

ii. For every $s \in S$ and every $x \in L$, $x \to s \in S$.

A sublocale is always nonempty, since $1 = \prod \phi \in S$. The least sublocale $\{1\}$ will be denoted by $\mathbf{0}$

Definition 1.3.18. [39] Let L be a locale and $a \in L$. The open sublocale associated with a is defined by $o(a) = \{a \to x, x \in L\}$. The closed sublocale is the compliment of open sublocale and it is defined by $c(a) = \{x \in L, a \sqsubseteq x\} = \uparrow a$.

Proposition 1.3.19. *[39] Let L be a locale. A subset $S \subseteq L$ is a sublocale if and only if it is a locale in the induced order and the embedding map $j : S \subseteq L$ is a localic map.*

Definition 1.3.20. [39] A nucleus in a locale L is a mapping $v : L \to L$ such that

i. $a \sqsubseteq v(a)$,

ii. $a \sqsubseteq b \Rightarrow v(a) \sqsubseteq v(b)$

iii. $v(v(a)) = v(a)$ and

iv. $v(a \sqcap b) = v(a) \sqcap v(b)$.

Sublocales of a locale L have alternate representations in[39].

i. Sublocales of a locale L are represented as onto frame homomorphism $g : L \to M$, a sublocale homomorphism. The translation between sublocale homomorphism to sublocales and vice versa is as follows.

$h \mapsto h_*[M]$ for an onto $h : L \to M$ and h_* is its right adjoint, and

$S \mapsto j_S^* : L \to S$ for $j_S : S \subseteq L$.

ii. Sublocales of a locale can also be represented using frame congruence. A sublocale homomorphism $g : L \to M$ induces a frame congruence $E_g = \{(x, y) : g(x) = g(y)\}$ and a frame congruence gives rise to a sublocale homomorphism $x \mapsto Ex : L \to L/E$, where L/E denotes the quotient frame defined by the congruence E, and Ex denotes the E-class.

iii. Sublocales of a locale can also be represented using nucleus. The translation between nuclei and frame congruence resp. sublocale homomorphism is straight forward:

$v \mapsto E_v = \{(x, y) : v(x) = v(y)\}$,

$E \mapsto v_E = (x \mapsto \bigsqcup Ex) : L \to L$;

$v \mapsto v_h = v$ restricted to $L \to v[L]$,

$h \mapsto v_h = (x \mapsto h_* h(x)) : L \to L$

We can relate sublocales and nuclei directly. For a sublocale $S \subseteq L$, set $v_S(a) = j_S^*(a) = \bigsqcap\{s \in S : a \sqsubseteq s\}$ and for a nucleus $v : L \to L$, set $S_v = v[L]$.

Proposition 1.3.21. *[39] The formula $S \mapsto v_S$ and $v \mapsto S_v$ constitute a one-one correspondence between subloales of L and nuclei.*

Definition 1.3.22. [39] A sublocale S of a locale L is said to be dense if it contains 0_L.

Definition 1.3.23. [39] The closure of a sublocale S of a locale L is the least closed sublocale of L containing S, given by the formula $\bar{S} = \uparrow(\bigsqcap S)$.

Definition 1.3.24. [39] A cover of a locale L is a subset $A \subseteq L$ such that $\bigsqcup A = 1$. A subcover of a cover A is a subset $B \subseteq A$ such that $\bigsqcup B = 1$. A locale is said to be compact if each cover has a finite subcover.

Definition 1.3.25. [39] A localic map $f : L \to M$ is said to be closed if the image of each closed sublocale is closed.

Definition 1.3.26. [39] A localic map $f : L \to M$ is said to be open if the image $f[S]$ of each open sublocale $S \subseteq L$ is open.

1.3.3 Separation Axioms

As in classical topology, the point free topology have separation axioms. Subfit and fit correspond to T_1 axiom of classical topology.

Definition 1.3.27. [39] A locale L is said to be subfit if for $a, b \in L, a \not\sqsubseteq b$, then $\exists c \in L$, such that $a \sqcup c = 1$ and $b \sqcup c \neq 1$.

Definition 1.3.28. [39] A locale L is said to be fit if for $a, b \in L, a \not\sqsubseteq b$, then $\exists c \in L$, such that $a \sqcup c = 1$ and $c \to b \nleqslant b$.

Definition 1.3.29. [39] A frame L is called I-Hausdorff whenever the diagonal $\Delta : L \to L \bigoplus L$ is a closed localic map.

Definition 1.3.30. [10] A locale L is said to have $S_2^{'}$ property if for any $a, b \in L$, if $a \sqcup b = 1$ with $a \neq 1$ and $b \neq 1$, then there exist $u, v \in L$ with $u \sqcap v = 0$, $v \not\sqsubseteq a$, $u \not\sqsubseteq b$.

Definition 1.3.31. [39] In a locale L, for $a, b \in L$, we say that a is rather below b, denoted by $a \prec b$, if there exist $c \in L$ such that $a \sqcap c = 0$ and $c \sqcup b = 1$.

Definition 1.3.32. [39] A locale L is said to be regular if $a = \bigsqcup\{x : x \prec a\}$ for every $a \in L$.

Definition 1.3.33. [39] Let a, b be elements of a locale L. We say that a is completely below b and write $a \prec\prec b$ if there are $a_r \in L$ (r rational, $0 \leqslant r \leqslant 1$) such that $a_0 = a, a_1 = b$ and $a_r \prec a_s$ for $r < s$.

Definition 1.3.34. [39] A locale L is said to be completely regular if $a = \bigsqcup\{x : x \prec\prec a\}$ for every $a \in L$.

Definition 1.3.35. [10] A locale L is called normal if it satisfies the condition: If $a \sqcup b = 1$, then there exist $u, v \in L$ such that $a \sqcup v = 1, u \sqcup b = 1, u \sqcap v = 0$.

1.4. Cryptography

Public key cryptography mainly depends on two types of computational problems. One is the problem of factorization of integers and other is discrete logarithm problem in groups. Diffie Hellman key exchange protocol [7] is based on discrete logarithm problem.

Definition 1.4.1. [34] Public key cryptography, or, asymmetric cryptography, is any cryptographic system that uses pairs of keys: *public keys* which may be disseminated widely, and *private keys* which are known only to the owner.

In a public key encryption system any person can encrypt the message using the receiver's public key. That encrypted message can only be decrypted with the receiver's private key [34].

We define two party key exchange protocol as a sequence of calculation and transmission between two parties, most commonly referred to as Alice and Bob.

Definition 1.4.2. [34]**Key Exchange Protocol(KEP)**

i. Setup:

An initial handshake is performed, and protocol parameters specified.

ii. Generation of public/private keys

Both parties generate ephemeral key pairs (k_s^A, k_p^A) and (k_s^B, k_p^B) respectively.

iii. Exchange of public keys:

The parties exchnage their public keys k_p^A, k_p^B.

iv. Calculating the shared keys:

Alice uses the recieved public key k_p^B and her own key pair to calculate a shared key (shared secret)K_A. Bob uses k_p^A and his own key pair to calculate K_B.

Correctness of a protocol is given if $K_A = K_B$ for all possible key pairs.

The following definition discuss about two different attacker model for our framework.

Definition 1.4.3. [34] i. Passive Attacker/Eavesdropper

A passive attacker gathers all the information that is sent between the parties involved in the protocol and tries to infer information about the shared secret. This attacker has no means of interfering with the transmission and can not alter or disrupt them.

ii Active Attacker/Main -in-the -Middle:

An active attacker not only sees all the transmission between the parties but also has the ability to alter or disrupt the information in transit or inject his own information into the channel.

Definition 1.4.4. [34] **Diffie-Hellman key exchange protocol**

i. Setup:

The protocol parameters are negotiated. These include the group G of order n with generator g.

ii. Generation of public/private keys:

Both parties, Alice and Bob, choose secret elements $a, b \in Z_n$ respectively as their secret keys and calculate their public keys as

$$p_A = g^a$$

$$p_B = g^b.$$

iii. Exchange of public keys:

Alice and Bob exchange their public keys p_A, p_B.

iv. Calculating the shared key:

Alice, upon recieving p_B from Bob, calculates

$$K_A = p_B^a$$

Bob similarly computes

$$K_B = p_A^b$$

Correctness follows from the commutativity in Z_n, since

$K_A = (g^b)^a = g^{ab} = (g^a)^b = K_B$.

Definition 1.4.5. [34] **ElGamal Encryption based on Diffie-Hellman KEP**

The public parameters include a cyclic group G of order n together with a generator g.

i. Alice generate a static key pair by uniformly at random choosing the secret key $a \in Z_n$ and calculating the public key $g^a \in G$. She publishes her public key g^a.

ii. For every message m_i, Bob uniformly at random chooses an element $b_i \in Z_n$ and calculates the pair $(g^{b_i}, m_i.g^{ab_i})$ using Alice's public key.

iii. Upon receiving a pair $(g^{b_i}, m_i.g^{ab_i})$, Alice uses her secret key to calculate $(g^{b_i})^a$ and multiplies the second component by its inverse, hence attaining m_i.

Chapter 2

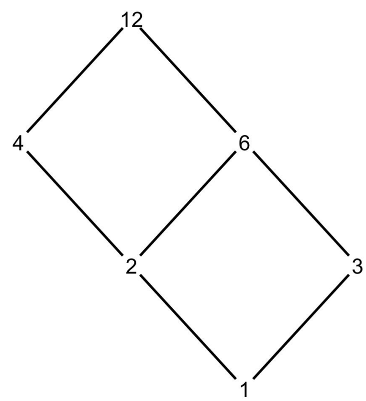
Unique Sublocales from Ideals of a Locale

In a locale L, if I is an ideal, which is closed under arbitrary join, then we can construct a complete lattice $M = \{I_a; a \in L\}$ of ideals of L with the property $I \subseteq I_a$ for all $a \in L$. M induces a frame congruence R_I on L and R_I determines a sublocale S of L. The topological properties such as subfit, fit, S_2', regularity, normality and compactness of the sublocale S of L thus constructed can be obtained using the class of core elements of L with respect to I. On the other hand, from a sublocale S of a locale L, an ideal I_S which is closed under arbitrary join can be obtained. It is proved that the sublocale constructed using the congruence R_{I_S} as above is embeddable in the given sublocale S.

2.1. Ideal $I_a, a \in L$

Definition 2.1.1. Given an ideal I of a locale L, for each $a \in L$, define $I_a = \{x \in L : a \sqcap x \in I\}$.

Examples 2.1.2. *1. Let the locale L be given as follows.*



Let $I = \{1, 2\}$. Then I is an ideal in L.

$I_3 = I_6 = \{1, 2, 4\}$, $I_4 = \{1, 2, 3, 6\}$, $I_2 = I_1 = L$ and $I_{12} = I$.

*2. Let $f : L \to L$ be a morphism in **Frm** and $b \in L$. Then $(f)_b = \{x \in L : \Sigma_{f(x)} \subseteq \Sigma_b\}$ is an ideal in L. For each $a \in L$, let $\langle a \rangle_f = \{x \in L : a \sqcap x \in (f)_b\} = \{x \in L : \Sigma_{f(a \sqcap x)} \subseteq \Sigma_b\}$. Then $\langle a \rangle_f$ is ideal in L for all $a \in L$.*

This section discusses various properties of the collection I_a.

Proposition 2.1.3. *Let L be a locale and let I be any ideal in L. For each $a \in L$, I_a is an ideal in L.*

Proof. Since $0 = 0 \sqcap a \in I$, $0 \in I_a$. Hence I_a is nonempty.

Let $x, y \in I_a$. Then $a \sqcap x, a \sqcap y \in I$.

Since I is closed under finite join, $a \sqcap (x \sqcup y) = (a \sqcap x) \sqcup (a \sqcap y) \in I$.

Therefore $x \sqcup y \in I_a$. Hence I_a is a subjoin semilattice of L.

Let $x \in I_a$ and $y \in L$ such that $y \sqsubseteq x$.

Since $x \in I_a$, $a \sqcap x \in I$.

$y \sqsubseteq x$ implies $y \sqcap a \sqsubseteq x \sqcap a$. Since I is a lower set, $y \sqcap a \in I$.

Hence $y \in I_a$. Thus I_a is a lower set. □

Definition 2.1.4. Let I be an ideal in a locale L. An element $a \notin I$ in L is said to be partially prime to the ideal L if for any $x \in L$, $a \sqcap x \in I$ implies $x \in I$.

Example 2.1.5. *Let L be a chain and $a \sqsubseteq b$. Then b is partially prime to the ideal $\downarrow a$.*

The next proposition gives a sufficient condition for the ideal I_a to be prime.

Proposition 2.1.6. *If I is a prime ideal in a locale L, then I_a is prime ideal for $a \in L$. If $a \in L$ is partially prime to the ideal I and I_a is prime, then I is prime.*

Proof. Let I be a prime ideal and let $x \sqcap y \in I_a$. Then $a \sqcap (x \sqcap y) \in I$.

Since I is prime, either $a \sqcap x \in I$, or $y \in I$.

If $a \sqcap x \in I$, then $x \in I_a$.

If $y \in I$, then $a \sqcap y \in I$ and hence $y \in I_a$.

Conversely let a be partially prime to I and I_a be prime ideal in L.

Let $x \sqcap y \in I$. Then we have $a \sqcap (x \sqcap y) \in I$. Hence $x \sqcap y \in I_a$.

Since I_a is prime, either $x \in I_a$ or $y \in I_a$.

That is either $a \sqcap x \in I$ or $a \sqcap y \in I$.

Since a is partially prime to I, either $x \in I$ or $y \in I$. □

In 2.1.2 (2), if Σ_b is meet-irreducible element in the spectrum $Sp(L)$ of L, then $(f)_b$ is a prime ideal in L. Then for each $a \in L$, $\langle a \rangle_f$ is prime ideal.

Proposition 2.1.7. *Let L be a locale and I be any ideal in L.*

i. If $a, b \in L$ with $a \sqsubseteq b$, then $I_b \subseteq I_a$

ii. $I \subseteq I_a$ for every $a \in L$

iii. $I_a = L$ if and only if $a \in I$

iv. $I_1 = I$.

Proof. Let L be a locale and I be any ideal in L.

i. Let $a \sqsubseteq b$. Then $x \in I_b$ implies $b \sqcap x \in I$.

Since $a \sqcap x \sqsubseteq b \sqcap x, a \sqcap x \in I$.

Hence $x \in I_b$ implies $x \in I_a$. Thus $I_b \subseteq I_a$.

ii. Let $x \in I$. Since I is a lower set, $a \sqcap x \in I$ for all $a \in L$.

Thus $x \in I_a$ for all $a \in L$. Hence $I \subseteq I_a$ for all $a \in L$.

iii. Let $I_a = L$. Then $1 \in I_a = L$. Thus $a = a \sqcap 1 \in I$.

Hence $I_a = L$ implies $a \in I$.

Conversely assume $a \in I$. Then for any $x \in L$, $a \sqcap x \in I$.

Hence $x \in I_a$ for all $x \in L$. Thus $I_a = L$.

iv. $I_1 = \{x \in L : x \sqcap 1 \in I\} = \{x \in L : x \in I\} = I$. □

The above proposition compares the ideals I_a for $a \in L$ and gives a hint to construct the concept of core element.

Proposition 2.1.8. *Let L be a locale and let I be any ideal in L.*

i. For any $a, b \in L$, $I_a \cap I_b = I_{a \sqcup b}$.

ii. For any $a, b \in L$, $I_a \cup I_b \subseteq I_{a \sqcap b}$. If $a \sqcap b$ is partially prime to I, then $I_a \cup I_b = I_{a \sqcap b}$.

Proof. Let L be a locale and let I be any ideal in L.

i. $x \in I_a \cap I_b$ if and only if $a \sqcap x \in I$ and $b \sqcap x \in I$

if and only if $x \sqcap (a \sqcup b) = (x \sqcap a) \sqcup (x \sqcap b) \in I$

if and only if $x \in I_{a \sqcup b}$

Hence $I_a \cap I_b = I_{a \sqcup b}$.

ii. $x \in I_a \cup I_b$ implies $a \sqcap x \in I$ or $b \sqcap x \in I$.

Then $(a \sqcap x) \sqcap (b \sqcap x) = (a \sqcap b) \sqcap x \in I$.

Thus $x \in I_a \cup I_b$ implies $x \in I_{a \sqcap b}$. Hence $I_a \cup I_b \subseteq I_{a \sqcap b}$.

Let $a \sqcap b$ be partially prime to I. Then $x \in I_{a \sqcap b}$ implies $(a \sqcap b) \sqcap x \in I$.

Since $a \sqcap b$ is partially prime to I, $x \in I$.

Hence $x \in I_a \cup I_b$. □

Proposition 2.1.9. *Let the ideal I in a locale L be closed under arbitrary join, then the set $M = \{I_a; a \in L\}$ is a complete lattice under the partial order inclusion.*

Proof. By 2.1.8, M is closed under finite intersection.

Suppose I is closed under arbitrary join and let $I_{a_\alpha} \in M$, $\alpha \in J$, for some index set J.

Then $x \in \bigcap_{\alpha \in J} I_{a_\alpha}$ if and only if $x \in I_{a_\alpha}$ for all $\alpha \in J$

if and only if $x \sqcap a_\alpha \in I$ for all $\alpha \in J$

if and only if $\bigsqcup_{\alpha \in J}(x \sqcap a_\alpha) = x \sqcap \bigsqcup a_\alpha \in I$

if and only if $x \in I_{\bigsqcup a_\alpha}$.

Hence $\bigcap_{\alpha \in J} I_{a_\alpha} = I_{\bigsqcup a_\alpha} \in M$. Also $I_0 = L$ is the top element.
Therefore M is a complete semilattice with top and bottom elements. Hence M is a complete lattice. □

Proposition 2.1.10. *If the ideal I in a locale L is closed under arbitrary join, then there is a complete join semilattice homomorphism from the locale L to the complete lattice $M = (\{I_a; a \in L\}, \supseteq)$.*

Proof. Order $M = \{I_a; a \in L\}$ as $I_a \sqsubseteq I_b$ if and only if $I_a \supseteq I_b$.
With respect to the order $\sqsubseteq$, we have $I_a \sqcup I_b = I_a \cap I_b$.
Thus $M = (\{I_a; a \in L\}, \supseteq)$ is a complete lattice with bottom element $I_0 = L$ and top element $I_1 = I$.
Define $f : L \to M$ by $f(a) = I_a$.
$f(\bigsqcup a_\alpha) = I_{\bigsqcup a_\alpha} = \bigcap I_{a_\alpha} = \bigsqcup f(a_\alpha)$ and $f(0) = I_0 = L$. □

2.2. Sublocales from ideals of a locale

The work in this section explains a method of construction of sublocales using ideals of a locale L.

Lemma 2.2.1. *If I is an ideal of a locale having the property that I is closed under arbitrary join. Then for any $a, b, c \in L$ and $A \subseteq L$, we have*
i. $I_a = I_b$ implies $I_{a \sqcap c} = I_{b \sqcap c}$.
ii. $I_a = I_b$ implies $I_{a \sqcup \bigsqcup A} = I_{b \sqcup \bigsqcup A}$.

Proof. Let the ideal I of locale L is closed under arbitrary join.
i. Let $a, b, c \in L$ and $I_a = I_b$.

$x \in I_{a \sqcap c}$ if and only if $a \sqcap (c \sqcap x) = (a \sqcap c) \sqcap x \in I$
if and only if $c \sqcap x \in I_a = I_b$
if and only if $b \sqcap (c \sqcap x) = x \sqcap (b \sqcap c) \in I$
if and only if $x \in I_{b \sqcap c}$.

Therefore $I_a = I_b$ implies $I_{a \sqcap c} = I_{b \sqcap c}$.

ii. Let $I_a = I_b$ and $A \subseteq L$.

$$
\begin{aligned}
x \in I_{a \sqcup \bigsqcup A} & \text{ if and only if } x \sqcap (a \sqcup \bigsqcup A) = x \sqcap \bigsqcup (a \sqcup y) = \bigsqcup x \sqcap (a \sqcup y) \in I \\
& \text{ if and only if } x \sqcap (a \sqcup y) = (x \sqcap a) \sqcup (x \sqcap y) \in I \text{ for all } y \in A \\
& \text{ if and only if } x \sqcap a \in I \text{ and } x \sqcap y \in I \text{ for all } y \in A \\
& \text{ if and only if } x \in I_a = I_b \text{ and } x \sqcap y \in I \text{ for all } y \in A \\
& \text{ if and only if } x \sqcap b \in I \text{ and } x \sqcap \bigsqcup y = \bigsqcup (x \sqcap y) \in I \\
& \text{ if and only if } (x \sqcap b) \sqcup (x \sqcap \bigsqcup A) = x \sqcap (b \sqcup \bigsqcup A) \in I \\
& \text{ if and only if } x \in I_{b \sqcup \bigsqcup A}.
\end{aligned}
$$

Hence $I_a = I_b$ implies $I_{a \sqcup \bigsqcup A} = I_{b \sqcup \bigsqcup A}$. □

Definition 2.2.2. Let I be an ideal of a locale L having the property that I is closed under arbitrary join. Define a relation R_I on L by $(a, b) \in R_I$ if and only if $I_a = I_b$.

The following proposition is a direct consequence of above lemma.

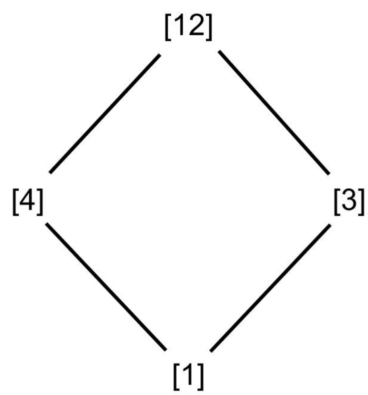
Proposition 2.2.3. *Let I be an ideal of a locale L having the property that I is closed under arbitrary join. The binary relation R_I defined on L is a frame congruence on L.*

Proof. The binary relation R_I defined by $(a, b) \in R_I$ if and only if $I_a = I_b$ is an equivalence relation on L. If $(a, b) \in R_I$, by above lemma $(a \sqcap c, b \sqcap c) \in R_I$ and $(a \sqcup \bigsqcup A, b \sqcup \bigsqcup A) \in R_I$. Hence R_I is a congruence relation on L. □

Since R_I is a congruence on L, by [8], L/R_I is a frame with respect to the partial order $[x] \sqsubseteq [y]$ if and only if $x \sqsubseteq y$ in L.

In example 2.1.2 (1), the congruence R_I gives $[1] = \{1, 2\}, [3] = \{3, 6\}$,

$[4] = \{4\}$ and $[12] = \{12\}$ and the quotient frame L/R_I is given below.



Lemma 2.2.4. *Let I be an ideal of a locale L having the property that I is closed under arbitrary join. There exist a bijection between the locale L/R_I and the complete lattice $M = (\{I_a : a \in L\}, \supseteq)$.*

Proof. The function $f : L/R_I \to M$ defined by $f([a]) = I_a$ is a bijection. □

Lemma 2.2.5. *If I is a prime ideal in a locale L, then $I_a = L$ for all $a \in I$ and $I_b = I$ for all $b \notin I$.*

Proof. Let I be a prime ideal. By Proposition 2.1.7, $I_a = L$ for all $a \in I$.

Let $b \notin I$. Then $I_b = \{x \in L : b \sqcap x \in I\}$.

If $x \in I_b$, then $b \sqcap x \in I$.

Since I is prime and $b \sqcap x \in I$, we have $x \in I$. Therefore $I_b \subseteq I$.

Hence $I_b = I$ for all $b \notin I$. □

Proposition 2.2.6. *If I is a prime ideal in a locale L, then the sublocale (quotient frame) L/R_I is isomorphic to the two element locale **2**.*

Proof. By above lemma, if I is prime $I_a = I_0$ for all $a \in I$ and $I_a = I_1$ for all $a \notin I$. Hence $L/R_I = \{[0], [1]\}$ which is isomorphic to the locale $\mathbf{2}$. □

Corollary 2.2.7. *Let the locale L be a chain and I be any ideal of L. Then the sublocale L/R_I is isomorphic to the two element locale $\mathbf{2}$.*

Proof. Let L be a chain. Then every ideal of I is principal and prime. □

Remark. Given an ideal I that is closed under arbitrary join, we get a frame congruence on L and hence a sublocale of L.

In example 2.1.2(1), the sublocale corresponding to the ideal $I = \{1, 2\}$ is the closed sublocale $c(2) =\uparrow 2$.

Lemma 2.2.8. *Let c be a meet-irreducible element of a locale L. Then the ideal $I =\downarrow (c)$ is prime.*

Proof. Let $x \sqcap y \in I$. That is $x \sqcap y \sqsubseteq c$.
Since c is meet irreducible, either $x \sqsubseteq c$ or $y \sqsubseteq c$. So either $x \in I$ or $y \in I$.
Hence I is prime. □

Proposition 2.2.9. *Let c be a meet-irreducible element of a locale L and let $I =\downarrow (c)$. Let S be the sublocale corresponding to the ideal I. Then S is closed if and only if c is maximal element of the locale L.*

Proof. Since c is meet-irreducible element of the locale L, by above lemma ideal I is prime.
By lemma 2.2.5, $I_a = L, \forall a \in I$ and $I_a = I, \forall a \notin I$.
Then by construction, the corresponding sublocale $S = \{c, 1\}$.
Assume S is closed. Then $S = (\uparrow \bigsqcap S) {=}\uparrow (c) = \{c, 1\}$.

Thus there exist no element b such that $c \subsetneq b \subsetneq 1$.

Hence c is maximal element of the locale L.

Conversely assume c is maximal element of the locale L.

Then $\uparrow c = \{c, 1\} = S$. Hence the sublocale S is closed. □

2.3. Ideals from Sublocales of a locale

Given a sublocale S of a locale L, we construct the ideal I_S, which is closed under arbitrary join. In this section we show that the sublocale constructed using the congruence R_{I_S} is embeddable in the sublocale S of L.

Proposition 2.3.1. *Let S be a sublocale of L and $j : S \to L$ be the inclusion map. Then $kerj_S^* = \{x \in L : j_S^*(x) = \sqcap S\}$ is an ideal of L and $kerj_S^*$ is closed under arbitrary join.*

Proof. $j_S^*(x) = \sqcap\{s \in S : x \sqsubseteq s\}$.

Then $j_S^*(0) = \sqcap S$. So $0 \in kerj_S^*$ and hence $kerj_S^*$ is nonempty.

Let $x \in kerj_S^*$ and $y \in L$ such that $y \sqsubseteq x$.

Then $j_S^*(y) = j_S^*(y \sqcap x) = j_S^*(y) \sqcap j_S^*(x) = \sqcap S$.

Thus $y \in kerj_S^*$. Hence $kerj_S^*$ is a lower set.

Let $x_i \in kerj_S^*$ for $i \in I$. Then we have $j_S^*(x_i) = \sqcap S$ for all $i \in I$.

Also $j_S^*(\sqcup x_i) = \sqcup j_S^*(x_i) = \sqcup \sqcap S = \sqcap S$. Thus $\sqcup x_i \in kerj_S^*$.

Hence $kerj_S^*$ is an ideal which is closed under arbitrary join. □

Denote the ideal $kerj_S^*$ by I_S. Let L/R_{I_S} be the corresponding quotient frame.

Proposition 2.3.2. *Let S be a sublocale of a locale L. If $\sqcap S$ is a meet-irreducible element of the locale L, then the ideal I_S is prime.*

Proof. Let $x \sqcap y \in I_S$. Then $j_S^*(x \sqcap y) = \bigsqcap S$. That is $j_S^*(x) \sqcap j_S^*(y) = \bigsqcap S$.
Since $\bigsqcap S$ is meet-irreducible element, either $j_S^*(x) = \bigsqcap S$ or $j_S^*(y) = \bigsqcap S$.
Hence either $x \in I_S$ or $y \in I_S$. Thus the ideal I_S is prime. □

Proposition 2.3.3. *A sublocale S of a locale L is dense in L if and only if the ideal I_S is trivial.*

Proof. Let the sublocale S be dense in L. Then $0 \in S$ and hence $\bigsqcap S = 0$.
Then $I_S = \{x \in L : j_S^*(x) = \bigsqcap S = 0\}$.
Since j_S^* is a nucleus on L, we have $x \sqsubseteq j_S^*(x)$ for all $x \in L$.
$y \in I_S$ if and only if $y \sqsubseteq j_S^*(y) = 0$.
Hence $I_S = \{0\}$, the trivial ideal.
Conversely let the ideal I_S is trivial.
By Proposition 2.1.7, $I_a = L$ if and only if $a \in I$.
Since I_S is trivial ideal, $I_a = L$ if and only if $a = 0$.
So $[0] = \{0\}$ and hence $0 \in S$.
Thus the sublocale S is dense in L. □

From the above proposition it is clear that sublocales which are not dense in the corresponding locale gives non trivial ideals.

Proposition 2.3.4. *If S is a closed sublocale of the locale L, then the ideal I_S is principal.*

Proof. Let $S = C(a) = \uparrow(a)$ be a closed sublocale of L.

Then the corresponding nucleus j_S^* is of the form $j_S^*(x) = a \sqcup x$ for all $x \in L$.

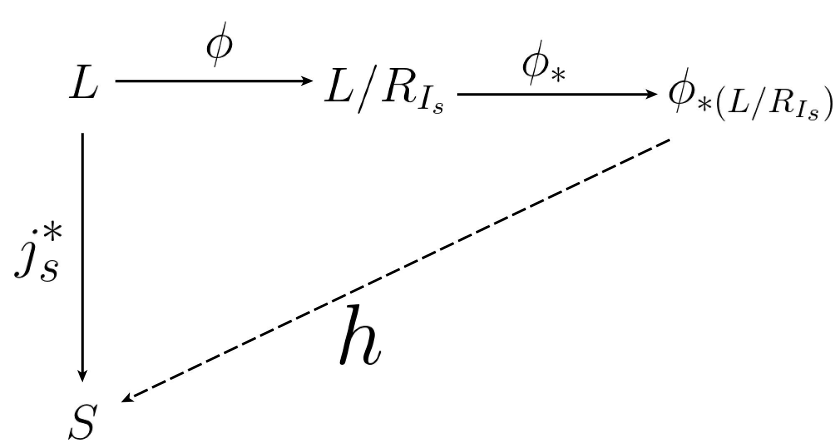
$$\begin{aligned} I_S &= ker j_S^* = \{x \in L : j_S^*(x) = \bigsqcap S = a\} \\ &= \{x \in L; a \sqcup x = a\} = \{x \in L : x \sqsubseteq a\} \\ &= \downarrow(a) \end{aligned}$$

Thus the ideal I_S is principal. □

Proposition 2.3.5. *Let S be a sublocale of a locale L. Then the sublocale constructed using the congruence R_{I_S} is embeddable in S.*

Proof. Let S be a sublocale of a locale L and let L/R_{I_S} be the quotient frame constructed using the congruence R_{I_S} in L. Let $\phi : L \to L/R_{I_S}$ be the corresponding extremal epimorphism in **Frm**. Then $\phi_*(L/R_{I_S})$ is the sublocale generated by the congruence R_{I_S}. We will show that the sublocale $\phi_*(L/R_{I_S})$ is embeddable in the sublocale S.

Let $y \in \phi_*(L/R_{I_S})$, then $y = \phi_*([x])$ for some $x \in L$. Thus y can be written as $y = \phi_*(\phi(x))$ for some $x \in L$. Define $h : \phi_*(L/R_{I_S}) \to S$ by $h(y) = j_S^*(x)$. Then the following triangle commutes.



The map $h : \phi_*(L/R_{I_S}) \to S$ is a one-one map. Hence the sublocale $\phi_*(L/R_{I_S})$ is embeddable in the sublocale S. □

2.4. Core element with respect to an ideal I

Let L be a locale and $I \subseteq L$ be an ideal, which is closed under arbitrary join. The concept of core element with respect to the ideal I is introduced in this section.

Definition 2.4.1. An element $a \in L$ is called core element with respect to the ideal I if $I_a = I$. Let us denote the set of core elements of L by Ç.

By proposition 2.1.7 (iv), 1 ∈Ç. Hence Ç is nonempty.

Proposition 2.4.2. *For any ideal I of a locale L , we have the following*

i. Ç is a congruence class with respect to R_I.

ii. Ç is closed under finite meet and arbitrary join.

iii. Ç is a filter of L .

iv. If I is prime, Ç is a completely prime filter.

Proof. Let Ç be the set of core elements of a locale L.

i. By proposition 2.1.7 (iv), 1 ∈Ç.

We will show that the equivalence class of 1 with respect to R_I is Ç.

$[1]_{R_I} = \{t \in L : (1, t) \in R_I\} = \{t \in L : I_t = I_1\} = \{t \in L : I_t = I\} =$ Ç.

ii. Let $x, y \in$ Ç. Then by above part, $x, y \in [1]_{R_I}$ so that $(1, x) \in R_I$ and $(1, y) \in R_I$.

Since R_I is a congruence, $(1, x) \in R_I$ implies $(1 \sqcap y, x \sqcap y) \in R_I$. That is $(y, x \sqcap y) \in R_I$.

Since R_I is an equivalence relation, $(1, y) \in R_I, (y, x \sqcap y) \in R_I$ implies $(1, x \sqcap y) \in R_I$.

Hence $x \sqcap y \in [1]_{R_I} =$ Ç. Thus Ç is closed under finite meet.

Now let $S = \{x_i; i \in J\} \subseteq$Ç. Then we have $(1, x_i) \in R_I$ for every $i \in J$.

Since R_I is a congruence, we have $(1 \sqcup \bigsqcup S, x_i \sqcup \bigsqcup S) = (1, \bigsqcup S) \in R_I$.

Hence $\bigsqcup S \in [1]_{R_I} =$ Ç. Thus Ç is closed under arbitrary join.

iii. By proposition 2.1.7 (iv), $1 \in$Ç. By above part Ç is closed under finite meet.

Let $x \in$Ç and $y \in L$ be such that $x \sqsubseteq y$. Since $x \in$ Ç, we have $I_x = I$.

By proposition 2.1.7 (i), since $x \sqsubseteq y$, $I_y \subseteq I_x = I$.

Also by proposition 2.1.7 (ii), $I \subseteq I_y$. Hence $I_y = I$. Thus $y \in$Ç. Hence Ç is a filter in L.

iv. Let I be prime ideal. Then by Lemma 2.2.5, Ç $= \{x \in L : x \notin I\}$.

Let $\bigsqcup x_\alpha \in$ Ç. Then $\bigsqcup x_\alpha \notin I$. Since I is closed under arbitrary join, $x_\alpha \notin I$ for some α. Hence $x_\alpha \in$Ç and so Ç is completely prime filter of L. □

Theorem 2.4.3. *Let I be an ideal of a locale L. Then the sublocale (quotient frame) L/R_I is a Boolean algebra if and only if for each $x \in L$, there exist $y \in L$ such that $x \sqcap y \in I$ and $x \sqcup y \in$ Ç.*

Proof. Let $x \in L$. Then $[x] \in L/R_I$.

The sublocale L/R_I is a Boolean algebra if and only if there exist $[y] \in L/R_I$ such that $[x] \sqcap [y] = [0]$ and $[x] \sqcup [y] = [1]$.

That is if and only if $[x \sqcap y] = [0]$, $[x \sqcup y] = [1]$ or $I_{x \sqcap y} = I_0 = L$ and $I_{x \sqcup y} = I_1 =$ Ç.

Hence by proposition 2.1.7, $x \sqcap y \in I$ and $x \sqcup y \in$ Ç. □

Theorem 2.4.4. *Let I be an ideal of a locale L. If L/R_I is a Boolean algebra, then R_I is the largest congruence relation having congruence class Ç.*

Proof. Clearly R_I is a congruence with Ç as a congruence class.

Let θ be any other congruence with Ç as a congruence class and let $(x, y) \in \theta$.

Then for any $a \in L$, we have $(x, y) \in \theta$ implies $(x \sqcup a, y \sqcup a) \in \theta$.

Hence $x \sqcup a \in$Ç if and only if $y \sqcup a \in$ Ç. That is $I_{x \sqcup a} = I$ if and only if $I_{y \sqcup a} = I$.

Then by proposition 2.1.8, we have $I_x \cap I_a = I$ if and only if $I_y \cap I_a = I$.

Since L/R_I is a Boolean algebra, by above theorem, there exist $x', a' \in L$ such that $x \sqcap x', a \sqcap a' \in I$ and $I_{x \sqcup x'} = I$, $I_{a \sqcup a'} = I$.

Since $x \sqcap x', a \sqcap a' \in I$, we have $x' \in I_x$ and $a' \in I_a$.

Thus $x' \sqcap a' \in I_x \cap I_a = I_{x \sqcup a} = I$.

$x' \sqcap a' \in I$, implies $a' \in I_{x'}$.

Similarly, we get $a' \in I_{y'}$ for suitable $y' \in L$.

Thus we have $a' \in I_{x'}$ if and only if $a' \in I_{y'}$. Thus $I_{x'} = I_{y'}$ or $(x', y') \in R_I$.

Hence $x' \in$Ç if and only if $y' \in$ Ç. That is $I_{x'} = I$ if and only if $I_{y'} = I$.

Hence $I_{x \sqcup x'} = I_x$ if and only if $I_{y \sqcup y'} = I_y$. Thus $I_x = I$ if and only if $I_y = I$.

Hence $I_x = I_y$. Thus $(x, y) \in R_I$. □

Proposition 2.4.5. *The quotient frame (sublocale) L/R_I is subfit if and only if for every $a, b \in L$ with $a \not\sqsubseteq b$, there exist $c \in L$ such that $a \sqcup c \in$ Ç, $b \sqcup c \notin$ Ç.*

Proof. Assume the quotient frame L/R_I satisfies subfit property.

Let $a, b \in L$ with $a \not\sqsubseteq b$. Then $[a] \not\sqsubseteq [b]$ in L/R_I.

Since L/R_I is a subfit, there exist $[c] \in L/R_I$ such that $[a] \sqcup [c] = [1]$ and $[b] \sqcup [c] \neq [1]$.

Hence by proposition 2.4.2, $a \sqcup c \in$ Ç, $b \sqcup c \notin$ Ç.

For converse, let $[a], [b] \in L/R_I$ such that $[a] \not\sqsubseteq [b]$. Then $a, b \in L$ with $a \not\sqsubseteq b$.

By assumption there exist $c \in L$ such that $a \sqcup c \in$ Ç, $b \sqcup c \notin$ Ç.

But $a \sqcup c \in$ Ç if and only if $[a \sqcup c] = [a] \sqcup [c] = [1]$.

Hence the quotient frame L/R_I is a subfit frame. □

Proposition 2.4.6. *The quotient frame (sublocale) L/R_I is fit if and only if for every $a, b \in L$ with $a \not\sqsubseteq b$, there exist $c, d \in L$ such that $a \sqcup c \in$ Ç, $c \sqcap d \sqsubseteq b, d \not\sqsubseteq b$.*

Proof. Suppose the quotient frame L/R_I is fit.

Let $a, b \in L$ with $a \nsqsubseteq b$. Then $[a], [b] \in L/R_I$ with $[a] \nsqsubseteq [b]$.

Since L/R_I is fit, there exist $[c] \in L/R_I$ such that $[a] \sqcup [c] = [1]$ and $[c] \to [b] \nsqsubseteq [b]$.

But $[a] \sqcup [c] = [1]$ if and only if $a \sqcup c \in$ Ç.

Also $[c] \to [b] \nsqsubseteq [b]$ if and only if there exist $[d] \in L/R_I$such $[d] \sqcap [c] \sqsubseteq [b]$ and $[d] \nsqsubseteq [b]$.

That is if and only if there exist $d \in L$ such that $c \sqcap d \sqsubseteq b, d \nsqsubseteq b$.

For converse, let $[a], [b] \in L/R_I$ with $[a] \nsqsubseteq [b]$. Then $a, b \in L$ with $a \nsqsubseteq b$.

By assumption there exist there exist $c, d \in L$ such that $a \sqcup c \in$ Ç, $c \sqcap d \sqsubseteq b, d \nsqsubseteq b$.

Then $[c], [d] \in L/R_I$ with $[a] \sqcup [c] = [1]$ and $[c] \to [b] \nsqsubseteq [b]$.

Hence the quotient frame L/R_I is fit. □

Proposition 2.4.7. *The quotient frame (sublocale) L/R_I is S_2' if and only if for every $a, b \in L$ with $a \sqcup b \in$ Ç, $a, b \notin$ Ç, there exist $u, v \in L$ such that $a \nsqsubseteq u, b \nsqsubseteq v$ and $u \sqcap v \in I$.*

Proof. Suppose the locale L/R_I is S_2'.

Let $a, b \in L$ with $a \sqcup b \in$ Ç, $a, b \notin$ Ç. Then $[a], [b] \in L/R_I$ with

$[a] \sqcup [c] = [1], [a] \neq [1], [b] \neq [1]$.

Since the locale L/R_I is S_2', there exist $[u], [v] \in L/R_I$ such that

$[a] \nsqsubseteq [u], [b] \nsqsubseteq [v], [u] \sqcap [v] = [0]$.

But $[u] \sqcap [v] = [0]$ if and only if $u \sqcap v \in I$.

In a similar manner we can prove the converse. □

Lemma 2.4.8. *$[a] \prec [b] \in L/R_I$ if and only if there exist $c \in L$ such that $a \sqcap c \in I$ and $b \sqcup c \in$ Ç.*

Proof. $[a] \prec [b] \in L/R_I$ if and only if there exist $[c] \in L/R_I$ such that $[a] \sqcap [c] = [0]$

and $[b] \sqcup [c] = [1]$. But $[a] \sqcap [c] = [0]$ if and only if $a \sqcap c \in I$ and $[b] \sqcup [c] = [1]$ if and only if $b \sqcup c \in$ Ç. Hence the result. □

Proposition 2.4.9. *The quotient frame (sublocale) L/R_I is regular if and only if for every $a \in L$ there exist $x_i, b_i \in L$ for every $i \in J$, where J is an indexing set, such that $I_{\bigsqcup x_i} = I_a, x_i \sqcap b_i \in I$ and $a \sqcup b_i \in$Ç.*

Proof. The quotient frame (sublocale) L/R_I is regular if and only if for every $[a] \in L/R_I$, there exist $[x_i] \in L/R_I$ such that $[a] = [\sqcup x_i]$ with $[x_i] \prec [a]$.
But $[a] = [\bigsqcup x_i]$ if and only if $I_{\bigsqcup x_i} = I_a$.
Also by above lemma, $[x_i] \prec [a]$ if and only if there exist $b_i \in L$ such that $x_i \sqcap b_i \in I$ and $a \sqcup b_i \in$Ç. □

Proposition 2.4.10. *The quotient frame (sublocale) L/R_I is normal if and only if for every $a, b \in L$ with $a \sqcup b \in$ Ç, there exist $u, v \in L$ such that $a \sqcup v \in$ Ç, $b \sqcup u \in$ Ç, $u \sqcap v \in I$.*

Proof. The quotient frame L/R_I is normal if and only if for every $[a], [b] \in L/R_I$ with $[a] \sqcup [b] = [1]$, there exist $[u], [v] \in L/R_I$ such that
$[u] \sqcap [v] = [0]$ and $[a] \sqcup [v] = [1] = [b] \sqcup [u]$.
But $[u] \sqcap [v] = [0]$ if and only $u \sqcap v \in I$ and $[a] \sqcup [v] = [1] = [b] \sqcup [u]$ if and only if $a \sqcup v \in$ Ç, $b \sqcup u \in$ Ç. □

Definition 2.4.11. A filter F in a locale L is said to be weekly completely prime if $\bigsqcup a_\alpha \in F$, there exist $\alpha_1, \alpha_2, \alpha_3............\alpha_n$ such that $a_{\alpha_1} \sqcup a_{\alpha_2} \sqcup a_{\alpha_3} \sqcup ...... \sqcup a_{\alpha_n} \in F$.

Proposition 2.4.12. *The quotient frame (sublocale) L/R_I is compact if and only if the filter Ç is weekly completely prime.*

Proof. Assume the quotient frame L/R_I is compact.

Let $\bigsqcup a_\alpha \in$ Ç. Then $[\bigsqcup a_\alpha] = \bigsqcup [a_\alpha] = [1]$.

Thus $\{[a_\alpha] : \alpha \in J\}$ is a cover for the locale L/R_I.

Since the frame L/R_I is compact, there exist $\alpha_1, \alpha_2, ...\alpha_n \in J$ such that

$[a_{\alpha_1}] \sqcup [a_{\alpha_2}] \sqcup .... \sqcup [a_{\alpha_n}] = [a_{\alpha_1} \sqcup a_{\alpha_2} \sqcup .... \sqcup a_{\alpha_n}] = [1]$.

Thus $a_{\alpha_1} \sqcup a_{\alpha_2} \sqcup a_{\alpha_3} \sqcup .... \sqcup a_{\alpha_n} \in$ Ç. Hence the filter Ç is weekly completely prime.

In a similar manner we can prove the converse. □

Chapter 3

An Embedding Theorem for Locales

For $a, b \in L$ the collection $[a, \Sigma_b] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_b\}$ are ideals in $O(L)$. We claim that for each $a \in L$, the collection $J_a = \{[a, \Sigma_b] : b \in L\}$ is a spatial locale of pseudo subframes of $O(L)$. Defining proper congruences on L and $O(L)$, we have derived an embedding theorem for locale L. Finally the collection $B = \{J_a, a \in L\}$ forms a full subcategory of the category **Loc**. The coproduct $J = \coprod J_a$ satisfies the separation axioms subfit and normality if and only if each J_a is subfit and normal respectively.

3.1. Ideals $[a, \Sigma_b]$ for a,b $\in$ L

Let L be a locale and $O(L)$ denote the collection of all order preserving maps on L. That is $O(L)$=$\{f; f : L \to L \textit{ is order preserving}\}$. Define a relation $\leqslant$ on $O(L)$ by $f \leqslant g$ if and only if $f(a) \sqsubseteq g(a) \;\; \forall a \in L$. Then the relation $\leqslant$ is a partial order on

$O(L)$. If $\sqcup$ and $\sqcap$ denote join and meet with respect to the partial order $\sqsubseteq$ on L, then $f \vee g, f \wedge g : L \to L$ defined by $(f \vee g)(a) = f(a) \sqcup g(a)$ and $(f \wedge g)(a) = f(a) \sqcap g(a)$ represents join and meet in $O(L)$ with respect to the partial order $\leqslant$. Also infinite distributivity of $\wedge$ over $\bigvee$ follows from the infinite distributivity of $\sqcap$ over $\bigsqcup$. Hence $O(L)$ is a locale with bottom $\mathbf{0}$ and top $\mathbf{1}$, where $\mathbf{0}, \mathbf{1} : L \to L$ are defined by $\mathbf{0}(a) = 0$ and $\mathbf{1}(a) = 1 \forall a \in L$.

Definition 3.1.1. Let L be a locale. For $a, b \in L$, define
$[a, \Sigma_b]$=$\{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_b\}$.

Some simple properties of $[a, \Sigma_b]$ have been verified in the following lemmas.

Lemma 3.1.2. *Let L be a locale and $a, b \in L$*

i. $[a, \Sigma_b]$ is an ideal for all $a, b \in L$.

ii. If Σ_b is meet-irreducible element of $\Omega(Sp(L))$, then $[a, \Sigma_b]$ is prime ideal.

Proof. Let L be a locale and $a, b \in L$

i. Since $\Sigma_{\mathbf{0}(a)}$=$\Sigma_0 = \phi \subseteq \Sigma_b$, we get $\mathbf{0} \in [a, \Sigma_b]$ for all $a, b \in L$. Hence $[a, \Sigma_b]$ is nonempty.

Let $f, g \in [a, \Sigma_b]$.

$$\begin{aligned} f, g \in [a, \Sigma_b] &\Rightarrow \Sigma_{f(a)} \subseteq \Sigma_b \text{ and } \Sigma_{g(a)} \subseteq \Sigma_b \\ &\Rightarrow \Sigma_{f(a)} \cup \Sigma_{g(a)} \subseteq \Sigma_b \\ &\Rightarrow \Sigma_{(f \vee g(a))} = \Sigma_{f(a) \sqcup g(a)} \subseteq \Sigma_b \\ &\Rightarrow f \vee g \in [a, \Sigma_b] \end{aligned}$$

Hence $[a, \Sigma_b]$ is a sub-join semilattice.

Now let $f \in [a, \Sigma_b]$ and $g \leqslant f$ in $O(L)$.

$g \leqslant f \Rightarrow g(a) \sqsubseteq f(a) \Rightarrow \Sigma_{g(a)} \subseteq \Sigma_{f(a)}$.

$g \leqslant f$ and $f \in [a, \Sigma_b] \Rightarrow \Sigma_{g(a)} \subseteq \Sigma_{f(a)} \subseteq \Sigma_b \Rightarrow g \in [a, \Sigma_b]$.

Hence $[a, \Sigma_b]$ is an ideal.

ii. Assume Σ_b is a meet-irreducible element of $\Omega(Sp(L))$ and $f \wedge g \in [a, \Sigma_b]$.

$$f \wedge g \in [a, \Sigma_b] \Rightarrow \Sigma_{(f \wedge g)(a)} \subseteq \Sigma_b$$
$$\Rightarrow \Sigma_{f(a)} \cap \Sigma_{g(a)} \subseteq \Sigma_b$$

Since Σ_b is meet-irreducible element of $\Omega(Sp(L))$, either $\Sigma_{f(a)} \subseteq \Sigma_b$ or $\Sigma_{g(a)} \subseteq \Sigma_b$. That is either $f \in [a, \Sigma_b]$ or $g \in [a, \Sigma_b]$. Hence $[a, \Sigma_b]$ is a prime ideal. □

In [24] Johnstone has defined lattice without bottom element or Top element as pseudo lattice. Using the same terminology we can define pseudo subframe as follows.

Definition 3.1.3. Pseudo subframe M of a frame L is a subset M of L which is closed under all joins and nonempty finite meets so that $\mathbf{1}_L \notin M$.

Proposition 3.1.4. *Let L be a locale and $a, b \in L$,then $[a, \Sigma_b]$ is a pseudo subframe of $O(L)$.*

Proof. Let I be a nonempty indexed set and let $f_i \in [a, \Sigma_b] \quad \forall i \in I$.

$$f_i \in [a, \Sigma_b] \Rightarrow \Sigma_{f_i(a)} \subseteq \Sigma_b, \quad \forall i \in I$$
$$\Rightarrow \bigcup \Sigma_{f_i(a)} \subseteq \Sigma_b \text{ and } \Sigma_{f_i(a)} \cap \Sigma_{f_j(a)} \subseteq \Sigma_b$$
$$\Rightarrow \Sigma_{\bigsqcup f_i(a)} \subseteq \Sigma_b \text{ and } \Sigma_{f_i(a) \sqcap f_j(a)} \subseteq \Sigma_b$$
$$\Rightarrow \bigvee f_i \in [a, \Sigma_b] \text{ and } f_i \wedge f_j \in [a, \Sigma_b]$$

Hence $[a, \Sigma_b]$ is a complete lattice.

Also $[a, \Sigma_b]$ satisfies infinite distributive law as $O(L)$ satisfies the same.

But $\mathbf{1} \notin [a, \Sigma_b]$ if $b \neq 1$.

Hence $[a, \Sigma_b]$ is a pseudo subframe of $O(L)$ and $[a, \Sigma_1]$ is the locale $O(L)$. □

Lemma 3.1.5. *Let L be a locale and $a_1, a_2, b_1, b_2 \in L$.*

i. If $a_1 \sqsubseteq a_2$, then $[a_1, \Sigma_b] \supseteq [a_2, \Sigma_b]$.

ii. If $b_1 \sqsubseteq b_2$, then $[a, \Sigma_{b_1}] \subseteq [a, \Sigma_{b_2}]$.

Proof. Let L be a locale and $a_1, a_2, b_1, b_2 \in L$.

i. Suppose $a_1 \sqsubseteq a_2$. Then $f(a_1) \sqsubseteq f(a_2) \quad \forall f \in O(L)$.

$$\begin{aligned} f \in [a_2, \Sigma_b] &\Rightarrow \Sigma_{f(a_2)} \subseteq \Sigma_b \\ &\Rightarrow \Sigma_{f(a_1)} \subseteq \Sigma_{f(a_2)} \subseteq \Sigma_b \\ &\Rightarrow f \in [a_1, \Sigma_b] \end{aligned}$$

Hence $[a_1, \Sigma_b] \supseteq [a_2, \Sigma_b]$.

ii. Let $b_1 \sqsubseteq b_2$. Then $\Sigma_{b_1} \subseteq \Sigma_{b_2}$.

$$\begin{aligned} f \in [a, \Sigma_{b_1}] &\Rightarrow \Sigma_{f(a)} \subseteq \Sigma_{b_1} \subseteq \Sigma_{b_2} \\ &\Rightarrow f \in [a, \Sigma_{b_2}] \end{aligned}$$

Hence $[a, \Sigma_{b_1}] \subseteq [a, \Sigma_{b_2}]$. □

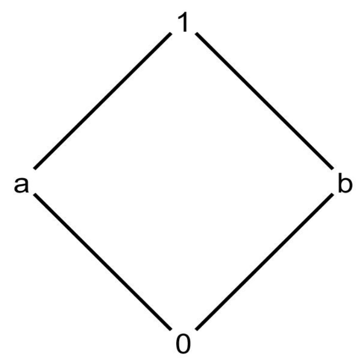
Remark. f_b denotes the constant function on the locale L with the value b. That is $f_b(x) = b \ \ \forall x \in L$.

Lemma 3.1.6. *Let L be a locale. For $a, b, c \in L$, $[a, \Sigma_b] = [a, \Sigma_c]$ if and only if $\Sigma_b = \Sigma_c$.*

Proof. If $\Sigma_b = \Sigma_c$, then clearly $[a, \Sigma_b] = [a, \Sigma_c]$.

Conversely let $[a, \Sigma_b] = [a, \Sigma_c]$. Since $\Sigma_{f_b(a)} \subseteq \Sigma_b$, $f_b \in [a, \Sigma_b] = [a, \Sigma_c]$. Then $\Sigma_{f_b(a)} \subseteq \Sigma_c$. That is $\Sigma_b \subseteq \Sigma_c$. In a similar manner $\Sigma_c \subseteq \Sigma_b$. Hence $\Sigma_b = \Sigma_c$. □

Examples 3.1.7. *1. Let the locale L be given as follows.*



Then $\Sigma_0 = \phi$, $\Sigma_a = \{F_1\}$, $\Sigma_b = \{F_2\}$, $\Sigma_1 = \{F_1, F_2\}$, *where completely prime filters* F_1 *and* F_2 *are given by* $F_1 = \{a, 1\}$, $F_2 = \{b, 1\}$.

Then $O(L) = \{f_1, f_2, ..., f_{33}\}$, *where the order preserving maps* $f_i : L \to L$ *is given by the following formulas.*

$$f_1(x) = \begin{cases} 0 & \text{if } x{=}0,a,b,1 \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if } x{=}0,a,b \\ a & \text{if } x{=}1 \end{cases}$$

$$f_3(x) = \begin{cases} 0 & \text{if } x{=}0,a,b \\ b & \text{if } x{=}1 \end{cases}$$

$$f_4(x) = \begin{cases} 0 & \text{if } x{=}0,a,b \\ 1 & \text{if } x{=}1 \end{cases}$$

$$f_5(x) = \begin{cases} 0 & \textit{if x=0,a} \\ a & \textit{if x=b,1} \end{cases}$$

$$f_6(x) = \begin{cases} 0 & \textit{if x=0,a} \\ a & \textit{if x=b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_7(x) = \begin{cases} 0 & \textit{if x=0,a} \\ b & \textit{if x=b,1} \end{cases}$$

$$f_8(x) = \begin{cases} 0 & \textit{if x=0,a} \\ b & \textit{if x=b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_9(x) = \begin{cases} 0 & \textit{if x=0,b} \\ a & \textit{if x=a,1} \end{cases}$$

$$f_{10}(x) = \begin{cases} 0 & \textit{if x=0} \\ a & \textit{if x=a,b,1} \end{cases}$$

$$f_{11}(x) = \begin{cases} x & \textit{if x=0,a,b,1} \end{cases}$$

$$f_{12}(x) = \begin{cases} 0 & \textit{if x=0} \\ a & \textit{if x=a} \\ 1 & \textit{if x=b,1} \end{cases}$$

$$f_{13}(x) = \begin{cases} 0 & \textit{if x=0,b} \\ b & \textit{if x=a,1} \end{cases}$$

$$f_{14}(x) = \begin{cases} 0 & \textit{if x=0,b} \\ b & \textit{if x=a} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{15}(x) = \begin{cases} 0 & \textit{if x=0} \\ b & \textit{if x=a} \\ a & \textit{if x=b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{16}(x) = \begin{cases} 0 & \textit{if x=0} \\ b & \textit{if x=a,b,1} \end{cases}$$

$$f_{17}(x) = \begin{cases} 0 & \textit{if x=0} \\ b & \textit{if x=a,b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{18}(x) = \begin{cases} 0 & \textit{if x=0} \\ b & \textit{if x=a} \\ 1 & \textit{if x=b,1} \end{cases}$$

$$f_{19}(x) = \begin{cases} 0 & \textit{if x=0,b} \\ a & \textit{if x=a} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{20}(x) = \begin{cases} 0 & \textit{if x=0} \\ a & \textit{if x=a,b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{21}(x) = \begin{cases} 0 & \textit{if x=0,b} \\ 1 & \textit{if x=a,1} \end{cases}$$

$$f_{22}(x) = \begin{cases} 0 & \textit{if x=0} \\ 1 & \textit{if x=a,b,1} \end{cases}$$

$$f_{23}(x) = \begin{cases} a & \textit{if x=0,a,b,1} \end{cases}$$

$$f_{24}(x) = \begin{cases} a & \textit{if x=0,a,b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{25}(x) = \begin{cases} a & \textit{if x=0,a} \\ 1 & \textit{if x=b,1} \end{cases}$$

$$f_{26}(x) = \begin{cases} a & \textit{if x=0} \\ 1 & \textit{if x=a,b,1} \end{cases}$$

$$f_{27}(x) = \begin{cases} b & \textit{if x=0,a,b,1} \end{cases}$$

$$f_{28}(x) = \begin{cases} b & \textit{if x=0,a,b} \\ 1 & \textit{if x=1} \end{cases}$$

$$f_{29}(x) = \begin{cases} b & \textit{if x=0,a} \\ 1 & \textit{if x=b,1} \end{cases}$$

$$f_{30}(x) = \begin{cases} b & \textit{if x=0} \\ 1 & \textit{if x=a,b,1} \end{cases}$$

$$f_{31}(x) = \begin{cases} 1 & \textit{if x=0,a,b,1} \end{cases}$$

$$f_{32}(x) = \begin{cases} a & \textit{if x=0,b} \\ 1 & \textit{if x=a,1} \end{cases}$$

$$f_{33}(x) = \begin{cases} b & \textit{if x=0,b} \\ 1 & \textit{if x=a,1} \end{cases}$$

$[a, \Sigma_0] = \{f \in O(L) : \Sigma_{f(a)} \subseteq \Sigma_0\}$ $=\{f \in O(L) : \Sigma_{f(a)} = \phi\}$

$=\{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$

$[a, \Sigma_a] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{19}, f_{20}, f_{23}, f_{24}, f_{25}\}$

$[a, \Sigma_b] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{27}, f_{28}, f_{29}\}$

$[a, \Sigma_1] = [a, \Sigma_{a \sqcup b}] = O(L)$

2. For non spatial example, consider L = the Boolean algebra of all regularly open subsets of the real line R. Then for any regularly open subsets U, V we have

$[U, \Sigma_V] = \{f \in O(L); \Sigma_{f(U)} \subseteq \Sigma_V\} = O(L)$, *since* $Sp(L) = \phi$.

3.2. The locale $J_a, a \in L$

Let L be a locale. Fix some $a \in L$, and let J_a=$\{[a, \Sigma_b] : b \in L\}$. Define binary relations $\curlywedge, \curlyvee$ on J_a by $[a, \Sigma_b] \curlywedge [a, \Sigma_c] = [a, \Sigma_{b \sqcap c}]$ and $[a, \Sigma_b] \curlyvee [a, \Sigma_c] = [a, \Sigma_{b \sqcup c}]$. Then $(J_a, \curlywedge)$ and $(J_a, \curlyvee)$ are commutative monoids in which every element is idempotent. Also

$[a, \Sigma_b] \curlyvee ([a, \Sigma_b] \curlywedge [a, \Sigma_c]) = [a, \Sigma_b] \curlyvee [a, \Sigma_{b \sqcap c}] = [a, \Sigma_{b \sqcup (b \sqcap c)}] = [a, \Sigma_b]$ and

$[a, \Sigma_b] \curlywedge ([a, \Sigma_b] \curlyvee [a, \Sigma_c]) = [a, \Sigma_b] \curlywedge [a, \Sigma_{b \sqcup c}] = [a, \Sigma_{b \sqcap (b \sqcup c)}] = [a, \Sigma_b]$.

Thus absorption laws are satisfied and hence J_a is a lattice. Since $\bigsqcup a_i$, for $a_i \in L$ exist, J_a is a complete lattice.

$$\begin{aligned} [a, \Sigma_b] \curlywedge \bigvee [a, \Sigma_{c_i}] &= [a, \Sigma_b] \curlywedge [a, \Sigma_{\bigsqcup c_i}] = [a, \Sigma_{b \sqcap \bigsqcup c_i}] = [a, \Sigma_{\bigsqcup b \sqcap c_i}] \\ &= \bigvee [a, \Sigma_{b \sqcap c_i}] = \bigvee ([a, \Sigma_b] \curlywedge [a, \Sigma_{c_i}]) \end{aligned}$$

Hence J_a satisfies infinite distributive law. Thus J_a is a locale of pseudo subframes of $O(L)$ with top element $[a, \Sigma_1]$ and bottom element $[a, \Sigma_0]$.

From example 3.1.7(2), we get the locale J_U = The one point locale O, for all $U \in L$.

From example 3.1.7(1), we get the locales $J_0 = \{[0, \Sigma_b] : b \in L\}$,

$J_a = \{[a, \Sigma_b] : b \in L\}$, $J_b = \{[b, \Sigma_a] : a \in L\}$ and $J_1 = \{[1, \Sigma_b] : b \in L\}$.

Proposition 3.2.1. *The locale J_a is compact if and only if $Sp(L)$ is compact.*

Proof. Assume J_a is compact. We have to show that $Sp(L)$ is compact.

Let $\Sigma_1 = \bigcup_{i \in I} \Sigma_{b_i}$. Then $\Sigma_1 = \Sigma_{\bigsqcup b_i}$.

By Lemma 3.1.6, $\curlyvee[a, \Sigma_{b_i}] = [a, \Sigma_{\bigsqcup b_i}] = [a, \Sigma_1]$. Hence $\{[a, \Sigma_{b_i}]; i \in I\}$ is a cover for J_a.

Since J_a is compact, we have $[a, \Sigma_{b_1}] \curlyvee [a, \Sigma_{b_2}] \curlyvee .... \curlyvee [a, \Sigma_{b_n}] = [a, \Sigma_1]$ for some $b_1, b_2, ...b_n \in L$. That is $[a, \Sigma_{b_1 \sqcup b_2 \sqcup b_3 \sqcup .....b_n}] = [a, \Sigma_1]$.

Then $\Sigma_{b_1 \sqcup b_2 \sqcup b_3 \sqcup .....b_n} = \Sigma_1$, using lemma 3.1.6. Hence $Sp(L)$ is compact.

Conversely assume $Sp(L)$ is compact. Let $\{[a, \Sigma_{b_i}]; i \in I\}$ be a cover of J_a.

That is $\curlyvee[a, \Sigma_{b_i}] = [a, \Sigma_1]$ or $[a, \Sigma_{\bigsqcup b_i}] = [a, \Sigma_1]$. This gives $\Sigma_{\bigsqcup b_i} = \Sigma_1$.

Since $Sp(L)$ is compact, we have $\Sigma_1 = \Sigma_{b_1 \sqcup b_2 \sqcup ... \sqcup b_n}$.

Hence $[a, \Sigma_{b_i \sqcup b_2 \sqcup ...... \sqcup b_n}] = [a, \Sigma_1]$. Thus J_a is compact. □

Isomorphism of J_a with a quotient locale of L

Define a relation $\sim_a$ on L by b$\sim_a$c if $[a, \Sigma_b] = [a, \Sigma_c]$. Clearly the relation $\sim_a$ is an equivalence relation. Let (b,c)$\in \sim_a$. We claim that $(b \sqcap d, c \sqcap d) \in \sim_a$ and $(b \sqcup \bigsqcup S, c \sqcup \bigsqcup S) \in \sim_a$.

$[a, \Sigma_{b \sqcap d}] = [a, \Sigma_b] \curlywedge [a, \Sigma_d] = [a, \Sigma_c] \curlywedge [a, \Sigma_d] = [a, \Sigma_{c \sqcap d}]$.

Hence $(b \sqcap d, c \sqcap d) \in \sim_a$.

$[a, \Sigma_{b \sqcup \bigsqcup S}] = [a, \Sigma_b] \curlyvee [a, \Sigma_{\bigsqcup S}] = [a, \Sigma_c] \curlyvee [a, \Sigma_{\bigsqcup S}] = [a, \Sigma_{c \sqcup \bigsqcup S}]$.

Hence $(b \sqcup \bigsqcup S, c \sqcup \bigsqcup S) \in \sim_a$. Thus $\sim_a$ is a frame congruence on L.

Then by [8], $L/\sim_a$ is a quotient frame (sublocale) of L with respect to the partial order $[a] \sqsubseteq [b]$ iff $a \sqsubseteq b$.

Note that if L is spatial, then $L/\sim_a = L$ for all $a \in L$.

Define $\psi_a : L/\sim_a \rightarrow J_a$ by $\psi_a([b]) = [a, \Sigma_b]$. Then

$\psi_a([b] \sqcap [c]) = \psi_a([b \sqcap c]) = [a, \Sigma_{b \sqcap c}] = [a, \Sigma_b] \curlywedge [a, \Sigma_c] = \psi_a([b]) \curlywedge \psi_a([c])$ and $\psi_a(\bigsqcup[b_i]) = \psi_a([\bigsqcup b_i]) = [a, \Sigma_{\bigsqcup b_i}] = \curlyvee[a, \Sigma_{b_i}] = \curlyvee \psi_a([b_i])$.

Hence ψ_a is a frame homomorphism.

Also ψ_a is one-one and onto. Thus ψ_a is an isomorphism in the category **Frm**. Since the isomorphism is a self dual property, ψ_a is an isomorphism in the category **Loc**. Thus the locale J_a is isomorphic to a sublocale of the locale L.

Congruence on $O(L)$

Define a relation R_a on $O(L)$ by fR_ag if $\Sigma_{f(a)} = \Sigma_{g(a)}$. Then R_a is an equivalence relation. Suppose fR_ag. Then

$\Sigma_{(f\wedge h)(a)} = \Sigma_{f(a)\sqcap h(a)} = \Sigma_{f(a)} \cap \Sigma_{h(a)} = \Sigma_{g(a)} \cap \Sigma_{h(a)} = \Sigma_{g(a)\sqcap h(a)} = \Sigma_{(g\wedge h)(a)}$.

Thus $f \wedge h \ \ R_a \ \ g \wedge h$.

$\Sigma_{(f\vee\bigvee f_i)(a)} = \Sigma_{f(a)} \cup \Sigma_{\bigsqcup f_i(a)} = \Sigma_{g(a)} \cup \Sigma_{\bigsqcup f_i(a)} = \Sigma_{g(a)\sqcup\bigsqcup f_i(a)} = \Sigma_{(g\vee\bigvee f_i)(a)}$.

Hence $f \vee \ \bigvee f_i \ \ R_a \ \ g \vee \ \bigvee f_i$. Thus R_a is a congruence on $O(L)$.

Then by [8], $O(L)/R_a$ is a quotient frame (sublocale) of $O(L)$ with respect to the partial order $[f] \leqslant [g]$ in $O(L)/R_a$ if and only if $f \leqslant g$ in $O(L)$.

Example 3.2.2. *In example 3.1.7(1), we have* $O(L)/R_a = \{[f_1], [f_9], [f_{13}], [f_{21}]\}$*,where*

$[f_1] = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8\}$

$[f_9] = \{f_9, f_{10}, f_{11}, f_{12}, f_{19}, f_{20}, f_{23}, f_{24}, f_{25}\}$

$[f_{13}] = \{f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{27}, f_{28}\}$

$[f_{21}] = \{f_{21}, f_{22}, f_{26}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}\}$

Lemma 3.2.3. *If* $(f, g) \in R_a$*, then* $(f(a), g(a)) \in \sim_a$.

Proof. Let $(f, g) \in R_a$. Then $\Sigma_{f(a)} = \Sigma_{g(a)}$.

By lemma 3.1.6, $[a, \Sigma_{f(a)}] = [a, \Sigma_{g(a)}]$. Hence $(f(a), g(a)) \in \sim_a$. □

Proposition 3.2.4. *The quotient frame(sublocale)* $L/\sim_a$ *of* L *is isomorphic to the quotient frame(sublocale)* $O(L)/R_a$ *of* $O(L)$.

Proof. Define the map $\sigma : O(L)/R_a \to L/\sim_a$ by $\sigma([f]) = [f(a)]$.

$$\begin{aligned}\sigma([f]) = \sigma([g]) &\Rightarrow [f(a)] = [g(a)]\\ &\Rightarrow [a, \Sigma_{f(a)}] = [a, \Sigma_{g(a)}]\\ &\Rightarrow \Sigma_{f(a)} = \Sigma_{g(a)}, \text{ by lemma 3.1.6}\\ &\Rightarrow [f] = [g].\end{aligned}$$

Thus the map σ is one one.

Also for each $[b] \in L/\sim_a$, $\sigma([f_b]) = [f_b(a)] = [b]$. Thus σ is onto.

$\sigma(\bigvee[f_i]) = \sigma([\bigvee f_i]) = [(\bigvee f_i)(a)] = [\bigsqcup(f_i(a))] = \bigsqcup[f_i(a)] = \bigsqcup\sigma([f_i])$ and

$$\begin{aligned}\sigma([f] \wedge [g]) &= \sigma([f \wedge g]) = [(f \wedge g)(a)] = [f(a) \sqcap g(a)]\\ &= [f(a)] \sqcap [g(a)] = \sigma(f) \sqcap \sigma(g)\end{aligned}$$

Hence σ is an isomorphism in **Frm**. Thus a sublocale of L is isomorphic to a sublocale of $O(L)$ □

Theorem 3.2.5. ***Embedding Theorem for locale L*** *A sublocale of the locale L can be embedded as a sublocale of $O(L)$. If the locale L is spatial, then L can be embedded as a sublocale of $O(L)$.*

Proof. Define $G : O(L) \to O(L)/R_a$ by $G(f) = [f]$. Then G is an onto frame homomorphism.

Consider $\sigma \circ G : O(L) \to L/\sim_a$ where $\sigma : O(L)/R_a \to L/\sim_a$ is the isomorphism in 3.2.4. For each $[b] \in L$ we have $f_b \in O(L)$ such that $(\sigma \circ G)(f_b) = [b]$.

Thus the map $\sigma \circ G$ is onto.

Also since G, σ are frame homomorphisms, $\sigma \circ G$ is a frame homomorphism.

Since $\sigma \circ G$ is an onto frame homomorphism, its adjoint δ is a one one localic map from the sublocale $L/\sim_a$ of L to $O(L)$.

If L is spatial, $L/\sim_a = L$ and so L can be embedded as a sublocale of $O(L)$ □

Localic map from $Sp(L)$ to $O(L)/R_a$

Define $\phi_a : O(L)/R_a \to Sp(L)$ by $\phi_a([f]) = \Sigma_{f(a)}$. Then

$\phi_a([f] \wedge [g]) = \Sigma_{f\wedge g(a)} = \Sigma_{f(a)} \cap \Sigma_{g(a)} = \phi_a([f]) \cap \phi_a([g])$ and

$\phi_a(\bigvee [f]_i) = \phi_a([\bigvee f_i]) = \Sigma_{(\bigvee f_i)(a)} = \bigcup \Sigma_{f_i(a)}$.

Hence ϕ_a is a frame homomorphism and its adjoint ϕ_a^* is a localic map from $Sp(L)$ to O(L)/R_a.

Lemma 3.2.6. *If $f \in [a, \Sigma_b]$, then [f]$\in [a, \Sigma_b]$.*

Proof. Let $f \in [a, \Sigma_b]$ and let $g \in [f]$. Then we have $\Sigma_{f(a)} = \Sigma_{g(a)}$.

Since $f \in [a, \Sigma_b]$, $\Sigma_{f(a)} \subseteq \Sigma_b$, which implies $\Sigma_{g(a)} \subseteq \Sigma_b$. Hence $[f] \in [a, \Sigma_b]$. □

Proposition 3.2.7. *The locale J_a is subfit if and only if for every $[a, \Sigma_b]$ in J_a with $[a, \Sigma_b] \nleqslant [a, \Sigma_c]$, there exists $\Sigma_d \in Sp(L)$ such that $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$ and $[\mathbf{1}] \nsubseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$.*

Proof. Suppose locale J_a is a subfit. Let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with $[a, \Sigma_b] \nleqslant [a, \Sigma_c]$.

Since J_a is a subfit, there exist $[a, \Sigma_d]$ such that $[a, \Sigma_b] \curlyvee [a, \Sigma_d] = [a, \Sigma_1]$ and $[a, \Sigma_c] \curlyvee [a, \Sigma_d] \neq [a, \Sigma_1]$.

Since $[a, \Sigma_{b\sqcup d}] = [a, \Sigma_1]$, we have $\mathbf{1} \in [a, \Sigma_{b\sqcup d}]$. Then by lemma 3.2.6, $[\mathbf{1}] \in [a, \Sigma_{b\sqcup d}]$.

Thus $\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_{b\sqcup d}$. Hence $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$.

Also, if $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$, then $\mathbf{1} \in [a, \Sigma_{c\sqcup d}]$, a contradiction.

Hence $[\mathbf{1}] \nsubseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$.

Conversely, suppose $[a, \Sigma_b] \nleqslant [a, \Sigma_c]$. Then by hypothesis, $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_b \cup \Sigma_d)$.

That is $\phi_a([\mathbf{1}]) \in \Sigma_{b\sqcup d}$. Thus $\mathbf{1} \in [a, \Sigma_{b\sqcup d}]$ and hence $[a, \Sigma_{b\sqcup d}] = [a, \Sigma_1]$.

If $[a, \Sigma_{c\sqcup d}] = [a, \Sigma_1]$, then $[\mathbf{1}] \subseteq \phi_a^{-1}(\Sigma_c \cup \Sigma_d)$, a contradiction. Hence J_a is subfit. □

Proposition 3.2.8. *The locale J_a has S_2' property if and only if for every $[a, \Sigma_b] \neq [a, \Sigma_1]$, $[a, \Sigma_c] \neq [a, \Sigma_1]$ in J_a with $[a, \Sigma_{b\sqcup c}] = [a, \Sigma_1]$, there exist*

$\Sigma_d, \Sigma_e \in \Omega(Sp(L))$ *such that* $\Sigma_e \nsubseteq \Sigma_b, \Sigma_d \nsubseteq \Sigma_c$ *and* $\phi_a([f]) \subseteq \Sigma_d \sqcap \Sigma_e$*implies that* $f \in [\boldsymbol{0}]$.

Proof. Suppose J_a has S_2' property. Let $[a, \Sigma_b] \curlyvee [a, \Sigma_c] = [a, \Sigma_1]$. Then there exist $[a, \Sigma_d], [a, \Sigma_e] \in J_a$ such that $[a, \Sigma_d] \curlywedge [a, \Sigma_e] = [a, \Sigma_0]$,$[a, \Sigma_e] \nleqslant [a, \Sigma_b]$ and $[a, \Sigma_d] \nleqslant [a, \Sigma_c]$. Then $\Sigma_d, \Sigma_e \in \Omega(Sp(L))$. If $\Sigma_e \subseteq \Sigma_b$, then $[a, \Sigma_e] \subseteq [a, \Sigma_b]$, a contradiction.

Hence $\Sigma_e \nsubseteq \Sigma_b$. Similarly we can prove that $\Sigma_d \nsubseteq \Sigma_c$.

Let $\phi_a([f]) \subseteq \Sigma_d \cap \Sigma_e$. Then $\Sigma_{f(a)} \subseteq \Sigma_{d \sqcap e}$.

Hence $f \in [a, \Sigma_{d \sqcap e}] = [a, \Sigma_0]$. Thus $f \in [\mathbf{0}]$.

Hence $\phi_a([f]) \subseteq \Sigma_d \cap \Sigma_e$ implies $f \in [\mathbf{0}]$.

Conversely, let $[a, \Sigma_b] \neq [a, \Sigma_1], [a, \Sigma_c] \neq [a, \Sigma_1]$ with $[a, \Sigma_{b \sqcup c}] = [a, \Sigma_1]$.

Then by assumption, there exist $\Sigma_d, \Sigma_e \in \Omega(Sp(L))$ with $\Sigma_e \nsubseteq \Sigma_b$, $\Sigma_d \nsubseteq \Sigma_c$ and $\phi_a([f]) \subseteq \Sigma_d \cap \Sigma_e$ implies that $f \in [\mathbf{0}]$.

Also $f \in [a, \Sigma_{d \sqcap e}]$, implies $\Sigma_{f(a)} \subseteq \Sigma_{d \sqcap e}$.

Hence $\phi_a([f]) \subseteq \Sigma_d \cap \Sigma_e \Rightarrow f \in [\mathbf{0}] \Rightarrow f \in [a, \Sigma_0]$.

Hence $[a, \Sigma_d] \curlywedge [a, \Sigma_e] = [a, \Sigma_0]$.

Also since $\Sigma_e \nsubseteq \Sigma_b$, $\Sigma_d \nsubseteq \Sigma_c$, $[a, \Sigma_e] \nleqslant [a, \Sigma_b]$ and $[a, \Sigma_d] \nleqslant [a, \Sigma_c]$. Thus J_a has S_2' property. □

Lemma 3.2.9. *If* $b \prec c$ *in* L*, then* $[a, \Sigma_b] \prec [a, \Sigma_c]$ *in* J_a*.*

Proof. Suppose $b \prec c$ in L. Then there exist $d \in L$ such that $b \sqcap d = 0$ and $c \sqcup d = 1$. We have

$[a, \Sigma_b] \curlywedge [a, \Sigma_d] = [a, \Sigma_{b \sqcap d}] = [a, \Sigma_0]$ and $[a, \Sigma_c] \curlyvee [a, \Sigma_d] = [a, \Sigma_{c \sqcup d}] = [a, \Sigma_1]$.

Hence $[a, \Sigma_b] \prec [a, \Sigma_c]$ in J_a. □

Proposition 3.2.10. *If* L *is a regular locale, then* J_a *is a regular locale.*

Proof. Proof follows directly from the above lemma. □

Proposition 3.2.11. *Locale J_a is normal if and only if for every $[a,\Sigma_b]$, $[a,\Sigma_c] \in J_a$ with $[a,\Sigma_b] \curlyvee [a,\Sigma_c] = [a,\Sigma_1]$, there exist $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$ such that $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$, $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$ and $\phi_a([f]) \in \Sigma_{u\sqcap v} \Rightarrow f \in [\mathbf{0}]$.*

Proof. Suppose J_a is normal. Let $[a,\Sigma_b],[a,\Sigma_c] \in J_a$ with $[a,\Sigma_b] \curlyvee [a,\Sigma_c] = [a,\Sigma_1]$. Since J_a is normal, there exist $[a,\Sigma_u],[a,\Sigma_v] \in J_a$ such that
$[a,\Sigma_b] \curlyvee [a,\Sigma_u] = [a,\Sigma_1]$, $[a,\Sigma_c] \curlyvee [a,\Sigma_v] = [a,\Sigma_1]$ and $[a,\Sigma_u] \curlywedge [a,\Sigma_v] = [a,\Sigma_0]$.
Then $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$.
Also $\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_b \cup \Sigma_u$. Hence $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$.
Similarly $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$. Now $\phi_a([f]) \in \Sigma_{u\sqcap v}$ implies $\Sigma_{f(a)} \subseteq \Sigma_{u\sqcap v}$.
Hence $f \in [a,\Sigma_{u\sqcap v}] = [a,\Sigma_0]$, which implies $\Sigma_{f(a)} \subseteq \Sigma_0$. Hence $f \in [\mathbf{0}]$.
Conversely, let $[a,\Sigma_b],[a,\Sigma_c] \in J_a$ with $[a,\Sigma_b] \curlyvee [a,\Sigma_c] = [a,\Sigma_1]$. By assumption, there exist $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$ such that $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$, $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_v \cup \Sigma_c)$ and $\phi_a([f]) \in \Sigma_{u\sqcap v} \Rightarrow f \in [\mathbf{0}]$. $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$.
Since $\Sigma_u, \Sigma_v \in \Omega(Sp(L))$, $[a,\Sigma_u],[a,\Sigma_v] \in J_a$. Also $[\mathbf{1}] \in \phi_a^{-1}(\Sigma_u \cup \Sigma_b)$ implies $\phi_a([\mathbf{1}]) = \Sigma_{\mathbf{1}(a)} \subseteq \Sigma_{u\sqcup b}$. Thus $\mathbf{1} \in [a,\Sigma_{u\sqcup b}]$. Hence $[a,\Sigma_{u\sqcup b}] = [a,\Sigma_1]$.
Similarly we can prove that $[a,\Sigma_{v\sqcup c}] = [a,\Sigma_1]$.
Also $f \in [a,\Sigma_{u\sqcap v}]$ implies $\Sigma_{f(a)} = \phi_a([f]) \subseteq \Sigma_{u\sqcap v}$. Then by assumption $f \in [a,\Sigma_0]$.
Hence $[a,\Sigma_{u\sqcap v}] = [a,\Sigma_0]$. □

Proposition 3.2.12. *The locale J_a is Boolean if and only if for each $b \in L$, Σ_b is a clopen subset of $\Omega(Sp(L))$.*

Proof. Suppose the locale J_a is Boolean and let $\Sigma_b \in \Omega(Sp(L))$.
Then $[a,\Sigma_b] \in J_a$. Since the locale J_a is Boolean, there exist $[a,\Sigma_c] \in J_a$ such that $[a,\Sigma_b] \curlywedge [a,\Sigma_c] = [a,\Sigma_0]$ and $[a,\Sigma_b] \curlyvee [a,\Sigma_c] = [a,\Sigma_1]$.

$[a, \Sigma_{b \sqcap c}] = [a, \Sigma_0]$ implies $\Sigma_{b \sqcap c} = \Sigma_0$ by lemma 3.1.6. Hence $\Sigma_b \cap \Sigma_c = \phi$.

$[a, \Sigma_{b \sqcup c}] = [a, \Sigma_1]$ implies that $\Sigma_{b \sqcup c} = \Sigma_1$. Hence $\Sigma_b \cup \Sigma_c = \Sigma_1$.

Thus $\Sigma_c \in \Omega(Sp(L))$ is the compliment of Σ_b. Hence Σ_b is both closed and open.

Conversely assume that each Σ_b is clopen and let $[a, \Sigma_b] \in J_a$.

Then $\Sigma_b \in \Omega(Sp(L))$. Since Σ_b is clopen, we have $\Sigma_c = (\Sigma_b)^c \in \Omega(Sp(L))$.

Then $[a, \Sigma_c] \in J_a$.

$[a, \Sigma_b] \curlywedge [a, \Sigma_c] = [a, \Sigma_{b \sqcap c}] = [a, \Sigma_b \cap \Sigma_c] = [a, \phi] = [a, \Sigma_0]$.

$[a, \Sigma_b] \curlyvee [a, \Sigma_c] = [a, \Sigma_{b \sqcup c}] = [a, \Sigma_b \cup \Sigma_c] = [a, \Sigma_1]$.

Hence the locale J_a is Boolean. □

3.3. Coproduct of the locales $J_a, a \in L$

In [39], if $L_i, i \in I$ are locales, then the cartesian product $\prod L_i$ together with component wise ordering is a locale. Since each $J_a, a \in L$ is a locale, $J = \prod J_a$ is a locale together with the map $p_a : J_a \to J$, $a \in L$, defined by $p_a([a, \Sigma_b]) = \prod[b, \Sigma_x]$ where $[b, \Sigma_x] = [b, \Sigma_1]$ for all $b \neq a$ and $[a, \Sigma_x] = [a, \Sigma_b]$. Then $(p_a : J_a \to J)_{a \in L}$ is the coproduct of locales J_a.

Notation Any element of the coproduct locale J is denoted by $\prod[a, \Sigma_{x_a}]$, where $[a, \Sigma_{x_a}] \in J_a$.

Proposition 3.3.1. *The locale J is subfit if and only if each J_a is a subfit.*

Proof. Suppose each J_a is subfit and let $A = \prod[a, \Sigma_{x_a}], B = \prod[a, \Sigma_{y_a}] \in J$ such that $A \nleqslant B$.

Then there exist $d \in L$ such that $[d, \Sigma_{x_d}] \nleqslant [d, \Sigma_{y_d}]$.

Since J_d is subfit, there exist $[d, \Sigma_z]$ such that $[d, \Sigma_{x_d}] \curlyvee [d, \Sigma_z] = [d, \Sigma_1]$ and

$[d, \Sigma_{y_d}] \curlyvee [d, \Sigma_z] \neq [a, \Sigma_1]$.

Take $C \in J$ as $C = \prod[a, \Sigma_{z_a}]$ where $[a, \Sigma_{z_a}] = [a, \Sigma_1]$ for $a \neq d$ and $[d, \Sigma_{z_d}] = [d, \Sigma_z]$.

Then we have $A \curlyvee C = 1_J$ and $B \curlyvee C \neq 1_J$. Hence J is a subfit locale.

Conversely assume that J is a subfit locale and let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with

$[a, \Sigma_b] \nleqslant [a, \Sigma_c]$.

Then let $A = \prod[b, \Sigma_{x_b}], B = \prod[b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_1]$, $[b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and $[a, \Sigma_{x_a}] = [a, \Sigma_b]$ and $[a, \Sigma_{y_a}] = [a, \Sigma_c]$.

Then $A, B \in J$ is such that $A \nleqslant B$. Since J is a subfit locale there exist

$C = \prod[b, \Sigma_{z_b}] \in J$ such that $A \curlyvee C = 1_J$ and $B \curlyvee C \neq 1_J$.

Then we must have $[a, \Sigma_b] \curlyvee [a, \Sigma_{z_a}] = [a, \Sigma_1]$ and $[a, \Sigma_c] \curlyvee [a, \Sigma_{z_a}] \neq [a, \Sigma_1]$.

Thus J_a is a subfit locale. □

Proposition 3.3.2. *If the locale J has S_2' property, then each $J_a, a \in L$ has S_2' property.*

Proof. Suppose that the locale J has S_2' property and let $[a, \Sigma_b] \neq [a, \Sigma_1]$,

$[a, \Sigma_c] \neq [a, \Sigma_1] \in J_a$ with $[a, \Sigma_b] \curlyvee [a, \Sigma_c] = [a, \Sigma_1]$.

Let $A = \prod[b, \Sigma_{x_b}], B = \prod[b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and $[a, \Sigma_{x_a}] = [a, \Sigma_b]$ and $[a, \Sigma_{y_a}] = [a, \Sigma_c]$.

Then we have $A, B \neq 1_J \in J$ with $A \curlyvee B = 1_J$.

Since J has S_2' property, there exist $U = \prod[b, \Sigma_{u_b}], V = \prod[b, \Sigma_{v_b}] \in J$ such that $U \curlywedge V = 0_J, U \nleqslant B, V \nleqslant A$.

Thus $[a, \Sigma_{v_a}] \nleqslant [a, \Sigma_b]$, $[a, \Sigma_{u_a}] \nleqslant [a, \Sigma_c]$ and $[a, \Sigma_{v_a}] \curlywedge [a, \Sigma_{u_a}] = [a, \Sigma_0]$.

Hence J_a has S_2' property. □

Proposition 3.3.3. *The locale J_a is normal if and only if J is normal.*

Proof. Suppose that each J_a is normal and let $A = \prod[b, \Sigma_{x_b}]$, $B = \prod[b, \Sigma_{y_b}] \in J$ such that $A \curlyvee B = 1_J$.

Then $[a, \Sigma_{x_a}], [a, \Sigma_{y_a}] \in J_a$ with $[a, \Sigma_{x_a}] \curlyvee [a, \Sigma_{y_a}] = [a, \Sigma_1]$ for all $a \in L$.

Since J_a is normal, there exist $[a, \Sigma_{u_a}], [a, \Sigma_{v_a}] \in J_a$ such that

$[a, \Sigma_{x_a}] \curlyvee [a, \Sigma_{v_a}] = [a, \Sigma_1]$ and $[a, \Sigma_{y_a}] \curlyvee [a, \Sigma_{u_a}] = [a, \Sigma_1], [a, \Sigma_{v_a}] \curlyvee [a, \Sigma_{u_a}] = [a, \Sigma_0]$.

Let $U = \prod [b, \Sigma_{u_b}], V = \prod [b, \Sigma_{v_b}]$.

Then $U, V \in J$ such that $A \curlyvee V = 1_J, B \curlyvee U = 1_J, U \curlywedge V = 0_J$. Hence J is normal.

Conversely assume that J is normal and let $[a, \Sigma_b], [a, \Sigma_c] \in J_a$ with

$[a, \Sigma_b] \curlyvee [a, \Sigma_c] = [a, \Sigma_1]$.

Consider $A = \prod [b, \Sigma_{x_b}], B = \prod [b, \Sigma_{y_b}]$ where $[b, \Sigma_{x_b}] = [b, \Sigma_{y_b}] = [b, \Sigma_1]$ for $b \neq a$ and $[a, \Sigma_{x_a}] = [a, \Sigma_b], [a, \Sigma_{y_a}] = [a, \Sigma_c]$.

Then $A, B \in J$ with $A \curlyvee B = 1_J$.

Since J is normal, there exist $U = \prod [b, \Sigma_{u_b}], V = \prod [b, \Sigma_{v_b}]$ such that $A \curlyvee V = 1_J$, $B \curlyvee U = 1_J, U \curlywedge V = 0_J$.

Then $[a, \Sigma_b] \curlyvee [a, \Sigma_{v_a}] = [a, \Sigma_1], \quad [a, \Sigma_c] \curlyvee [a, \Sigma_{u_a}] = [a, \Sigma_1]$, and

$[a, \Sigma_{v_a}] \curlywedge [a, \Sigma_{u_a}] = [a, \Sigma_0]$.

Hence J_a is normal □

3.4. Filters $\langle a, \Sigma_b \rangle$ for a,b $\in$ L

Definition 3.4.1. Let L be a locale. For each $a, b \in L$, define

$\langle a, \Sigma_b \rangle = \{f \in O(L) : \Sigma_{f(a)} \supseteq \Sigma_b\}$.

Example 3.4.2. *In example 3.1.7*

$$\begin{aligned}
\langle a, \Sigma_0 \rangle &= \{f \in O(L) : \Sigma_{f(a)} \supseteq \Sigma_0\} \\
&= \{f \in O(L) : \Sigma_{f(a)} \supseteq \phi\} = O(L) \\
\langle a, \Sigma_a \rangle &= \{f_9, f_{10}, f_{11}, f_{12}, f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}, f_{26}, f_{30}, f_{31}, f_{32}, f_{33}\} \\
\langle a, \Sigma_b \rangle &= \{f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{21}, f_{22}, f_{26}, f_{27}, f_{28}, f_{29}, f_{30}, f_{31}, f_{32}, f_{33}\} \\
\langle a, \Sigma_1 \rangle &= \{f_{21}, f_{22}, f_{26}, f_{30}, f_{31}, f_{32}, f_{33}\}
\end{aligned}$$

Some simple properties of $\langle a, \Sigma_b \rangle$ have been verified in the following propositions.

Proposition 3.4.3. *For each $a, b \in L$*

i. $\langle a, \Sigma_b \rangle$ is closed under finite meet and join.

ii. $\langle a, \Sigma_b \rangle$ is an upper set.

iii. $\langle a, \Sigma_b \rangle$ is a filter in L.

iv. If Σ_b is a join-irreducible element of $\Omega(Sp(L))$, then $\langle a, \Sigma_b \rangle$ is a prime filter in $O(L)$.

Proof. i. Let f,g $\in \langle a, \Sigma_b \rangle$.

$$\begin{aligned}
f, g \in \langle a, \Sigma_b \rangle &\Rightarrow \Sigma_{f(a)} \supseteq \Sigma_b \text{ and } \Sigma_{g(a)} \supseteq \Sigma_b \\
&\Rightarrow \Sigma_{f(a)} \cup \Sigma_{g(a)} \supseteq \Sigma_b \text{ and } \Sigma_{f(a)} \cap \Sigma_{g(a)} \supseteq \Sigma_b \\
&\Rightarrow \Sigma_{f(a) \sqcup g(a)} \supseteq \Sigma_b \text{and } \Sigma_{f(a) \sqcap g(a)} \supseteq \Sigma_b \\
&\Rightarrow \Sigma_{(f \vee g(a))} \supseteq \Sigma_b \text{ and } \Sigma_{(f \wedge g(a))} \supseteq \Sigma_b \\
&\Rightarrow f \vee g, f \wedge g \in \langle a, \Sigma_b \rangle
\end{aligned}$$

ii. Let $g \in \langle a, \Sigma_b \rangle$ and $g \leqslant f$ in $O(L)$.

$g \in \langle a, \Sigma_b \rangle, g \leqslant f \Rightarrow g(a) \sqsubseteq f(a)$

$$\Rightarrow \Sigma_{g(a)} \subseteq \Sigma_{f(a)}$$

$$\Rightarrow \Sigma_b \subseteq \Sigma_{g(a)} \subseteq \Sigma_{f(a)}$$

$$\Rightarrow f \in \langle a, \Sigma_b \rangle$$

Hence $\langle a, \Sigma_b \rangle$ is an upper set.

iii. Proof follows directly from part i, ii.

iv. Assume Σ_b is a join-irreducible element of $\Omega(Sp(L))$. Let $f \vee g \in \langle a, \Sigma_b \rangle$.

$$f \vee g \in \langle a, \Sigma_b \rangle \Rightarrow \Sigma_{(f \vee g)(a)} \supseteq \Sigma_b$$

$$\Rightarrow \Sigma_{f(a)} \cup \Sigma_{g(a)} \supseteq \Sigma_b$$

Since Σ_b is join-irreducible, either $\Sigma_{f(a)} \supseteq \Sigma_b$ or $\Sigma_{g(a)} \supseteq \Sigma_b$. Hence $f \in \langle a, \Sigma_b \rangle$ or $g \in \langle a, \Sigma_b \rangle$. Thus $\langle a, \Sigma_b \rangle$ is a prime filter. □

Proposition 3.4.4. *Let L be a locale and $a_1, a_2, b_1, b_2, a, b \in L$.*

i. If $a_1 \sqsubseteq a_2$, then $\langle a_1, \Sigma_b \rangle \subseteq \langle a_2, \Sigma_b \rangle$.

ii. If $b_1 \sqsubseteq b_2$, then $\langle a, \Sigma_{b_1} \rangle \supseteq \langle a, \Sigma_{b_2} \rangle$

Proof. Let L be a locale and $a_1, a_2, b_1, b_2, a, b \in L$.

i. Suppose $a_1 \sqsubseteq a_2$. Then $f(a_1) \sqsubseteq f(a_2) \quad \forall f \in O(L)$.

$$f \in \langle a_1, \Sigma_b \rangle \Rightarrow \Sigma_{f(a_1)} \supseteq \Sigma_b$$

$$\Rightarrow \Sigma_{f(a_2)} \supseteq \Sigma_{f(a_1)} \supseteq \Sigma_b$$

$$\Rightarrow f \in \langle a_2, \Sigma_b \rangle$$

Therefore $\langle a_1, \Sigma_b \rangle \subseteq \langle a_2, \Sigma_b \rangle$.

ii. Let $b_1 \sqsubseteq b_2$. Then $\Sigma_{b_1} \subset \Sigma_{b_2}$.

$$f \in \langle a, \Sigma_{b_2} \rangle \Rightarrow \Sigma_{f(a)} \supseteq \Sigma_{b_2} \supseteq \Sigma_{b_1}.$$

$$\Rightarrow f \in \langle a, \Sigma_{b_1} \rangle$$

Therefore $\langle a, \Sigma_{b_1} \rangle \supseteq \langle a, \Sigma_{b_2} \rangle$. □

Proposition 3.4.5. *For any $a, b, c \in L$, $\langle a, \Sigma_b \rangle = \langle a, \Sigma_c \rangle$ if and only if $\Sigma_b = \Sigma_c$.*

Proof. If $\Sigma_b = \Sigma_c$, then clearly $\langle a, \Sigma_b \rangle = \langle a, \Sigma_c \rangle$.
Conversely let $\langle a, \Sigma_b \rangle = \langle a, \Sigma_c \rangle$. Since $\Sigma_{f_b(a)} \supseteq \Sigma_b$, $f_b \in \langle a, \Sigma_b \rangle = \langle a, \Sigma_c \rangle$. Then $\Sigma_{f_b(a)} \supseteq \Sigma_c$. That is $\Sigma_b \subseteq \Sigma_c$. In a similar manner $\Sigma_c \subseteq \Sigma_b$. Hence $\Sigma_b = \Sigma_c$. □

Proposition 3.4.6. *Let Σ_b be compact element in $Sp(L)$. Then for each $a \in L, \langle a, \Sigma_b \rangle$ is completely prime filter in $O(L)$.*

Proof. By proposition 5.5.5, $\langle a, \Sigma_b \rangle$ is a filter in $O(L)$. Let $\bigvee f_\alpha \in \langle a, \Sigma_b \rangle$. Then we have $\Sigma_{\bigvee f_\alpha}(a) \supseteq \Sigma_b$ or $\bigcup \Sigma_{f_\alpha(a)} \supseteq \Sigma_b$. Since Σ_b is a compact element in $Sp(L)$, there is some β such that $\Sigma_{f_\beta}(a) \supseteq \Sigma_b$. Hence $f_\beta \in \langle a, \Sigma_b \rangle$. Thus for each $a \in L, \langle a, \Sigma_b \rangle$ is completely prime filter. □

Proposition 3.4.7. *Let Σ_b be a join-irreducible element in $\Omega(Sp(L))$. If $f \in O(L)$ immediately preceeds $g \in O(L)$ with $\Sigma_{f(a)} \neq \Sigma_{g(a)}$ and $\Sigma_b = \Sigma_{g(a)} \backslash \{F\}$ for some $F \in Sp(L)$, then $\langle a, \Sigma_b \rangle$ is a slicing filter in $O(L)$.*

Proof. Let Σ_b be a join-irreducible element in $\Omega(Sp(L))$. Then by Proposition5.5.5, $\langle a, \Sigma_b \rangle$ is a prime filter in $O(L)$. Since $\Sigma_b = \Sigma_{g(a)} \backslash \{F\}, \Sigma_{g(a)} \supseteq \Sigma_b$. Hence
$g \in \langle a, \Sigma_b \rangle$. As f immediately preceeds g and $\Sigma_{f(a)} \neq \Sigma_{g(a)}$, $\Sigma_{f(a)} \subseteq \Sigma_{g(a)}$. Since $\Sigma_b = \Sigma_{g(a)} \backslash \{F\}, \Sigma_{f(a)} \not\supseteq \Sigma_b$. Hence $f \notin \langle a, \Sigma_b \rangle$. Hence $\langle a, \Sigma_b \rangle$ is a slicing filter in $O(L)$. □

Proposition 3.4.8. *For a fixed $a \in L$, $\cap \langle a, \Sigma_{b_\alpha} \rangle = \langle a, \Sigma_{\bigsqcup b_\alpha} \rangle$.*

Proof. Fix some $a \in L$.

$$
\begin{aligned}
f \in \cap\langle a, \Sigma_{b_\alpha}\rangle &\Rightarrow f \in \langle a, \Sigma_{b_\alpha}\rangle \text{ for all } \alpha \\
&\Rightarrow \Sigma_{f(a)} \supseteq \Sigma_{b_\alpha} \text{ for all } \alpha \\
&\Rightarrow \Sigma_{f(a)} \supseteq \bigcup \Sigma_{b_\alpha} = \Sigma_{\bigsqcup b_\alpha} \\
&\Rightarrow \cap\langle a, \Sigma_{b_\alpha}\rangle \subseteq \langle a, \Sigma_{\bigsqcup b_\alpha}\rangle.
\end{aligned}
$$

Now let $f \in \langle a, \Sigma_{\bigsqcup b_\alpha}\rangle$.

$$
\begin{aligned}
f \in \langle a, \Sigma_{\bigsqcup b_\alpha}\rangle &\Rightarrow \Sigma_{f(a)} \supseteq \Sigma_{\bigsqcup b_\alpha} = \bigcup \Sigma_{b_\alpha} \\
&\Rightarrow \Sigma_{f(a)} \supseteq \Sigma_{b_\alpha} \text{ for all } \alpha \\
&\Rightarrow f \in \langle a, \Sigma_{b_\alpha}\rangle \text{ for all } \alpha \\
&\Rightarrow f \in \cap\langle a, \Sigma_{b_\alpha}\rangle
\end{aligned}
$$

So $\langle a, \Sigma_{\bigsqcup b_\alpha}\rangle \supseteq \cap\langle a, \Sigma_{b_\alpha}\rangle$.

Hence $\langle a, \Sigma_{\bigsqcup b_\alpha}\rangle = \cap\langle a, \Sigma_{b_\alpha}\rangle$. □

Proposition 3.4.9. *For each $a \in L$, let $S_a = \{\langle a, \Sigma_b\rangle\ b \in L\}$. Then S_a is a complete lattice under the partial order $\subseteq$.*

Proof. By above proposition S_a is a complete meet semilattice with top element $\langle a, \Sigma_0\rangle$. Since every complete semilattice with top and bottom is a complete lattice, S_a is a complete lattice under the partial order $\subseteq$. □

3.5. Construction of subspace of $Sp(O(L))$ using $\langle a, \Sigma_b\rangle, a, b \in L$

Proposition 3.5.1. *Let Σ_b be a compact open set in $Sp(L)$ and let $Y_b = \{\langle a, \Sigma_b\rangle : a \in L\}$. Then $(Y_b, \Omega(Sp(O(L)))/Y_b)$ is a compact subspace of spectrum $Sp(O(L))$.*

Proof. Since Σ_b is a compact element of $Sp(L)$, for all $a \in L$, the filter $\langle a, \Sigma_b \rangle$ is completely prime. Hence $Y_b \subseteq Sp(O(L))$ and $\Omega(Sp(O(L)))/Y_b$ is the subspace topology on Y_b. Hence $(Y_b, \Omega(Sp(O(L)))/Y_b)$ is a subspace of spectrum $Sp(O(L))$.

Let $\{\Sigma_{f_\alpha} : \alpha \in I\}$ be an open cover of Y_b. Then we have $Y_b \subseteq \bigcup \Sigma_{f_\alpha} = \Sigma_{\bigvee f_\alpha}$. Then the element $\langle 0, \Sigma_b \rangle$ of Y_b is in $\Sigma_{\bigvee f_\alpha}$. Hence $\bigvee f_\alpha \in \langle 0, \Sigma_b \rangle$ and so $\bigcup \Sigma_{f_\alpha(0)} = \Sigma_{\sqcup f_\alpha(0)} \supseteq \Sigma_b$. Since Σ_b is compact there exist $\beta \in I$ such that $\Sigma_{f_\beta(0)} \supseteq \Sigma_b$. Then $\langle 0, \Sigma_b \rangle \in \Sigma_{f_\beta}$. Since $f_\beta(a) \geqslant f_\beta(0)$, $\Sigma_{f_\beta(a)} \supseteq \Sigma_b$ for all $a \in L$ and so $Y_b \subseteq \Sigma_{f_\beta}$. Hence Y_b is compact. □

Proposition 3.5.2. *Let Σ_b be a compact open set in $Sp(L)$ and let $Y_b = \{\langle a, \Sigma_b \rangle : a \in L\}$. Then $(Y_b, \Omega(Sp(O(L)))/Y_b)$ satisfies T_0 axiom.*

Proof. Let $\langle a_1, \Sigma_b \rangle, \langle a_2, \Sigma_b \rangle \in Y_b$ such that $\langle a_1, \Sigma_b \rangle \neq \langle a_2, \Sigma_b \rangle$. So there is atleast one $f \in O(L)$ such that f is in one of them and not in other.

Let $f \in \langle a_1, \Sigma_b \rangle$ and $f \notin \langle a_2, \Sigma_b \rangle$. Then $\langle a_1, \Sigma_b \rangle \in \Sigma_f \cap Y_b$ and $\langle a_2, \Sigma_b \rangle \notin \Sigma_f \cap Y_b$. Hence $(Y_b, \Omega(Sp(O(L)))/Y_b)$ satisfies T_0 axiom. □

Proposition 3.5.3. *Let Σ_b be a compact open set in $Sp(L)$ and let $Y_b = \{\langle a, \Sigma_b \rangle : a \in L\}$. Then $(Y_b, \Omega(Sp(O(L)))/Y_b)$ is connected.*

Proof. Let $Y_b = (\Sigma_f \cap Y_b) \cup (\Sigma_g \cap Y_b)$, where $(\Sigma_f \cap Y_b), (\Sigma_g \cap Y_b)$ are nonempty open subsets of $(Y_b, \Omega(Sp(O(L)))/Y_b)$.

So there is $\langle a_1, \Sigma_b \rangle, \langle a_2, \Sigma_b \rangle \in Y_b$ such that $\langle a_1, \Sigma_b \rangle \in \Sigma_f \cap Y_b, \langle a_2, \Sigma_b \rangle \in \Sigma_g \cap Y_b$. Then $\Sigma_{f(a_1)} \supseteq \Sigma_b, \Sigma_{g(a_2)} \supseteq \Sigma_b$. Since $f, g \in O(L)$, $f(a_1) \sqsubseteq f(1), g(a_2) \sqsubseteq g(1)$. Hence $\Sigma_{f(1)} \supseteq \Sigma_b, \Sigma_{g(1)} \supseteq \Sigma_b$. Hence $\langle 1, \Sigma_b \rangle \in (\Sigma_f \cap Y_b) \cap (\Sigma_g \cap Y_b)$. So $\Sigma_f \cap Y_b, \Sigma_g \cap Y_b$ cannot be disjoint. Hence $(Y_b, \Omega(Sp(O(L)))/Y_b)$ is connected. □

Chapter 4

The Concept of L-slice for a locale L

Given a locale L and a join semilattice J with bottom element 0_J, we have introduced a new concept of an action σ of locale L on join semilattice J together with a set of conditions. The pair (σ, J) is called L-slice. L-slice, though algebraic in nature adopts properties of L through the action σ.

4.1. L-Slices

This section discusses the concept of L-slice and some of its properties.

Definition 4.1.1. Let L be a locale and J be join semilattice with bottom element 0_J. By the "action of L on J"we mean a function $\sigma : L \times J \to J$ such that the following conditions are satisfied.

i. $\sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2)$ for all $a \in L, x_1, x_2 \in J$.

ii. $\sigma(a, 0_J) = 0_J$ for all $a \in L$.

iii. $\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x))$ for all $a, b \in L, x \in J$.

iv. $\sigma(1_L, x) = x$ and $\sigma(0_L, x) = 0_J$ for all $x \in J$.

v. $\sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x)$ for $a, b \in L, x \in J$.

If σ is an action of the locale L on a join semilattice J, then we call (σ, J) as L-slice.

Next propsoition gives sufficient conditon for a subset $S \subseteq O(L)$, to be an L-slice.

Proposition 4.1.2. *Let L be a locale, and let S be a set of order preserving maps $L \to L$ such that :*

i. The constant map $\boldsymbol{0} \in S$ ($\boldsymbol{0}$ takes everything to 0).

ii. If $f, g \in S$, then $f \vee g \in S$.

iii. For all $a \in L$ and for all $f \in S$, the meet of the constant map $\boldsymbol{a}$ and f is in S (i.e. $f \sqcap \boldsymbol{a} \in S$).

Then the map $\sigma : L \times S \to S$ defined by $\sigma(a, f)(x) = f(x) \sqcap a$ is an action of L on S.

Proof. By the hypothesis, S is a join semilattice with bottom element $\mathbf{0}$ and the map σ is well defined.

i. $\sigma(a, f \vee g)(x) = (f \vee g)(x) \sqcap a = (f(x) \sqcup g(x)) \sqcap a = (f(x) \sqcap a) \sqcup (g(x) \sqcap a)$
$$= \sigma(a, f)(x) \sqcup \sigma(a, g)(x) = (\sigma(a, f) \vee \sigma(a, g))(x)$$

ii. $\sigma(a, \mathbf{0})(x) = \mathbf{0}(x) \sqcap a = 0 \sqcap a = 0 = \mathbf{0}(x)$

iii. $\sigma(a \sqcap b, f)(x) = f(x) \sqcap (a \sqcap b) = a \sqcap (f(x) \sqcap b)$
$$= a \sqcap \sigma(b, f)(x) = \sigma(a, \sigma(b, f))(x) = \sigma(b, \sigma(a, f))(x)$$

iv. $\sigma(1_L, f)(x) = f(x) \sqcap 1_L = f(x)$

$\sigma(0_L, f)(x) = f(x) \sqcap 0_L = \mathbf{0}(x)$

v. $\sigma(a \sqcup b, f)(x) = f(x) \sqcap (a \sqcup b) = (f(x) \sqcap a) \sqcup (f(x) \sqcap b)$

$$= \sigma(a, f)(x) \sqcup \sigma(b, f)(x) = (\sigma(a, f) \vee \sigma(b, f))(x)$$

Hence (σ, S) is an L-slice. □

Examples 4.1.3. *1. Let L be a locale and I be any ideal of L. Consider each $x \in I$ as constant map $\boldsymbol{x} : L \to L$. Then by proposition 4.1.2, (σ, I) is an L-slice. In particular (σ, L) is an L-slice.*

2. Let the locale L be a chain with Top and Bottom elements and J be any join semilattice with bottom element. Define $\sigma : L \times J \to J$ by $\sigma(a, j) = j \;\; \forall a \neq 0$ and $\sigma(0_L, j) = 0_J$. Then σ is an action of L on J and (σ, J) is an L-slice.

Proposition 4.1.4. *The product of two L-slices of a locale L is an L-slice.*

Proof. Let $(\sigma_1, J_1), (\sigma_2, J_2)$ be two L-slices of a locale L. Since J_1, J_2 are join semilattices with bottom elements, $J_1 \times J_2$ is join semilattice with bottom $(0_{J_1}, 0_{J_2})$.
Define $\sigma : L \times (J_1 \times J_2) \to J_1 \times J_2$ by $\sigma(a, (x_1, x_2)) = (\sigma_1(a, x_1), \sigma_2(a, x_2))$. Then

i. $\sigma(a, (x_1, y_1) \vee (x_2, y_2)) = \sigma(a, (x_1 \vee x_2, y_1 \vee y_2)) = (\sigma_1(a, x_1 \vee x_2), \sigma_2(a, y_1 \vee y_2))$

$$= (\sigma_1(a, x_1) \vee \sigma_1(a, x_2), \sigma_2(a, y_1) \vee \sigma_2(a, y_2))$$
$$= (\sigma_1(a, x_1), \sigma_2(a, y_1)) \vee (\sigma_1(a, x_2), \sigma_2(a, y_2))$$
$$= \sigma(a, (x_1, y_1)) \vee \sigma(a, (x_2, y_2))$$

ii. $\sigma(a, (0_{J_1}, 0_{J_2})) = (\sigma_1(a, 0_{J_1}), \sigma_2(a, 0_{J_2})) = (0_{J_1}, 0_{J_2})$

iii. $\sigma(a \sqcap b, (x, y)) = (\sigma_1(a \sqcap b, x), \sigma_2(a \sqcap b, y)) = (\sigma_1(a, \sigma_1(b, x)), \sigma_2(a, \sigma_2(b, y)))$

$$= \sigma(a, (\sigma_1(b, x), \sigma_2(b, y))) = \sigma(a, \sigma(b, (x, y)))$$

iv. $\sigma(1_L, (x, y)) = (\sigma_1(1_L, x), \sigma_2(1_L, y)) = (x, y)$

$\sigma(0_L, (x, y)) = (\sigma_1(0_L, x), \sigma_2(0_L, y)) = (0_{J_1}, 0_{J_2})$

v. $\sigma(a_1 \sqcup a_2, (x, y)) = (\sigma_1(a_1 \sqcup a_2, x), \sigma_2(a_1 \sqcup a_2, y))$

$$= (\sigma_1(a_1, x) \vee \sigma_1(a_2, x), \sigma_2(a_1, y) \vee \sigma_2(a_2, y))$$

$$= (\sigma_1(a_1, x), \sigma_2(a_1, y)) \vee (\sigma_1(a_2, x), \sigma_2(a_2, y))$$

$$= \sigma(a_1, (x, y)) \vee \sigma(a_2, (x, y))$$

Thus σ is an action on $J_1 \times J_2$ and $(\sigma, J_1 \times J_2)$ is a L-slice of locale L. □

Definition 4.1.5. Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if

i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ for all $x_1, x_2 \in J$.

ii. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Remark. More about L-slice homomorphism are studied in chapter 5.

4.2. L-Subslice

Definition 4.2.1. Let (σ, J) be an L-slice of a locale L. A subjoin semilattice J' of J is said to be L-subslice of J if J' is closed under action by elements of L.

Examples 4.2.2. *1. Let L be a locale and $O(L)$ denotes the collection of all order preserving maps on L. Then $(\sigma, O(L))$ is an L-slice, where $\sigma : L \times O(L) \to O(L)$ is defined by $\sigma(a, f) = f_a$, where $f_a : L \to L$ is defined by $f_a(x) = f(x) \sqcap a$. Let $K = \{f \in O(L) : f(x) \sqsubseteq x, \forall x \in L\}$. Then (σ, K) is an L-subslice of the L-slice $(\sigma, O(L))$.*

2. Let (σ, J) be an L-slice and let $x \in (\sigma, J)$. Define $\langle x \rangle = \{\sigma(a, x); a \in L\}$. Then $(\sigma, \langle x \rangle)$ is an L-subslice of (σ, J) and it is the smallest L-subslice of (σ, J) containing x.

Proposition 4.2.3. *The intersection of any family of L-subslices of an L-slice (σ, J) is again an L-subslice of (σ, J).*

Proof. Let (σ, J) be an L-slice and let $\{(\sigma, J_\alpha)\}$ be any collection of L-subslices of (σ, J). Then $\bigcap J_\alpha$ is a sub join semilattice of J.
Let $a \in L$ and $x \in \bigcap J_\alpha$. $x \in \bigcap J_\alpha$ implies that $x \in J_\alpha$ for every α.
Since each (σ, J_α) is an L-subslice of (σ, J), we have $\sigma(a, x) \in J_\alpha$ for every α.
Hence $\sigma(a, x) \in \bigcap J_\alpha$. This shows that $(\sigma, \bigcap J_\alpha)$ is an L-subslice of (σ, J). □

Remark. Union of two L-subslices of an L-slice (σ, J) need not be an L-subslice of (σ, J) as union of two subjoin semilattices of J need not be a subjoin semilattice. If (σ, J') and (σ, J'') be two L-subslices of the L-slice (σ, J), define
$J' \bigvee J'' = \{x \vee y : x \in J', y \in J''\}$. Then we can show that $(\sigma, J' \bigvee J'')$ is an L-subslice of (σ, J).

Proposition 4.2.4. *Let (σ, J') and (σ, J'') be two L-subslices of the L-slice (σ, J) of a locale L, then $(\sigma, J' \bigvee J'')$ is an L-subslice of the L-slice (σ, J) and it is the smallest L-subslice of (σ, J) containing both (σ, J') and (σ, J'').*

Proof. Since (σ, J') and (σ, J'') are L-subslices of the L-slice (σ, J), J' and J'' are subjoin semilattices of J and so is $J' \bigvee J''$. Let $x \vee y \in J' \bigvee J''$ and $a \in L$.
$\sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) \in J' \bigvee J''$.
Hence $(\sigma, J' \bigvee J'')$ is an L-subslice of the L-slice (σ, J).
For each $x \in (\sigma, J')$, $x = x \vee 0 \in (\sigma, J' \bigvee J'')$. Hence $(\sigma, J') \subseteq (\sigma, J' \bigvee J'')$.
Similarly $(\sigma, J'') \subseteq (\sigma, J' \bigvee J'')$.
Let (σ, J_1) be any other L-subslice of the L-slice (σ, J) such that
$(\sigma, J') \subseteq (\sigma, J_1)$ and $(\sigma, J'') \subseteq (\sigma, J_1)$.
For any $z \in (\sigma, J' \bigvee J'')$, there exist $x \in (\sigma, J')$ and $y \in (\sigma, J'')$ such that $z = x \vee y$.

Since $x, y \in (\sigma, J_1)$, $z = x \vee y \in (\sigma, J_1)$.

Hence $(\sigma, J' \bigvee J'')$ is the smallest L-subslice of (σ, J) containing both (σ, J') and (σ, J''). □

Proposition 4.2.5. *Let (σ, J') be an L-subslice of the L-slice (σ, J) for a locale L. For any $a \in L$, let $\sigma(a, J') = \{\sigma(a, x) : x \in J'\}$. Then $(\sigma, \sigma(a, J'))$ is an L-subslice of (σ, J).*

Proof. Since (σ, J') is an L-subslice of $(\sigma, J), \sigma(a, J') \subseteq J'$.

Let $\sigma(a, x), \sigma(a, y) \in \sigma(a, J')$ and $b \in L$. Then $x, y \in (\sigma, J')$. Since (σ, J') is an L-subslice of $(\sigma, J), x \vee y, \sigma(b, x) \in (\sigma, J')$.

Thus $\sigma(a, x) \vee \sigma(a, y) = \sigma(a, x \vee y) \in \sigma(a, J')$ and

$\sigma(b, \sigma(a, x)) = \sigma(a, \sigma(b, x)) \in \sigma(a, J')$.

Hence for any $a \in L, (\sigma, \sigma(a, J'))$ is an L-subslice of (σ, J). □

Lemma 4.2.6. *Let $(\sigma, J_1), (\sigma, J_2)$ be two L-subslices of the L-slice (σ, J) for a locale L. Then for any $a \in L$, $\sigma(a, J_1 \bigvee J_2) = \sigma(a, J_1) \bigvee \sigma(a, J_2)$.*

Proof. Let $x \in \sigma(a, J_1 \bigvee J_2)$. Then $x = \sigma(a, j_1 \vee j_2)$.

That is $x = \sigma(a, j_1) \vee \sigma(a, j_2) \in \sigma(a, J_1) \bigvee \sigma(a, J_2)$.

Hence $\sigma(a, J_1 \bigvee J_2) \subseteq \sigma(a, J_1) \bigvee \sigma(a, J_2)$.

$y \in \sigma(a, J_1) \bigvee \sigma(a, J_2)$ implies $y = \sigma(a, j_1) \vee \sigma(a, j_2)$.

Thus $y = \sigma(a, j_1 \vee j_2) \in \sigma(a, J_1 \bigvee J_2)$.

Hence $\sigma(a, J_1) \bigvee \sigma(a, J_2) \subseteq \sigma(a, J_1 \bigvee J_2)$. This completes the proof. □

4.3. Factor Slice

Let (σ, J') be an L-subslice of the L-slice (σ, J) for a locale L and let $x \in (\sigma, J')$. Define $x \vee J' = \{x \vee y; y \in (\sigma, J')\}$. We will study various propertis of $x \vee J'$ and will make use of them to define the factor slice.

Lemma 4.3.1. *Let (σ, J') be an L-subslice of the L-slice (σ, J) and let $x \in (\sigma, J)$. Then $x \vee J' \subseteq J'$ if and only if $x \in (\sigma, J')$.*

Proof. First let $x \vee J' \subseteq J'$. Since (σ, J') is an L-subslice of $(\sigma, J), 0 \in (\sigma, J')$. Hence $x = x \vee 0 \in x \vee J' \subseteq J'$. Thus $x \in (\sigma, J')$.
Conversely let $x \in (\sigma, J')$. Then for any $y \in (\sigma, J')$, since $x \in J'$, $x \vee y \in (\sigma, J')$. Hence $x \vee J' \subseteq J'$. □

Proposition 4.3.2. *Let $(\sigma, J_1), (\sigma, J_2)$ be two L-subslices of the L-slice (σ, J) such that $J_1 \subseteq J_2$ and let $x \in (\sigma, J)$. Then $x \vee J_1 \subseteq x \vee J_2$.*

Proof. Let $x \in (\sigma, J)$ and $J_1 \subseteq J_2$. Let $y \in x \vee J_1$, then $y = x \vee j$ for some $j \in J_1$. Since $J_1 \subseteq J_2$, $j \in J_2$ and so $y = x \vee j \in x \vee J_2$. Hence $x \vee J_1 \subseteq x \vee J_2$. □

Let (σ, J') be an L-subslice of the L-slice (σ, J) for a locale L. Consider the set $J/J' = \{x \vee J' : x \in (\sigma, J)\}$. We will prove that $(\delta, J/J')$ is an L-slice, where the action $\delta : L \times J/J' \to J/J'$ is defined by $\delta(a, x \vee J') = \sigma(a, x) \vee J'$.

Proposition 4.3.3. *Let (σ, J') be an L-subslice of the L-slice (σ, J). Then $(\delta, J/J')$ is an L-slice.*

Proof. Let $(x \vee J') \veebar (y \vee J') = (x \vee y) \vee J'$. Then $(J/J', \veebar, J')$ is a join semilattice with bottom element J'. We will show that δ is an action on J/J'.

i. $$\begin{aligned}\delta(a, (x \vee J') \veebar (y \vee J')) &= \delta(a, (x \vee y) \vee J') = \sigma(a, x \vee y) \vee J' \\ &= (\sigma(a, x) \vee \sigma(a, y)) \vee J' = (\sigma(a, x) \vee J') \veebar (\sigma(a, y) \vee J') \\ &= \delta(a, x \vee J') \veebar \delta(a, y \vee J')\end{aligned}$$

ii. $\delta(a, J') = \delta(a, 0 \vee J') = \sigma(a, 0_J) \vee J' = 0_J \vee J' = J'$

iii. $\delta(a \sqcap b, x \vee J') = \sigma(a \sqcap b, x) \vee J' = \sigma(a, \sigma(b, x)) \vee J'$

$$= \delta(a, \sigma(b, x) \vee J') = \delta(a, \delta(b, x \vee J'))$$

iv. $\delta(1_L, x \vee J') = \sigma(1_L, x) \vee J' = x \vee J'$

$\delta(0_L, x \vee J') = \sigma(0_L, x) \vee J' = 0_J \vee J' = J'$

v. $\delta(a \sqcup b, x \vee J') = \sigma(a \sqcup b, x) \vee J' = (\sigma(a, x) \vee \sigma(b, x)) \vee J'$

$$= (\sigma(a, x) \vee J') \veebar (\sigma(b, x) \vee J') = \delta(a, x \vee J') \veebar \delta(b, x \vee J')$$

Hence $(\delta, J/J')$ is an L-slice. □

Definition 4.3.4. The L-slice $(\delta, J/J')$ described in proposition 4.3.3 is called factor of L-slice (σ, J) with respect to the subslice (σ, J').

Proposition 4.3.5. *Let (σ, J) be an L-slice of a locale L and (σ, J') be L-subslice of (σ, J). Then the map $\phi : (\sigma, J) \to (\delta, J/J')$ defined by $\phi(x) = x \vee J'$ is an L-slice homomorphism.*

Proof. $\phi(x \vee y) = (x \vee y) \vee J' = (x \vee J') \veebar (y \vee J') = \phi(x) \veebar \phi(y)$ and
$\phi(\sigma(a, x)) = \sigma(a, x) \vee J' = \delta(a, x \vee J') = \delta(a, \phi(x))$.
Hence ϕ is an L-slice homomorphism. □

The L-slice homomorphism $\phi : J \to J/J'$ of proposition 4.3.5 is called canonical L-slice homomorphism from an L-slice to its factor slice.

4.4. L-slice congruence

In this section we define congruence R on L-slice for a locale L and discuss its various properties. For each congruence R on an L-slice (σ, J) of a locale, we prove that

$(\gamma, J/R)$ is an L-slice, where J/R denotes the collection of all equivalence classes with respect to the relation R and $\gamma : L \times J/R \to J/R$ is defined by $\gamma(a, [x]) = [\sigma(a, x)]$.

Definition 4.4.1. Let (σ, J) be an L-slice of a locale L. An equivalence relation R on (σ, J) is called an L-slice congruence if

i. xRy implies $x \vee zRy \vee z$ for any $x, y, z \in (\sigma, J)$

ii. xRy implies $\sigma(a, x)R\sigma(a, y)$ for all $a \in L, x, y \in (\sigma, J)$.

Proposition 4.4.2. *Let $(\sigma, J), (\mu, K)$ be two L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then the relation R on (σ, J) defined by xRy if and only if $f(x) = f(y)$ is a congruence on (σ, J).*

Proof. Clearly, the relation R is an equivalence relation on (σ, J). Let xRy and $z \in (\sigma, J)$. Then we have $f(x) = f(y)$.

$f(x \vee z) = f(x) \vee f(z) = f(y) \vee f(z) = f(y \vee z)$. Hence $x \vee zRy \vee z$.

For any $a \in L$, $f(\sigma(a, x)) = \mu(a, f(x)) = \mu(a, f(y)) = f(\sigma(a, y))$. So $\sigma(a, x)R\sigma(a, y)$.

Hence R is a congruence on (σ, J). □

Definition 4.4.3. The L-slice congruence R discussed in proposition 4.4.2 is called natural congruence associated with the L-slice homomorphism $f : (\sigma, J) \to (\mu, K)$.

Definition 4.4.4. Let R, R' be two L-slice congruences on an L-slice (σ, J) of a locale L. We say that the congruence R is weaker than the congruence R', or R' is stronger than R, if for any $x, y \in (\sigma, J)$, $xR'y$ whenever xRy and we write $R \subseteq R'$.

Two L-slice congruences R, R' on an L-slice (σ, J) are equivalent if $R \subseteq R'$ and $R' \subseteq R$.

Proposition 4.4.5. *Let $(\sigma, J), (\mu, K)$ be two L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then the relation R' on (σ, J)*

defined by $xR'y$ *if and only if* $\mu(a, f(x)) = \mu(a, f(y))$ *for some* $a \in L$ *is a congruence on* (σ, J) *and is stronger than the natural congruence* R *on* (σ, J).

Proof. The relation R' is clearly reflexive and symmetric.

For transitivity, let $xR'y$, $yR'z$. Then there exist $a, b \in L$ such that $\mu(a, f(x)) = \mu(a, f(y))$ and $\mu(b, f(y)) = \mu(b, f(z))$.

$$\begin{aligned} \mu(a \sqcap b, f(x)) &= \mu(b, \mu(a, f(x))) = \mu(b, \mu(a, f(y))) \\ &= \mu(a, \mu(b, f(y))) = \mu(a, \mu(b, f(z))) \\ &= \mu(a \sqcap b, f(z)) \end{aligned}$$

Hence $xR'z$. So the relation R' is an equivalence relation.

Let $xR'y$, then there exist $a \in L$ such that $\mu(a, f(x)) = \mu(a, f(y))$ and let $z \in (\sigma, J), b \in L$.

$$\begin{aligned} \mu(a, f(x \vee z)) &= \mu(a, f(x) \vee f(z)) = \mu(a, f(x)) \vee \mu(a, f(z)) \\ &= \mu(a, f(y)) \vee \mu(a, f(z)) = \mu(a, f(y) \vee f(z)) \\ &= \mu(a, f(y \vee z)) \end{aligned}$$

Hence $x \vee zR'y \vee z$.

$$\begin{aligned} \mu(a, f(\sigma(b, x))) &= \mu(a, \mu(b, f(x))) = \mu(b, \mu(a, f(x))) \\ &= \mu(b, \mu(a, f(y))) = \mu(a, \mu(b, f(y))) \\ &= \mu(a, f(\sigma(b, y))) \end{aligned}$$

So $\sigma(b, x)R'\sigma(b, y)$. Hence R' is a congruence on (σ, J).

Suppose $x, y \in (\sigma, J)$ such that xRy where R is the natural congruence induced by the L-slice homomorphism $f : (\sigma, J) \to (\mu, K)$.

Then $f(x) = f(y)$ or $\mu(1, f(x)) = \mu(1, f(y))$. Hence $xR'y$.

So R' is stronger than R. □

Proposition 4.4.6. *Let (σ, J) be an L-slice of a locale L and let $\{R_\alpha\}$ be an arbitrary collection of congruences on (σ, J). Then $\cap R_\alpha$ is a congruence on (σ, J).*

Proof. $x \cap R_\alpha y$ if and only if $xR_\alpha y$ for all α. Clearly $\cap R_\alpha$ is an equivalence relation. Let $x \cap R_\alpha y$ and let $z \in (\sigma, J)$, $a \in L$. $x \cap R_\alpha y$ implies $xR_\alpha y$ for all α. Since each R_α is a congruence on (σ, J), $x \vee zR_\alpha y \vee z$ and $\sigma(a, x)R_\alpha\sigma(a, y)$ for all α. So $x \vee z \cap R_\alpha y \vee z$ and $\sigma(a, x) \cap R_\alpha\sigma(a, y)$. Hence $\cap R_\alpha$ is a congruence on (σ, J). □

Proposition 4.4.7. *Let R be a congruence on the L-slice (σ, J) of a locale L. For each $a \in L$, the relation R_a defined by xR_ay if and only if $\sigma(a, x)R\sigma(a, y)$ is a congruence on the L-slice (σ, J) and it is stronger than the congruence R.*

Proof. For each $a \in L$, the relation R_a is an equivalence relation. Let xR_ay and let $z \in (\sigma, J), b \in L$. Since xR_ay, we have $\sigma(a, x)R\sigma(a, y)$.

Since R is a congruence on $(\sigma, J), \sigma(a, x) \vee \sigma(a, z)R\sigma(a, y) \vee \sigma(a, z)$ and $\sigma(b, \sigma(a, x))R\sigma(b, \sigma(a, y))$.

That is we have $\sigma(a, x \vee z)R\sigma(a, y \vee z)$ and $\sigma(a, \sigma(b, x))R\sigma(a, \sigma(b, y))$.

So $x \vee zR_ay \vee z$ and $\sigma(b, x)R_a\sigma(b, y)$. Hence R_a is a congruence on (σ, J).

Let xRy, then by definition of congruence, $\sigma(a, x)R\sigma(a, y)$ and so xR_ay. Hence the congruence R_a is stronger than the congruence R. □

Remark. The relation Φ on the L-slice (σ, J) defined by $x\Phi y$ for all $x, y \in (\sigma, J)$ is a congruence on the L-slice (σ, J).

Proposition 4.4.8. *Let R, R', Φ be congruences on an L-slice (σ, J) of a locale L and let $a, b \in L$.*

i. $(R \cap R')_a = R_a \cap R'_a$.

ii. $\Phi_a = \Phi$ *for all* $a \in L$.

iii. $R_{a \sqcap b} = (R_a)_b = (R_b)_a$.

iv. $R_1 = R$ *and* $R_0 = \Phi$.

Proof. Let R, R', Φ be congruences on an L-slice (σ, J) and let $a, b \in L$.

i. $x(R \cap R')_a y$ if and only if $\sigma(a,x) R \cap R' \sigma(a,y)$

if and only if $\sigma(a,x) R \sigma(a,y)$ and $\sigma(a,x) R' \sigma(a,y)$

if and only if $x R_a y$ and $x R'_a y$

if and only if $x R_a \cap R'_a y$.

Hence $(R \cap R')_a = R_a \cap R'_a$.

ii. $x \Phi_a y$ if and only if $\sigma(a,x) \Phi \sigma(a,y)$

if and only if $x \Phi y$.

Hence $\Phi_a = \Phi$ for all $a \in L$.

iii. $x R_{a \sqcap b} y$ if and only if $\sigma(a \sqcap b, x) R \sigma(a \sqcap b, y)$

if and only if $\sigma(a, \sigma(b,x)) R \sigma(a, \sigma(b,y))$

if and only if $\sigma(b,x) R_a \sigma(b,y)$

if and only if $x(R_a)_b y$

Hence $R_{a \sqcap b} = (R_a)_b = (R_b)_a$.

iv. $x R_1 y$ if and only if $\sigma(1,x) R \sigma(1,y)$

if and only if $x R y$.

Hence $R_1 = R$.

All elements of (σ, J) are related with the congruence R_0 since $\sigma(0, x)R\sigma(0, y)$ for all $x, y \in (\sigma, J)$. Hence $R_0 = \Phi$. □

Definition 4.4.9. A congruence R on an L-slice (σ, J) of a locale L with the property that $\sigma(a \sqcup b, x)R\sigma(a \sqcup b, y)$ if and only if $\sigma(a, x)R\sigma(a, y)$ and $\sigma(b, x)R\sigma(b, y)$ is called relative congruence.

Let $Con(J)$ denotes the collection of all relative congruences on the L-slice (σ, J) of a locale L. We will show that $Con(J)$ is an L-slice under the action $\psi : L \times Con(J) \to Con(J)$ defined by $\psi(a, R) = R_a$.

Proposition 4.4.10. *Let $Con(J)$ denotes the collection of all relative congruences on the L-slice (σ, J). Then $(\psi, Con(J))$ is an L-slice.*

Proof. Order $Con(J)$ by $R \leqslant R'$ if $R' \subseteq R$. Then $R \vee R' = R \cap R'$. Hence $Con(J)$ is a join semilattice with bottom element Φ.

Define $\psi : L \times Con(J) \to Con(J)$ by $\psi(a, R) = R_a$.

Then by proposition 4.4.8

i. $\psi(a, R \vee R') = \psi(a, R) \vee \psi(a, R')$

ii. $\psi(a, \Phi) = \Phi$

iii. $\psi(a \sqcap b, R) = \psi(a, \psi(b, R)) = \psi(b, \psi(a, R))$

iv. $\psi(1, R) = R, \psi(0, R) = \Phi$

Also since $Con(J)$ is a collection of relative congruences, we have

$R_{a \sqcup b} = R_a \cap R_b = R_a \vee R_b$. Hence

v. $\psi(a \sqcup b, R) = \psi(a, R) \vee \psi(b, R)$. Hence $(\psi, Con(J))$ is an L-slice. □

Let R be a congruence on (σ, J) and let J/R denotes the collection of all equivalence classes with respect to the relation R. Then J/R is a join semilattice with bottom element $[0_J]$, where the partial order $\leqslant$ on J/R is defined by $[x] \leqslant [y]$ if and only if $x \leqslant y$ in (σ, J). In the next proposition, we will show that $(\gamma, J/R)$ is an L-slice where the action $\gamma : L \times J/R \to J/R$ is defined by $\gamma(a, [x]) = [\sigma(a, x)]$.

Proposition 4.4.11. *If R is a congruence relation on (σ, J), then $(\gamma, J/R)$ is an L-slice.*

Proof. Clearly the mapping $\gamma : L \times J/R \to J/R$ defined by $\gamma(a, [x]) = [\sigma(a, x)]$ is well defined.

i. $\gamma(a, [x] \vee [y]) = \gamma(a, [x \vee y]) = [\sigma(a, x \vee y)] = [\sigma(a, x) \vee \sigma(a, y)]$
$= [\sigma(a, x)] \vee [\sigma(a, y)] = \gamma(a, [x]) \vee \gamma(a, [y])$.

ii. $\gamma(a, [0_J]) = [\sigma(a, 0_J)] = [0_J]$

iii. $\gamma(a \sqcap b, [x]) = [\sigma(a \sqcap b, x)] = [\sigma(a, \sigma(b, x))]$
$= \gamma(a, [\sigma(b, x)]) = \gamma(a, \gamma(b, [x]))$.

iv. $\gamma(1_L, [x]) = [\sigma(1_L, x)] = [x]$
$\gamma(0_L, [x]) = [\sigma(0_L, x)] = [0_J]$

v. $\gamma(a \sqcup b, [x]) = [\sigma(a \sqcup b, x)] = [\sigma(a, x) \vee \sigma(b, x)]$
$= [\sigma(a, x)] \vee [\sigma(b, x)] = \gamma(a, [x]) \vee \gamma(b, [x])$.

Hence γ is an action of L on J/R and $(\gamma, J/R)$ is an L-slice. $\square$

Definition 4.4.12. Let (σ, J) be an L-slice of a locale L and R be a congruence on (σ, J). Then the L-slice $(\gamma, J/R)$ described in proposition 4.4.11 is called quotient slice of L-slice (σ, J) with respect to the congruence R.

Proposition 4.4.13. *Let R be an L-slice congruence on an L-slice (σ, J) of a locale L and let $(\gamma, J/R)$ be the corresponding quotient slice. Then the map $\pi : (\sigma, J) \to (\gamma, J/R)$ defined by $\pi(x) = [x]$ is an onto L-slice homomorphism.*

Proof. For $x, y \in (\sigma, J), a \in L$, $\pi(x \vee y) = [x \vee y] = [x] \vee [y] = \pi(x) \vee \pi(y)$
$\pi(\sigma(a, x)) = [\sigma(a, x)] = \gamma(a, [x]) = \gamma(a, \pi(x))$.
Also for each $[x] \in (\gamma, J/R)$, there is an $x \in (\sigma, J)$ such that $\pi(x) = [x]$.
Thus $\pi : (\sigma, J) \to (\gamma, J/R)$ is an onto L-slice homomorphism. □

Definition 4.4.14. Let (σ, J) be an L-slice of a locale L. For each $a \in L$, the map $\sigma_a : (\sigma, J) \to (\sigma, J)$ defined by $\sigma_a(x) = \sigma(a, x)$ is an L-slice homomorphism.

Remark. More about L-slice homomorphism σ_a are studied in chapter 5.

Proposition 4.4.15. *Let R be a congruence on the L-slice (σ, J) of a locale L such that R and R_a are equivalent and let $(\gamma, J/R)$ be the quotient slice of (σ, J) with respect to R. Then the L-slice homomorphism $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ defined by $\gamma_a([x]) = \gamma(a, [x])$ is one-one.*

Proof. Let $[x], [y] \in J/R$ such that $\gamma_a([x]) = \gamma_a([y])$.
Then $[\sigma(a, x)] = [\sigma(a, y)]$. Hence $\sigma(a, x) R \sigma(a, y)$.
But since congruences R and R_a are equivalent, xRy and hence $[x] = [y]$.
Thus γ_a is one-one. □

Proposition 4.4.16. *Let R be a congruence on L-slice (σ, J) for a locale L such that R and R_a are equivalent and let $(\gamma, J/R)$ be the quotient slice of (σ, J) with respect to R. Then the natural congruence R' induced by L-slice homomorphism σ_a is weaker than the congruence R.*

Proof. Let R' be the natural congruence induced by the L-slice homomorphism σ_a and let $xR'y$. Then we have $\sigma_a(x) = \sigma_a(y)$.

Then $[\sigma(a, x)] = [\sigma(a, y)]$, where $[\ ]$is the equivalence class determined by the congruence R. Thus $\gamma_a[x] = \gamma_a[y]$. But since γ_a is one-one $[x] = [y]$. Hence xRy and so $R' \subseteq R$. □

Proposition 4.4.17. *Let R be a congruence on L-slice (σ, J) for a locale L and $(\gamma, J/R)$ be the corresponding quotient slice and let $a \in L$. If $\sigma_a : (\sigma, J) \to (\sigma, J)$ is onto, then $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto.*

Proof. Suppose $\sigma_a : (\sigma, J) \to (\sigma, J)$ is onto and let $[y] \in J/R$.

Then $y \in J$, and since σ_a is onto, there exist $x \in (\sigma, J)$ such that $\sigma_a(x) = y$.

Then $[\sigma_a(x)] = [\sigma(a, x)] = \gamma(a, [x]) = \gamma_a([x]) = [y]$.

Hence $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto. □

Proposition 4.4.18. *Let R be a congruence on L-slice (σ, J) for a locale L and $(\gamma, J/R)$ be the corresponding quotient slice and let $a \in L$. Then*

$\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto if and only if R has the property that for each $y \in (\sigma, J)$, there exist some $x \in (\sigma, J)$ such that $\sigma(a, x)Ry$.

Proof. First suppose $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto and let $y \in (\sigma, J)$.

Then $[y] \in (\gamma, J/R)$. Since $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto, there exist $[x] \in (\gamma, J/R)$ such that $\gamma_a([x]) = [y]$. That is $[\sigma(a, x)] = [y]$.

Hence $\sigma(a, x)Ry$.

For converse, let $[y] \in (\gamma, J/R)$. Then $y \in (\sigma, J)$ and by hypothesis there exist $x \in (\sigma, J)$ such that $\sigma(a, x)Ry$. Then $[\sigma(a, x)] = [y]$ or $\gamma_a([x]) = [y]$.

Hence $\gamma_a : (\gamma, J/R) \to (\gamma, J/R)$ is onto. □

4.5. Ideal

Definition 4.5.1. A subslice (σ, I) of an L-slice (σ, J) is said to be ideal of (σ, J) if $x \in (\sigma, I)$ and $y \in (\sigma, J)$ are such that $y \leqslant x$, then $y \in (\sigma, I)$.

Proposition 4.5.2. *Let (σ, J) be an L-slice of a locale L and $\{(\sigma, I_\alpha) : \alpha \in \Delta\}$ be a family of ideals of (σ, J) and let $I = \bigcap I_\alpha$. Then (σ, I) is an ideal of (σ, J).*

Proof. By Proposition 4.2.3, (σ, I) is a subslice of (σ, J). Let $x \in (\sigma, I)$ and let $y \in (\sigma, J)$ such that $y \leqslant x$. $x \in (\sigma, I)$ implies that $x \in (\sigma, I_\alpha)$ for all α. Since each (σ, I_α) is an ideal of (σ, J), $y \in (\sigma, I_\alpha)$ for all α. Hence $y \in (\sigma, I)$. Thus (σ, I) is an ideal of the L-slice (σ, J). □

Definition 4.5.3. An ideal (σ, I) of an L-slice (σ, J) is a prime ideal if it has the following properties:

i. If a and b are any two elements of L such that $\sigma(a \sqcap b, x) \in (\sigma, I)$, then either $\sigma(a, x) \in (\sigma, I)$ or $\sigma(b, x) \in (\sigma, I)$.

ii. (σ, I) is not equal to the whole slice (σ, J).

Definition 4.5.4. Let L be a locale and (σ, J) be an L-slice. An ideal (σ, I) of an L-slice (σ, J) is called minimal ideal if it properly contains no ideal other than the zero ideal $(\sigma, \{0\})$.

Proposition 4.5.5. *Any two distinct minimal ideal of an L-slice (σ, J) of a locale L are disjoint.*

Proof. Let $(\sigma, I), (\sigma, K)$ be any two distinct minimal ideals of the L-slice (σ, J). Since intersection of two ideals in (σ, J) is an ideal in (σ, J), $(\sigma, I \cap K)$ is an ideal in (σ, J) and $(\sigma, I \cap K) \subseteq (\sigma, I)$ and $(\sigma, I \cap K) \subseteq (\sigma, K)$. But since (σ, I) and (σ, K) are minimal, $(\sigma, I \cap K)$ is the zero ideal $(\sigma, \{0\})$. □

4.6. Annihilator

Proposition 4.6.1. *Let $(\sigma, J'), (\sigma, J'')$ be two L-subslices of the L-slice (σ, J) for a locale L. Then $\langle J', J''\rangle = \{a \in L : \sigma(a, J'') \subseteq (\sigma, J')\}$ is an ideal in L.*

Proof. Since $\sigma(0, J'') = \{0\} \subseteq J', 0 \in \langle J', J''\rangle$. Let $a, b \in \langle J', J''\rangle$.

Then $\sigma(a, J'') \subseteq (\sigma, J')$ and $\sigma(b, J'') \subseteq (\sigma, J')$.

Let $\sigma(a \vee b, x) \in \sigma(a \vee b, J'')$. Then $\sigma(a \vee b, x) = \sigma(a, x) \vee \sigma(b, x)$.

But $\sigma(a, x) \in \sigma(a, J'') \subseteq (\sigma, J')$ and $\sigma(b, x) \in \sigma(b, J'') \subseteq (\sigma, J')$.

Since (σ, J') is an L-subslice of $(\sigma, J), \sigma(a \vee b, x) = \sigma(a, x) \vee \sigma(b, x) \in (\sigma, J')$.

Thus $\sigma(a \vee b, J'') \subseteq (\sigma, J')$. Hence $a \vee b \in \langle J', J''\rangle$.

Let $a \in \langle J', J''\rangle$ and $b \in L$ such that $b \leqslant a$.

$\sigma(b, x) = \sigma(b \sqcap a, x) = \sigma(a, \sigma(b, x)) \in \sigma(a, J'') \subseteq (\sigma, J')$. Thus $\sigma(b, J'') \subseteq (\sigma, J')$ and so $b \in \langle J', J''\rangle$. Hence $\langle J', J''\rangle$ is an ideal in L. □

Definition 4.6.2. Let (σ, J) be an L-slice. The ideal $\langle 0, J\rangle$ of L is called the annihilator of the L-slice (σ, J) and is denoted by $Ann(J)$.

Proposition 4.6.3. *Let (σ, J) be an L-slice of a locale L. Then*

$Ann(J) = \{a \in L : \sigma_a = \boldsymbol{0}\}$.

Proof. Let (σ, J) be an L-slice of a locale L.

$a \in Ann(J)$ if and only if $\sigma(a, j) = 0$ for all $j \in (\sigma, J)$

if and only if $\sigma_a(j) = 0$ for all $j \in (\sigma, J)$

if and only if $\sigma_a = \mathbf{0}$.

Hence $Ann(J) = \{a \in L : \sigma_a = \mathbf{0}\}$. □

Proposition 4.6.4. *Let (σ, J) be an L-slice of a locale L. If the action $\sigma : L \times J \to J$ satisfies the property $\sigma(a, x) = 0$ implies $a = 0$ or $x = 0$, then $Ann(J)$ is a prime ideal.*

Proof. Let (σ, J) be an L-slice of a locale L and the action $\sigma : L \times J \to J$ satisfies the property $\sigma(a, x) = 0$ implies $a = 0$ or $x = 0$. By Proposition 4.6.1 $Ann(J)$ is an ideal in L.

Let $a \sqcap b \in Ann(J)$. If $a = 0$, then $a \in Ann(J)$ and so $Ann(J)$ is prime ideal.

Suppose $a \neq 0$. Then $a \sqcap b \in Ann(J)$ implies that $\sigma(a \sqcap b, J) = 0$.

That is $\sigma(a \sqcap b, j) = \sigma(a, \sigma(b, j)) = 0$ for all $j \in J$.

Since $a \neq 0$, by the property of σ, we have $\sigma(b, j) = 0$ for all $j \in J$. Hence $b \in Ann(J)$.

Thus $Ann(J)$ is a prime ideal in L. □

Definition 4.6.5. An L-slice (σ, J) of a locale L is said to be faithful if $Ann(J) = \{0\}$.

Example 4.6.6. *The L-slice $(\sqcap, L)$ is faithful.*

Proposition 4.6.7. *If $(\sigma, J_1), (\sigma, J_2)$ are two L-subslices of the L-slice (σ, J), then $Ann(J_1 \bigvee J_2) = Ann(J_1) \cap Ann(J_2)$.*

Proof. Let $a \in Ann(J_1 \bigvee J_2)$. Then we have $\sigma(a, J_1 \bigvee J_2) = \{0\}$.

Then by lemma 4.2.6, $\sigma(a, J_1) \bigvee \sigma(a, J_2) = \{0\}$. Then we must have $\sigma(a, J_1) = \{0\}$ and $\sigma(a, J_2) = \{0\}$. Thus $a \in Ann(J_1 \bigvee J_2)$ implies $a \in Ann(J_1) \cap Ann(J_2)$.

Hence $Ann(J_1 \bigvee J_2) \subseteq Ann(J_1) \cap Ann(J_2)$.

Let $b \in Ann(J_1) \cap Ann(J_2)$ implies $\sigma(b, J_1) = \{0\}$ and $\sigma(b, J_2) = \{0\}$.

Thus $\sigma(b, J_1) \bigvee \sigma(b, J_2) = \sigma(b, J_1 \bigvee J_2) = \{0\}$. This implies $b \in Ann(J_1) \bigvee Ann(J_2)$.

Hence $Ann(J_1 \bigvee J_2) = Ann(J_1) \cap Ann(J_2)$. □

4.7. Sublocale of a locale using its Slice

In this section we discuss a method of obtaining sublocale of a locale L from L-slice of its ideals. In 2.1.9, $M = \{I_a; a \in L\}$ is a complete lattice under the partial order $\supseteq$. Define $\sigma : L \times M \to M$ by $\sigma(a, I_b) = (I_b)_a$. In the next proposition, we will show that (σ, M) is an L-slice.

Lemma 4.7.1. *Let L be a locale and I be any ideal of L. For $a, b \in L$,* $(I_b)_a = (I_a)_b = I_{a \sqcap b}$.

Proof. Let $a, b \in L$

$$\begin{aligned} x \in (I_a)_b &\Leftrightarrow b \sqcap x \in I_a \\ &\Leftrightarrow a \sqcap (b \sqcap x) \in I \\ &\Leftrightarrow (a \sqcap b) \sqcap x \in I \\ &\Leftrightarrow x \in I_{a \sqcap b} \end{aligned}$$

Hence $(I_a)_b = I_{a \sqcap b}$. □

Proposition 4.7.2. *Let L be a locale and I be any ideal, which is closed under arbitrary join, of L. Then (σ, M) is an L-slice of L.*

Proof. $\sigma : L \times M \to M$ be defined by $\sigma(a, I_b) = (I_b)_a$.

i. $$\begin{aligned} \sigma(a \sqcup b, I_c) &= (I_c)_{a \sqcup b} = I_{c \sqcap (a \sqcup b)} = I_{(c \sqcap a) \sqcup (c \sqcap b)} \\ &= I_{c \sqcap a} \cap I_{c \sqcap b} = (I_c)_a \vee (I_c)_b \\ &= \sigma(a, I_c) \vee \sigma(b, I_c) \end{aligned}$$

ii. $$\begin{aligned} \sigma(a, I_b \vee I_c) &= \sigma(a, I_b \cap I_c) = \sigma(a, I_{b \sqcup c}) = (I_{b \sqcup c})_a = I_{a \sqcap (b \sqcup c)} \\ &= I_{(a \sqcap b) \sqcup (a \sqcap c)} = I_{a \sqcap b} \cap I_{a \sqcap c} = (I_b)_a \vee (I_c)_a \\ &= \sigma(a, I_b) \vee \sigma(a, I_c) \end{aligned}$$

iii. $\sigma(a, I_0) = (I_0)_a = I_{a \sqcap 0} = I_0$

iv. $\sigma(a \sqcap b, I_c) = (I_c)_{a \sqcap b} = I_{c \sqcap (a \sqcap b)} = (I_{c \sqcap a})_b = \sigma(b, I_{c \sqcap a})$

$$= \sigma(b, \sigma(a, I_c)) = \sigma(a, \sigma(b, I_c))$$

v. $\sigma(1, I_a) = (I_a)_1 = I_{a \sqcap 1} = I_a$

$\sigma(0, I_a) = (I_a)_0 = I_{a \sqcap 0} = I_0$

Hence (σ, M) is an L-slice. □

By 2.2.3, the sublocale L/R_I is determined by the congruence aRb if and only if $I_a = I_b$. But this is equivalent to the natural congruence associated with the L-slice homomorphism $\sigma_{I_1} = \sigma_I : (\sqcap, L) \to (\sigma, M)$. Hence the sublocale L/R_I can be represented as a quotient L-slice of $(\sqcap, L)$

Chapter 5

L-slice Homomorphisms and their properties

We have defined L-slice homomorphism between two L-slices of a locale L. It has been proved that the collection $(\delta, L-Hom(J,K))$ of all L-slice homomorphisms from (σ, J) to (μ, K) is an L-slice with respect to the action δ and that every L-slice (σ, J) is isomorphic to a subslice of $(\delta, L-Hom(L,J))$.

5.1. Properties of L-slice Homomorphism

Definition 5.1.1. Let (σ, J), (μ, K) be L-slices of a locale L. A map

$f : (\sigma, J) \to (\mu, K)$ is said to be L-slice homomorphism if

i. $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$ for all $x_1, x_2 \in (\sigma, J)$.

ii. $f(\sigma(a, x)) = \mu(a, f(x))$ for all $a \in L$ and all $x \in (\sigma, J)$.

Examples 5.1.2. *i. Let (σ, J) be an L-slice and (σ, J') be an L-subslice of (σ, J). Then the inclusion map $i : (\sigma, J') \to (\sigma, J)$ is an L-slice homomorphism.*

ii. Let $I = \downarrow(a), J = \downarrow(b)$ *be principal ideals of the locale* L*. Then* (σ, I)*,* (σ, J) *are L-slices. Then the map* $f : (\sigma, I) \to (\sigma, J)$ *defined by* $f(x) = x \sqcap b$ *is an L-slice homomorphism.*

Proposition 5.1.3. *If* $f : (\sigma, J) \to (\mu, K)$ *is a L-slice homomorphism, then* $f(0_J) = 0_K$*.*

Proof. Let (σ, J), (μ, K) be L-slices of a locale L and $f : (\sigma, J) \to (\mu, K)$ be a L-slice homomorphism.

$f(0_J) = f(\sigma(0, x)) = \mu(0, f(x)) = 0_K$. □

Proposition 5.1.4. *The composition of two L-slice homomorphisms is an L-slice homomorphism.*

Proof. Let $(\sigma, J_1), (\mu, J_2), (\delta, J_3)$ be L-slices of a locale L.

Let $f : (\sigma, J_1) \to (\mu, J_2)$ and $g : (\mu, J_2) \to (\delta, J_3)$ be L-slice homomorphisms and let $x_1, x_2 \in (\sigma, J_1)$.

$$\begin{aligned}(g \circ f)(x_1 \vee x_2) &= g(f(x_1 \vee x_2)) = g(f(x_1) \vee f(x_2)) \\ &= g(f(x_1)) \vee g(f(x_2)) = (g \circ f)(x_1) \vee (g \circ f)(x_2)\end{aligned}$$

Let $a \in L$ and $x \in (\sigma, J_1)$.

$$\begin{aligned}(g \circ f)(\sigma(a, x)) &= g(f(\sigma(a, x))) = g(\mu(a, f(x))) = \delta(a, g(f(x))) \\ &= \delta(a, (g \circ f)(x))\end{aligned}$$

Thus $g \circ f$ is an L-slice homomorphism. □

Proposition 5.1.5. *Let* (σ, J)*,*(μ, K) *be L-slices of a locale* L *and let* $f : (\sigma, J) \to (\mu, K)$ *be L-slice homomorphism. Let* (σ, J') *be an L-subslice of* (σ, J)

and (μ, K') *be an L-subslice of* (μ, K).

i. Let $f(J') = \{f(x); x \in (\sigma, J')\}$. *Then* $(\mu, f(J'))$ *is an L-subslice of* (μ, K).

ii. Let $f^{-1}(K') = \{x \in (\sigma, J) : f(x) \in (\mu, K')\}$. *Then* $(\sigma, f^{-1}(K'))$ *is an L-subslice of* (σ, J).

iii. For any $x \in (\sigma, J), f(\langle x \rangle) = \langle f(x) \rangle$.

Proof. i. Let $f(x), f(y) \in (\mu, f(J'))$. Then $x, y \in (\sigma, J')$. Since (σ, J') is an L-subslice of $(\sigma, J), x \vee y \in (\sigma, J')$. Hence $f(x) \vee f(y) = f(x \vee y) \in (\mu, f(J'))$.
Let $a \in L$ and $f(x) \in (\mu, f(J'))$. Then $\mu(a, f(x)) = f(\sigma(a, x)) \in (\mu, f(J'))$. Hence $(\mu, f(J'))$ is an L-subslice of (μ, K).

ii. Let $x, y \in (\sigma, f^{-1}(K'))$. Then $f(x), f(y) \in (\mu, K')$. Since (μ, K') is an L-subslice of (μ, K), $f(x \vee y) = f(x) \vee f(y) \in (\mu, K')$ and $f(\sigma(a, x)) = \mu(a, f(x)) \in (\mu, K')$ for $a \in L$. Hence $x \vee y, \sigma(a, x) \in (\sigma, f^{-1}(K'))$. Thus $(\sigma, f^{-1}(K'))$ is an L-subslice of (σ, J).

iii. $y \in f(\langle x \rangle)$ if and only if $y = f(\sigma(a, x)) = \mu(a, f(x))$
if and only if $y \in \langle f(x) \rangle$

Hence $f(\langle x \rangle) = \langle f(x) \rangle$. □

Proposition 5.1.6. *Let* (σ, J)*,*(μ, K) *be L-slices of a locale L and* $f : (\sigma, J) \to (\mu, K)$ *be L-slice homomorphism.*

i. Let $kerf = \{x \in J : f(x) = 0_K\}$. *Then* $(\sigma, kerf)$ *is an ideal of* (σ, J).

ii. Let $imf = \{y \in K : y = f(x) \text{ for some } x \in (\sigma, J)\}$. *Then* (μ, imf) *is an L-subslice of* (μ, K).

Proof. Let (σ, J), (μ, K) be L-slices of a locale L.

i. Let $x_1, x_2 \in kerf$. Then we have $f(x_1) = f(x_2) = 0_K$.

$f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = 0_K$.

Thus $x_1 \vee x_2 \in kerf$. Hence $kerf$ is subjoin semilattice of J with bottom element 0_J.

Let $x \in kerf$ and $a \in L$. We have to show that $\sigma(a, x) \in (\sigma, kerf)$.

We have $f(\sigma(a, x)) = \mu(a, f(x)) = \mu(a, 0_K) = 0_K$. Hence $\sigma(a, x) \in (\sigma, kerf)$.

Thus $(\sigma, kerf)$ is a L-subslice of (σ, J).

Let $x \in (\sigma, kerf)$ and $y \in (\sigma, J)$ such that $y \leqslant x$. Since f preserves join, we have $f(y) \leqslant f(x)$. Thus $f(y) \leqslant 0_K$. So $y \in (\sigma, kerf)$. Hence $(\sigma, kerf)$ is an ideal of (σ, J).

ii. Since $0_K = f(0_J), 0_K \in imf$.

Let $y_1, y_2 \in imf$. Then $f(x_1) = y_1, f(x_2) = y_2$ for some $x_1, x_2 \in (\sigma, J)$.

$y_1 \vee y_2 = f(x_1) \vee f(x_2) = f(x_1 \vee x_2)$. Hence $y_1 \vee y_2 \in imf$. Thus imf is a subjoin semilattice of K with bottom element 0_K.

Let $a \in L$ and $y = f(x) \in imf$.

$\mu(a, y) = \mu(a, f(x)) = f(\sigma(a, x)) \in (\mu, imf)$.

Thus (μ, imf) is an L-subslice of (μ, K). □

Proposition 5.1.7. *Let (σ, J)be an L-slice of a locale L, $f : (\sigma, J) \to (\sigma, J)$ be an L-slice homomorphism and $Fix_f = \{x \in (\sigma, J) : f(x) = x\}$. Then (σ, Fix_f) is an L-subslice of (σ, J)*

Proof. Since $f(0_J) = 0_J, 0_J \in Fix_f$. Thus Fix_f is nonempty.

Let $x_1, x_2 \in Fix_f$. Then $f(x_1) = x_1, f(x_2) = x_2$.

$f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = x_1 \vee x_2$.

Hence $x_1 \vee x_2 \in Fix_f$ and Fix_f is a subjoin semilattice of J with bottom element 0_J.

Let $a \in L, x \in (\sigma, Fix_f)$. Now $f(\sigma(a, x)) = \sigma(a, f(x)) = \sigma(a, x)$.

Thus (σ, Fix_f) is an L-subslice of (σ, J). □

Proposition 5.1.8. *Let (σ, J), (μ, K) be L-slices of a locale L and let*

$f : (\sigma, J) \to (\mu, K)$ be L-slice homomorphism. If (μ, I) be an ideal of (μ, K), then $(\sigma, f^{-1}(I))$ is an ideal of (σ, J). In particular if (μ, I) is prime ideal, then $(\sigma, f^{-1}(I))$ is a prime ideal of (σ, J).

Proof. Let $x, y \in (\sigma, f^{-1}(I))$. Then $f(x), f(y) \in (\mu, I)$.
Since (μ, I) is an ideal, $f(x \vee y) = f(x) \vee f(y) \in (\mu, I)$. Thus $x \vee y \in f^{-1}(I)$. Hence $f^{-1}(I)$ is a subjoin semilattice of J.
Also for each $x \in (\sigma, f^{-1}(I))$ and $a \in L$, $f(\sigma(a, x)) = \mu(a, f(x)) \in (\mu, I)$.
Thus $\sigma(a, x) \in (\sigma, f^{-1}(I))$. Hence $(\sigma, f^{-1}(I))$ is a L-subslice of (σ, J).
Let $x \in (\sigma, f^{-1}(I))$ and $y \in (\sigma, J)$ such that $y \leqslant x$.
Since f preserves join, f preserves order. Hence $f(y) \leqslant f(x)$. Since $f(x) \in (\mu, I)$ and (μ, I) is an ideal of (μ, K), $f(y) \in (\mu, I)$. Hence $y \in (\sigma, f^{-1}(I))$. Thus $(\sigma, f^{-1}(I))$ is an ideal of (σ, J).
Now let (μ, I) be prime ideal of (μ, K).
Suppose $\sigma(a \sqcap b, x) \in (\sigma, f^{-1}(I))$, then $f(\sigma(a \sqcap b, x)) = \mu(a \sqcap b, f(x)) \in (\mu, I)$.
Since (μ, I) is prime, either $f(\sigma(a, x)) = \mu(a, f(x)) \in (\mu, I)$ or
$f(\sigma(b, x)) = \mu(b, f(x)) \in (\mu, I)$. So either $\sigma(a, x) \in (\sigma, f^{-1}(I))$ or $\sigma(b, x) \in (\sigma, f^{-1}(I))$.
Hence $(\sigma, f^{-1}(I))$ is a prime ideal of (σ, J). □

Proposition 5.1.9. *Let (σ, J), (μ, K) be L-slices and let $f : (\sigma, J) \to (\mu, K)$ be a bijective L-slice homomorphism. Then the map $f^{-1} : (\mu, K) \to (\sigma, J)$ is an L-slice homomorphism.*

Proof. Since inverse of a lattice homomorphism is a lattice homomorphism,
$f^{-1} : (\mu, K) \to (\sigma, J)$ preserves finite join. Let $y \in (\mu, K)$ and $a \in L$.
Then $y = f(x)$ for some $x \in (\sigma, J)$.
$f^{-1}(\mu(a, y)) = f^{-1}(\mu(a, f(x))) = f^{-1}(f(\sigma(a, x))) = \sigma(a, x) = \sigma(a, f^{-1}(y))$.

Hence the map $f^{-1} : (\mu, K) \to (\sigma, J)$ is an L-slice homomorphism. $\square$

Definition 5.1.10. Let (σ, J), (μ, K) be L-slices of a locale L. A map $f : (\sigma, J) \to (\mu, K)$ is said to be an L-slice isomorphism if

i. f is one-one

ii. f is onto

iii. f is an L-slice homomorphism.

Lemma 5.1.11. *Let (σ, J), (μ, K) be two L-slices of a locale L.*

i. The map $\mathbf{0}$:$(\sigma, J) \to (\mu, K)$ defined by $\mathbf{0}(x)$=0_K for $x \in (\sigma, J)$ is an L-slice homomorphism.

ii. If $f, g : (\sigma, J) \to (\mu, K)$ are L-slice homomorphism, then the map $f \vee g : (\sigma, J) \to (\mu, K)$ defined by $(f \vee g)(x) = f(x) \vee g(x)$ for $x \in (\sigma, J)$ is an L-slice homomorphism.

Proof. Let $x, y \in (\sigma, J)$ and $a \in L$.

i. $\mathbf{0}(x \vee y)$=0_K =$\mathbf{0}(x)\vee\mathbf{0}(y)$.

$\mathbf{0}(\sigma(a, x))$=0_K = $\mu(a, 0_K) = \mu(a, \mathbf{0}(x))$.

Thus $\mathbf{0}$ is an L-slice homomorphism.

ii. Let the map $f \vee g : (\sigma, J) \to (\mu, K)$ defined by $(f \vee g)(x) = f(x) \vee g(x)$.

$$\begin{aligned}
(f \vee g)(x \vee y) &= f(x \vee y) \vee g(x \vee y) = f(x) \vee f(y) \vee g(x) \vee g(y) \\
&= (f \vee g)(x) \vee (f \vee g)(y) \\
(f \vee g)(\sigma(a, x)) &= f(\sigma(a, x)) \vee g(\sigma(a, x)) = \mu(a, f(x)) \vee \mu(a, g(x)) \\
&= \mu(a, f(x) \vee g(x)) = \mu(a, (f \vee g)(x))
\end{aligned}$$

Hence $(f \vee g)$ is an L-slice homomorphism. $\square$

Proposition 5.1.12. *Let* (σ, J)*,*(μ, K) *be L-slices of a locale L and L- Hom(J,K) denote the collection of all L-slice homomorphisms from* (σ, J) *to* (μ, K)*. Then* $(\delta, L - Hom(J, K))$ *is an L-slice, where the action,*
$\delta : L \times L - Hom(J, K) \to L - Hom(J, K)$ *is defined by* $\delta(a, f)(x) = \mu(a, f(x))$ *for all* $x \in (\sigma, J)$*.*

Proof. The collection $L - Hom(J, K)$ is a poset under the partial order relation $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in (\sigma, J)$. With respect to this partial order, the map $(f \vee g) : (\sigma, J) \to (\mu, K)$defined by $(f \vee g)(x) = f(x) \vee g(x)$ for $x \in (\sigma, J)$ is the join for $f, g \in L - Hom(J, K)$. By lemma 5.1.11, $f \vee g \in L - Hom(J, K)$. Thus $L - Hom(J, K)$ is a join semilattice with bottom element $\mathbf{0}$.
Define a map $\delta : L \times L - Hom(J, K) \to L - Hom(J, K)$ as follows.
For each $a \in L$ and $f \in L - Hom(J, K)$ define $\delta(a, f) : (\sigma, J) \to (\mu, K)$ by
$\delta(a, f)(x) = \mu(a, f(x))$.

$$\begin{aligned} \delta(a, f)(x_1 \vee x_2) &= \mu(a, f(x_1 \vee x_2)) = \mu(a, f(x_1) \vee f(x_2)) \\ &= \mu(a, f(x_1)) \vee \mu(a, f(x_2)) = \delta(a, f)(x_1) \vee \delta(a, f)(x_2) \\ \delta(a, f)(\sigma(a, x)) &= \mu(a, f(\sigma(a, x))) = \mu(a, \mu(a, f(x))) = \mu(a, \delta(a, f)(x)) \end{aligned}$$

Hence $\delta(a, f)$ is a L-slice homomorphism and hence $\delta(a, f) \in L - Hom(J, K)$.
Also δ satisfies the following properties.

i. $$\begin{aligned} \delta(a, f_1 \vee f_2)(x) &= \mu(a, (f_1 \vee f_2)(x)) = \mu(a, f_1(x) \vee f_2(x)) \\ &= \mu(a, f_1(x)) \vee \mu(a, f_2(x)) = \delta(a, f_1)(x) \vee \delta(a, f_2)(x) \\ &= (\delta(a, f_1) \vee \delta(a, f_2))(x) \end{aligned}$$

That is $\delta(a, f_1 \vee f_2) = \delta(a, f_1) \vee \delta(a, f_2)$.

ii. $\delta(a, \mathbf{0})(x) = \mu(a, \mathbf{0}(x)) = \mu(a, 0_K) = 0_K = \mathbf{0}(x)$

That is $\delta(a, \mathbf{0}) = \mathbf{0}$.

iii. $\delta(a \sqcap b, f)(x) = \mu(a \sqcap b, f(x)) = \mu(a, \mu(b, f(x)))$

$$= \mu(a, \delta(b, f)(x)) = \delta(a, \delta(b, f))(x)$$

That is $\delta(a \sqcap b, f) = \delta(a, \delta(b, f))$.

iv. $\delta(1, f)(x) = \mu(1, f(x)) = f(x)$

Thus $\delta(1, f) = f$ and

$\delta(0_L, f)(x) = \mu(0_L, f(x)) = 0_K = \mathbf{0}(x)$

Thus $\delta(0_L, f) = \mathbf{0}$.

v. $\delta(a_1 \vee a_2, f)(x) = \mu(a_1 \vee a_2, f(x)) = \mu(a_1, f(x)) \vee \mu(a_2, f(x))$

$$= (\delta(a_1, f) \vee \delta(a_2, f))(x)$$

Thus δ is an action of the locale L on $L - Hom(J, K)$ and$(\delta, L - Hom(J, K))$ is an L-slice. □

Proposition 5.1.13. *i. Any L-slice homomorphism $v : (\sigma_1, J) \to (\sigma_2, K)$ induces an L-slice homomorphism $v' : (\delta_1, L - Hom(K, M)) \to (\delta_2, L - Hom(J, M))$ for any L-slice(σ_3, M).*
ii. Any L-slice homomorphism $u : (\sigma_1, J) \to (\sigma_2, K)$ induces an L-slice homomorphism $u' : (\mu_1, L - Hom(M, J)) \to (\mu_2, L - Hom(M, K))$ for any L-slice (σ_3, M).

Proof. Let $(\sigma_1, J), (\sigma_2, K), (\sigma_3, M)$ be L-slices.
i. Let $(\delta_1, L{-}Hom(K, M)), (\delta_2, L{-}Hom(J, M))$ be L-slices of L-slice homomorphisms and $v : (\sigma_1, J) \to (\sigma_2, K)$ be an L-slice homomorphism.

Define $v' : (\delta_1, L - Hom(K,M)) \to (\delta_2, L - Hom(J,M))$ by $v'(f) = f \circ v$ for all $f \in L - Hom(K,M)$. Let $f_1, f_2 \in L - Hom(K,M), x \in (\sigma_2, K), a \in L$.

$$\begin{aligned}
v'(f_1 \vee f_2)(x) &= ((f_1 \vee f_2) \circ v)(x) = (f_1 \vee f_2)(v(x)) \\
&= f_1(v(x)) \vee f_2(v(x)) = ((f_1 \circ v) \vee (f_2 \circ v))(x) \\
&= (v'(f_1) \vee v'(f_2))(x) \\
v'(f_1 \vee f_2) &= v'(f_1) \vee v'(f_2) \\
v'(\delta_1(a,f))(x) &= (\delta_1(a,f) \circ v)(x) = (\delta_1(a,f))(v(x)) \\
&= \sigma_3(a, f(v(x))) = \sigma_3(a, v'(f)(x)) \\
&= \delta_2(a, v'(f))(x) \\
v'(\delta_1(a,f)) &= \delta_2(a, v'(f))
\end{aligned}$$

Hence v' is a L-slice homomorphism from $(\delta_1, L - Hom(K,M)))$ to $(\delta_2, L - Hom(J,M))$.

ii. Let $(\mu_1, L - Hom(M,J)), (\mu_2, L - Hom(M,K))$ be L-slices of L-homomorphisms and $v : (\sigma_1, J) \to (\sigma_2, K)$ be an L-slice homomorphism.

Define $u' : (\mu_1, L - Hom(M,J)) \to (\mu_2, L - Hom(M,K))$ by $u'(g) = u \circ g$.

$$\begin{aligned}
u'(g_1 \vee g_2)(x) &= (u \circ (g_1 \vee g_2))(x) = u((g_1 \vee g_2)(x)) \\
&= u(g_1(x) \vee g_2(x)) = (u'(g_1) \vee u'(g_2))(x) \\
u'(g_1 \vee g_2) &= u'(g_1) \vee u'(g_2) \\
u'(\mu_1(a,f))(x) &= (u \circ \mu_1(a,f))(x) = u((\mu_1(a,f))(x)) \\
&= v(\sigma_1(a, f(x))) = \sigma_2(a, u(f(x))) \\
&= \sigma_2(a, u'(f))(x) = \mu_2(a, u'(f))(x) \\
u'(\mu_1(a,f)) &= \mu_2(a, u'(f))
\end{aligned}$$

Hence u' is a L-slice homomorphism from $(\mu_1, L{-}Hom(M,J))$ to $(\mu_2, L{-}Hom(M,K))$.

□

Proposition 5.1.14. *Let $(\sigma, J), (\mu, K), (\delta, M)$ be L-slices of a locale L.*
$u; (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism and
$u' : (\mu_1, L - Hom(M,J)) \to (\mu_2, L - Hom(M,K))$,
$v' : (\delta_1, L - Hom(K,M)) \to (\delta_2, L - Hom(J,M))$ be the induced L-slice homomorphism. If u is one-one, then u' is also one-one. Conversely if u' is one-one, then u is a monomorphism.

Proof. Assume u is one-one. Let $g, h \in L{-}Hom(M,J)$ such that $u'(g) = u'(h)$. Then we have $u \circ g = u \circ h$. That is $u(g(x)) = u(h(x))$ for all $x \in (\sigma, J)$. Since u is one-one, we get $g(x) = h(x)$ for all $x \in (\sigma, J)$. Hence $g = h$. Thus u' is one-one.

Conversely assume u' is one-one. Then we have $u'(g) = u'(h)$ implies $g = h$. That is $u \circ g = u \circ h$ implies $g = h$. Hence u is a monomorphism. □

Proposition 5.1.15. *Every L-slice (σ, J) is isomorphic to a subslice of L-Hom(L,J).*

Proof. Let (σ, J) be an L-slice and let $(\delta, L - Hom(L,J))$ be the L-slice of L-slice homomorphisms from $(\sqcap, L)$ to (σ, J).
Define a mapping $\psi : (\sigma, J) \to (\delta, L - Hom(L,J))$ as follows.
For each $x \in (\sigma, J)$, let $\psi(x) : (\sqcap, L) \to (\sigma, J)$ be defined by $\psi(x)(a) = \sigma(a, x)$, for all $a \in L$.

$$\begin{aligned}
\psi(x)(a_1 \sqcup a_2) &= \sigma(a_1 \sqcup a_2, x) = \sigma(a_1, x) \vee \sigma(a_2, x) \\
&= \psi(x)(a_1) \vee \psi(x)(a_2) \\
\psi(x)(a \sqcap b) &= \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) \\
&= \sigma(a, \psi(x)(b))
\end{aligned}$$

Thus $\psi(x)$ is an L-slice homomorphism from $(\sqcap, L)$ to (σ, J).

Hence $\psi(x) \in L - Hom(L, J)$.

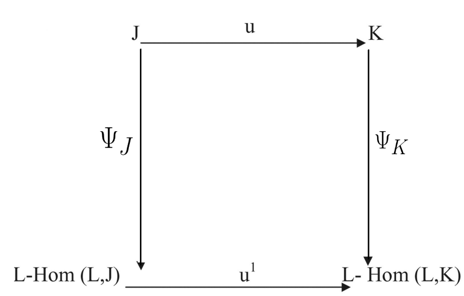
$$\begin{aligned}
\psi(x_1 \vee x_2)(a) &= \sigma(a, x_1 \vee x_2) = \sigma(a, x_1) \vee \sigma(a, x_2) \\
&= \psi(x_1)(a) \vee \psi(x_2)(a) = (\psi(x_1) \vee \psi(x_2))(a) \\
\psi(\sigma(a, x))(b) &= \sigma(b, \sigma(a, x)) = \sigma(a, \sigma(b, x)) \\
&= \sigma(a, \psi(x)(b)) = \delta(a, \psi(x))(b)
\end{aligned}$$

Hence ψ is an L-slice homomorphism.

Also $\psi(x) = \psi(y)$ implies that $\psi(x)(a) = \psi(y)(a)$ for all $a \in L$.

That is $\sigma(a, x) = \sigma(a, y)$ for all $a \in L$. In particular $\sigma(1, x) = \sigma(1, y)$ which implies $x = y$. Thus ψ is a one-one L-slice homomorphism from (σ, J) onto $(\delta, im\psi)$. Since $(\delta, im\psi)$ is a subslice of $(\delta, L - Hom(L, J))$, (σ, J) is isomorphic to a subslice of $(\delta, L - Hom(L, J))$. □

Proposition 5.1.16. *Let $(\sigma, J), (\mu, K)$ be L-slices. $u; (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism and $u' : (\mu_1, L - Hom(L, J)) \to (\mu_2, L - Hom(L, K))$ be the induced L-slice homomorphism. ψ_J, ψ_K be the L-slice homomorphisms from $(\sigma, J), (\mu, K)$ to $(\mu_1, L - Hom(L, J)), (\mu_2, L - Hom(L, K))$. Then the following rectangle commutes.*



Proof. For $x \in (\sigma, J), a \in L$,

$$\begin{aligned}
(\psi_K \circ u)(x)(a) &= (\psi_K(u(x)))(a) = \mu(a, u(x)) \\
((u' \circ \psi_J)(x))(a) &= (u'(\psi_J)(x))(a) = (u \circ \psi_J(x))(a) \\
&= u(\psi(x)(a)) = u(\sigma(a, x)) = \mu(a, u(x))
\end{aligned}$$

Hence $\psi_K \circ u = u' \circ \psi_J$ □

Proposition 5.1.17. ***L-slice Isomorphism theorem*** *Let $(\sigma, J), (\mu, K)$ be two L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be a L-slice homomorphism. Let R be the natural congruence associated with the L-slice homomorphism f. Then the quotient slice $(\gamma, J/R)$ of (σ, J) is isomorphic to the subslice (μ, imf) of the L-slice (μ, K).*

Proof. Let $(\sigma, J), (\mu, K)$ be two L-slices of a locale L and let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Define $\psi : (\gamma, J/R) \to (\mu, imf)$ by $\psi([x]) = f(x)$.

$$\begin{aligned}
\psi([x] \vee [y]) &= \psi([x \vee y]) = f(x \vee y) = f(x) \vee f(y) = \psi([x]) \vee \psi([y]) \\
\psi(\gamma(a, [x])) &= \psi([\sigma(a, x)]) = f(\sigma(a, x)) = \mu(a, f(x)) = \mu(a, \psi([x]))
\end{aligned}$$

Hence ψ is an L-slice homomorphism. Clearly ψ is one-one and onto.
Hence $\psi : (\gamma, J/R) \to (\mu, imf)$ is an L-slice isomorphism. □

5.2. Finitely Generated L-slice

The notion of finitely generated L-slice of a locale L is introduced and we have shown that every finitely generated L-slice (σ, J) of a locale L with n generators is isomorphic to the quotient slice of the L-slice $(\sqcap, L^n)$.

Definition 5.2.1. Let (σ, J) be an L-slice of a locale L. A subset S of (σ, J) is said to be span of the set $\{x_1, x_2, ...x_n\} \subseteq (\sigma, J)$ if each $x \in S$ can be written as $x = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$, where $a_i \in L$.

Proposition 5.2.2. *Let (σ, J) be an L-slice of a locale L and $\{x_1, x_2, ...x_n\} \subseteq (\sigma, J)$. Let $S = Span(\{x_1, x_2, ...x_n\})$. Then (σ, S) is a subslice of (σ, J).*

Proof. Let $x, y \in S$. Then there is $a_1, a_2, ..., a_n, b_1, b_2, ...b_n \in L$ such that

$$x = \bigvee_{i=1}^{n} \sigma(a_i, x_i), \quad y = \bigvee_{i=1}^{n} \sigma(b_i, x_i).$$

$$x \vee y = \bigvee_{i=1}^{n} \sigma(a_i, x_i) \vee \bigvee_{i=1}^{n} \sigma(b_i, x_i) = \bigvee_{i=1}^{n} (\sigma(a_i, x_i) \vee \sigma(b_i, x_i))$$

$$= \bigvee_{i=1}^{n} \sigma(a_i \sqcup b_i, x_i) \in S.$$ Therefore S is a subjoin semilattice of (σ, J).

Let $a \in L$. Then $$\sigma(a, x) = \sigma(a, \bigvee_{i=1}^{n} \sigma(a_i, x_i)) = \bigvee_{i=1}^{n} \sigma(a, \sigma(a_i, x_i))$$

$$= \bigvee_{i=1}^{n} \sigma(a \sqcap a_i, x_i) \in S.$$

Hence (σ, S) is a subslice of (σ, J). □

Definition 5.2.3. An L-slice (σ, J) of a locale L is said to be finitely generated if there is a finite subset $S \subseteq (\sigma, J)$ such that $(\sigma, J) = Span(S)$. Elements of S are called generators of the L-slice (σ, J).

An L-slice (σ, J) of a locale L is said to be generated by n elements if there is a finite subset $S \subseteq (\sigma, J)$ having n elements such that $(\sigma, J) = Span(S)$ and there is no subset $T \subseteq (\sigma, J)$ having less than n elements which spans the L-slice (σ, J).

Example 5.2.4. *If L is a locale, then $(\sqcap, L)$ is a finitely generated L-slice.*

Definition 5.2.5. An L-slice (σ, J) with a single generator x is called cyclic L-slice. (σ, J) is a cyclic L-slice if $(\sigma, \langle x \rangle) = (\sigma, J)$.

Proposition 5.2.6. *Let (σ, J) be an L-slice of a locale L and let S be a finite subset of (σ, J) such that $Span(S) = (\sigma, J)$. Then $Span(T) = (\sigma, J)$ for all subset T of (σ, J) such that $S \subseteq T$.*

Proof. Let $S = \{x_1, x_2, ...x_n\}$ be such that $Span(S) = (\sigma, J)$.
Then for any $x \in (\sigma, J)$, $x = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$.
If $z_i \in T$,then $x = \bigvee \sigma(b_i, z_i)$, where $b_i = a_i$ if $z_i \in S$ and $b_i = 0_L$ if $z_i \in T - S$. Hence $Span(T) = (\sigma, J)$. □

Proposition 5.2.7. *Let (σ, J) and (μ, J') be L-slices of a locale L, and let (σ, J) be finitely generated with generators $\{x_1, x_2, ...x_n\}$. If $f : (\sigma, J) \to (\mu, J')$ is an onto L-slice homomorphism, then (μ, J') is finitely generated.*

Proof. Let $y \in (\mu, J')$. There exist $x \in (\sigma, J)$ such that $y = f(x)$.
Since (σ, J) is finitely generated, there is $a_1, a_2, ...a_n \in L$ such that $x = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$.

$$\begin{aligned} y &= f(\bigvee_{i=1}^{n} \sigma(a_i, x_i)) \\ &= \bigvee_{i=1}^{n} f(\sigma(a_i, x_i)) \\ &= \bigvee_{i=1}^{n} \mu(a_i, f(x_i)) \end{aligned}$$

Therefore $\{f(x_1), f(x_2), ..., f(x_n)\}$ generates (μ, J'). □

Proposition 5.2.8. *Let (σ, J) be a finitely generated L-slice of a locale L with generators $\{x_1, x_2, ...x_n\}$. Then $\phi : (\sqcap, L^n) \to (\sigma, J)$ defined by $\phi(a_1, a_2, ...a_n) = \bigvee_{j=1}^{n} \sigma(a_j, x_j)$ is an onto L-slice homomorphism.*

Proof. By Proposition 4.1.4, $(\sqcap, L^n)$ is an L-slice of a locale L.

$$\begin{aligned}
\phi(\bigvee_{i=1}^{n}(a_{1i}, a_{2i}, ...a_{ni})) &= \phi(\bigvee_{i=1}^{n} a_{1i}, \bigvee_{i=1}^{n} a_{2i}, ... \bigvee_{i=1}^{n} a_{ni}) \\
&= \bigvee_{j=1}^{n} \sigma(\bigvee_{i=1}^{n} a_{ji}, x_j) \\
&= \bigvee_{j=1}^{n}\bigvee_{i=1}^{n} \sigma(a_{ji}, x_j) \\
&= \bigvee_{i=1}^{n}(\bigvee_{j=1}^{n} \sigma(a_{ji}, x_j) \\
&= \bigvee_{i=1}^{n}(\phi(a_{1i}, a_{2i}, ...a_{ni})
\end{aligned}$$

Thus ϕ preserves join.

$$\begin{aligned}
\phi((a \sqcap (a_1, a_2, ...a_n))) &= \phi(a \sqcap a_1, a \sqcap a_2, ...a \sqcap a_n) \\
&= \bigvee_{i=1}^{n} \sigma(a \sqcap a_i, x_i) \\
&= \bigvee_{i=1}^{n} \sigma(a, \sigma(a_i, x_i)) \\
&= \sigma(a, \bigvee_{i=1}^{n} \sigma(a_i, x_i)) \\
&= \sigma(a, \phi(a_1, a_2, ...a_n))
\end{aligned}$$

Hence ϕ is an L-slice homomorphism.
Let $y \in (\sigma, J)$. Then $y = \bigvee_{i=1}^{n} \sigma(a_i, x_i)$.
So $(a_1, a_2, ...a_n) \in (\sqcap, L^n)$ such that $\phi((a_1, a_2, ...a_n)) = y$. Hence ϕ is onto. □

Corollary 5.2.9. *Let (σ, J) be a finitely generated L-slice of a locale L with generators $\{x_1, x_2, ...x_n\}$. Then (σ, J) is isomorphic to the quotient L-slice $(\sqcap, L^n/R)$ of the product L-slice $(\sqcap, L^n)$.*

Proof. By proposition 5.2.8, $\phi : (\sqcap, L^n) \to (\sigma, J)$ defined by $\phi(a_1, a_2, ...a_n) = \bigvee \sigma(a_i, x_i)$ is an onto L-slice homomorphism.
Let R be the congruence xRy if and only if $\phi(x) = \phi(y)$. Then by isomorphism theorem for L-slices $im\phi = (\sigma, J)$ is isomorphic to the quotient L-slice $(\gamma, L^n/R)$. □

5.3. Properties of L-slice homomorphism σ_a

Definition 5.3.1. Let (σ, J) be an L-slice of a locale L. For each $a \in L$, define $\sigma_a : (\sigma, J) \to (\sigma, J)$ by $\sigma_a(x) = \sigma(a, x)$.

Proposition 5.3.2. *Let (σ, J) be an L-slice. For each $a \in L$, $\sigma_a : (\sigma, J) \to (\sigma, J)$ is an L-slice homomorphism.*

Proof. Let $x, y \in (\sigma, J), b \in L$.

$$\begin{aligned} \sigma_a(x \vee y) &= \sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) = \sigma_a(x) \vee \sigma_a(y) \\ \sigma_a(\sigma(b, x)) &= \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, \sigma_a(x)) \end{aligned}$$

Hence σ_a is an L-slice homomorphism. □

Proposition 5.3.3. *Let (σ, J) be an L-slice and $a \in L$*

i. $\sigma_a(x) \leqslant x$ for all $x \in (\sigma, J)$.

ii. If I is an ideal in (σ, J), then $\sigma_a(I) \subseteq I$.

Proof. i. $x = \sigma(1, x) = \sigma(a \sqcup 1, x) = \sigma(a, x) \vee \sigma(1, x) = \sigma_a(x) \vee x$.
Thus $\sigma_a(x) \leqslant x$ for all $x \in (\sigma, J)$.
ii. Let I be any ideal in (σ, J). For each $x \in I$, since $\sigma_a(x) \leqslant x$, $\sigma_a(x) \in I$. Hence $\sigma_a(I) \subseteq I$. □

Proposition 5.3.4. *Let (σ, J) be an L-slice of a locale with top element 1 and bottom element 0 and $a, b \in L$*

i. σ_0 is the zero map and σ_1 is the identity map on (σ, J).

ii. $\sigma_{a \sqcup b} = \sigma_a \vee \sigma_b$ and $\sigma_{a \sqcap b} = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$.

Proof. i. $\sigma_0(x) = \sigma(0, x) = 0_J$ for all $x \in J$. Hence σ_0 is the zero map on (σ, J).

$\sigma_1(x) = \sigma(1, x) = x$ for all $x \in J$. Hence σ_1 is the identity map on (σ, J).

ii. $\sigma_{a \sqcup b}(x) = \sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x) = \sigma_a(x) \vee \sigma_b(x) = (\sigma_a \vee \sigma_b)(x)$ for all $x \in (\sigma, J)$.

Hence $\sigma_{a \sqcup b} = \sigma_a \vee \sigma_b$.

$$\sigma_{a \sqcap b}(x) = \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, \sigma_b(x)) = \sigma_a(\sigma_b(x))$$
$$= (\sigma_a \circ \sigma_b)(x) = (\sigma_b \circ \sigma_a)(x) \text{ for all } x \in (\sigma, J).$$

Hence $\sigma_{a \sqcap b} = \sigma_a \circ \sigma_b = \sigma_b \circ \sigma_a$. □

Definition 5.3.5. Let $(X, \leqslant)$ be a poset.A map $f : X \to X$ is called interior operator if

i. f is order preserving

ii. $f(x) \leqslant x$ for all $x \in X$

iii. $f \circ f = f$.

Proposition 5.3.6. *Let (σ, J) be an L-slice. Then for each $a \in L, \sigma_a$ is an interior operator on (σ, J).*

Proof. Since for each $a \in L$, σ_a is an L-slice homomorphism, σ_a is order preserving. By proposition 5.3.3, $\sigma_a(x) \leqslant x$ for all $x \in L$ and by proposition 5.3.4, $\sigma_a \circ \sigma_a = \sigma_a$. Hence σ_a is an interior operator on (σ, J). □

Proposition 5.3.7. *The collection $M = \{\sigma_a : a \in L\}$ is a bounded distributive lattice and a subslice of $(\delta, L - Hom(J, J))$.*

Proof. $(M, \vee, \sigma_0)$ is a join semilattice with bottom element σ_0 and $(M, \circ, \sigma_1)$ is a meet semilattice with top element σ_1.

$\sigma_a \circ (\sigma_a \vee \sigma_b) = \sigma_a \circ \sigma_{a \sqcup b} = \sigma_{a \sqcap (a \sqcup b)} = \sigma_a$ and

$\sigma_a \vee (\sigma_a \circ \sigma_b) = \sigma_a \vee \sigma_{a \sqcap b} = \sigma_{a \sqcup (a \sqcap b)} = \sigma_a$

Thus absorption laws are satisfied and so M is a bounded lattice with top σ_1 and bottom σ_0. Also

$$\begin{aligned}
\sigma_a \circ (\sigma_b \vee \sigma_c) &= \sigma_a \circ (\sigma_{b \sqcup c}) = \sigma_{a \sqcap (b \sqcup c)} = \sigma_{(a \sqcap b) \sqcup (a \sqcap c)} \\
&= \sigma_{a \sqcap b} \vee \sigma_{a \sqcap c} = (\sigma_a \circ \sigma_b) \vee (\sigma_a \circ \sigma_c) \\
\sigma_a \vee (\sigma_b \circ \sigma_c) &= \sigma_a \vee \sigma_{b \sqcap c} = \sigma_{a \sqcup (b \sqcap c)} = \sigma_{(a \sqcup b) \sqcap (a \sqcup c)} \\
&= \sigma_{a \sqcup b} \circ \sigma_{a \sqcup c} = (\sigma_a \vee \sigma_b) \circ (\sigma_a \vee \sigma_c)
\end{aligned}$$

Hence M is a bounded distributive lattice. Clearly $M \subseteq L - Hom(J, J)$. Let $b \in L$ and $\sigma_a \in M$. Then

$\delta(b, \sigma_a)(x) = \sigma(b, \sigma_a(x)) = \sigma(b, \sigma(a, x)) = \sigma(b \sqcap a, x) = \sigma_{b \sqcap a}(x)$.

Thus M is closed under action by elements of L. Hence (δ, M) is a L-subslice of $(\delta, L - Hom(J, J)))$. □

Proposition 5.3.8. *There is an onto L-slice homomorphism from $(\sqcap, L)$ to (δ, M).*

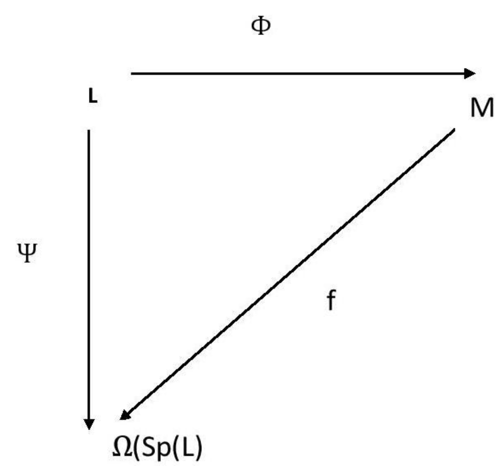
Proof. Define $\phi : (\sqcap, L) \to (\delta, M)$ by $\phi(a) = \sigma_a$.

$\phi(a \sqcup b) = \sigma_{a \sqcup b} = \sigma_a \vee \sigma_b = \phi(a) \vee \phi(b)$ and

$\phi(\sigma(a, b)) = \phi(a \sqcap b) = \sigma_{a \sqcap b} = \sigma_a \circ \sigma_b = \sigma(a, \sigma_b) = \sigma(a, \phi(b))$.

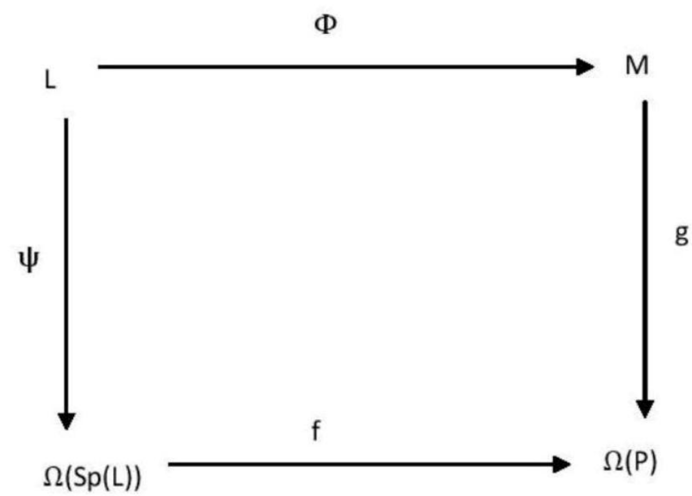
Surjection of ϕ is clear from the definition. Hence ϕ is an onto L-slice homomorphism from $(\sqcap, L)$ to (δ, M). □

Let ψ be natural frame homomorphism from L to $\Omega(Sp(L))$ and $\phi : (\sqcap, L) \to (\delta, M)$ is the L-slice homomorphism of proposition 5.3.8. Then there is a lattice homomorphism f from M to $\Omega(Sp(L))$ such that the following triangle commutes. The map f is defined by $f(\sigma_a) = \Sigma_a$



If L is a spatial locale, then ψ is one-one and so ϕ is one-one. Thus if L is spatial locale, L-slices $(\sigma, L), (\delta, M)$ are isomorphic.

Since M is a distributive lattice, by Priestley duality, the distributive lattice M is dual to a topological space P. Then there is a frame homomorphism f from $\Omega(Sp(L))$ to $\Omega(P)$ such that the following rectangle commutes. Then $f_* : \Omega(P) \to \Omega(Sp(L))$ is a localic map.



5.4. Fixed points with respect to the L-slice homomorphism σ_a

In this section we discuss some properties of the set $Fix_{\sigma_a} = \{x \in J : \sigma_a(x) = x\}$. We will show that $N = \{Fix_{\sigma_a}; a \in L\}$ together with an action γ is an L-slice.

Proposition 5.4.1. *For each $a \in L$, let $Fix_{\sigma_a} = \{x \in J : \sigma_a(x) = x\}$. Then (σ, Fix_{σ_a}) is a subslice of the L-slice (σ, J).*

Proof. Let $x, y \in Fix_{\sigma_a}$. Then $\sigma_a(x) = x, \sigma_a(y) = y$.

$$\sigma_a(x \vee y) = \sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) = \sigma_a(x) \vee \sigma_a(y) = x \vee y$$

So Fix_{σ_a} is a subjoin semilattice of (σ, J). Let $x \in Fix_{\sigma_a}$ and $b \in L$.

$$\sigma_a(\sigma(b, x)) = \sigma(a, \sigma(b, x)) = \sigma(b, \sigma(a, x)) = \sigma(b, x)$$

So $\sigma(b, x) \in (\sigma, Fix_{\sigma_a})$. Hence (σ, Fix_{σ_a}) is an L-subslice of (σ, J) □

Proposition 5.4.2. *Let (σ, J) be an L-slice of a locale L and $a \in L$. Then the following statements are equivalent.*

i. σ_a has the property that if $x \leqslant y \leqslant z$ with $\sigma_a(x) = x$ and $\sigma_a(z) = z$, then $\sigma_a(y) = y$.

ii. (σ, Fix_{σ_a}) is an ideal of (σ, J).

Proof. First assume statement i. By above theorem (σ, Fix_{σ_a}) is a subslice of (σ, J). Let $x \in (\sigma, Fix_{\sigma_a})$ and $y \in (\sigma, J)$ such that $y \leqslant x$. We have $\sigma_a(y) \leqslant y \leqslant x$ and $\sigma_a(\sigma_a(y)) = \sigma_a(y), \sigma_a(x) = x$. Then by assumption $\sigma_a(y) = y$. So $y \in (\sigma, Fix_{\sigma_a})$ and hence (σ, Fix_{σ_a}) is an ideal in (σ, J).

ii implies i follows directly from the definition of ideal of an L-slice. □

Proposition 5.4.3. *Let (σ, Fix_{σ_a}) is an ideal for some fixed $a \in L$. If σ_b or σ_c is one-one for all b, c with $\sigma(b \sqcap c, x) \in (\sigma, Fix_{\sigma_a})$, then (σ, Fix_{σ_a}) is a prime ideal.*

Proof. Let (σ, Fix_{σ_a}) is an ideal and $\sigma(b \sqcap c, x) \in (\sigma, Fix_{\sigma_a})$.

Then $\sigma_a(\sigma(b \sqcap c, x)) = \sigma(b \sqcap c, x)$ or $\sigma_a(\sigma_{b\sqcap c})(x) = \sigma_{b\sqcap c}(x)$.

Equivalently $(\sigma_a \circ \sigma_b \circ \sigma_c)(x) = (\sigma_b \circ \sigma_c)(x)$.

Suppose σ_c is one-one and $\sigma(b, x) \notin (\sigma, Fix_{\sigma_a})$. Then $\sigma_a(\sigma(b, x)) \neq \sigma(b, x)$.

That is $(\sigma_a \circ \sigma_b)(x) \neq \sigma_b(x)$. Since σ_c is one-one, $(\sigma_c \circ \sigma_a \circ \sigma_b)(x) \neq (\sigma_c \circ \sigma_b)(x)$

$(\sigma_a \circ \sigma_b \circ \sigma_c)(x) \neq (\sigma_b \circ \sigma_c)(x)$, which is a contradiction. Hence $\sigma(b, x) \in (\sigma, Fix_{\sigma_a})$.

Similarly if σ_b is one-one, then $\sigma(c, x) \in (\sigma, Fix_{\sigma_a})$. Hence (σ, Fix_{σ_a}) is a prime ideal. □

Proposition 5.4.4. *Consider the L-slice $(\sqcap, L)$. Then for each $a \in L, (\sigma, Fix_{\sigma_a})$ is a principal ideal.*

Proof.
$$Fix_{\sigma_a} = \{x \in L : \sigma_a(x) = x\} = \{x \in L : a \sqcap x = x\}$$
$$= \{x \in L : x \sqsubseteq a\} = \downarrow a$$

Hence (σ, Fix_{σ_a}) is a principal ideal. □

Proposition 5.4.5. *Let R be a congruence on an L-slice (σ, J) for a locale L and $(\gamma, J/R)$ be the quotient L-slice of (σ, J) with respect to the congruence R. Then*

i. $\{[x] : x \in Fix_{\sigma_a}\} \subseteq Fix_{\gamma_a}$

ii. $Fix_{\gamma_a} = \{[x]; \sigma(a, x)Rx\}$

iii. $\{[x] : x \in ker_{\sigma_a}\} \subseteq ker_{\gamma_a}$

iv. $ker_{\gamma_a} = \{[x]; \sigma(a, x)R0\}$.

Proof. i. Let $x \in Fix_{\sigma_a}$. Then $\sigma(a, x) = x$.

$\gamma_a([x]) = \gamma(a, [x]) = [\sigma(a, x)] = [x]$. So $[x] \in Fix_{\gamma_a}$.

Hence $\{[x] : x \in Fix_{\sigma_a}\} \subseteq Fix_{\gamma_a}$.

ii. $Fix_{\gamma_a} = \{[x]; \gamma_a([x]) = [x]\} = \{[x]; [\sigma(a, x)] = [x]\}$.

But $[\sigma(a, x)] = [x]$ if and only if $\sigma(a, x)Rx$. Hence $Fix_{\gamma_a} = \{[x]; \sigma(a, x)Rx\}$.

iii. Let $x \in ker_{\sigma_a}$. Then $\sigma_a(x) = 0_J$.

$\gamma_a([x]) = \gamma(a, [x]) = [\sigma(a, x)] = [0_J] = 0_{J/R}$. So $[x] \in ker_{\gamma_a}$.

Hence $\{[x] : x \in ker_{\sigma_a}\} \subseteq ker_{\gamma_a}$.

iv. $ker_{\gamma_a} = \{[x]; [\sigma(a, x)] = \gamma_a[x] = [0]\}$. But $[\sigma(a, x)] = [0]$ if and only if $\sigma(a, x)R0$.
Hence $ker_{\gamma_a} = \{[x]; \sigma(a, x)R0\}$. □

Proposition 5.4.6. *Let (σ, J) be an L-slice and $a, b \in L$*

i. If $a \sqsubseteq b$, then $Fix_{\sigma_a} \subseteq Fix_{\sigma_b}$.

ii. $Fix_{\sigma_0} = \{0_J\}$ and $Fix_{\sigma_1} = (\sigma, J)$.

iii. $Fix_{\sigma_{a \sqcap b}} = Fix_{\sigma_a} \cap Fix_{\sigma_a}$.

Proof. (σ, J) be an L-slice and $a, b \in L$

i. Let $a \sqsubseteq b$ and $x \in Fix_{\sigma_a}$, then $a \sqcap b = a$ and $\sigma_a(x) = x$.

$\sigma_a(x) = x$ implies that $\sigma(a, x) = \sigma(a \sqcap b, x) = x$.

Then we have $\sigma(b, \sigma(a, x)) = x$. Equivalently $\sigma_b(x) = x$ and so $x \in Fix_{\sigma_b}$.

Thus $Fix_{\sigma_a} \subseteq Fix_{\sigma_b}$.

ii. Since $\sigma_0(x) = 0_J$ for all $x \in J$, $Fix_{\sigma_0} = \{0_J\}$.

$Fix_{\sigma_1} = \{x \in (\sigma, J) : \sigma_1(x) = \sigma(1, x) = x\} = (\sigma, J)$.

iii. By part i, $Fix_{\sigma_{a \sqcap b}} \subseteq Fix_{\sigma_a} \cap Fix_{\sigma_b}$.

Now let $x \in Fix_{\sigma_a} \cap Fix_{\sigma_b}$. Then $\sigma_a(x) = \sigma_b(x) = x$.

$\sigma_{a \sqcap b}(x) = \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, \sigma_b(x)) = \sigma(a, x) = \sigma_a(x) = x$. Thus $x \in Fix_{\sigma_{a \sqcap b}}$ and hence $Fix_{\sigma_{a \sqcap b}} = Fix_{\sigma_a} \cap Fix_{\sigma_a}$. □

Proposition 5.4.7. *Let $(\sigma, J), (\mu, J)$ be two L-slices and let σ_a or μ_a is onto for some $a \in L$. Then $\sigma_a = \mu_a$ if and only if $Fix_{\sigma_a} = Fix_{\mu_a}$.*

Proof. Suppose σ_a is onto and $Fix_{\sigma_a} = Fix_{\mu_a}$. Let $y \in (\sigma, J)$. Then there exist $x \in (\sigma, J)$ such that $\sigma_a(x) = y$. Now

$$\begin{aligned} \sigma_a(y) &= \sigma(a, y) = \sigma(a, \sigma_a(x)) \\ &= \sigma(a, \sigma(a, x)) = \sigma(a, x) = y \end{aligned}$$

Hence $y \in Fix_{\sigma_a} = Fix_{\mu_a}$. So $\mu_a(y) = y$. Hence $\sigma_a(y) = \mu_a(y)$. Converse is simple. □

Proposition 5.4.8. *Let (σ, J) be an L-slice of a locale L and $N = \{Fix_{\sigma_a} : a \in L\}$. Define $Fix_{\sigma_a} \vee Fix_{\sigma_b} = Fix_{\sigma_{a \sqcup b}}, Fix_{\sigma_a} \wedge Fix_{\sigma_b} = Fix_{\sigma_{a \sqcap b}}$. Then $(N, \vee, \wedge)$ is a distributive lattice and an L-slice.*

Proof. It is easy to show that $(N, \vee, Fix_{\sigma_0}), (N, \wedge, Fix_{\sigma_1})$ are semilattices.

Also $Fix_{\sigma_a} \vee (Fix_{\sigma_a} \wedge Fix_{\sigma_b}) = Fix_{\sigma_a} \vee Fix_{\sigma_{a \sqcap b}} = Fix_{\sigma_{a \sqcup (a \sqcap b)}} = Fix_{\sigma_a}$ and

$Fix_{\sigma_a} \wedge (Fix_{\sigma_a} \vee Fix_{\sigma_b}) = Fix_{\sigma_a} \wedge Fix_{\sigma_{a \sqcup b}} = Fix_{\sigma_{a \sqcap (a \sqcup b)}} = Fix_{\sigma_a}$. Hence absorption laws are satisfied and so $(N, \vee, \wedge)$ is a lattice. Also we can verify distributive law easily. Define $\gamma : L \times N \to N$ by $\gamma(b, Fix_{\sigma_a}) = Fix_{\sigma_{a \sqcap b}}$.

i. $\gamma(b, Fix_{\sigma_a} \vee Fix_{\sigma_c}) = \gamma(b, Fix_{\sigma_{a \sqcup c}}) = Fix_{\sigma_{b \sqcap (a \sqcup c)}} = Fix_{\sigma_{(b \sqcap a) \sqcup (b \sqcap c)}}$

$$= Fix_{\sigma_{b \sqcap a}} \vee Fix_{\sigma_{b \sqcap c}} = \gamma(b, Fix_{\sigma_a}) \vee \gamma(b, Fix_{\sigma_c})$$

ii. $\gamma(b, Fix_{\sigma_0}) = Fix_{\sigma_{b \sqcap 0}} = Fix_{\sigma_0}$

iii. $\gamma(b \sqcap c, Fix_{\sigma_a}) = Fix_{\sigma_{(b \sqcap c) \sqcap a}} = Fix_{\sigma_{b \sqcap (c \sqcap a)}} = \gamma(b, Fix_{\sigma_{c \sqcap a}}) = \gamma(b, \gamma(c, Fix_{\sigma_a}))$

iv. $\gamma(1, Fix_{\sigma_a}) = Fix_{\sigma_{1 \sqcap a}} = Fix_{\sigma_a}$

$\gamma(0, Fix_{\sigma_a}) = Fix_{\sigma_{0 \sqcap a}} = Fix_{\sigma_0}$

v. $\gamma(b \sqcup c, Fix_{\sigma_a}) = Fix_{\sigma_{(b \sqcup c) \sqcap a}} = Fix_{\sigma_{(b \sqcap a) \sqcup (c \sqcap a)}} = Fix_{\sigma_{b \sqcap a}} \vee Fix_{\sigma_{c \sqcap a}}$

$$= \gamma(b, Fix_{\sigma_a}) \vee \gamma(c, Fix_{\sigma_a})$$

Hence (γ, N) is an L-slice. □

Proposition 5.4.9. *There is an onto L-slice homomorphism from the L-slices* (δ, M) *to* (γ, N).

Proof. The map $g : (\delta, M) \to (\gamma, N)$ defined by $g(\sigma_a) = Fix_{\sigma_a}$ is an onto L-slice homomorphism from (δ, M) to (γ, N). □

Since the composition of two L-slice homomorphism is again an L-slice homomorphism, $g \circ \phi$ is an L-slice homomorphism from $(\sqcap, L)$ to (γ, N).

5.5. Filters in L with respect to the slice (σ, J)

Let (σ, J) be an L-slice of a locale L and let $x \in (\sigma, J)$. In this section we discuss about the map $\sigma_x : (\sqcap, L) \to (\sigma, J)$ defined by $\sigma_x(a) = \sigma(a, x)$. We will show that for each $x \in (\sigma, J)$, σ_x is an L-slice homomorphism. We look into various properties of the collection $F_x = \{a \in L; \sigma_x(a) = x\}$.

Proposition 5.5.1. *Let (σ, J) be an L-slice of a locale L. For each $x \in J$, $\sigma_x : (\sqcap, L) \to (\sigma, J)$ is an L-slice homomorphism.*

Proof. Let $a, b \in L$

$$\begin{aligned} \sigma_x(a \sqcup b) &= \sigma(a \sqcup b, x) = \sigma(a, x) \vee \sigma(b, x) = \sigma_x(a) \vee \sigma_x(b) \\ \sigma_x(a \sqcap b) &= \sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, \sigma_x(b)) \end{aligned}$$

□

Remark. Let I be any ideal in the L-slice $(\sqcap, L)$. Since σ_x is an L-slice homomorrphism, $\sigma_x(I)$ is a subslice of (σ, J)

Proposition 5.5.2. *Let I, K be ideals of the L-slice $(\sqcap, L)$ and let $I \subseteq K$. Then $\sigma_x(I) \subseteq \sigma_x(K)$.*

Proof. Let I, K be ideals of a locale L such that $I \subseteq K$. Let $y \in \sigma_x(I)$. Then $y = \sigma_x(a)$ for some $a \in I \subseteq K$. Hence $y = \sigma_x(a) \in \sigma_x(K)$. Thus $\sigma_x(I) \subseteq \sigma_x(K)$. □

Proposition 5.5.3. *Let (σ, J) be an L-slice of a locale L and let $P = \{\sigma_x : x \in (\sigma, J)\}$. Then (δ, P) is an L- subslice of $(\delta, L - Hom(L, J))$.*

Proof. Let $\sigma_x, \sigma_y \in P$.

$$\begin{aligned} (\sigma_x \vee \sigma_y)(a) &= \sigma_x(a) \vee \sigma_y(a) = \sigma(a, x) \vee \sigma(a, y) \\ &= \sigma(a, x \vee y) = \sigma_{x \vee y}(a) \end{aligned}$$

Thus $\sigma_x \vee \sigma_x = \sigma_{x \vee y} \in P$. Hence P is a subjoin semilattice of $L - Hom(L, J)$.

Also $\sigma_0(a) = \sigma(a, 0) = 0 = \mathbf{0}(a)$. Hence σ_0 is the bottom element of P.

Also for $a, b \in L$ and $\sigma_x \in P$,

$$\delta(b, \sigma_x)(a) \;=\; \sigma(b, \sigma_x(a)) = \sigma(a, \sigma(b, x)) = \sigma_{\sigma(b,x)}(a)$$

Thus $\delta(b, \sigma_x) = \sigma_{\sigma(b,x)} \in P$. Hence (δ, P) is an L-subslice of $(\delta, L - Hom(L, J))$. □

Proposition 5.5.4. *Let (σ, J) be an L-slice of a locale L and let $P = \{\sigma_x : x \in (\sigma, J)\}$. Then the slices (σ, J) and (δ, P) are isomorphic.*

Proof. Define $\phi : (\sigma, J) \to (\delta, P)$ by $\phi(x) = \sigma_x$.

$\phi(x \vee y) = \sigma_{x \vee y} = \sigma_x \vee \sigma_y = \phi(x) \vee \phi(y)$ and

$\phi(\sigma(a, x)) = \sigma_{\sigma(a,x)} = \delta(a, \sigma_x) = \delta(a, \phi(x))$.

Thus ϕ is an L-slice homomorphism.

From the definition of P, clearly ϕ is onto.

Now let $\phi(x) = \phi(y)$. Then $\sigma_x = \sigma_y$, which implies $\sigma_x(a) = \sigma_y(a)$ for all $a \in L$.

In particular, $\sigma_x(1) = \sigma_y(1)$. Then $\sigma(1, x) = \sigma(1, y)$. So $x = y$ and hence ϕ is one-one.

Thus $\phi : (\sigma, J) \to (\delta, P)$ is an isomorphism. □

Proposition 5.5.5. *Let (σ, J) be an L-slice of a locale L. For each $x \in (\sigma, J)$, let $F_x = \{a \in L; \sigma(a, x) = x\}$. Then F_x is a filter in L.*

Proof. By Definition 4.1.1(iv), $1 \in F_x$. Hence F_x is nonempty.

Let $a, b \in F_x$. Then $\sigma(a, x) = x, \sigma(b, x) = x$.

$\sigma(a \sqcap b, x) = \sigma(a, \sigma(b, x)) = \sigma(a, x) = x$. Hence $a \sqcap b \in F_x$.

Let $a \in F_x$ and $c \in L$ such that $a \leqslant c$.

$\sigma(a, x) = \sigma(a \sqcap c, x) = \sigma(c \sqcap a, x) = \sigma(c, \sigma(a, x)) = \sigma(c, x)$. Hence $c \in F_x$.

Thus F_x is a filter in L. □

Proposition 5.5.6. *Let (σ, J) be an L-slice and $x \leqslant y \in (\sigma, J)$. Then*

i. $x \leqslant \sigma(a, y)$ for all $a \in F_x$.

ii. $\sigma(b, x) \leqslant y$ for all $b \in F_y$.

Proof. i. Let $x \leqslant y \in (\sigma, J)$ and $a \in F_x$. Then $\sigma(a, x) = x$.

$\sigma(a, y) = \sigma(a, x \vee y) = \sigma(a, x) \vee \sigma(a, y) = x \vee \sigma(a, y)$.

Hence $x \leqslant \sigma(a, y)$ for all $a \in F_x$.

ii. Let $b \in F_y$. Then $\sigma(b, y) = y$.

$\sigma(b, y) = \sigma(b, x \vee y) = \sigma(b, x) \vee \sigma(b, y) = \sigma(b, x) \vee y$.

Hence $\sigma(b, x) \leqslant y$ for all $b \in F_y$. □

Proposition 5.5.7. *The filter F_x is proper for $x \neq 0_J$.*

Proof. Suppose $x \neq 0_J$. If $0_L \in F_x$, then $\sigma_x(0_L) = x$, which implies $0_J = \sigma(0_L, x) = x$. Hence if $x \neq 0_J$, $0_L \notin F_x$ and so F_x is proper. □

Proposition 5.5.8. *Consider the L-slice $(\sqcap, L)$. Then for each $b \in (\sqcap, L), F_b$ is a closed sublocale of L.*

Proof. $F_b = \{a \in L : \sigma_b(a) = b\} = \{a \in L : a \sqcap b = b\} = \{a \in L : a \sqsupseteq b\}$

$= \uparrow b$.

Hence F_b is a closed sublocale of L. □

Proposition 5.5.9. *Let $x \in (\sigma, J)$ be join-irreducible element of (σ, J), then F_x is a prime filter in L.*

Proof. By proposition 5.5.5, F_x is a filter in L. Let $a \sqcup b \in F_x$.

Then $\sigma_x(a \sqcup b) = \sigma(a \sqcup b, x) = x$. That is $\sigma(a, x) \vee \sigma(b, x) = x$. Since x is join-irreducible, $x \leqslant \sigma(a, x)$ or $x \leqslant \sigma(b, x)$. But we have $\sigma(a, x) \leqslant x$ for all $a \in L$. Hence $\sigma(a, x) = x$ or $\sigma(b, x) = x$. Hence either $a \in F_x$ or $b \in F_x$.

Thus F_x is a prime filter in L. □

Definition 5.5.10. An element $x \in (\sigma, J)$ is said to be compact element of the L-slice (σ, J), if for any collection $\{a_\alpha\}$ of L whenever $\sigma(\sqcup a_\alpha, x) = x$, then there exist a finite sub collection $\{a_1, a_2, ...a_n\}$ of $\{a_\alpha\}$ such that $\sigma(a_1, x) \vee \sigma(a_2, x) \vee ... \vee \sigma(a_n, x) = x$. A slice (σ, J) is compact if each element $x \in (\sigma, J)$ is compact.

From the definition of compact L-slice, it is clear that every L-subslice of a compact L-slice is compact.

Example 5.5.11. *Let (σ, J) be any L-slice. Then 0_J is a compact element.*

Proposition 5.5.12. *Let L be a locale. If the L-slice $(\sqcap, L)$ is compact, then the locale L is compact.*

Proof. Suppose the L-slice $(\sqcap, L)$is compact and let $\{a_\alpha\} \in L$ such that $\sqcup a_\alpha = 1$. Then for any $b \in (\sqcap, L), (\sqcup a_\alpha) \sqcap b = 1 \sqcap b = b$. Since $(\sqcap, L)$ is compact, there exist a finite sub collection $\{a_1, a_2, ...a_n\}$ of $\{a_\alpha\}$ such that $(a_1 \sqcup a_2 \sqcup ... \sqcup a_n) \sqcap b = b$. In particular this is true for $b = 1$. Hence $(a_1 \sqcup a_2 \sqcup ... \sqcup a_n) \sqcap 1 = 1$. Then $(a_1 \sqcup a_2 \sqcup ... \sqcup a_n) = 1$ and hence the locale L is compact. □

The above proposition shows that the notion of compactness in the L-slice is stronger than the topological compactness and compactness in locale.

Proposition 5.5.13. *Let L be a compact locale and (σ, J) be an L-slice. Let $x \in (\sigma, J)$ be such that σ_x is one one. Then x is a compact element of the L-slice (σ, J).*

Proof. Let L be a compact locale and $x \in (\sigma, J)$. Suppose $\sigma(\sqcup a_\alpha, x) = x$.
That is $\sigma_x(\sqcup a_\alpha) = x = \sigma_x(1)$.
Since $\sigma_x : L \to J$ is one one, $\sqcup a_\alpha = 1$.
Since L is a compact locale, there exist a finite sub collection $a_1, a_2, ...a_n$ of $\{a_\alpha\}$ such that $a_1 \sqcup a_2 \sqcup ... \sqcup a_n = 1$.

Then $\sigma(a_1 \sqcup a_2 \sqcup ... \sqcup a_n, x) = \sigma(1, x) = x$.

Hence x is a compact element of the L-slice (σ, J). □

Corollary 5.5.14. *Let L be a compact locale and (σ, J) be an L-slice. If σ_x is one one for every $x \in (\sigma, J)$, then (σ, J) is a compact L-slice..*

Corollary 5.5.15. *Let L be a compact locale and let $\sqcap_x : L \to L$ is one one for every $x \in (\sqcap, L)$. Then the L-slice $(\sqcap, L)$ is compact.*

Proposition 5.5.16. *Let $(\sigma, J), (\mu, K)$ be L-slices of a locale L and $f : (\sigma, J) \to (\mu, K)$ be a one-one L-slice homomorphism. If x is a compact element of the L-slice (σ, J), then $f(x)$ is a compact element of the L-slice (μ, K).*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be a one-one L-slice homomorphism and let x be a compact element of the L-slice (σ, J). Let $\{a_\alpha\} \in L$ such that $\mu(\sqcup a_\alpha, f(x)) = f(x)$.

Then we have $f(\sigma(\sqcup a_\alpha, x)) = f(x)$.

Since f is a one-one L-slice homomorphism, $\sigma(\sqcup a_\alpha, x) = x$.

Then by compactness of the element $x \in (\sigma, J)$, there exist a finite sub collection $a_1, a_2, ...a_n$ of $\{a_\alpha\}$ such that $\sigma(a_1 \sqcup a_2 \sqcup .... \sqcup a_n, x) = x$.

Then we have $\mu(a_1 \sqcup a_2 \sqcup .... \sqcup a_n, f(x)) = f(\sigma(a_1 \sqcup a_2 \sqcup .... \sqcup a_n, x)) = f(x)$.

Hence $f(x)$ is a compact element of the L-slice (μ, K). □

Definition 5.5.17. A proper filter F in a locale L is partially completely prime filter if for any indexing set I and $a_i \in L, i \in I$, $\bigvee a_i \in F \Rightarrow \exists a_1, a_2....a_n$ such that $a_1 \vee a_2 \vee ... \vee a_n \in F$.

Proposition 5.5.18. *Let (σ, J) be an L-slice of a locale L and $x \in (\sigma, J)$. Then x is a compact element of the slice (σ, J) if and only if the filter F_x is partially completely prime.*

Proof. Suppose x is a compact element of the L-slice (σ, J). By proposition 5.5.5, F_x is a filter in L. Let $\sqcup a_\alpha \in F_x$. Then we have $\sigma(\sqcup a_\alpha, x) = x$.

Since x is a compact element, there is a finite collection $\{a_1, a_2, ...a_n\}$ of $\{a_\alpha\}$ such that $\sigma(a_1, x) \vee \sigma(a_2, x) \vee ... \vee \sigma(a_n, x) = x$.

That is $\sigma(a_1 \sqcup a_2 \sqcup ... \sqcup a_n, x) = x$. So $a_1 \sqcup a_2 \sqcup ... \sqcup a_n \in F_x$ and hence F_x is a partially completely prime filter.

Conversely assume F_x is partially completely prime. Let $\{a_\alpha\} \in L$ such that $\sigma(\sqcup a_\alpha, x) = x$. Then we have $\sqcup a_\alpha \in F_x$.

Since F_x is partially completely prime, there is a finite collection $\{a_1, a_2, ...a_n\}$ of $\{a_\alpha\}$ such that $a_1 \sqcup a_2 \sqcup ... \sqcup a_n \in F_x$.

Hence $\sigma(a_1, x) \vee \sigma(a_2, x) \vee ... \vee \sigma(a_n, x) = x$.

So x is a compact element of the L-slice (σ, J). □

Proposition 5.5.19. *Let $x \in (\sigma, J)$ be join-irreducible compact element of (σ, J), then F_x is a completely prime filter.*

Proof. Let $x \in (\sigma, J)$ be join-irreducible compact element of (σ, J) and let $\sqcup a_\alpha \in F_x$. Since x is a compact element, by proposition 5.5.18, F_x is a partially completely prime filter. Hence there is $a_1, a_2, ...a_n \in \{a_\alpha\}$ such that $a_1 \sqcup a_2 \sqcup ... \sqcup a_n \in F_x$.

Since x is join-irreducible element of (σ, J), there is some a_i such that $a_i \in F_x$. Hence F_x is completely prime filter. □

Definition 5.5.20. A compact element x of an L-slice (σ, J) of a locale L is said to be maximal compact element if the filter F_x associated with $x \in (\sigma, J)$ has the property that $F_x \subseteq F_y$ for all compact elements $y \in (\sigma, J)$.

Proposition 5.5.21. *If the collection of all compact elements of a locale forms a chain with maximal element x, then x is a maximal compact element of the L-slice $(\sqcap, L)$.*

Proof. By Proposition 5.5.8, $F_x =\uparrow x$.

Let $y \in L$ be any other compact element of the L-slice $(\sqcap, L)$.

Since $y \leqslant x, F_x =\uparrow x \subseteq\uparrow y = F_y$. Hence x is a maximal compact element of the L-slice $(\sqcap, L)$. □

Proposition 5.5.22. *Let $F = \{a \in L : \sigma(a, x) = x \ \ \forall x \in (\sigma, J)\}$. Then $F = \bigcap F_x$ and F is a filter in L*

Proof. Let $a \in F$. Then $\sigma_x(a) = \sigma(a, x)$ for all $x \in (\sigma, J)$. So $a \in F_x$ for all $x \in (\sigma, J)$. Hence $F \subseteq \bigcap F_x$. In a similar way we can show that $\bigcap F_x \subseteq F$. Hence $F = \bigcap F_x$. Since the intersection of filters of L is a filter in L, F is a filter in L. □

Construction of Sublocale of locale L with respect to L-slice (σ, J)

Proposition 5.5.23. *Let $Y = \{F_x : x \text{ is join irreducible and compact in } (\sigma, J)\}$. Then $(Y, \Omega(Sp(L))/Y)$ is a topological space.*

Proof. If $x \in (\sigma, J)$ is join-irreducible compact element, then by proposition 5.5.19, F_x is completely prime filter and so $Y \subseteq Sp(L)$. Then $\Omega(Sp(L))/Y$ is the subspace topology on Y and $(Y, \Omega(Sp(L))/Y)$ is a topological space. □

Examples 5.5.24. *i. Let $(\sqcap, L)$ be an L-slice. By Proposition 5.5.8, we have $F_b =\uparrow b$, a principal filter of L. Then $Y = \{\uparrow b : b \in (\sqcap, L) is \ \ join\!-\!irreducible \ \ compact \ \ element\}$. But $b \in (\sqcap, L)$ is join-irreducible and compact element if and only $\uparrow b$ is completely prime filter in the locale L. Hence $Y = Sp(L)$.*

ii. Let L be a locale and let I be an ideal of L.Consider the L-slice $(\sqcap, I)$. As in the *case of above example,*

$Y = \{\uparrow b : \uparrow b$ *is a completely prime filter in* $L, b \in (\sqcap, I)\} \subsetneq Sp(L)$. *Hence* $(Y, \Omega(Sp(L))/Y)$ *is a proper subspace of* $Sp(L)$.

iii. Let the L-slice (σ, J) *is the L-slice of example 4.2.2(ii). For* $x \in (\sigma, J)$, $F_x = L \backslash \{0_L\}$. *Hence* $Y = L \backslash \{0_L\}$. *Let* $b \in L$ *be minimal element of L, then* $Y = L \backslash \{0_L\} = \Sigma_b$. *So* Y *is an open subset of* $Sp(L)$ *and* $\Omega(Sp(L))/Y$ *is isomorphic to* **2**.

The subspace $(Y, \Omega(Sp(L))/Y)$ depends on the L-slice (σ, J). If Y is an open set in $Sp(L)$, then $\Omega(Sp(L))/Y$ is a sublocale of the locale $\Omega(Sp(L))$ and hence a sublocale of L. If $Y = Sp(L)$, the points of L is completely determined by the L-slice (σ, J).

Proposition 5.5.25. *If the L-slice* (σ, J) *of a locale L has a maximal compact irreducible element z, then*
$(Y = \{F_x : x$ *is join irreducible and compact in* $(\sigma, J)\}, \Omega(Sp(L))/Y)$ *is a compact subspace of spectrum* $Sp(L)$ *of the locale L*

Proof. Let $\{\Sigma_{a_\alpha} : \alpha \in I\}$ be an open cover for Y. Then $Y \subseteq \bigcup \Sigma_{a_\alpha} = \Sigma_{\sqcup a_\alpha}$. Since $F_z \in Y, F_z \in \Sigma_{\sqcup a_\alpha}$ or $\sqcup a_\alpha \in F_z$. Since F_z is a completely prime filter, there is some $\beta \in I$ such that $a_\beta \in F_z$. Then $F_z \in \Sigma_{a_\beta}$. Also for any $F_x \in Y$, we have $a_\beta \in F_z \subseteq F_x$. Hence $Y \subseteq \Sigma_{a_\beta}$ and so Y is compact. □

5.6. Weak S-module

Given a complete semiring $(S, +, ., 0_S, 1_S)$, where finite product $\cdot$ distribute over infinite sum $+$, and a monoid $(M, *, 0_M)$, a weak S-module is introduced to be an action of S on $(M, *, 0_M)$. We have defined weak S-module homomorphism between two weak S-modules $(\delta, M), (\gamma, N)$ and it is proved that if $(N, *')$ is commutative, then the collection of all weak S-module homomorphisms from (δ, M) to (γ, N) is a weak S-module.

Definition 5.6.1. A semiring is a triple $(S, +, \cdot)$, where S is a set and $+$ and $\cdot$ are binary operations, such that $+$ is commutative, both $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws holds for all $x, y, z \in S$.

i. $x.(y + z) = x.y + x.z$.

ii. $(x + y).z = (x.z) + (y.z)$.

If $(S, \cdot)$ is a monoid, then $(S, +, \cdot)$ is a semiring with 1.

Definition 5.6.2. A complete semiring is a semiring for which the addition monoid is a complete monoid and the following infinitary distributive laws hold

$\Sigma(a.a_i) = a.\Sigma a_i$ and $\Sigma(a_i.a) = (\Sigma a_i).a$.

Definition 5.6.3. A topological semiring is a semiring S togather with a topology under which the semiring operations are continuous.

Definition 5.6.4. Let $(S, +, ., 0_S, 1_S)$ be a complete semiring where finite . distribute over infinite + and let $(M, *, 0_M)$ be a monoid. By an action of S on M, we mean a function $\delta : S \times M \to M$ such that the following conditions are satisfied.

i. $\delta(r + s, x) = \delta(r, x) * \delta(s, x)$ for all $r, s \in S, x \in M$

ii. $\delta(r, x * y) = \delta(r, x) * \delta(r, y)$

iii. $\delta(r, 0_M) = 0_M$

iv. $\delta(r.s, x) = \delta(r, \delta(s, x))$

v. $\delta(0_R, x) = 0_M$ and $\delta(1_R, x) = x$.

If δ is an action of S on M, we call (δ, M) as a weak S-module.

Note If $(S, \cdot)$ is commutative, then $\delta(r.s, x) = \delta(r, \delta(s, x)) = \delta(s, \delta(r, x))$.

Example 5.6.5. *Every L-slice is an example for weak L-module.*

Definition 5.6.6. Let (δ, M) be a weak S-module, a submonoid M' of M is said to be a weak S-submodule of (δ, M) if M' is closed under action by elements of S.

Definition 5.6.7. A weak S-module homomorphism between weak S-modules $(\delta, M), (\gamma, N)$ is a map $g : (\delta, M) \to (\gamma, N)$ such that

i. $g(x * y) = g(x) *' g(y)$

ii. $g(\delta(r, x)) = \gamma(r, g(x))$ for all $x, y \in M, r \in S$.

Proposition 5.6.8. *Composition of two weak S-module homomorphisms is a weak S-module homomorphism.*

Proof. Let $g : (\delta, M) \to (\delta', M')$, $h : (\delta', M') \to (\delta'', M'')$ be two weak S-module homomorphisms.

$$\begin{aligned}
(h \circ g)(x * y) &= h(g(x * y)) = h(g(x) *' g(y)) = h(g(x)) *'' h(g(y)) \\
(h \circ g)(\delta(r, x)) &= h(g(\delta(r, x))) = h(\delta'(r, g(x))) \\
&= \delta''(r, h(g(x))) = \delta''(r, (h \circ g)(x))
\end{aligned}$$

Hence $h \circ g$ is a weak S-module homomorphism. □

Proposition 5.6.9. *Let $(\delta, M), (\gamma, N)$ be two weak S-modules. Then*

i. The map $\boldsymbol{0}$: $(\delta, M) \to (\gamma, N)$ defined by $\boldsymbol{0}(x) = 0_N$ for all $x \in (\delta, M)$ is a weak S-module homomorphism.

ii. If $f, g : (\delta, M) \to (\gamma, N)$ are two weak S-module homomorphisms and $(N, *')$ is *commutative, then* $f * g : (\delta, M) \to (\gamma, N)$ *defined by* $f * g(x) = f(x) *' g(x)$ *is a weak* S*-module homomorphism.*

iii. If $f : (\delta, M) \to (\gamma, N)$ *be a weak* S*-module homomorphism. Then for any* $r \in S$*, the map* $\eta(r, f) : (\delta, M) \to (\gamma, N)$ *defined by* $\eta(r, f)(x) = \gamma(r, f(x))$ *is a weak* S*-module homomorphism.*

Proof. Let $(\delta, M), (\gamma, N)$ be two weak S-modules.

i. Let $x, y \in (\delta, M), r \in S$

$$\begin{aligned} \mathbf{0}(x * y) &= 0_N = \mathbf{0}(x) *' \mathbf{0}(y) \\ \mathbf{0}(\delta(r, x)) &= 0_N = \gamma(r, 0_N) = \gamma(r, \mathbf{0}(x)) \end{aligned}$$

Hence $\mathbf{0}$ is a weak S-module homomorphism.

ii. Let $f, g : (\delta, M) \to (\gamma, N)$ be two weak S-module homomorphism.

$$\begin{aligned} (f * g)(x * y) &= f(x * y) *' g(x * y) = f(x) *' f(y) *' g(x) *' g(y) \\ &= f(x) *' g(x) *' f(y) *' g(y) = (f * g)(x) *' (f * g)(y) \\ (f * g)(\delta(r, x)) &= f(\delta(r, x)) *' g(\delta(r, x)) = \gamma(r, f(x)) *' \gamma(r, g(x)) \\ &= \gamma(r, f(x) *' g(x)) = \gamma(r, (f * g(x))) \end{aligned}$$

Hence $f * g$ is a weak S-module homomorphism.

iii. Let $r \in S$

$$\begin{aligned}
\eta(r,f)(x*y) &= \gamma(r,f(x*y)) = \gamma(r,f(x)*'f(y)) \\
&= \gamma(r,f(x))*'\gamma(r,f(y)) = \eta(a,f)(x)*'\eta(a,g)(x) \\
\eta(r,f)(\delta(s,x)) &= \gamma(r,f(\delta(s,x))) = \gamma(r,\gamma(s,f(x))) \\
&= \gamma(s,\gamma(r,f(x))) = \gamma(s,\eta(r,f)(x)
\end{aligned}$$

Hence $\eta(r,f)$ is a weak S-module homomorphism. □

Proposition 5.6.10. *Let $(\delta,M),(\gamma,N)$ be two weak S-modules, where $(N,*')$ is commutative. Then the collection Δ of all weak S-module homomorphisms from (δ,M) to (γ,N) is weak S-module.*

Proof. For any $f,g \in \Delta$ define $f*g : (\delta,M) \to (\gamma,N)$ by $f*g(x) = f(x)*'g(x)$. Then $(\Delta,*)$ is a monoid. Define $\eta : S\times\Delta \to \Delta$ as a map $\eta(r,f) : (\delta,M) \to (\gamma,N)$ by $\eta(r,f)(x) = \gamma(r,f(x))$. Then η is an action of S on Δ.
Let $r,s \in S, x \in (\delta,M)$

i. $\eta(r+s,f)(x) = \gamma(r+s,f(x)) = \gamma(r+s,f(x)) = \gamma(r,f(x))*'\gamma(s,f(x))$
$= \eta(r,f)(x)*'\eta(s,f)(x) = \eta(r,f)*\eta(s,f)(x)$

ii. $\eta(r,f*g)(x) = \gamma(r,f*g(x)) = \gamma(r,f(x)*'g(x)$
$= \gamma(r,f(x))*'\gamma(r,g(x)) = (\eta(r,f)*\eta(r,g))(x)$

iii. $\eta(r,\mathbf{0})(x) = \gamma(r,\mathbf{0}(x)) = \gamma(r,0_N) = 0_N = \mathbf{0}(x)$

iv. $\eta(r.s,f)(x) = \gamma(r.s,f(x)) = \gamma(r,\gamma(s,f(x)))$
$= \gamma(r,\eta(s,f)(x) = \eta(r,\eta(s,f))(x)$

v. $\eta(0_S,f)(x) = \gamma(0_S,f(x)) = 0_N = \mathbf{0}(x)$

Hence (η, Δ) is a weak S-module homomorphism. □

Proposition 5.6.11. *Let $f : (\delta, M) \to (\gamma, N)$ be a weak S-module homomorphism.*

i. $kerf = \{x \in (\delta, M) : f(x) = 0_N\}$ is a weak S-submodule of (δ, M).

ii. $imf = \{y \in (\gamma, N) : y = f(x) for \ \ some \ \ x \in (\delta, M)\}$ is a weak S-submodule of (γ, N).

Proof. Let $f : (\delta, M) \to (\gamma, N)$ be a weak S-module homomorphism.

i. Since $f(0_M) = 0_N$, $0_M \in kerf$. Let $x, y \in kerf$. Then $f(x) = f(y) = 0_N$.

$$\begin{aligned} f(x * y) &= f(x) *' f(y) = 0_N *' 0_N = 0_N \\ f(\delta(r, x)) &= \gamma(r, f(x)) = \gamma(r, 0_N) = 0_N \end{aligned}$$

So $kerf$ is a a weak S-submodule of (δ, M).

ii. Since $f(0_M) = 0_N, 0_N \in imf$. Let $x', y' \in imf$. Then $x, y \in (\delta, M)$ such that $f(x) = x', f(y) = y'$.

$$\begin{aligned} x' *' y' &= f(x) *' f(y) = f(x * y) \in imf \\ \gamma(r, x') &= \gamma(r, f(x)) = f(\gamma(r, x)) \in imf \end{aligned}$$

Hence imf is a weak S-submodule of (γ, N). □

Proposition 5.6.12. *Let $f : (\delta, M) \to (\delta, M)$ be a weak S-module homomorphism. Then $F = \{x \in (\delta, M) : f(x) = x\}$ is a weak S-submodule.*

Proof. Let $f : (\delta, M) \to (\delta, M)$ be a weak S-module homomorphism. Since $f(0_M) = 0_M, 0_M \in F$. Let $x, y \in F$. Then $f(x * y) = f(x) * f(y) = x * y$. Hence $x * y \in F$. Thus $(F, *)$ is a submonoid of $(M, *)$.

Let $r \in S, x \in F$. Then $f(\delta(r,x)) = \delta(r, f(x)) = \delta(r,x)$. Thus $\delta(r,x) \in F$. Hence (δ, F) is a weak S-submodule of (δ, M). □

Definition 5.6.13. A weak S-module $(\delta, M, *)$, is said to be finitely generated if there exist $x_1, x_2, ...x_n \in (\delta, M)$ such that each $x \in (\delta, M)$ can be written as
$x = \delta(r_1, x_1) * \delta(r_2, x_2) * .... * \delta(r_n, x_n)$, where $r_1, r_2, ...r_n \in S$.

Definition 5.6.14. A topological weak S-module is a (δ, M, τ), where τ is a topology on (δ, M) such that

i. $* : M \times M \to M$ is continuous

ii. $\delta_a : M \to M$ defined by $\delta_a(x) = \delta(a, x)$ is continuous for every $a \in S$.

Definition 5.6.15. A morphism between topological weak S-module (δ, M, τ_1), (γ, M, τ_2) is a map $h : (\delta, M, \tau_1) \to (\gamma, M, \tau_2)$ such that

i. $h(x * y) = h(x) * h(y)$

ii. $h(\delta(a, x)) = \gamma(a, h(x))$ for all $x, y \in M, a \in S$.

iii. h is continuous.

5.7. Relation between the categories L-slice and TopWMod

Let **L-slice** denotes the category of L-slices and L-slice homomorphisms.

Proposition 5.7.1. *Let $f : (\sigma, J) \to (\mu, K)$ be an injective L-slice homomorphism. If image $imf = \downarrow z$, where $z \in (\mu, K)$ is a maximal element of (μ, K), then f is a section in the category **L-slice**.*

Proof. Define $g : (\mu, K) \to (\sigma, J)$ as follows.

Let $y \in (\mu, K)$. If $y \in imf$, then $y = f(x)$ for a unique $x \in (\sigma, J)$. Then define

$g(y) = x$. If $y \notin imf$, define $g(y) = 0_J$. Then $g : (\mu, K) \to (\sigma, J)$ is an L-slice homomorphism and $(g \circ f)(x) = x$, for all $x \in (\sigma, J)$. Hence f is a section in the category **L-slice**. □

Proposition 5.7.2. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is a retraction in the category **L-slice** if and only if f is onto.*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be a retraction in the category **L-slice**.
Let $y \in (\mu, K)$. Since $f : (\sigma, J) \to (\mu, K)$ is a retraction, there is an L-slice homomorphism $g : (\mu, K) \to (\sigma, J)$ such that $(f \circ g) = I$. Hence $f(g(y)) = y$ and so f is onto. Conversely let $f : (\sigma, J) \to (\mu, K)$ be on onto L-slice homomorphism. For each $y \in (\mu, K)$, there is some $x \in (\sigma, J)$ such that $y = f(x)$. Define $g : (\mu, K) \to (\sigma, J)$ by $g(f(x)) = x$. Then we have g is an L-slice homomorphism and $(f \circ g)(y) = y$, for all $y \in (\mu, K)$. Hence f is a retraction in the category **L-slice**. □

In a similar manner we can show the following propositions.

Proposition 5.7.3. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is a monomorphism in the category **L-slice** if and only if f is injective.*

Proposition 5.7.4. *Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism. Then f is an epimorphism in the category **L-slice** if and only if f is surjective.*

Topological weak L-module associated with a L-slice

Let (σ, J) be an L-slice with bottom element $\mathbf{0}$. Let $Pt(J) = \{\downarrow x : x \in (\sigma, J)\}$. Define the binary operation $*$ on $Pt(J)$ by $\downarrow x * \downarrow y = \downarrow x \vee y$. Then $(Pt(J), *, \mathbf{0})$ is a commutative monoid. Define $\delta : L \times Pt(J) \to Pt(J)$ by $\delta(a, \downarrow x) = \downarrow \sigma(a, x)$. In the next proposition, we will show that δ is an action of the semiring L on the monoid $(Pt(J), *, \mathbf{0})$.

Proposition 5.7.5. *$(\delta, Pt(J))$ is a weak L-module.*

Proof. $\delta : L \times Pt(J) \to Pt(J)$ is defined by $\delta(a, \downarrow x) = \downarrow \sigma(a, x)$.

i. $\delta(a + b, \downarrow x) = \delta(a \sqcup b, \downarrow x) = \downarrow \sigma(a \sqcup b, x) = \downarrow (\sigma(a, x) \vee \sigma(b, x))$

$$= \downarrow \sigma(a, x) * \downarrow \sigma(b, x) = \delta(a, \downarrow x) * \delta(b, \downarrow x)$$

ii. $\delta(a, \downarrow x * \downarrow y) = \delta(a, \downarrow x \vee y) = \downarrow \sigma(a, x \vee y) = \downarrow (\sigma(a, x) \vee \sigma(a, y))$

$$= \downarrow \sigma(a, x) * \downarrow \sigma(a, y) = \delta(a, \downarrow x) * \delta(a, \downarrow y)$$

iii. $\delta(a, \mathbf{0}) = \downarrow \sigma(a, 0) = \downarrow 0 = \mathbf{0}$

iv. $\delta(a.b, \downarrow x) = \delta(a \sqcap b, x) = \downarrow \sigma(a \sqcap b, x) = \downarrow \sigma(a, \sigma(b, x))$

$$= \delta(a, \downarrow \sigma(b, x)) = \delta(a, \delta(b, \downarrow x))$$

v. $\delta(0, \downarrow x) = \downarrow \sigma(0, x) = \downarrow 0 = \mathbf{0}$

$\delta(1, \downarrow x) = \downarrow \sigma(1, x) = \downarrow x$

Hence $(\delta, Pt(J))$ is a weak L-module. □

For each $x \in (\sigma, J)$ define $\lambda_x = \{\downarrow y \in Pt(J) : x \in \downarrow y\}$.

Proposition 5.7.6. *Let (σ, J) be an L-slice and $x, y \in (\sigma, J)$. Then*

i. $\lambda_{\mathbf{0}} = Pt(J)$

ii. $\lambda_x \cap \lambda_y = \lambda_{x \vee y}$.

Proof. i. $\lambda_{\mathbf{0}} = \{\downarrow y \in Pt(J) : \mathbf{0} \in \downarrow y\}$. Since ideal of a slice is closed under taking lower elements, $\mathbf{0} \in \downarrow y$, for every $\downarrow y \in Pt(J)$. Hence $\lambda_{\mathbf{0}} = Pt(J)$.

ii. $\downarrow z \in \lambda_x \cap \lambda_y \Rightarrow \downarrow z \in \lambda_x$ and $\downarrow z \in \lambda_y$

$\Rightarrow x \in \downarrow z$ and $y \in \downarrow z$

$\Rightarrow x \leqslant z$ and $y \leqslant z$

$\Rightarrow x \vee y \leqslant z$

$\Rightarrow \downarrow z \in \lambda_{x \vee y}$

Hence $\lambda_x \cap \lambda_y \subseteq \lambda_{x \vee y}$.

$\downarrow z \in \lambda_{x \vee y} \Rightarrow x \vee y \leqslant z$

$\Rightarrow x, y \leqslant x \vee y \leqslant z$

$\Rightarrow \downarrow z \in \lambda_x$ and $\downarrow z \in \lambda_y$

$\Rightarrow \downarrow z \in \lambda_x \cap \lambda_y$

Thus $\lambda_{x \vee y} \subseteq \lambda_x \cap \lambda_y$. Hence $\lambda_x \cap \lambda_y = \lambda_{x \vee y}$. □

By above proposition $B = \{\lambda_x : x \in J\}$ is closed under finite intersection and hence B is a base for a unique topology τ on Pt(J).

Proposition 5.7.7. *$(\delta, Pt(J), \tau)$ is a topological weak L-module.*

Proof. We have $(\delta, Pt(J))$ is a weak L-module.

Let $f : Pt(J) \times Pt(J) \rightarrow Pt(J)$ be defined by $f(\downarrow x, \downarrow y) = \downarrow x * \downarrow y = \downarrow x \vee y$. We will show that f is continuous with respect to the topology τ.

Let U be any open set containing $f(\downarrow x, \downarrow y) = \downarrow x * \downarrow y = \downarrow x \vee y$. Then there exist a basic open set λ_z such that $\downarrow x \vee y \in \lambda_z$ and $\lambda_z \subseteq U$.

$\downarrow x \vee y \in \lambda_z$ implies that $x \vee y \geqslant z$. By construction λ_x, λ_y are open set containing $\downarrow x, \downarrow y$ respectively. Now we will show that $f(\lambda_x \times \lambda_y) \subseteq U$.

Let $\downarrow a \in \lambda_x, \downarrow b \in \lambda_y$. Then $x \leqslant a, y \leqslant b$. $f(\downarrow a, \downarrow b) = \downarrow a * \downarrow b = \downarrow a \vee b$.

But $x \leqslant a, y \leqslant b$ implies that $x \vee y \leqslant a \vee b$. Hence $z \leqslant x \vee y \leqslant a \vee b$ or $z \leqslant a \vee b$.

Hence $\downarrow a \vee b \in \lambda_z$. Thus $f(\lambda_x \times \lambda_y) \subseteq \lambda_z \subseteq U$. Hence $f : Pt(J) \times Pt(J) \rightarrow Pt(J)$ is continuous with respect to the topology τ.

Now we will show that for every $a \in L$ the map $\delta_a : Pt(J) \to Pt(J)$ defined by $\delta_a(\downarrow x) = \delta(a, \downarrow x) = \downarrow \sigma(a, x)$ is continuous. For any basic open set λ_x,

$$\begin{aligned}
\delta_a^{-1}(\lambda_x) &= \{\downarrow z \in Pt(J) : \delta_a(\downarrow z) = \delta(a, \downarrow z) \in \lambda_x\} \\
&= \{\downarrow z \in Pt(J) : \downarrow \sigma(a, z) \in \lambda_x\} \\
&= \{\downarrow z \in Pt(J) : x \leqslant \sigma(a, z) \leqslant z\} \\
&= \lambda_x
\end{aligned}$$

Thus δ_a is continuous with respect to the topology τ. Hence $(\delta, Pt(J), \tau)$ is a topological weak L-module. □

Proposition 5.7.8. *If $f : (\sigma, J) \to (\mu, K)$ is an L-slice homomorphism, then there is a morphism $\phi : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ in the category **TopWMod** of topological weak L-modules.*

Proof. Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism.
Define $\phi : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ by $\phi(\downarrow y) = \downarrow f^{-1}(y)$.

$$\begin{aligned}
\phi(\downarrow y *' \downarrow z) &= \phi(\downarrow y \vee' z) = \downarrow f^{-1}(y \vee' z) = \downarrow f^{-1}(y) \vee f^{-1}(z) \\
&= \downarrow f^{-1}(y) * \downarrow f^{-1}(z) = \phi(\downarrow y) * \phi(\downarrow z) \\
\phi(\rho(a, \downarrow x)) &= \phi(\downarrow \mu(a, x)) = \downarrow f^{-1}(\mu(a, x)) = \downarrow \sigma(a, f^{-1}(x)) \\
&= \delta(a, \downarrow f^{-1}(x)) = \delta(a, \phi(\downarrow x))
\end{aligned}$$

Now we will show that the map ϕ is continuous. Let λ_z be an open set containing $\phi(\downarrow x) = \downarrow f^{-1}(x)$. Then $\downarrow f^{-1}(x) \in \lambda_z$ and so $z \leqslant f^{-1}(x)$. Thus $f(z) \leqslant x$ and so $\downarrow x \in \lambda_{f(z)}$. Thus $\lambda_{f(z)}$ is an open set containing $\downarrow x$. We will show that $\phi(\lambda_{f(z)}) \subseteq \lambda_z$. Let $\downarrow a \in \lambda_{f(z)}$. Then $f(z) \leqslant a$. Now $\phi(\downarrow a) = \downarrow f^{-1}(a)$. $f(z) \leqslant a$ implies that

$z \leqslant f^{-1}(a)$. Hence $\downarrow f^{-1}(a) \in \lambda_z$. Thus $\phi(\lambda_{f(z)}) \subseteq \lambda_z$. Hence ϕ is continuous. Thus ϕ is a morphism in the category **TopWMod**. □

Proposition 5.7.9. *There is contravariant functor from the category* ***L-slice*** *to the category* ***TopWMod****.*

Proof. Define $\Psi : Ob(\textbf{L-slice}) \to Ob(\textbf{TopWMod})$ by $\Psi(J) = Pt(J)$.
Also define $\Psi : Mor(\textbf{L-slice}) \to Mor(\textbf{TopWMod})$ as follows.
Let $f : (\sigma, J) \to (\mu, K)$ be an L-slice homomorphism.
Define $\Psi(f) : (\rho, Pt(K), \tau_2) \to (\delta, Pt(J), \tau_1)$ by $\Psi(f)(\downarrow x) = \downarrow f^{-1}(x)$. Then by above proposition $\Psi(f) \in Mor(\textbf{TopWMod})$.
If $f : (\sigma, J) \to (\mu, K)$ and $g : (\mu, K) \to (\upsilon, K')$ be L-slice homomorphisms.

$$\begin{aligned} \Psi(g \circ f)(\downarrow x) &= \downarrow (g \circ f)^{-1}(x) = \downarrow f^{-1}(g^{-1}(x)) = \Psi(f)(\downarrow g^{-1}(x)) \\ &= \Psi(f)(\Psi(g)(\downarrow x)) = \Psi(g) \circ \Psi(f)(\downarrow x) \end{aligned}$$

Let $id : (\sigma, J) \to (\sigma, J)$ be an identity morphism in **L-slice**.
Then $\Psi(id)(\downarrow x) = \downarrow id^{-1}(x) = \downarrow x$. Hence $\Psi(id)$ is an identity morphism in **TopWMod**. This shows that Ψ is a contravariant functor from the category **L-slice** to the category **TopWMod**. □

Proposition 5.7.10. *The functor Ψ maps the subcategory* ***FinL-slice*** *of finitely generated L-slices of* ***L-slice*** *into the subcategory* ***FinTopWMod*** *of finitely generated topological weak modules of* ***TopWMon****.*

Proof. If the L-slice (σ, J) is finitely generated, then the weak L-module $(\delta, Pt(J))$ is finitely generated. Hence Ψ maps the subcategory **FinL-slice** into the subcategory **FinTopWMod**. □

Chapter 6

Extended Diffie Hellman Key Exchange Protocol Using L-Slices of a Locale L

A basic principle of cryptography is formulated by Auguste Kerckhoffs [38] in 1883 and is reformulated by Claude Shannon [44]. In a cryptosystem, the only unknown to an attacker is the key used. The cryptosystem is constituted with attacker model in mind in order to make the system more secure, but depends on changing keys on regular basis. The problem of providing both parties with secret key beforehand for any secured communication, had its first solution provided by Whitfield Diffie and Martin Hellman [7].

In this chapter we have developed a key exchange protocol that utilizes the concept of L-slices for the generation of secret and public keys. The L-slice and its properties are utilized to extend the existing Diffie Hellman key exchange protocol that uses groups in algebra to the background of L-slices of a locale L. A modification is given to the extended Diffie Hellman key exchange protocol using L-slices of a locale L inorder to give optimum security to the system.

6.1. Key Exchange Protocol based on L-slice

In this section, we present an extension of Diffie-Hellman key exchange protocol[7] to the back ground of L-slices of a locale L.

Key Exchange Protocol based on L-slice

0. Setup:

Alice and Bob concur on the protocol specifies and these involve the L-slice (σ, J), $x \in (\sigma, J)$.

1. Generation of secret and public keys

Both parties select $i_A = a \in L, i_B = b \in L$ as their secret keys. Their public keys are the maps

$$\begin{aligned} c_A &= \sigma(a, x) \\ c_B &= \sigma(b, x) \end{aligned}$$

2. Interchange public keys

Alice and Bob interchange their public keys c_A, c_B.

3. Computing the shared key

After getting c_B from Bob, Alice computes

$$K_A = \sigma(a, c_B) = \sigma(a, \sigma(b, x)) = \sigma(a \sqcap b, x)$$

Bob similarly computes,

$$K_B = \sigma(b, c_A) = \sigma(b, \sigma(a, x)) = \sigma(a \sqcap b, x)$$

The correctness of protocol follows as $K_A = K_B$.

Mathematical aspects of above protocol

From Proposition 5.5.5 for each $x \in (\sigma, J)$, $F_x = \{l \in L; \sigma(l, x) = x\}$ is a filter in L. Thus it is hard to find a particular $l \in L$ such that $\sigma(l, x) = x$. In the same way, given $x, y \in (\sigma, J)$, it is hard to find $l \in L$ with $\sigma(l, x) = y$. Hence recovering secret keys i_A, i_B from public keys c_A, c_B is very hard, as the action $\sigma : L \times J \to J$ is not an invertible function.

The task of recovering the secret key from public key is equivalent to the following problem.

Problem

Given an L-slice (σ, J) and two elements $y, z \in (\sigma, J)$, find an element $l \in L$ such that $\sigma(l, y) = z$.

The solution of this problem is not necessarily unique and we define the set of all solutions as $L_{\sigma_y} = \{l \in L : \sigma_y(l) = z\} = \{l \in L : \sigma(l, y) = z\}$. The solution set L_{σ_y} represents the level set of the L-slice homomorphism $\sigma_y : (\sqcap, L) \to (\sigma, J)$.

If the attacker Eve get some $\alpha \in L$ such that $\sigma(a, x) = \sigma(\alpha, x)$, then Eve can calculate the shared secret $K_a = K_b$ from it as follows. Using the public key of Bob, Eve calculates,

$$\sigma(\alpha, \sigma(b, x)) = \sigma(b, \sigma(\alpha, x)) = \sigma(b, \sigma(a, x)) = \sigma(a \sqcap b, x) = K_a$$

If the level set L_{σ_x} contains more elements, then the probability of getting $\alpha \in L$ such that $\sigma(a, x) = \sigma(\alpha, x)$ is high. Thus the security of the system depends on the cardinality of L_{σ_x}.

If cardinality of $L_{\sigma_x} > n$, where n is a small integer, we can increase the security by adopting the following method for generation of keys.

Let R be the natural congruence associated with the L-slice homomorphism $\sigma_x : (\sqcap, L) \to (\sigma, J)$ and let $(\sqcap, L/R)$ be the corresponding quotient L-slice of the L-slice $(\sqcap, L)$. Then (δ, J) is an L/R-slice, where $\delta : L/R \times J \to J$ is defined by $\delta([a], x) = \sigma(a, x)$.

If cardinality of $L_{\sigma_x} > n$, where n is a small integer, we use modified form of Key Exchange Protocol based on L-slice of a locale L.

6.2. Modified key exchange protocol

i. Setup:

Alice and Bob concur on the protocol specifies, these involve the L/R-slice (δ, J), $x \in (\delta, J)$.

ii. Generation of Public/Private keys

Both parties select $i_A = [a], i_B = [b] \in L/R$ as their secret keys. Their public keys are the maps

$$
\begin{aligned}
c_A &= \delta([a], x) = \sigma(a, x) \\
c_B &= \delta([b], x) = \sigma(b, x)
\end{aligned}
$$

iii. Interchange of public keys

Alice and Bob interchange their common keys c_A, c_B.

iv. Computinging the shared key

After getting c_B from Bob, Alice computes

$$K_A = \delta([a], c_B) = \sigma(a, c_B) = \sigma(a, \delta([b], x)) = \sigma(a, \sigma(b, x)) = \sigma(a \sqcap b, x)$$

Bob similarly computes,

$$K_B = \delta([b], c_A) = \sigma(b, c_A) = \sigma(b, \delta([a], x)) = \sigma(b, \sigma(a, x)) = \sigma(a \sqcap b, x)$$

In this case the problem of retrieving secret keys from common key is equivalent to the problem.

Problem

Given an L/R-slice (δ, J) and two elements $x, z \in (\sigma, J)$, find an element $l \in L$ such that $\sigma(l, x) = z$. The solution set of this problem is
$L/R_{\delta_x} = \{[l] \in L/R : \sigma_x([l]) = z\} = \{[l] \in L/R : \sigma(l, x) = z\}$ and this set contains a unique element of L/R. Hence the probability of getting $[\alpha] \in L/R$ such that $\sigma(\alpha, x) = \sigma(a, x)$ is very small and so it is very hard to find shared secret from common keys.

We can extend ElGamal encryption [14] procedure into L-slice background as follows.

Extension of ElGamal Encryption

Let (σ, J) be an L-slice,where J is a vector lattice. Then we can extend ElGamal encryption [14] to the background of L-slice as follows.

i. The secret and common keys are generated and exchanged using the key exchange protocol discussed in 3.1.

ii. For every message m_i, Bob calculates the pair $(\sigma(b, x), m_i + \sigma(a \sqcap b, x))$

iii. Alice can decrypt the message using $m_i = m_i + \sigma(a \sqcap b, x) - \sigma(a \sqcap b, x)$

Conclusion

In the existing context of theory of topological semigroups, topological groups, topological lattices, topological vector spaces and so on, the development of these theories pertain to points, their neighbourhoods and their local behaviour, where as in the set up of the theory of locales which are also called generalized topological spaces, we have the background of point free topology. This framework is used, in the study in our thesis, to develop the notion of an action σ of a locale L on a join semilattice J with bottom element 0_J to form the entity (σ, J), which we call L-slice, that has properties which could be studied algebraically as well as topologically.

Various properties of an L-slice (σ, J) of a locale L are investigated. The action σ of the locale L on the join semilattice J is utilized to construct sublocales of L. An L-slice congruence R is defined and a quotient L-slice $(\gamma, J/R)$ with respect to the congruence R is obtained. An Isomorphism theorem for L-slices of the locale L is derived and as an application, it is proved that every finitely generated L-slice with n generators is isomorphic to the quotient slice of $(\sqcap, L^n)$.

For $a \in L, x \in J$, various properties of the L-slice homomorphisms $\sigma_a : (\sigma, J) \to (\sigma, J), \sigma_x : (\sqcap, L) \to (\sigma, J)$ have been studied. Properties of the fixed set of $\sigma_a : (\sigma, J) \to (\sigma, J)$ are discussed. The property of compactness in the L-slice (σ, J) is defined and is characterized in terms of the filter $F_x = \{a \in L : \sigma_x(a) = x\}$.

It has been shown that L-slice compactness is stronger than topological compactness and localic compactness.

It is known that there is a contravariant functor from the category **JSLat** of join semilattice with 0, and semilattice homomorphism to the category **iTopMon** of idempotent topological monoids, and continuous monoid homomorphisms. In this study, the existence of a contravariant functor from the category **L-slice** of L-slices and L-slice homomorphisms to the category **TopWMod** of topological weak modules and continuous weak module homomorphisms has been established.

Several intermediary results were obtained during the above studies.

As an application, a key exchange protocol that uses the concept of L-slice for generation of secret and public keys are developed. As the action σ of a locale L on a join semilattice J is not an invertible function, it is very hard to find secret keys from publicly known common keys. Hence this method gives a more secure cryptosystem.

There is ample scope for further studies in the background of above investigations. The topological properties such as separation axioms, countable compactness, connectedness etc. are to be analyzed in the context of L-slice. Viewing topology as theory of information, the properties of L-slice could be used as an effective tool in image processing and mathematical morphology. The connection between the concept of L-slice and semantics of programming language could be developed.

Research Papers

1. Sabna K.S, Mangalambal N.R, *An Embedding Theorem for Locales*, Global Journal of Pure and Applied Mathematics, Vol 13, Number 7 (2017) , Research India Publication, 35193530 .

2. Sabna K.S, Mangalambal N.R, *Vietoris Locale-Using Spectrum*, IOSR Journal of Mathematics, Vol 12, Issue 6, Ver 1, Nov-Dec.2016, 1-3.

3. Sabna K.S, Mangalambal N.R, *New ideals containing the kernel of a frame homomorphism*, International Journal of Theoretical and Computational Mathematics, Vol 2, No.2, November 2016, ISSN:2395-6607, 29-33.

4. Mary Elizabeth Antony, Sabna K.S, Mangalambal N.R, *Some notes on Second countability in Frames*, IOSR Journal of Mathematics, Vol 9, Issue 2, Nov-Dec.2013, 29-32.

5. Sabna K.S, Mangalambal N.R, *Compact Subspace of Spectrum of O(L) from spectrum of a locale L*, accepted to be published as a chapter in the book titled *Advanced Mathematics : Theory and Applications (AMTA)*, Research India Publication.

6. Sabna K.S, Mangalambal N.R, *Unique Sublocales from Ideals of a Locale*, Communicated.

7. Sabna K.S, Mangalambal N.R, *Fixed points with respect to the L-slice homomorphism* σ_a, Communicated.

8. Sabna K.S, Mangalambal N.R, *A Subspace of Spectrum of L with respect to L-slice for a locale L*, Communicated.

9. Sabna K.S, Mangalambal N.R, *An Isomorphism theorem for L-Slice of a locale L*, Communicated.

10. Sabna K.S, Mangalambal N.R, *Relation between the categories **L-slice** and **Top-WMod***, Communicated.

11. Sabna K.S, Mangalambal N.R, *Extended Diffie Hellman Key Exchange Protocol Using L-Slices of a Locale L*, Communicated.

12. Sabna K.S, Mangalambal N.R, *Compact, Connected,* T_0 *Subspace of Spectrum of* $O(L)$ *from spectrum of a locale L*, Communicated.

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