Ph.D. THESIS **MATHEMATICS**

A STUDY OF BOUNDARY REPRESENTATIONS AND HYPERRIGIDITY OF OPERATOR SPACES AND OPERATOR SYSTEMS

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CERTIFICATE

I hereby certify that the thesis entitled "**A STUDY OF BOUNDARY REPRESENTATIONS AND HYPERRIGIDITY OF OPERATOR SPACES AND OPERATOR SYSTEMS**" is a bonafide work carried out by **Mr. Arunkumar C.S.,** under my guidance for the award of Degree of Ph.D., in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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(Research Supervisor)

DECLARATION

I hereby declare that the thesis, entitled "**A STUDY OF BOUNDARY REPRESENTATIONS AND HYPERRIGIDITY OF OPERATOR SPACES AND OPERATOR SYSTEMS**" is based on the original work done by me under the supervision of Dr. A. K. Vijayarajan, Professor, Kerala School of Mathematics and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Contents

Chapter 1

Introduction

1.1 Motivation and Survey of Literature

Noncommutative approximation and extremal theories initiated by Arveson [10] in the context of operator systems in C^*-algebras have seen tremendous growth in the recent past. The extremal theory concerning Choquet boundary of uniform algebras is a very important tool in classical analysis. Let Ω be a compact Hausdorff space and $\mathcal{U}$ be a uniform algebra in $C(\Omega)$. A point $\omega \in \Omega$ is said to be a Choquet boundary point of $\mathcal{U}$ if the evaluation function δ_ω corresponding to ω has unique extension from $\mathcal{U}$ to $C(\Omega)$. The Choquet boundary theory has a lot of applications to other areas of mathematics such as approximation theory, measure theory, Markov process, several variable complex analysis and etc.

Arveson [4] presented the non-commutative counterpart of Choquet boundary of function systems for operator systems in C^*-algebras and called it as boundary representations. Let M be a linear subspace of a C^*-algebra B and $\rho : B \to B(H)$ be an irreducible representation. Then ρ is said to be a boundary representation for M if the completely positive map $\rho_{\restriction M}$ has exactly one completely positive extension to B, namely ρ itself. Arveson introduced boundary representations to analyse a general problem that to what extent does an algebra of operators on a Hilbert space determine the structure of the C^*-algebra it generates. Arveson obtained a nice characterization of boundary representations and showed that it has interesting applications to operator theory [5]. Later, he found that boundary representations can be used to construct non commutative Silov boundary. More precisely, the C^*-algebra generated by the direct sum of images of boundary representations enjoys certain universal property and gives a realization of the C^*-envelope(the non-commutative Silov boundary). Even so, the existence of boundary representations was left open for many years.

Hamana [35, 36] in 1979 proved the existence of C^*-envelope for operator systems and existence of triple envelope for operator spaces. His proofs rely on the theory of injective envelopes but not the notion of boundary representations. In 1998, an algebraic characterization of boundary representations in terms of Hilbert modules was given by Muhly and Solel [53]. They actually studied representations with unique extension property by dropping the irreducibility condition from the definition of boundary representations. Using Hamana's technique, Muhly and Solel proved that representations with unique extension property for operator algebras are presicely those completely contractive representations that determine modules that are both orthogonally projective and orthogonally injective.

Inspired by the papers of Agler [1] on model theory, Dritschel and McCullough [27] in 2005 come up with an important development in this direction by showing that every UCP map of an operator system into the concrete C^*-algebra $B(H)$ can be dilated to a maximal UCP-map. This yeilds another proof of the realization of the C^*-envelop without using the theory of injective envelops. But the existence of boundary representations were not yet proved.

In the separable case, the existence of sufficiently many boundary representations to completely norm the operator system was resolved by Arveson [8] in 2008. That is, for operator system S in a C^*-algebra A and every $[a_{ij}] \in M_n(S)$ we have

$$\|[a_{ij}]\| = \sup_{\pi} \|\pi([a_{ij}])\| \;\; \forall\, n \geq 1,$$

where the supremum is taken all over the boundary representations π of $C^*(S)$ for S. Here the key ideas were the disintegration theory of C^*-algebras and the work of Dritschel and McCullough mentioned above.

Kleski [45] in 2014 showed that in the above result the 'sup' can be substituted by 'max'. This would imply that the non commutative Choquet boundary is a boundary in the classical sense.

In 2015, Davidson and Kennedy [23] established the existence of boundary representations. They proved that an operator system S that generates the C^*-algebra $C^*(S)$ has sufficiently many boundary representations to completely norm it. In fact their results don't assumes separability.

In 2017, Magajna investigated boundary representations in the setting of Hilbert C^*-modules over abelian von Neumann algebras and used it to examine C^*-extreme points. In the same spirit of Davidson and Kennedy [23], Magajna proved that certain pure maps will have appropriate dilation to boundary representations and hence established the natural analogue of Arveson's conjecture for certain operator systems in the new context: let $\mathcal{Z}$ be an abelian von Neumann algebra and S be a central operator system generating a C^*-algebra A, then the $\mathcal{Z}$-boundary representations of A for S on self dual C^*-modules over $\mathcal{Z}$ completely norm S.

Fuller, Hartz and Lupini [33] in 2018 initiate the study of non commutative Choquet boundary in the setting of operator spaces. The operator space counterpart of Arveson's conjecture is then settled: any operator space is completely normed by its boundary representations. Their result provides a clear-cut interpretation of the non commutative Silov boundary of operator spaces in terms of non commutative choquet boundary. They also introduced matrix convexity for operator spaces and proved the following dilation theoretic result that connects extreme points with boundary representations. The extreme points of the rectangular matrix convex set of all completely contractive maps from an operator space X into $B(H, K)$ can be dilated to boundary representations of the operator space X.

In 2011, Arveson introduced non-commutative Korovkin sets and studied its connections with boundary representations.

Korovkin theorem [47] concerns the convergence of positive linear maps on function algebras. The classical Korovkin theorem is as follows: for each $n \in \mathbb{N}$, let

$\Phi_n : C[0,1] \to C[0,1]$ be a positive linear map. If $\lim_{n\to\infty} ||\Phi_n(f) - f|| = 0$ for every $f \in \{1, x, x^2\}$, then $\lim_{n\to\infty} ||\Phi_n(f) - f|| = 0$ for every $f \in C[0,1]$. The set $G = \{1, x, x^2\}$ is called Korovkin set in $C[0,1]$. One may appreciate this theorem for its potentiality and powerfulness as well as for the simplicity of its proof. The Korovkin theorem impressed many mathematicians as the positive approximation has been instrumental in approximation theory, and it emerges naturally in various problems that use an approximation of continuous functions.

During the last fifty years, several authors have studied Korovkin's theorems in different settings. Concurrently, prolific and powerful relations of Korovkin's theory have been discovered with several fields of analysis, such as functional analysis, probability theory, partial differential equations, etc.

An important development was the revelation of geometric theory of Korovkin's sets by Saskin [70] in 1966 and Wulbert [81] in 1968. The article [12] of Berens and Lorentz sorted out the growth of the theory in detail. Also, some special topics of the theory were elucidated in the papers [2, 3].

The investigation of non commutative Korovkin's theorems for the setting of C^*-algebras was started by Priestley [63] in 1976. He proved the following; for a unital C^*-algebra A , if $\{\psi_m\}_{m\in\mathbb{N}}$ is a sequence of positive linear maps from A to A satisfying $\psi_m(1) \leq 1$ for all m, then

$$J = \{x \in A : x = x^*, \psi_m(x) \to x, \psi_m(x^2) \to x^2\}$$

is a J^*-algebra (Jordan $*$-algebra), where the convergence can be in operator norm topology(or in weak operator topology, or in strong operator topology). When A is the C^*-algebra of trace class operators on $B(H)$, the above theorem is also true for the trace norm convergence.

Later, several other authors studied Korovkin's theorem to different settings. The articles [49, 57, 64, 78, 80] are also worth mentioning in this context.

Saskin [70] in 1966 studied the geometric theory of Korovkin sets in the classical case. He established a very strong relation between Korovkin sets and the choquet boundary of function systems as follows. Let $C(\Omega)$ be the algebra of all continuous complex valued functions on a compact Hausdorff space Ω. Let $1 \in S \subseteq C(\Omega)$ and S separates Ω. Then S is a Korovkin set if and only if the choquet boundary of the function system generated by S is whole of Ω.

Arveson [10] in 2011 introduced the non commutative counterpart of Korovkin set, which he called a *hyperrigid set*. Let S be a finite or countably infinite and $A = C^*(S)$. Then S is called hyperrigid if for every faithful representation $A \subseteq B(H)$ of A and every sequence of UCP maps $\psi_m : B(H) \to B(H)$, $m = 1, 2 \cdots$,

$$\lim_{m\to\infty} \|\psi_m(s) - s\| = 0, \ \forall s \in S \Rightarrow \lim_{m\to\infty} \|\psi_m(x) - x\| = 0, \ \forall x \in A.$$

A direct computation shows that a set S is hyperrigid if and only if the subspace spanned by $S \cup S^*$ is hyperrigid. In other words, the set S is hyperrigid if and only if the operator system generated by S is hyperrigid. For this reason we consider operator systems instead of arbitrary subsets of a C^*-algebra. Arveson [10] proved that if a separable operator system S is hyperrigid in $C^*(S)$ then, every irreducible representation of $C^*(S)$ is a boundary representation for S. Along the lines of Saskin's theorem in the classical setting, Arveson [10] formulated *hyperrigidity conjecture* as follows: if every irreducible representation of $C^*(S)$ is a boundary representation for S, then S is hyperrigid. In the same paper, he established the conjecture in a particular case, namely for C^*-algebras with countable spectrum.

Kleski [46] in 2014 explored the extent to which the noncommutative Choquet boundary determines the hyperrigidity for Type I C^*-algebras. He discussed Arveson's hyperrigidity conjecture and obtained structural information about Type I C^*-algebras generated by operator systems when every irreducible representation is a boundary representation for the operator system.

Kennedy and Shalit [44] in 2015 had shown that, in the appropriate context, Arveson's hyperriridity conjecture is equivalent to the *Arveson-Douglas essential normality conjecture* involving quotient modules of the Drury-Arveson space. This equivalence also makes the hyperrigidity conjecture all the more enthralling.

Davidson and Kennedy [24] in 2016 established the hyperrigidity conjecture for function systems in $C(\Omega)$.

Cloautre [18,19] in 2018 analysed states on C^*-algebra to investigate hyperrigidity of operator systems. He [18] studied extension and restriction properties for states and provided supporting evidence for Arveson's hyperrigidity conjecture. Also, Clouatre introduced, what are called *unperforated pairs* of subspaces in C^*-algebras and showed that unperforated pairs of operator spaces constitute a tool that can be used to capture the information about states, and hence to study hyperrigidity. In the succeeding paper

[19] Cloautre investigated the concept of various *peaking* phenomena for states on a C^*-algebra. The idea was to localize the C^*-algebra at a given state using a sequence of elements of the C^*-algebra called the characteristic sequences and verified a local version of Arveson's Hyperrigidity conjecture using that localization procedure.

Salomon [68] in 2018 studied hyperrigidity in the frame of graph C^*-algebras. For a row-finite directed graph $\overrightarrow{G}$ without isolated vertices, the set V_E of isometries assigned to the edge set is hyperrigid in the Cuntz-Krieger algebra associated with $\overrightarrow{G}$. Salomon introduced the property of rigidity at 0 and using this he is able to connect hyperrigidity with unique extension property. He also showed that the C^*-envelope of the operator algebra generated by V_E can be identified with the Cuntz-Krieger algebra associated with $\overrightarrow{G}$.

Motivated by the example of operator system spanned by the unilateral right shift operator on a separable Hilbert space, Shankar and et. al. [56] in 2018 introduced an interesting weaker notion of boundary representations and hyperrigidity called weak boundary representations and quasi hyperigidity, respectively. The Saskin's theorem in the new setting was proved for a particular class of C^*-algebras. Also, Shankar and Vijayarajan [72] established the Hilbert module characterisation of hyperrigidity in a special case using the techniques of Muhly and Solel [52]. Also, they tried to tackle Arveson's hyperrigidity conjecture in these weaker settings but not able to solve fully.

In literature, there were other weaker notions of hyperrigid sets. Limaye and Namboodiri [51] in 1984 introduced weak Korovkin set in $B(H)$ by replacing norm convergence by weak operator topology convergence. Inspired by Arveson's boundary theorem, they proved that; an irreducible set of operators S in $B(H)$ is a weak Korovkin set if and only if identity is in the non commutative Choquet boundary of S.

Namboodiri [54] gave a short survey of the developments Korovkin-type theory in the non-commutative setting. Namboodiri [55] in 2012, introduced weak hyperrigid sets where the norm convergence is replaced by weak convergence, and established certain relationship of weak hyperrigid operator systems with the non commutative boundary.

However, the Arveson's hyperrigidity conjecture is still not fully settled and is an active research area. This motivate us to study boundary representations and hyperrigidity. Once a conjecture is not proved, one may introduce weaker notions of the original notions and use it to analyse the conjecture. In this line of thoughts we study weak boundary representations and quasi hyperrigidity. Another interesting line of research is to study the conjecture in new settings of current interest. As the notion of hyperrigidity is not previously explored in the non self adjoint setting of operator spaces, we study boundary representations and hyperrigidity for operator spaces. Also we explored boundary representations for the setting of spaces of unbounded operators, whereas due to the unavailability of BW-topology the notion of hyperrigidity in this case calls for future study.

1.2 Organisation of the Thesis

Boundary representations and hyperrigidity introduced by Arveson [4, 10] are well studied in the literature for the setting of operator systems in C^*-algebras. This thesis studies a weaker notion of boundary representations and hyperrigidity for operator systems, boundary representations and hyperrigidity for operator spaces, and boundary representations for spaces of unbounded operators(that is, in the context of locally C^*-algebras).

In Chapter 1, we indicate the motivation for the thesis problem and a brief survey of the literature of the previous works on the topics of boundary representations and hyperrigidity by various authors.

In Chapter 2, we recall the preliminaries from the theory of C^*-algebras and completely positive maps for our discussions on boundary representations and hyperrigidity. In Section 2.1, we recall the definitions of the notions of C^*-algebras, operator systems, operator spaces and the representations of C^*-algebras. In Section 2.2, we introduce the notion of ternary ring of operators and their representations. In Section 2.3, we explain the concept of completely positive maps on operator systems and completely contractive maps on operator spaces. The well known theorems like Stinespring's theorem and Arveson's extension theorem for completely positive maps are cited. In Section 2.4, we provide Paulsen's construction of an operator system, called Paulsen system associated to each operator space. In Section 2.5, we introduce the notion of Choquet boundary in the classical setting and the notion of non-commutative

Choquet boundary (boundary representations) for the non-commutative setting. We cite the major developments in proving the existence of boundary representations. In Section 2.6, we state the theorems of Korovkin and Saskin in the classical setting. The concept of Korovkin sets and its non-commutative counter part, called hyperrigid sets are explained. The Arveson's hyperrigidity conjecture connecting boundary representations and hyperrigidity are discussed. Also, we pointed out the various partial answers to hyperrigidity conjecture existing in the literature.

In Chapter 3, we study the amplifications of a weaker notion of boundary representations and hyperrigidity for operator systems in C^*-algebras. In Section 3.1, we recall the tensor products of different notions such as C^*-algebras, operator systems, CP-maps and representations. In Section 3.2, we describe the notions of weak unique extension property and tensor products. In Section 3.3, we study amplifications of weak boundary representations and quasi hyperrigidity [56] for operator systems in C^*-algebras. It's shown that an operator system is quasi hyperrigid if and only if all of its amplifications are quasi hyperrigid. This actually gives a partial answer to the following question: Two operator systems S_1 and S_2 are quasi hyperrigid in their generated C^*-algebras if and only if the tensor product $S_1 \otimes S_2$ is quasi hyperrigid in its generated C^*-algebra ?

In Chapter 4, we study the concept of boundary representations and hyperrigidity for operator spaces. In Section 4.1, we handle boundary representation for operator spaces and that of the Paulsen system. We deduce that boundary representations of an operator space are in one to one correspondence with the boundary representations of the associated Paulsen system. In Section 4.2, we introduce weak boundary representation for operator spaces and prove that a weak boundary representation for an operator space induces a weak boundary representation for the corresponding Paulsen system and vice versa. In section 4.3, finite representation for operator spaces and separating operator spaces are introduced. We prove a characterisation theorem for boundary representations; a map ϕ is a boundary representation for an operator space X if and only if ϕ is a rectangular operator extreme point for X, ϕ is a finite representation for X and X separates ϕ. In section 4.4, we introduce the notion of rectangular hyperrigidity for operator spaces in ternary ring of operators(TRO). We prove that if an operator space is rectangular hyperrigid in the TRO generated by the operator space, then every irreducible representation of the TRO is a boundary representation for the operator space. A partial answer for the converse of the above result is also provided which is a version of classical Saskin's theorem in this setting. A relation between rectangular hyperrigidity of an operator space and hyperrigidity of the corresponding Paulsen system is also established.

In Chapter 5, we initiate a study of non-commutative choquet boundary in the setting of spaces unbounded operators and in a more general setting of locally C^*-algebras. Section 5.1 contains preliminary definitions and basic results on locally C^*-algebras and local completely positive maps. In Section 5.2 we prove an analogue of Arveson's extension theorem for local contractive maps on unital linear subspaces of locally C^*-algebras. The Section 5.3 describes the connections between purity of local completely positive maps with irreducible representations. We prove that a local completely positive map is pure if and only if its minimal Stinespring representation is irreducible. In Section 5.4 we introduce unique extension property and boundary representations for unital subspaces in locally C^*-algebras. Examples for the new notions are provided. We have shown that local boundary representations are intrinsic invariants for local operator systems. That is; let S_1 and S_2 be unital linear subspaces of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let $\phi : S_1 \to S_2$ be a unital surjective local completely isometric linear map. Then for every local boundary representation π_1 of $\mathcal{A}_1$, there exists a local boundary representation π_2 of $\mathcal{A}_2$ such that $\pi_2 \circ \phi(a) = \pi_1(a) \; \forall \; a \in S_1$. Also, we characterise boundary representations in terms of purity and linear extreme points of certain convex sets.

In Chapter 6, we consider some questions for further study. The questions are stated concisely.

Chapter 2

Preliminaries

In this chapter, we introduce basic definitions and preliminary results required for the entire discussions in the further chapters. Specifically, here we recall the notions of C^*-algebras and completely positive maps, and standard theorems such as Arveson extension theorem and Stinespring's dilation theorem for CP maps. Also, the notion of Choquet boundary and Korovkin theory in the classical setting as well as in the setting of operator systems in C^*-algebras are explained.

2.1 C*-algebras

We shall always confine ourselves to vector spaces over the field of complex numbers $\mathbb{C}$. We begin with the definition of a Banach algebra.

A *Banach algebra* B is a complete normed linear space B with an additional structure of well defined multiplication in B, denoted by $(b_1, b_2) \to b_1b_2$, which satisfies the following conditions, for all $\lambda \in \mathbb{C}, b_1, b_2, b_3 \in A$:

i. $(b_1b_2)b_3 = b_1(b_2b_3)$;

ii. $(\lambda b_1 + b_2)b_3 = \lambda b_1b_3 + b_2b_3$, $b_3(\lambda b_1 + b_2) = \lambda b_3b_1 + b_3b_2$;

iv. $||b_1b_2|| \leq ||b_1||||b_2|| \; \forall b_1, b_2 \in B$.

A *Banach* **-algebra* B is a Banach algebra B with an involution map $b \to b^*$ that satisfies the following conditions for all $b_1, b_2 \in B, \lambda \in \mathbb{C}$:

i. $(b_1^*)^* = b_1$;

ii. $(b_1 + b_2)^* = b_1^* + b_1^*$;

iii. $(b_1b_2)^* = b_2^*b_1^*$;

iv. $(\lambda b_1)^* = \overline{\lambda}b_1^*$.

A Banach *-algebra B is called a C^**-algebra if it satisfies the* C^**-identity:*

$$||b^*b|| = ||b||^2 \quad \forall b \in B.$$

The following are examples of C^**-algebras.*

Let $C(\Omega)$ *be the algebra of all complex valued continuous functions on a compact Hausdorff space* Ω*, with supremum norm. The map* $f \to \overline{f}$ *is an involution on* $C(\Omega)$*. Thus the algebra* $C(\Omega)$ *is a* C^**-algebra. In fact* $C(\Omega)$ *is a commutative* C^**-algebra.*

The space $B(H)$ of all bounded operators on a complex Hilbert spaces H is a C^*-algebra with involution : $A \to A^*$, where A^* is the adjoint of the operator A.

A subset S_0 of a C^*-algebra A is called self adjoint if $a^* \in S_0$ for each $a \in S_0$.

A self adjoint unital linear subspace S of a C^*-algebra A is called an operator system.

An operator system in $B(H)$ is known as a concrete operator system. An abstract definition of operator systems was designed by Choi Effros [22] in the setting of $*$-vector spaces, and their representation theorem for abstract operator systems provides a concrete realisation of abstract operator systems as operator systems in $B(H)$.

A concrete operator space is a subspace of $B(H)$. Ruan [66] proposed an axiomatic definition for abstract characterisation of operator spaces. The representation theorem of Ruan [66] shows that each abstract operator space can be regarded as a concrete operators space. In this thesis, we mostly consider operator spaces in $B(H, K)$ with the norm induced by the inclusion $B(H, K) \hookrightarrow B(H \oplus K)$, where H and K are Hilbert spaces.

Let A be a C^*-algebra and $\pi : A \to B(H)$ be a homomorphism of the algebra A. Then π is called a representation of A if it preserves adjoint, that is $\pi(a^*) = \pi(a)^*$ for all $a \in A$. A representation π is called non-degenerate if $[\pi(A)H] = H$, where $[\pi(A)H] = \overline{span}\{\pi(a)\xi : a \in A, \xi \in H\}$. The representation π is called faithful if it is injective.

Let H_0 be a subspace of H and $a \in A$. We say H_0 is an invariant subspace for $\pi(a)$ if $\pi(a)(H_0) \subseteq H_0$. A subspace H_0 is called reducing subspace for $\pi(a)$ if $\pi(a)(H_0) \subseteq H_0$ and $\pi(a)(H_0^{\perp}) \subseteq H_0^{\perp}$. Also, if H_0 is an invariant subspace for all $\pi(a)$, $a \in A$, then we say H_0 is invariant subspace for $\pi(A)$. Similarly we define reducing subspaces for $\pi(A)$. The representation π is called irreducible if $\pi(A)$ has no nontrivial proper closed invariant subspaces. In other words, the only invariant closed subspaces of $\pi(A)$ are $H_0 = 0$ and $H_0 = H$. The set of all unitary equivalence classes of irreducible representations of A on a Hilbert space H is called the spectrum of A and we denote it by $\hat{A}$.

2.2 Ternary ring of operators

A ternary ring of operators (TRO) between Hilbert spaces H and K is a norm closed subspace T of $B(H, K)$ such that $xy^*z \in T$ for all $x, y, z \in T$. A TRO T always carries an operator space structure as a closed subspace of $B(H, K)$. A triple morphism between TRO's T_1 and T_2 is a linear map $\phi : T_1 \to T_2$ such that $\phi(xy^*z) = \phi(x)\phi(y)^*\phi(z)$ for all $x, y, z \in T_1$.

Let T be a TRO then $TT^* := \overline{lin}\{xy^* : x, y \in T\}$ is called the left C^*-algebra of T and similarly, $T^*T := \overline{lin}\{x^*y : x, y \in T\}$ the called as right C^*-algebra of T.

From [13, 8.1.17] the linking algebra $\mathcal{L}(T)$ is defined to be the set of 2×2 *matrices:*

$$\mathcal{L}(T) := \begin{bmatrix} TT^* & T \\ T^* & T^*T \end{bmatrix}.$$

Thus any TRO T can be seen as the 1-2 corner of its linking algebra $\mathcal{L}(T)$. Note that the linking algebra $\mathcal{L}(T)$ is a C^*-algbara.

A triple morphism $\theta : T \to B(H, K)$ induces a $*$-homomorphism $\omega : \mathcal{L}(T) \to B(K \oplus H)$ on the linking algebra [37, Proposition 2.1] such that

$$\omega = \begin{bmatrix} \omega_1 & \theta \\ \theta^* & \omega_2 \end{bmatrix}$$

where, $\omega_1 : TT^* \to B(K)$ and $\omega_2 : T^*T \to B(H)$ are $*$-representations satisfying $\omega_1(xy^*) = \theta(x)\theta(y)^*$ and $\omega_2(x^*y) = \theta(x)^*\theta(y)$ for all $x, y \in T$. Conversely [14, Proposition 3.1.2], if $\omega : \mathcal{L}(T) \to B(L)$ is a $*$-homomorphism of the C^*-algebra $\mathcal{L}(T)$, then there exists Hilbert spaces H, K such that $L = K \oplus H$ and there is a triple morphism $\theta : T \to B(H, K)$ with

$$\omega = \begin{bmatrix} \omega_1 & \theta \\ \theta^* & \omega_2 \end{bmatrix}$$

where, $\omega_1 : TT^* \to B(K)$ and $\omega_2 : T^*T \to B(H)$ are $*$-representations satisfying

$\omega_1(xy^*) = \theta(x)\theta(y)^*$ and $\omega_2(x^*y) = \theta(x)^*\theta(y)$ for all $x, y \in T$. Therefore, there is a 1-1 correspondence between the representations of a TRO and the representations of its linking algebra.

The notions of nondegenerate, irreducible and faithful representations of TRO's are natural generalizations from the C^*-algebras. A representation of a TRO T is a triple morphism $\phi : T \to B(H, K)$ for some Hilbert spaces H and K. A representation $\phi : T \to B(H, K)$ is nondegenerate if, whenever p, q are projections in $B(H)$ and $B(K)$, respectively, such that $q\phi(x) = \phi(x)p = 0$ for every $x \in T$, one has $p = 0$ and $q = 0$ (equivalently, if $\overline{\phi(T)H} = K$ and $\overline{\phi(T)^*K} = H$). Let $H_1 \subseteq H$ and $K_1 \subseteq K$ be closed subspaces, then (H_1, K_1) is said to be ϕ-invariant if $\phi(T)H_1 \subseteq K_1$ and $\phi(T)^*K_1 \subseteq H_1$. A representation $\phi : T \to B(H, K)$ is irreducible if, whenever p, q are projections in $B(H)$ and $B(K)$, respectively, such that $q\phi(x) = \phi(x)p$ for every $x \in T$, one has $p = 0$ and $q = 0$, or $p = 1$ and $q = 1$ (equivalently, if $(0, 0)$ and (H, K) are the only ϕ-invariant pairs). Finally ϕ is called faithful if it is injective or, equivalently, completely isometric. A TRO $T \subset B(H, K)$ is said to act nondegenerately or irreducibly if the corresponding inclusion representation is nondegenerate or irreducible, respectively. A representation of a TRO is nondegenerate (irreducible) if and only if the representation of its linking algebra is nondegenerate (irreducible) [14, Lemma 3.1.4, Lemma 3.1.5]. Let $\phi_i : T \to B(H_i, K_i),\ i = 1, 2$, be the representations. The representations ϕ_1 and ϕ_2 are said to be unitarily equivalent if there exists unitary operators $u_1 : H_1 \to H_2$ and $u_2 : K_1 \to K_2$ such that

$\phi_1(t) = u_2^*\phi_2(t)u_1$ for all $t \in T$. We refer to [13, 14] for a nice account on TRO's and the representation theory of TRO's.

2.3 Completely positive maps and completely contractive maps

Let A be a C^*-algebra and $M_n(A)$ be the set of all $n \times n$ matrices with matrix elements from A. The space $M_n(A)$ possess a canonical C^*-algebra structure in the following way. By GNS representation theorem, the C^*-algebra A can be viewed as a C^*-subalgebra of $B(H)$ for some Hilbert space H. Then $M_n(A)$ can be identified as a $*$-subalgebra of the C^*-algebra $M_n(B(H)) \equiv B(H^{(n)})$, where $H^{(n)} = H \oplus H \oplus \cdots \oplus H$ (n times). Direct computations shows that the copy of $M_n(A)$ under this identification is a C^*-algebra and hence $M_n(A)$ is a C^*-algebra.

Let S be an operator system in A. Naturally $M_n(S)$ is an operator system in $M_n(A)$. An element $a \in A$ is called self adjoint if $a = a^*$. A self adjoint element a is called positive if the spectrum of a is contained in $[0, \infty)$. Equivalently, the element a is positive if $a = b^*b$ for some $b \in A$. Let $S \subseteq A$ be an operator system in A. An element $a \in S$ is called positive if a is positive in A.

Let B be another C^*-algebra and M be a linear subspace of A. Let $\psi : M \to B$

be a linear map. Then ψ *induces a map* $\psi_n : M_n(M) \to M_n(B)$ *given by*

$$\psi_n([x_{ij}]) = [\psi(x_{ij})] \;\; \textit{for} \;\; [x_{ij}] \in M_n(M).$$

The map ψ_n *is sometimes called the* n*-amplification of the map* ψ*. In general the adverb completely for some property of* ψ *means that all the maps* $\{\psi_n\}$ *having that property.*

Let $\psi : S \to B$ *be a linear map. The map* ψ *is called positive if* $\psi(a) \geq 0$ *in* B *whenever* $a \geq 0$ *in* S*. The map* ϕ *is called* n*-positive if* ϕ_n *is positive, and we call* ϕ *completely positive(CP) if* ϕ *is* n*-positive for all* n*.*

The following two theorems are the backbone of our work. The first one is the celebrated theorem of Stinespring that classifies completely positive maps on C^**-algebras into the concrete* C^**-algebra* $B(H)$*.*

Theorem 2.3.1. *[58]* ***(Stinespring's dilation theorem)*** *Let A be a unital* C^**-algebra and* $\phi : A \to B(H)$ *be a completely positive map. Then there is a representation* $\pi : A \to B(K)$ *for some Hilbert space* K *with* $H \subseteq K$*, and a bounded operator* $V : H \to K$ *such that*

$$\phi(a) = V^*\pi(a)V, \;\; a \in A.$$

Moreover, in this theorem if ϕ *is unital, then* V *is an isometry.*

Let $K_1 = [\pi(A)VH]$*. Then the restriction* π_1 *of* π *to* K_1 *is also a representation satisfying condition* $\phi(a) = V^*\pi_1(a)V,\ a \in A$*. The representation* π_1 *is called a* *minimal Stinespring's dilation of ϕ. Without loss of generality we may assume that $[\pi(A)VH] = K$.*

Theorem 2.3.2. *[4] (**Arveson's extension theorem**) Let A be a C^*-algebra, and let S be an operator system in A. For a completely positive map $\phi : S \to B(H)$ there exists a completely positive map $\tilde{\phi} : A \to B(H)$ such that $\tilde{\phi}_{|S} = \phi$. That is, ϕ can be extended to a completely positive map on A.*

Let X and Y be operator spaces in the TROs T and T' respectively. A linear map $\phi : X \to Y$ is called contractive if $\|\phi(x)\| \le \|x\|\ \forall x \in X$. We say the map ϕ is completely contractive(CC) if the induced map ϕ_n is contractive, for all $n \in \mathbb{N}$.

2.4 Paulsen System

Given an operator space $X \subset B(H, K)$, we can assign an operator system $\mathcal{S}(X) \subset B(K \oplus H)$. This operator system is called the Paulsen system [58, Lemma 8.1] of X. $\mathcal{S}(X)$ is defined to be the space of operators

$$\left\{ \begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} : x, y \in X, \lambda, \mu \in \mathbb{C} \right\}$$

where I_H and I_K denote the identity operators on H and K respectively. Any completely contractive map $\phi : X \to B(\tilde{H}, \tilde{K})$ on the operator space X induces canoni-

cally a unital completely positive map $\mathcal{S}(\phi) : \mathcal{S}(X) \to B(\tilde{K} \oplus \tilde{H})$ *defined by*

$$\mathcal{S}(\phi)(\begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix}) = \begin{bmatrix} \lambda I_{\tilde{K}} & \phi(x) \\ \phi(y)^* & \mu I_{\tilde{H}} \end{bmatrix}.$$

Let T *be the TRO containing* X *as a generating subspace. Suppose* $\mathcal{A} = C^*(\mathcal{S}(X))$ *is the* C^**-algebra generated by* $\mathcal{S}(X)$*, then*

$$\mathcal{A} = \left\{ \begin{bmatrix} TT^* + \lambda I_K & T \\ T^* & T^*T + \mu I_H \end{bmatrix} : \lambda, \mu \in \mathbb{C} \right\}.$$

Observe that, $\mathcal{A}$ *is a unitalization of the linking algebra* $\mathcal{L}(T)$ *of* T*. Thus, there is a 1-1 correspondence between the representations of TRO* T *and the* $*$*-representations of the* C^**-algebra* $\mathcal{A}$*.*

2.5 Boundary representations

The concept of boundary representation is the non commutative remodelling of the classical notion of Choquet boundary of uniform algebras in $C(\Omega)$*. Let us first recall the classical notion.*

Let $S \subset C(\Omega)$ *be a unital separating subspace of* $C(\Omega)$*. Let* S^* *denote the set of all complex valued bounded linear functionals on* S*. For* $x \in \Omega$*, consider the evaluation* *linear functional* $\delta_x : S \to \mathbb{C}$ *defined by* $\delta_x(f) = f(x)$, $\forall f \in S$. *Clearly,* $\delta_x \in S^*$. *In fact* $\|\delta_x\| = 1$. *Also, if* $f \geq 0$ *in* S, *then* $\delta_x(f) = f(x) \geq 0$. *That is,* δ_x *is a positive linear functional. As the definition of* δ_x *make sense in the whole of* $C(\Omega)$, *we may view* δ_x *to be a positive linear functional on* $C(\Omega)$.

Definition 2.5.1. *Let* S *be a closed subspace of* $C(\Omega)$ *that separates* Ω. *The Choquet boundary* $\partial_c(S)$ *of* S *is defined as*

$$\partial_c(S) = \{x \in \Omega : \delta_{x|_S} \textit{ has one and only one positive linear extension to } C(\Omega)\}.$$

Let $\Delta \subset \Omega$. *Then* Δ *is called a boundary for* S *if for each* $f \in S$ $\exists$ $y \in \Delta$ *such that* $|f(x)| = \|f\|$. *Direct computations shows that* $\partial_c S \subset \Omega$ *is a boundary for* S. *The smallest closed boundary for* S *is called the Silov boundary. Interestingly, Choquet boundary is dense in Silov boundary [60, Proposition 6.4]. This justifies the terminology 'boundary' for* ∂S.

The non-commutative counter-part of Choquet boundary points is introduced by Arveson [4] in the context of operator systems in C^**-algebras, and called it boundary representations.*

Let S *be an operator system* S *in a* C^**-algebra* A. *We use* $C^*(S)$ *to denote the* C^**-algebra generated by* S *in* A. *If* S *generates* A, *then we write* $A = C^*(S)$.

Definition 2.5.2. *[10] Let* $A = C^*(S)$ *and let* $\pi : A \to B(H)$ *be a representation of* A.

The representation π possess unique extension property (UEP) for S if $\psi : A \to B(H)$ is any unital completely positive (UCP) map with the property that $\psi(x) = \pi(x)$ for each $x \in S$, then $\psi(a) = \pi(a)$ for every $a \in A$.

Definition 2.5.3. *[10] Let $A = C^*(S)$ and let $\pi : A \to B(H)$ be an irreducible representation of A. Then π is called a boundary representation for S if π has UEP for S.*

It is well-known that the irreducible representations of a commutative C^*-algebra $A = C(\Omega)$ are all one one dimensional. More precisely, the irreducible representations of $C(\Omega)$ are all multiplicative linear functionals from $C(\Omega)$ into $\mathbb{C}$. By Gelfand theory of commutative C^*-algebra, the multiplicative linear functionals are precisely evaluation maps δ_x, $x \in \Omega$. Thus, boundary representations of $S \subseteq C(\Omega)$ are all evaluation maps with unique extension property. Hence the notion of boundary representation is a generalisation of the notion of Choquet boundary point in the classical setting. And the set of all boundary representations of an operator system in C^*-algebras is a non-commutative counter-part of Choquet boundary of function systems in $C(\Omega)$.

Let $\partial_c(S) = \{\pi_\alpha : \alpha \in I\}$ be the collection of all boundary representations for S. We say that the operator systems S has sufficiently many boundary representations if

$$\|[x_{ij}]\| = \sup_{\alpha \in I} \|\pi_\alpha^{(n)}([x_{ij}])\| \; \forall \; [x_{ij}] \in M_n(S), \;\; \forall \; n \in \mathbb{N}.$$

Definition 2.5.4. *[27] The C^*-envelope $C_e^*(S)$ of an operator system S is the unique* *smallest C^*-algebra amongst those C^*-algebras $\mathcal{B}$ for which there is a completely isometric homomorphism $\phi : C^*(S) \to \mathcal{B}$. Similarly, one can define the C^*-envelope of an operator algebra.*

Arveson [4] suggested that there exists sufficiently many boundary representations for an operator algebra $\mathcal{A}$. In such a case, Arveson displayed that the C^*-algebra generated by the direct sum of ranges of boundary representations has a useful universal property, and confers a visualization of the C^*-envelope of $\mathcal{A}$. A central problem left open in Arveson's work [4] is the existence of boundary representations in general, although existence of boundary representations in several standard examples were demonstrated. Hence, the proof for the existence of the C^*-envelope for general operator algebras were unknown.

Hamana [36] proved the existence of the C^*-envelope of operator systems using the theory of injectivity but not boundary representations. Using the theory of dilations, Dritschel and McCullough [27] demonstrated the existence of C^*-envelope but existence of boundary representations were not shown. Arveson [8] re-examined the query of the existence of boundary representations by utilizing the techniques of Dritschel and McCullough and disintegration theory. In the separable case, Arveson proved his conjecture.

Theorem 2.5.1. *[8] Every separable operator system $S \subseteq C^*(S)$ has sufficiently many boundary representations.*

Later, Davidson and Kennedy [23] completely settled Arveson's conjecture by showing that every operator system has sufficiently many boundary representations to generate its C^*-envelope.

2.6 Hyperrigidity

A powerful theorem in classical approximation theory is the famous Korovkin theorem [48] which concerns convergence of positive linear maps on function algebras. The theorem consolidate different approximation theorem of continuous functions such as Bernstein approximation theorem and Weierstrass approximation theorem, etc..

Theorem 2.6.1. *[12] **(Korovkin's Theorem)** For each $m \in \mathbb{N}$, let $\psi_m : C[0,1] \to C[0,1]$ be a positive linear map. If $\lim_{m\to\infty} ||\psi_m(f) - f|| = 0$ for every $f \in \{1, x, x^2\}$, then $\lim_{m\to\infty} ||\psi_m(f) - f|| = 0$ for every $f \in C[0,1]$.*

Definition 2.6.1. *A set $S \subseteq C[a,b]$ is said to be a Korovkin set if for any sequence of positive linear maps $\psi_m : C[a,b] \to C[a,b]$, $\lim_{m\to\infty} ||\psi_m(f) - f|| = 0 \ \forall f \in S \implies \lim_{m\to\infty} ||\psi_m(f) - f|| = 0 \ \forall f \in C[0,1]$.*

Remark 2.6.1. *By Korovkin theorem the set $\{1, x, x^2\}$ is a Korovkin set for $C([0,1])$.*

In 1966, Saskin [70] identified a strong relation between Korovkin sets in $C[0,1]$ and their Choquet boundaries.

Theorem 2.6.2. *[12] **(Saskin's Theorem)** Let $1 \in S_0 \subseteq C(\Omega)$ and S_0 separates the points of Ω. Then the following two statements are equivalent:*

(i) S_0 is a Korovkin set for $C(\Omega)$

(ii) the Choquet boundary $\partial_c(S) = \Omega$, where $S =$ linear span(S_0).

In 2008, Arveson [10] proposed the concept of non-commutative analogue of Korovkin sets and that he named as hyperrigid sets.

Definition 2.6.2. *[10] Let A be a C^*-algebra and S be a generating set for A. Then S is said to be hyperrigid if for all faithful representation from A to $B(H)$ and any sequence of UCP maps $\psi_m : B(H) \to B(H)$, $m = 1, 2, ...,$*

$$\lim_{m\to\infty} ||\psi_m(s) - s|| = 0, \forall\, s \in S \Rightarrow \lim_{m\to\infty} ||\psi_m(a) - a|| = 0, \forall\, a \in A.$$

Direct computations show that a set $S_0 \subseteq A$ is hyperrigid if and only if the subspace spanned by $S_0 \cup S_0^*$ is hyperrigid. Hence, hyperrigidity is treated as a notion of operator systems in C^*-algebras.

The following characterization theorem was given by Arveson that connects the notion of hyperrigidity with the notion of boundary representation.

Theorem 2.6.3. *[10] Let S be an operator system and $A = C^*(S)$. If S is separable, then the assertions given below are identical to each other.*

(i) S *is hyperrigid.*

(ii) $\forall$ *separable non-degenerate representation* $\rho : A \to B(H)$ *and every sequence* $\psi_m : A \to B(H)$ *of UCP maps,*

$$\lim_{m\to\infty} ||\psi_m(x) - \rho(x)|| = 0, \forall\, x \in S \Rightarrow \lim_{m\to\infty} ||\psi_m(a) - \rho(a)|| = 0, \forall\, a \in A.$$

(iii) $\forall$ *separable non-degenerate representation* $\rho : A \to B(H)$*,* $\rho_{|S}$ *has the UEP.*

(iv) *For any* C^**-algebra* B *with a unital homomorphism* $\omega : A \to B$ *and a UCP map* $\psi : B \to B$

$$\psi(b) = b\ \ \forall\, b \in \omega(S) \Rightarrow \psi(b) = b\ \ \forall\, a \in \omega(A).$$

Arveson [10], supplied lot of examples for hyperrigid operator systems using the above characterisation theorem. For example; consider V *to be the Volterra integral operator acting on* $H = L^2[0,1]$*, then the operator system generated by* $S = \{V, V^2\}$ *is hyperrigid [10, Theorem 1.7].*

The following two theorems are applications of the Theorem 2.6.3. These theorems also provides different classes of examples for hyperrigid operator systems.

Theorem 2.6.4. *[10] For isometries* $W_1, W_2, ..., W_m$ *in* $B(H)$ *the operator system spanned by* $\{W_1, ..., W_m, W_1W_1^*, ..., W_mW_m^*\}$ *is hyperrigid in* $A = C^*(W_1, \cdots W_m)$*.*

Theorem 2.6.5. *[10] Let $a \in B(H)$ be such that $a = a^*$ and let A be the C^*-algebra generated by 1 and a. Let S_o and S be the operator systems spanned by the sets $\{1, a\}$ and $\{1, a, a^2\}$, respectively. If the spectrum of a has at least 3 points, then*

(i) S is hyperrigid in A, while

(ii) S_0 is not hyperrigid in A.

Along the lines of Saskin theorem 2.6.2, Arveson tried to find the possible relationship between the notion of hyperrigidity and the notion of boundary representations.

Corollary 2.6.1. *[10] Assume that S is separable and S is hyperrigid in $A = C^*(S)$. Then each irreducible representation of A is a boundary representation for S.*

Proof of the Corollary 2.6.1 follows from the equivalent condition (i) and (iii) of Theorem 2.6.3.

The converse of the Corollary 2.6.1 is called Arveson's hyperrigidity conjecture.

Conjecture 2.6.1. *[10] Let S be a separable operator system in A such that $A = C^*(S)$. If every irreducible representation of A is a boundary representation for S, then S is hyperrigid.*

In some special cases the Conjecture 2.6.1 is proved. Arveson [10] demonstrated the conjecture for C^*-algebras with countable number of in-equivalent irreducible representations. Davidson and Kennedy [24] demonstrated the Conjecture 2.6.1 for the function systems.

Arveson's hyperrigidity conjecture is not fully settled till now and is an active research topic.

Chapter 3

Amplifications of weak boundary representations and quasi hyperrigidity

An interesting weaker notion of boundary representations and hyperrigidity called weak boundary representations and quasi hyperrigidity is introduced in [56]. In this chapter, we study the amplifications of a weak boundary representations and quasi hyperrigidity for operator systems in C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra and S be an operator system in $\mathcal{A}$. It is shown that, an irreducible representation π of $\mathcal{A}$ is a weak boundary representation for S if and only if its n-amplification $\pi^{(n)}$ is a weak boundary representation for the operator system $M_n(S)$ for any $n \geq 2$. Also, we deduce that the operator system S is quasi hyperrigid in $\mathcal{A}$ if and only if the operator system $M_n(S)$ is quasi hyperrigid in $M_n(\mathcal{A})$ for any $n \geq 2$.

3.1 Tensor products

In this section, we recall the basic theory of tensor products of C^*-algebras, operator systems and related notions useful to our discussions.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras. Let $\mathcal{A}_1 \otimes \mathcal{A}_2$ denotes the algebraic tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. The space $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a $*$-algebra with respect to the following natural multiplication and involution

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$$

$$(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*.$$

A norm $\|.\|_\gamma$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ is called a C^*-cross norm if it satisfies the following for all $a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2$ and $\forall a, b \in \mathcal{A}_1 \otimes \mathcal{A}_2$

$$\|a_1 \otimes a_2\|_\gamma = \|a_1\|\|a_2\|$$

$$\|ab\|_\gamma \leq \|a\|\|b\|$$

$$\|a^* a\|_\gamma = \|a\|_\gamma^2 = \|a^*\|_\gamma^2.$$

Let $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2$ denotes the closure of $\mathcal{A}_1 \otimes \mathcal{A}_2$ when the later is provided with a C^*-cross norm γ. In general, there are many possible C^*-cross norms on $\mathcal{A}_1 \otimes \mathcal{A}_2$. For more *details we refer [34, 79].*

A C^*-algebra $\mathcal{A}_1$ is called nuclear if, for every C^*-algebra $\mathcal{A}_2$, there is a unique C^*-cross norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Among the nuclear C^*-algebras are finite dimensional C^*-algebras, commutative ones, GCR algebras and type I C^*-algebras. In this chapter, our main interest is in the semi-classical case of $\mathcal{A} \otimes M_n(\mathbb{C})$ with the unique C^*-cross norm induced from the C^*-norm of $\mathcal{A}$.

There is a natural way to define tensor product of representations via the tensor product of Hilbert spaces. Let H_1 and H_2 be Hilbert spaces. The tensor product $H_1 \otimes H_2$ of H_1 and H_2 is defined in the following way. The space $H_1 \otimes H_2$ is the completion of the inner product space $H_1 \otimes^v H_2$ with respect to the inner product

$$\left\langle \sum_j h_j \otimes g_j, \sum_i h_i' \otimes g_i' \right\rangle = \sum_{i,j} \langle h_j, h_i' \rangle \langle g_j, g_i' \rangle$$

where $H_1 \otimes^v H_2$ is the algebraic tensor product of the vector spaces H_1 and H_2. Let $\pi_1 : \mathcal{A}_1 \to B(H_1)$ and $\pi_2 : \mathcal{A}_2 \to B(H_2)$ be representations. Then $\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes_\gamma \mathcal{A}_2 \to B(H_1 \otimes H_2)$ given by

$$(\pi_1 \otimes \pi_2)(a_1 \otimes a_2) = \pi_1(a_1) \otimes \pi_2(a_2)$$

is a representation of $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_1$.

Similar to the case of tensor products of representations we can define tensor prod-

uct of CP-maps. Let S_i be an operator systems in $\mathcal{A}_i$ for $i = 1, 2$. Then the algebraic tensor product $S_1 \otimes S_2$ is an operator system in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Now, let $\phi_i : S_i \to \mathcal{B}_i$ is a CP-maps for $i = 1, 2$, where $\mathcal{B}_i$, $i = 1, 2$ are C^*-algebras. Then the tensor product $\phi_1 \otimes \phi_2$ of ϕ_1 and ϕ_2 is a CP-map and is given by

$$(\phi_1 \otimes \phi_2)(a_1 \otimes a_2) = \phi_1(a_1) \otimes \phi_2(a_2).$$

Note that the n-amplification $\phi^{(n)}$ of a CP-map ϕ is nothing but the tensor product of the CP-maps ϕ and $I^{(n)}$, where $I^{(n)}$ is the identity map on the matrix algebra $M_n(\mathbb{C})$.

3.2 Weak unique extension property

While the unique extension property (UEP) of Arveson demands uniqueness among the set of all unital completely positive extensions, demanding uniqueness only among the smaller class of conjugates of the representation by isometries merits attention. Here we recall the following definition.

Definition 3.2.1. *[56] Let S be an operator system generating the C^*-algebra $C^*(S)$ and H be a Hilbert space. A representation $\pi : C^*(S) \to B(H)$ is said to have weak unique extension property (WUEP) for S if the only completely positive extension of $\pi|_S$ of the form $V^*\pi(.)V$ is π itself, where V is an isometry on H.*

In the result below, we note that the WUEP property for representations is invariant under unitary conjugation.

Proposition 3.2.1. *Let $\mathcal{A}$ be a C^*-algebra and S be an operator system in $\mathcal{A}$ such that $\mathcal{A} = C^*(S)$. Let $\pi : \mathcal{A} \to B(H)$ be a representation with WUEP for S and $U : K \to H$ be a unitary, for some Hilbert space K. Then the representation $\rho(.) = U^*\pi(.)U$ also has WUEP for S.*

Proof. Let V be an isometry on K with the property that

$$V^*\rho(a)V = \rho(a) \ \ \forall a \in S.$$

Then $V^*U^*\pi(a)UV = U^*\pi(a)U \quad \forall a \in S$ will give

$$UV^*U^*\pi(a)UVU^* = \pi(a) \ \ \forall a \in S.$$

Since π has WUEP for S and UV^*U^* is an isometry on H,

$$UV^*U^*\pi(a)UVU^* = \pi(a) \ \ \forall a \in \mathcal{A}.$$

Hence $V^*\rho(a)V = \rho(a)\ \forall a \in \mathcal{A}$. □

Remark 3.2.1. *If the operator systems S_1 and S_2 generates the C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, then the operator system $S_1 \otimes S_2$ is a generating set for the C^*-algebra*

$\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2$, where $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2$ is the closure of the algebraic tensor product $\mathcal{A}_1 \otimes \mathcal{A}_2$ in any C^*-cross norm γ.

The following result examines the WUEP property for tensor product of representations. Let S_1 and S_2 be operator systems generates C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Let γ be a C^*-cross norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$.

Proposition 3.2.2. *Let $\pi_i : \mathcal{A}_i \to B(H_i)$, $i = 1, 2$ be representations of the C^*-algebra $\mathcal{A}_i = C^*(S_i)$. If the representation $\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes_\gamma \mathcal{A}_2 \to B(H_1 \otimes H_2)$ has WUEP for $S_1 \otimes S_2$, then π_1 has WUEP for S_1 and π_2 has WUEP for S_2.*

Proof. Assume that $\pi_1 \otimes \pi_2$ has WUEP for $S_1 \otimes S_2$. If π_1 does not have WUEP for S_1, then $\pi_{1|_{S_1}}$ has a completely positive extension $\phi_1 = V^*\pi_1 V$ other than π_1, where V is an isometry on H_1. Then $\phi = \phi_1 \otimes \pi_2$ is a completely positive extension of $\pi_1 \otimes \pi_{2|_{S_1 \otimes S_2}}$ and

$$\phi = \phi_1 \otimes \pi_2 = V^*\pi_1 V \otimes \pi_2 = (V^* \otimes I)(\pi_1 \otimes \pi_2)(V \otimes I)$$

where $V \otimes I$ is an isometry on $H_1 \otimes H_2$. This is a contradiction to the assumption that $\pi_1 \otimes \pi_2$ has WUEP for $S_1 \otimes S_2$. Hence π_1 has WUEP for S_1. Similarly π_2 has WUEP for S_2. $\square$

Remark 3.2.2. *It is not clear to us whether the converse of the Proposition 3.2.2 is true or not.*

3.3 Amplifications of weak boundary representations

In this section we study amplifications of weak boundary representations and quasi hyperrigidity. Let us recall the definition of weak boundary representation which is introduced in [56].

Definition 3.3.1. *[56] Let $\mathcal{A}$ be a unital C^*-algebra and S be an operator system in $\mathcal{A}$ such that $\mathcal{A} = C^*(S)$. An irreducible representation $\pi : \mathcal{A} \to B(H)$ is called a weak boundary representation for S of $\mathcal{A}$ if $\pi_{|S}$ has a unique completely positive extension of the form $V^*\pi V$, namely π itself, where V is an isometry on H.*

Example 3.3.1. *Let H be an infinite dimensional separable Hilbert space and S be the operator system generated by V, where V is the unilateral right shift in $B(H)$. Let $\mathcal{A} = C^*(S)$ be the C^*-algebra generated by S. Consider the inclusion map π from $\mathcal{A}$ to $B(H)$, which is clearly a representation of $\mathcal{A}$ on H. Then the map $\phi(.) = V^*\pi(.)V$ from $\mathcal{A}$ to $B(H)$ is a completely positive extension of $\pi_{|S}$ and it is different from π. Thus, π is not a weak boundary representation for S.*

Remark 3.3.1. *By the same argument in Example 3.3.1, one can generalise that example to any infinite dimensional Hilbert space H and any isometry on H which is not a unitary.*

Theorem 3.3.1. *Let $\pi_i : \mathcal{A}_i \to B(H_i)$, $i = 1, 2$ be irreducible representations of the C^*-algebra $\mathcal{A}_i = C^*(S_i)$. If the representation $\pi_1 \otimes \pi_2 : \mathcal{A}_1 \otimes_\gamma \mathcal{A}_2 \to B(H_1 \otimes H_2)$ is a* *weak boundary representation for $S_1 \otimes S_2$, then π_1 is a weak boundary representation for S_1 and π_2 is a weak boundary representation for S_2.*

Proof. Follows directly from Proposition 3.2.2. □

While the above result was easy to prove, the question of whether the tensor product of weak boundary representations inherits the property from individual representations is more involved which we examine in the particular case when $\mathcal{A}_2 = M_n(\mathbb{C})$. We use the following lemma from [38] to prove the main theorem of this paper. The central tool in the following lemma is the notion of $n \times n$ matrix units in a C^*-algebra [38].

Definition 3.3.2. *[38] Let $\mathcal{A}$ be a C^*-algebra with identity e. A set of elements $\{F_{ij}\}$, $i, j = 1, 2, \cdots n$, is a set of $n \times n$ matrix units if it satisfies the following three conditions;*

1. $F_{ij} = F_{ji}^* \; \forall \, i, j$
2. $F_{ij}F_{kl} = \delta_{jk}F_{il} \; \forall \, i, j, k, l$
3. $\sum_{i=1}^{n} F_{ii} = e$.

Lemma 3.3.1. *[38] Let H be a Hilbert space and $\mathcal{A}$ be a C^*-algebra. Let $\{F_{ij}\}$ be a set of $n \times n$ matrix units in $\mathcal{A}$, and M be a closed subspace of H with P the orthogonal projection on M. For a representation $\pi : \mathcal{A} \to B(H)$, if $\{P\pi(F_{ij})P\}$ is a set of matrix units (in the C^*-algebra $C^*(P\pi(\mathcal{A})P)$, with unit P) then M is invariant under each $\pi(F_{ij})$.*

Now, let us fix some notations which are used in the proof of the following theorem. We denote $n \times n$ identity matrix in $M_n(\mathbb{C})$ by I_n and we use $I^{(n)}$ to denote the identity representation of $M_n(\mathbb{C})$. For a representation $\pi : \mathcal{A} \to B(H)$, $\pi^{(n)}$ be the induced representation $\pi \otimes I^{(n)}$ of $M_n(\mathcal{A})$. Further we make use of the set of matrix units $\{F_{ij}\} = \{e \otimes E_{ij}\} = \{E_{ij}(e)\}$ in $\mathcal{A} \otimes M_n(\mathbb{C})$, where e is the identity of $\mathcal{A}$ and the set $\{E_{ij}\}$ is the standard matrix units of $M_n(\mathbb{C})$.

Remark 3.3.2. *It is clear that if S is an operator system in $\mathcal{A}$, then $M_n(S)$ is an operator system in $M_n(\mathcal{A})$, though $M_n(\mathcal{A})$ contains more general operator systems. In the following we identify $\mathcal{A} \otimes M_n(\mathbb{C})$ with $M_n(\mathcal{A})$. Starting from an operator system S_n in $M_n(\mathcal{A})$, it is clear that matrix entries of elements of S_n will form an operator system in $\mathcal{A}$.*

Theorem 3.3.2. *Let $\mathcal{A}$ be a C^*-algebra with unit e. Let S_n be an operator system in $\mathcal{A} \otimes M_n(\mathbb{C})$ that contains the set of matrix units $E_{ij}(e)$, $i, j = 1, 2, \cdots, n$. Let S be the set of elements in $\mathcal{A}$ which appears as a matrix entry in some element of S_n. Then an irreducible representation π of $\mathcal{A}$ on H is a weak boundary representation for S if and only if $\pi^{(n)}$ is a weak boundary representation for S_n for any $n \geq 2$.*

Proof. We first show the trivial implication that if $\pi^{(n)}$ is a weak boundary representation for S_n then π is a weak boundary representation for S. Let v_1 be an isometry on H such that

$$v_1^* \pi v_{1_{|S}} = \pi_{|S}.$$

Then $v_1 \otimes I_n$ is an isometry on $H \otimes \mathbb{C}^n$, where I_n is the $n \times n$ identity matrix. Also,

$$(v_1^* \otimes I_n)\pi^{(n)}(v_1^* \otimes I_n)_{|S} = \pi^{(n)}_{|S}.$$

As $\pi^{(n)}$ is a weak boundary representation we have $v_1 \otimes I_n$ is a unitary and consequently v_1 is a unitary. Hence π is a weak boundary representation of $\mathcal{A}$ for S.

Conversely, assume that π is a weak boundary representation of $\mathcal{A}$ for S. Let V be an isometry on $H^{(n)} = H \oplus H \oplus \cdots \oplus H$ such that

$$V^*\pi^{(n)}V_{|S_n} = \pi^{(n)}_{|S_n}.$$

Denote $F_{ij} = E_{ij}(e)$. Let $P = VV^*$ and M be the range of P. We first claim that M is invariant under the operators $\pi^{(n)}(F_{ij}), i, j = 1, 2, \cdots, n$. In view of the above lemma, it suffices to show that $\{P\pi^{(n)}(F_{ij})P\}$ is a set of matrix units. Since $\pi^{(n)}(F_{ij}) \in S_n$, we have

$$\begin{aligned} P\pi^{(n)}(F_{ij})P &= VV^*\pi^{(n)}(F_{ij})VV^* \\ &= V\pi^{(n)}(F_{ij})V^*. \end{aligned}$$

Using the last expressions and the fact that V is a unitary from $H^{(n)}$ to M, we can directly verify that $\{P\pi^{(n)}(F_{ij})P\}$ is a set of matrix units. Then by Lemma 3.3.1, M is invariant under each $\pi^{(n)}(F_{ij})$. Equivalently V commutes with each $\pi^{(n)}(F_{ij})$.

Now we claim that V can be factorised into $v_o \otimes I_n$ where v_0 is an isometry on H and I_n is the $n \times n$ identity matrix. Since V commutes with $\pi^{(n)}(F_{ij}) = G_{ij}$ where G_{ij} is the matrix unit of $B(H^{(n)})$, we have $VG_{ij} = G_{ij}V$. This implies that

$$v_{ij} = \begin{cases} v_{11} \text{ if } i = j \\ \\ 0 \quad \text{if } i \neq j \end{cases} .$$

Thus $V = v_0 \otimes I_n$, where $v_0 = v_{11}$.

Now, we show that $v_0^* \pi v_{0_{|S}} = \pi_{|S}$. For that, we use the following observation. Consider E_{11}, the first matrix unit of $M_n(\mathbb{C})$. Then,

$$\begin{aligned} V^* \pi^{(n)}(a \otimes E_{11})V &= \pi^{(n)}(a \otimes E_{11}) \\ (v_0^* \otimes I_n)(\pi \otimes I^{(n)})(a \otimes E_{11})(v_0 \otimes I^{(n)}) &= \pi \otimes I^{(n)}(a \otimes E_{11}) \\ (v_0^* \otimes I_n)(\pi(a) \otimes E_{11})(v_0 \otimes I^{(n)}) &= \pi(a) \otimes E_{11} \\ v_0^* \pi(a) v_0 \otimes E_{11} &= \pi(a) \otimes E_{11} \\ v_0^* \pi(a) v_0 &= \pi(a). \end{aligned}$$

Thus to prove $v_0^* \pi v_{0_{|S}} = \pi_{|S}$ for each $a \in S$, it is enough to prove that

$$V^* \pi^{(n)}(a \otimes E_{11})V = \pi^{(n)}(a \otimes E_{11})$$

for each $a \in S$. To prove $V^* \pi^{(n)}(a \otimes E_{11})V = \pi^{(n)}(a \otimes E_{11})$, we use the fact that

$\pi^{(n)}(F_{ij})$ commutes with $V = v_0 \otimes I_n$. Choose an element $x \in S$ such that a is the $(i, j)^{th}$ entry of x for some i and j. Direct multiplication shows that

$$E_{11}(a) = F_{1i} x F_{j1}.$$

Hence $\pi^{(n)}(E_{11}(a)) = \pi^{(n)}(F_{1i})\pi^{(n)}(x)\pi^{(n)}(F_{j1})$. Then, using the fact that $\pi^{(n)}(F_{ij})$ commutes with P, we have

$$P\pi^{(n)}(E_{11}(a))P = P\pi^{(n)}(F_{1i})P\pi^{(n)}(x)P\pi^{(n)}(F_{j1})P.$$

Then using $V^*\pi^{(n)}V_{|S_n} = \pi^{(n)}_{|S_n}$, we obtain

$$\begin{aligned} V^*\pi^{(n)}V(a \otimes E_{11}) &= V^*P\pi^{(n)}PV(E_{11}(a)) \\ &= V^*P\pi^{(n)}(E_{11}(a))PV \\ &= V^*\pi^{(n)}(F_{1i})VV^*\pi^{(n)}(x)VV^*\pi^{(n)}(F_{j1}) \\ &= \pi^{(n)}(F_{1i})\pi^{(n)}(x)\pi^{(n)}(F_{j1}) \\ &= \pi^{(n)}(E_{11}(a)) \\ &= \pi^{(n)}(a \otimes E_{11}). \end{aligned}$$

As π is a weak boundary representation, we have $v_0^*\pi(b)v_0 = \pi(b)$ for every $b \in \mathcal{A}$.

Then on $\mathcal{A} \otimes M_n(\mathbb{C})$,

$$v_0^*\pi v_0 \otimes I^{(n)} = \pi \otimes I^{(n)}$$
$$v_0^*\pi v_0 \otimes I_n I^{(n)} I_n = \pi \otimes I^{(n)}$$
$$(v_0^* \otimes I_n)(\pi \otimes I^{(n)})(v_0 \otimes I_n) = \pi \otimes I^{(n)}$$
$$V^*\pi^{(n)}V = \pi^{(n)}.$$

Thus $\pi^{(n)}$ is a weak boundary representation of $\mathcal{A} \otimes M_n(\mathbb{C})$ for S_n. □

Corollary 3.3.1. *Let S be an operator system in a C^*-algebra $\mathcal{A}$ such that $\mathcal{A} = C^*(S)$. Let $\pi : \mathcal{A} \to B(H)$ be a representation of $\mathcal{A}$. Then π is a weak boundary representation of $\mathcal{A}$ for S if and only if $\pi^{(n)}$ is a weak boundary representation of $M_n(\mathcal{A})$ for $M_n(S)$ for any $n \geq 2$.*

Proof. Immediately follows by taking $S_n = M_n(S)$ in Theorem 3.3.2. □

An operator system S in a C^*-algebra A is said to be quasi hyperrigid [56], if for every unital representation π of A on a Hilbert space H and for any isometry V on H the condition $V^*\pi(s)V = \pi(s)$ for all $s \in S$ implies that $V^*\pi(a)V = \pi(a)$ for all $a \in A$. Now, we establishes the relation between the quasi hyperrigidity of an operator system S_n in $\mathcal{A} \otimes M_n(\mathbb{C})$ with the quasi hyperrigidity of the corresponding operator system S(considered in Theorem 3.3.2) in $\mathcal{A}$.

Theorem 3.3.3. *Let $\mathcal{A}$ be a C^*-algebra with unit e. Let S be an operator system in $\mathcal{A} \otimes M_n(\mathbb{C})$ that contains the set of matrix units $E_{ij}(e)$, $i, j = 1, 2, \cdots, n$. Let S be the set of elements in $\mathcal{A}$ which appears as a matrix entry in some element of S. Then S is quasi hyperrigid in $\mathcal{A}$ if and only if S_n is quasi hyperrigid in $\mathcal{A} \otimes M_n(\mathbb{C})$ for any $n \geq 2$.*

Proof. Proof follows from Theorem 3.3.2 and the fact that π is an irreducible representation of $\mathcal{A}$ if and only if $\pi^{(n)}$ is an irreducible representation of $\mathcal{A} \otimes M_n(\mathbb{C})$. □

Corollary 3.3.2. *Let S be an operator system in a C^*-algebra $\mathcal{A}$ such that $\mathcal{A} = C^*(S)$. Then the operator system S is quasi hyperrigid in $\mathcal{A}$ if and only if the operator system $M_n(S)$ is quasi hyperrigid in $M_n(\mathcal{A})$ for any $n \geq 2$.*

Proof. Follows from Theorem 3.3.3 by taking $S_n = M_n(S)$. □

Now, observe that if S' is an operator system in $M_n(\mathbb{C})$ that contains all the usual matrix units of $M_n(\mathbb{C})$, then $C^*(S') = M_n(\mathbb{C})$ and S' is quasi hyperrigid $M_n(\mathbb{C})$. Thus an operator system S is quasi hyperrigid in $\mathcal{A}$ if and only if the operator system $S \otimes S'$ is quasi hyperrigid in $\mathcal{A} \otimes M_n(\mathbb{C})$. This leads to the following natural question:

Problem 3.3.1. *For operator systems S_1 and S_2 and the generated C^*-algebras $\mathcal{A}_i = C^*(S_i)$, $i = 1, 2$, is it true that $S_1 \otimes S_2$ is quasi hyperrigid in $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2$ if and only if S_i is quasi hyperrigid in $\mathcal{A}_i$ for each $i = 1, 2$?*

Example 3.3.2. *Let V be the unilateral right shift operator on a separable Hilbert space H and I be the identity operator on H. Let X be the linear subspace spanned by $\{I, V\}$ and $S(X)$ be the Paulsen system associated with X. That is,*

$$S(X) = \left\{ \begin{bmatrix} a_1 I & a_2 I + a_3 V \\ a_4 I + a_5 V^* & a_6 I \end{bmatrix} : a_i \in \mathbb{C} \, \textit{for} \; i = 1, 2, \cdots 6 \right\}.$$

Note that the operator system S spanned by the entries of the elements of $S(X)$ is nothing but the operator system spanned by V, that is,

$$S = \{b_1 I + b_2 V + b_3 V^* : b_1, b_2, b_3 \in \mathbb{C}\}.$$

Let $\mathcal{A}_1 = C^*(S)$, $\mathcal{A}_2 = C^*(S(X))$ and ψ be the identity representation of $\mathcal{A}_1$. The restriction of identity representation $\psi_{|S}$ has more than one UCP extensions, namely, ψ and $V^*\psi(.)V$ [56, Example 3.4]. Thus ψ is not a weak boundary representation for S and hence S is not quasi hyperrigid in $\mathcal{A}_1$. Now, by Theorem 3.3.2 the identity representation of $\mathcal{A}_2$ is not a weak boundary representation for the Paulsen system $S(X)$ and hence the Paulsen system $\mathcal{S}(X)$ is not quasi hyperrigid in the generated C^*-algebra.

Example 3.3.3. *Let V be the unilateral right shift operator on a separable Hilbert space H and I be the identity operator on H. Let X be the linear subspace spanned* *by* $\{I, V, VV^*\}$ *and* $S(X)$ *be the Paulsen system associated with* X. *That is,*

$$S(X) = \left\{ \begin{bmatrix} a_1 I & a_2 I + a_3 V + a_4 VV^* \\ a_5 I + a_6 V^* + a_7 VV^* & a_8 I \end{bmatrix} : a_i \in \mathbb{C} \, \textit{for} \; i = 1, \cdots 8 \right\}.$$

Note that the operator system S *spanned by the entries of the elements of* $S(X)$ *is nothing but the operator system spanned by* V *and* VV^*, *that is,*

$$S = \{b_1 I + b_2 V + b_3 V^* + b_4 VV^* : b_i \in \mathbb{C}, i = 1, \cdots, 4\}.$$

Let $\mathcal{A}_1 = C^*(S)$, $\mathcal{A}_2 = C^*(S(X))$. *Then by [10, Theorem 3.3],* S *is hyperrigid in* $\mathcal{A}_1$ *and hence* S *is quasi hyperrigid in* $\mathcal{A}_1$. *By Theorem 3.3.3, the Paulsen system* $\mathcal{S}(X)$ *is quasi hyperrigid* $\mathcal{A}_2$. *Thus all irreducible representations of* $\mathcal{A}_2$ *are weak boundary representations for the operator system* $\mathcal{S}(X)$.

Chapter 4

Boundary representations and hyperrigidity for operator spaces

In seeking to understand the structure of a concretely represented unital operator system, operator spaces and operator algebras, a powerful tool is Arveson's non-commutative Choquet boundary. The building blocks for this non-commutative boundary are certain special representations of $C^*(S)$ called boundary representations. In this chapter, we explore connections between boundary representations of operator spaces and those of the associated Paulsen systems. Using the notions of finite representation and separating property which we introduced, boundary representations for operator spaces are characterized. This characterisation theorem would imply that boundary representations are extreme points of certain convex sets and hence they are actually on the "boundary" of some sets. We also introduce weak boundary for op-

erator spaces. Rectangular hyperrigidity of operator spaces introduced here is used to establish an analogue of Saskin's theorem in the setting of operator spaces in finite dimensions.

4.1 Boundary representations for operator spaces

In this section, we study noncommutative choquet boundary for operator spaces. Ideally, all aspects of the non-commutative Choquet boundary of X could be understood by applying Arveson's usual machinery to $\mathcal{S}(X)$. For this to be fully realized however, we would need to know that the correspondence between X and $\mathcal{S}(X)$ preserves boundary representations. One direction was established in the work of Fuller–Hartz–Lupini [33], while the other is a main result of this section. First, let us recall the definitions of the required ingredients from [33].

Let X be an operator space and T be a TRO containing X as a generating subspace.

Definition 4.1.1. *[33] Let $\psi : X \to B(H, K)$ be a nondegenerate linear map. Then ψ is called a rectangular operator state if*

$$\|\psi\|_{cb} = 1.$$

Note that a rectangular operator state is a CC-map. Let ϕ be a rectangular operator *state. Then $\|\phi\|_{cb} = 1$, that is,*

$$\sup_{n\in\mathbb{N}} \|\phi^{(n)}\| = 1.$$

Thus $\|\phi^{(n)}\| \leq 1$ for every n. Therefore ϕ is a CC-map.

Definition 4.1.2. *[33] Let $\phi : X \to B(H,K)$ be rectangular operator state. A dilation of ϕ be a rectangular operator state $\psi : X \to B(\tilde{H},\tilde{K})$ such that there exists isometries $u : H \to \tilde{H}$ and $v : K \to \tilde{K}$ satisfying the condition*

$$v^*\psi(a)u = \phi(a) \quad \forall a \in X.$$

Similar to Arveson's extension theorem for CP-maps on operators systems with codomain $B(H)$, the CC-maps on operator spaces with codomain $B(H,K)$ also admits an extension theorem, called Haagerup-Paulsen-Wittstock- extension theorem.

Theorem 4.1.1. *[58, Theorem 8.2] Let X be an operator space in a C^*-algebra $\mathcal{A}$ and $\psi : X \to B(H,K)$ is a CB-map. Then there exists a CB-map $\tilde{\psi} : \mathcal{A} \to B(H,K)$ that extends ψ, with $\|\tilde{\psi}\|_{cb} = \|\psi\|_{cb}$.*

In particular any CC-map on X with codomain $B(H,K)$ can be extended to a CC-map from T to $B(H,K)$, in a cb-norm preserving manner. It follows that any rectangular operator state on the operator space X can be extended to a rectangular operator state on the TRO T. In general a rectangular operator state on the operator *space X can be extended to a rectangular operator state on T in several ways. The rectangular operator states with such an extension is unique is of our interest.*

Definition 4.1.3. *[33] A rectangular operator state ψ on X has the unique extension property if $\tilde{\psi}$ is any rectangular operator state on T such that $\tilde{\psi}_{|X} = \psi$, then $\tilde{\psi}$ is a triple morphism of T.*

Definition 4.1.4. *[33] A rectangular operator state $\phi : X \to B(H, K)$ is a boundary representation for X if it has unique extension property and the unique extension of ϕ to T is an irreducible representation of T.*

In our context, we mostly begin with a representation of T, rather than a rectangular operator state on X. Thus, we modify the definition of boundary representations to representations.

Definition 4.1.5. *An irreducible representation $\pi : T \to B(H, K)$ of T called a boundary representation for the operator space X if*

(i) $\pi_{|X}$ is a rectangular operator state on X, and

(ii) $\pi_{|X}$ has unique extension property.

Remark 4.1.1. *We note the close connections between the two Definitions 4.1.4 and 4.1.5. If $\phi : X \to B(H, K)$ is a boundary representation in the sense of Definition 4.1.4, then ϕ extends to an irreducible representation of T, say $\tilde{\phi}$. Direct verification shows that the representation $\tilde{\phi}$ is a boundary representation in the sense of Definition*

4.1.5. Conversely, if a representation $\pi : T \to B(H, K)$ is a boundary representation for X in the sense of 4.1.5, then $\pi_{|X}$ is a rectangular operator state on X and $\pi_{|X}$ is a boundary representation in the sense of Definition 4.1.4.

After introducing the Definition 4.1.4 of boundary representations for operator spaces, Fuller-Hartz-Lupini [33, Theorem 1.9] established the natural analogue of Arveson's conjecture: any operator space is completely normed by its boundary representations.

Theorem 4.1.2. *[33, Theorem 1.9] An operator space is completely normed by its boundary representations.*

The Theorem 4.1.2 actually give rises to an explicit description of the triple envelope of operator spaces, that is, the triple envelope is the direct sum of images boundary representations.

The proof of the Theorem 4.1.2 essentially uses the following Proposition [33, Proposition 1.8]. They proved that a boundary representation of the Paulsen system induces a boundary representation of the operator space.

Proposition 4.1.1. *[33, Proposition 1.8] Suppose $\theta : \mathcal{S}(X) \to B(L)$ is a boundary representation for the Paulsen system $\mathcal{S}(X)$ associated with X. Then we can decompose the Hilbert space L as an orthogonal direct sum $H \oplus K$ in such a way that $\theta = \mathcal{S}(\psi)$ for some rectangular operator state $\psi : X \to B(H, K)$ and ψ on X is a boundary representations.*

Here, we establish the converse of the Proposition 4.1.1.

Theorem 4.1.3. *If a rectangular operator state $\phi : X \to B(H, K)$ is a boundary representation for X, then $\mathcal{S}(\phi)$ is a boundary representation for $\mathcal{S}(X)$.*

Proof. Assume that the rectangular operator state $\phi : X \to B(H, K)$ is a boundary representation for X. Let $\theta : T \to B(H, K)$ be the irreducible representation such that $\theta_{|X} = \phi$. Let $\mathcal{A}$ be the C^*-algebra generated by $\mathcal{S}(X)$ inside $B(K \oplus H)$, then we have

$$\mathcal{A} = \left\{ \begin{bmatrix} TT^* + \lambda I_K & T \\ T^* & T^*T + \mu I_H \end{bmatrix} : \lambda, \mu \in \mathbb{C} \right\}$$

and

$$\omega = \begin{bmatrix} \omega_1 & \theta \\ \theta^* & \omega_2 \end{bmatrix}$$

is a unital representation of $\mathcal{A}$ on $K \oplus H$ such that $\omega_{|\mathcal{S}(X)} = \mathcal{S}(\phi)$, where ω_1 and ω_2 are the representations corresponding to θ of the respective C^*-algebras.

We claim that ω is irreducible. Let P be a non zero projection in $B(K \oplus H)$ that commutes with $\omega(\mathcal{A})$. In particular, P commutes with

$$\omega\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} I_K & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \omega\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & I_H \end{bmatrix}.$$

Therefore, $P = p \oplus q$ where p is a projection on K and q is a projection on H. Thus,

$$(p \oplus q)\omega(x) = \omega(x)(p \oplus q) \text{ for every } x \in \mathcal{A}$$

which implies that

$$p\theta(a) = \theta(a)q \text{ for every } a \in T.$$

Since θ is an irreducible representation of T, it follows that $p = I_K$ and $q = I_H$ and therefore $P = I_{K\oplus H}$. Hence ω is an irreducible representation.

Now, to prove $\mathcal{S}(\phi)$ is a boundary representation for $\mathcal{S}(X)$, it is enough to prove the following. If $\Phi : \mathcal{A} \to B(K \oplus H)$ is any unital completely positive map with the property $\Phi_{|_{\mathcal{S}(X)}} = \mathcal{S}(\phi)$, then $\Phi = \omega$. Let Φ be a such a map. By Stinespring's dilation theorem

$$\Phi(a) = V^*\rho(a)V, \ \ a \in \mathcal{A}$$

where $\rho : \mathcal{A} \to B(L)$ is the minimal Stinespring representation and $V : K \oplus H \to L$ is an isometry. Thus,

$$\omega_{|_{\mathcal{S}(X)}} = \Phi_{|_{\mathcal{S}(X)}} = V^*\rho(\cdot)V_{|_{\mathcal{S}(X)}}.$$

Since ρ is a unital representation on L, we can decompose $L = K_\rho \oplus H_\rho$, where K_ρ is the range of the orthogonal projection $\rho\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$ and H_ρ is the range of the orthogonal projection $\rho\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$. Then with respect to this decomposition one has that

$$\rho = \begin{bmatrix} \sigma_1 & \eta \\ \eta^* & \sigma_2 \end{bmatrix}$$

where $\eta : T \to B(H_\rho, K_\rho)$ is a triple morphism and σ_1, σ_2 are unital representations of the respective C^*-algebras.

We claim that $V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}$, for isometries $v_1 : K \to K_\rho$ and $v_2 : H \to H_\rho$. We have

$$\begin{aligned} V^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V =& V^* \begin{bmatrix} \sigma_1(1) & 0 \\ 0 & 0 \end{bmatrix} V = V^* \rho\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) V \\ =& \Phi \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \mathcal{S}(\phi) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Similarly,

$$V^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since V is an isometry, we must have $V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}$ for isometries v_1 and v_2. Using

$\mathcal{S}(\theta)_{|\mathcal{S}(X)} = \mathcal{S}(\phi) = \Phi_{|X} = V^*\rho V$, we have

$$\theta(x) = v_1^*\eta(x)v_2 \,\forall\, x \in X.$$

Our assumption θ is a boundary representation for X implies that

$$\theta(t) = v_1^*\eta(t)v_2 \,\forall\, t \in T.$$

Using [33, Proposition 1.6], θ is maximal implies that η is a trivial dilation. We have

$$\eta(t) = q\eta(t)p + (1-q)\eta(t)(1-p)$$

for every $t \in T$, where $p = v_2v_2^*$ and $q = v_1v_1^*$. The above equation implies that $\eta(T)v_2H \subseteq v_1K$ and $\eta(T)^*v_1K \subseteq v_2H$. Using the minimality assumption [33, Page 142] of η, we have K_ρ is the closed linear span of $\eta(T)\eta(T)^*v_1K \cup \eta(T)v_2H$ and H_ρ is the closed linear span of $\eta(T)^*\eta(T)v_2H \cup \eta(T)^*v_1K$. Straightforward verification shows that $\eta(T)\eta(T)^*v_1K\cup\eta(T)v_2H \subseteq v_1K$ and $\eta(T)^*\eta(T)v_2H\cup\eta(T)^*v_1K \subseteq v_2H$, therefore $K_\rho = v_1K$ and $H_\rho = v_2H$. Thus, v_1 and v_2 are onto. Since v_1 and v_2 are isometries and onto implies that v_1 and v_2 are unitary, therefore V is unitary.

Now, we have ρ is a representation and V is unitary, thus the equation $\Phi(a) = V^*\rho(a)V,\ \ a \in \mathcal{A}$ implies that Φ is a representation on $\mathcal{A}$. Since ω and Φ are representations on $\mathcal{A} = C^*(\mathcal{S}(X))$ and $\omega_{|\mathcal{S}(X)} = \Phi_{|\mathcal{S}(X)}$, we have $\Phi = \omega$. □

We give a couple of examples to illustrate the above theorem.

Example 4.1.1. *Let $X \subset B(H, K)$ be an operator space such that the TRO T generated by X acts irreducibly and such that $T \cap \mathcal{K}(H, K) \neq \{0\}$. Then the identity representation of T is a boundary representation for X if and only if the identity representation of $C^*(\mathcal{S}(X))$ is a boundary representation for $\mathcal{S}(X)$. To see this, first assume that the identity representation of T is a boundary representation for X. Then by rectangular boundary theorem [33, Theorem 1.17] the quotient map $B(H, K) \to B(H, K)/\mathcal{K}(H, K)$ is not completely isometric on X. Using the same line of argument in the proof of the converse part of [33, Theorem 1.17], we see that the quotient map $B(K \oplus H) \to B(K \oplus H)/\mathcal{K}(K \oplus H)$ is not completely isometry on $\mathcal{S}(X)$. Then by Arveson's boundary theorem [5, Theorem 2.1.1], the identity representation of $C^*(\mathcal{S}(X))$ is a boundary representation for $\mathcal{S}(X)$.*

Conversely, if the identity representation of $C^*(\mathcal{S}(X))$ is a boundary representation for $\mathcal{S}(X)$, then by [33, Proposition 1.8], identity representation of T is a boundary representation for X.

Example 4.1.2. *Let R be an operator system in $B(H)$ and let $A = C^*(R)$ be the C^*-algebra generated by R. In particular, R is an operator space and $A = C^*(R)$ is itself a TRO generated by R. We have $C^*(\mathcal{S}(R)) = M_2(A)$. Using Hopenwasser's result [38] , we can conclude that if π is a boundary representation of A for R then $\mathcal{S}(\pi)$ is a boundary representation of $C^*(\mathcal{S}(R))$ for $\mathcal{S}(R)$.*

Corollary 4.1.1. *If a rectangular operator state $\phi : X \to B(H, K)$ has the unique extension property for X, then $\mathcal{S}(\phi)$ has the unique extension property for $\mathcal{S}(X)$.*

Proof. The proof follows from the same line argument in Theorem 4.1.3 without the irreducibility assumption. □

Proposition 4.1.2. *Suppose $\theta : \mathcal{S}(X) \to B(L)$ has the unique extension property on the Paulsen system $\mathcal{S}(X)$ associated with X. Then one can decompose L as an orthogonal direct sum $K \oplus H$ in such a way that $\theta = \mathcal{S}(\psi)$ for some rectangular operator state $\psi : X \to B(H, K)$ and ψ on X has the unique extension property.*

Proof. The proof follows as in [33, Proposition 1.8]. □

Theorem 4.1.4. *Let $\theta : \mathcal{S}(X) \to B(L)$ be a CP-map. Then θ is a boundary representation for the operator system $\mathcal{S}(X)$ if and only if there exists Hilbert spaces H,K, and a boundary representation $\psi : X \to B(H, K)$ such that $\theta = \mathcal{S}(\psi)$ and $L = K \oplus H$.*

Proof. Suppose that $\theta : \mathcal{S}(X) \to B(L)$ is a boundary representation for the operator system $\mathcal{S}(X)$. Then by Proposition 4.1.1, we can decompose the Hilbert space L as $K \oplus H$ and there is a boundary representation $\psi : X \to B(H, K)$ such that $\theta = \mathcal{S}(\psi)$.

Conversely, assume that there exists Hilbert spaces H,K, and a boundary representation $\psi : X \to B(H, K)$ such that $\theta = \mathcal{S}(\psi)$. Then by Theorem 4.1.3, θ is a boundary representation for X. □

4.2 Weak boundary representations for operator spaces

Recently, Namboodiri, Pramod, Shankar, and Vijayarajan [56] introduced a notion of weak boundary representation, which is a weaker notion than Arveson's [4] boundary representation for operator systems. They studied relations of weak boundary representation with quasi hyperrigidity of operator systems in [56]. Here we introduce the notion of weak boundary representations for operator spaces as follows:

Definition 4.2.1. *Let $X \subset B(H,K)$ be an operator space and T be a TRO containing X as a generating subspace. An irreducible triple morphism $\psi : T \to B(H,K)$ is called a weak boundary representation for X if $\psi_{|X}$ has a unique rectangular operator state extension of the form $v^*\psi u$, namely ψ itself, where $v : H \to H$ and $u : K \to K$ are isometries.*

Suppose X is operator system, $H = K$, $v = u$, then the above notion of weak boundary representation recovers the weak boundary representation for operator systems. We can observe that all the boundary representations are weak boundary representations for operator spaces.

Now, we investigate the relations between weak boundary representation of an operator space and the weak boundary representation of it's Paulsen system.

Proposition 4.2.1. *Suppose $\omega : C^*(\mathcal{S}(X)) \to B(L_\omega)$ is a weak boundary representation for the Paulsen system $\mathcal{S}(X)$ associated with X. Then one can decompose L_ω*

as an orthogonal direct sum $K_\omega \oplus H_\omega$ *in such a way that* $\omega = \mathcal{S}(\theta)$ *for some weak boundary representation* $\theta : T \to B(H_\omega, K_\omega)$ *for* X*.*

Proof. Since ω is an irreducible representation of $C^*(\mathcal{S}(X))$ on L_ω, we can decompose $L_\omega = K_\omega \oplus H_\omega$ such that

$$\omega = \begin{bmatrix} \omega_1 & \theta \\ \theta^* & \omega_2 \end{bmatrix},$$

where $\theta : T \to B(H_\omega, K_\omega)$ is an irreducible representation.

Now, we will prove that θ is weak boundary representation for X. Let $u : H_\omega \to H_\omega$ and $v : K_\omega \to K_\omega$ be isometries such that $v^*\theta(a)u = \theta(a)\ \forall\ a \in X$. For every $a, b \in X$,

$$\begin{aligned}
\begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \omega \begin{bmatrix} \lambda_1 I_K & a \\ b^* & \lambda_2 I_H \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} &= \begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \lambda_1 I_K & \theta(a) \\ \theta(b)^* & \lambda_2 I_H \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 I_K & v^*\theta(a)u \\ (v^*\theta(b)u)^* & \lambda_2 I_H \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 I_K & \theta(a) \\ \theta(b)^* & \lambda_2 I_H \end{bmatrix}.
\end{aligned}$$

Since $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$ is an isometry on $K_\omega \oplus H_\omega$ and ω is a weak boundary representation for $\mathcal{S}(X)$ implies that for all $a \in T$,

$$\begin{aligned}
\begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \omega\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} &= \omega\left(\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}\right) \\
\begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix}\begin{bmatrix} 0 & \theta(a) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} &= \begin{bmatrix} 0 & \theta(a) \\ 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & v^*\theta(a)u \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & \theta(a) \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

From the last equality $v^*\theta(a)u = \theta(a)$, for every $a \in T$. Thus θ a is weak boundary representation for X. □

Proposition 4.2.2. *If $\theta : T \to B(H, K)$ is a weak boundary representation for X then, the corresponding representation ω of $C^*(\mathcal{S}(X))$ on $B(K \oplus H)$ is a weak boundary representation for the Paulsen system $\mathcal{S}(X)$.*

Proof. Arguing as in the proof of Theorem 4.1.3, we have ω is irreducible representation. Let V be an isometry on $K \oplus H$ such that $V^*\omega V_{|_{\mathcal{S}(X)}} = \omega_{|_{\mathcal{S}(X)}}$. As

$$V^*\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$V^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

we can factorize $V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}$, where v_1 and v_2 are isometries on K and H respectively. Also we have $v_1^* \theta v_{2|_X} = \theta_{|X}$. Our assumption, θ is a weak boundary representation for X implies that $v_1^* \theta(t) v_2 = \theta(t)$ for all $t \in T$. Thus, we have $q\theta(t)p = \theta(t)$ for all $t \in T$, where q and p be the projections onto range of v_1 and range of v_2 respectively. Now, for each $t \in T$,

$$q\theta(t) = q(q\theta(t)p) = q\theta(t)p = (q\theta(t)p)p = \theta(t)p.$$

Since θ is irreducible, we have $p = I_H$ and $q = I_K$. Therefore v_1 and v_2 are unitaries. Consequently, V is a unitary. Thus $V^*\omega V$ is a representation of $C^*(\mathcal{S}(X))$. Since $V^*\omega V = \omega$ on $\mathcal{S}(X)$, we must have that $V^*\omega V = \omega$ on $C^*(\mathcal{S}(X))$. $\square$

4.3 Characterisation of boundary representations for operator spaces

In this section, we give a characterisation of boundary representations for operator spaces in terms of extreme points of certain noncommutative convex sets and a couple *of other notions.*

Arveson [4] introduced the notion of finite representations in the setting of subalgebras of C^*-algebras. Namboodiri, Pramod, Shankar and Vijayarajan [56] explored the relation between finite representations and weak boundary representations in the context of operator systems. Here, we introduce the notion of finite representation in the setting of operator spaces.

Definition 4.3.1. *Let X be an operator space generating a TRO T. Let $\phi : T \to B(H, K)$ be a representation. We say that ϕ is a finite representation for X if for every isometries $u : H \to H$ and $v : K \to K$, the condition $v^*\phi(x)u = \phi(x)$, for all $x \in X$ implies that u and v are unitaries.*

It is clear that, when X is operator system, $H = K$, $v = u$, the above notion of finite representation recovers the Arveson's notion of finite representation.

Proposition 4.3.1. *Let $\omega : C^*(\mathcal{S}(X)) \to B(L)$ be a finite representation for Paulsen system $\mathcal{S}(X)$ associated with X. Then one can decompose L as an orthogonal direct sum $K \oplus H$ in such a way that $\omega = \mathcal{S}(\phi)$ for some finite representation $\phi : T \to B(H, K)$ for X.*

Proof. We can get triple morphism $\phi : T \to B(H, K)$ as in the proof of [33, Proposition 1.8]. Now, we will prove that ϕ is a finite representation for X.

Let $u : H \to H$ and $v : K \to K$ are isometries such that $v^*\phi(x)u = \phi(x)\ \forall\, x \in X$.

Then for all $x, y \in X$

$$\begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \omega \left(\begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} \right) \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \lambda I_K & \phi(x) \\ \phi(y)^* & \mu I_H \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$$
$$= \begin{bmatrix} \lambda I_K & v^*\phi(x)u \\ u^*\phi(y)^*v & \mu I_H \end{bmatrix}$$
$$= \mathcal{S}(\phi) \left(\begin{bmatrix} \lambda I_K & x \\ y^* & \mu I_H \end{bmatrix} \right).$$

Thus, $\begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \omega \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}_{|\mathcal{S}(X)} = \omega_{|\mathcal{S}(X)}$. Since ω is a finite representation for $\mathcal{S}(X)$, we have $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$ is a unitary and consequently u and v are unitaries. Hence ϕ is a finite representation. □

Proposition 4.3.2. *If $\phi : T \to B(H, K)$ is a finite representation for X then, $\omega = \begin{bmatrix} \omega_1 & \phi \\ \phi^* & \omega_2 \end{bmatrix} : C^*(\mathcal{S}(X)) \to B(K \oplus H)$ is finite representation for $\mathcal{S}(X)$.*

Proof. Arguing as the in the proof of Theorem 4.1.3, we have ω is a representation. Let $V : K \oplus H \to K \oplus H$ be an isometry such that $V^*\omega(a)V = \omega(a)$ for all $a \in \mathcal{S}(X)$.

Since

$$V^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we can decompose V as $\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}$, where $v : K \to K$ and $u : H \to H$ are isometries. For $a \in \mathcal{S}(X)$, we have

$$V^*\omega(a)V = \begin{bmatrix} v^* & 0 \\ 0 & u^* \end{bmatrix} \begin{bmatrix} \omega_1 & \phi \\ \phi^* & \omega_2 \end{bmatrix} (a) \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} \omega_1 & \phi \\ \phi^* & \omega_2 \end{bmatrix} (a) = \omega(a).$$

Therefore $v^*\phi(x)u = \phi(x)$ for every $x \in X$. Since ϕ is a finite representation for X implies that v and u are unitaries. Thus V is a unitary. Hence ω is a finite representation for $\mathcal{S}(X)$. □

The following theorem shows a relation between finite representations with weak boundary representations.

Theorem 4.3.1. *Let X be an operator space generating a TRO T. Let ϕ be an irreducible representation of T. Then ϕ is a finite representation for X if and only if ϕ is a weak boundary representation for X*

Proof. Suppose $\phi : T \to B(H, K)$ is an irreducible finite representation for X then by Proposition 4.3.2, we have $\omega : C^*(\mathcal{S}(X)) \to B(K \oplus H)$ is an irreducible finite representation for Paulsen system $\mathcal{S}(X)$. Using [56, Proposition 3.5], we get ω is a weak boundary representation for $\mathcal{S}(X)$ of $C^*(\mathcal{S}(X))$. Therefore, Proposition 4.2.1 implies that ϕ is a weak boundary representation for X.

Conversely, suppose $\phi : T \to B(H, K)$ is a weak boundary representation for X of T, then by Proposition 4.2.2, we have $\omega : C^*(\mathcal{S}(X)) \to B(K \oplus H)$ is a weak boundary representation for $\mathcal{S}(X)$. Using [56, Proposition 3.5], we get ω is an irreducible finite representation for $\mathcal{S}(X)$. Therefore, Proposition 4.3.1 implies that ϕ is an irreducible finite representation for X.

□

Arveson [4] introduced the notion of separating subalgbras to characterize boundary representations in the context of subalgebras of C^*-algebas. Pramod, Shankar and Vijayarajan [62] studied the separating notion in the setting of operator systems and explored the relation with boundary representations. Here, we introduce the notion of separating operator space as follows:

Definition 4.3.2. *Let X be an operator space generating a TRO T. Let $\phi : T \to B(H, K)$ be an irreducible representation. We say that X separates ϕ if for every irreducible representation $\psi : T \to B(\tilde{H}, \tilde{K})$ and isometries $u : H \to \tilde{H}$ and $v : K \to \tilde{K}$, $v^*\psi(x)u = \phi(x)$, for all $x \in X$ implies that ϕ and ψ are unitarily equivalent. Also, the operator space X is called a separating operator space if it separates every* *irreducible representations of T.*

It is clear that, when X is an operator system, $H = K$, $v = u$, the notion of separating operator space recovers the notion of separating operator system.

Proposition 4.3.3. *Suppose $\omega : C^*(\mathcal{S}(X)) \to B(L)$ is an irreducible representation and $\mathcal{S}(X)$ separates ω. Then one can decompose L as an orthogonal direct sum $K \oplus H$ in such a way that $\omega = \mathcal{S}(\phi)$ for some irreducible representation $\phi : T \to B(H, K)$ and X separates ϕ.*

Proof. Existence and irreduciblity of a triple morphism $\phi : T \to B(H, K)$ follows from the proof of [33, Proposition 1.8]. Now we will prove that X separates ϕ.

Let $\theta : T \to B(H_\theta, K_\theta)$ be an irreducible representation of T such that $v_1^*\theta(x)v_2 = \phi(x)$ for all $x \in X$, where $v_1 : K \to K_\theta$ and $v_2 : H \to H_\theta$ are isometries. Let $\rho : C^*(\mathcal{S}(X)) \to B(K_\theta \oplus H_\theta)$ be the irreducible representation of $C^*(\mathcal{S}(X))$ corresponding to θ. Then for the isometry $V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}$ we have $V^*\rho V_{|_{\mathcal{S}(X)}} = \omega_{|_{\mathcal{S}(X)}}$. Since $\mathcal{S}(X)$ separates ω, there exists a unitary $U : K \oplus H \to K_\theta \oplus H_\theta$ such that $U^*\rho(a)U = \omega(a)$ for all $a \in C^*(\mathcal{S}(X))$. Using

$$U^* \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } U^* \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} U = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

we can factories U as $\begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$. Thus $u_1^*\theta(t)u_2 = \phi(t)$ for every $t \in T$. Hence X separates ϕ. $\square$

Proposition 4.3.4. *If $\phi : T \to B(H, K)$ is an irreducible representation such that X separates ϕ, then the corresponding representation $\omega : C^*(\mathcal{S}(X)) \to B(K \oplus H)$ is an irreducible representation such that $\mathcal{S}(X)$ separates ω.*

Proof. Let $\phi : T \to B(H, K)$ be an irreducible representation. Then the corresponding representation ω of the C^*-algebra $C^*(\mathcal{S}(X))$ on $B(K \oplus H)$ can be written as

$$\omega = \begin{bmatrix} \omega_1 & \phi \\ \phi^* & \omega_2 \end{bmatrix}.$$

Using the same line of argument as in the proof of Theorem 4.1.3, we have ω is an irreducible representation. We will prove that $\mathcal{S}(X)$ separates ω.

Let $\rho : C^*(\mathcal{S}(X)) \to B(L)$ be an irreducible representation such that $V^*\rho V_{|\mathcal{S}(X)} = \omega_{|\mathcal{S}(X)}$ for some isometry $V : K \oplus H \to L$. Since ρ is an irreducible representation of $C^*(\mathcal{S}(X))$, we can decompose $L = K_\rho \oplus H_\rho$ such that

$$\rho = \begin{bmatrix} \rho_1 & \theta \\ \theta^* & \rho_2 \end{bmatrix},$$

where $\theta : T \to B(H_\rho, K_\rho)$ is an irreducible representation of T. Also, we have

$$V = \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix},$$

where $v_1 : K \to K_\rho$ and $v_2 : H \to H_\rho$ are isometries. Substituting the expressions of ρ and V in the equation $V^*\rho(\cdot)V = \omega(\cdot)$, we obtain $v_1^*\theta(x)v_2 = \phi(x)$ for all $x \in X$. Our assumption that X separates ϕ implies that there exists unitaries u_1 and u_2 such that $u_1^*\theta(t)u_2 = \phi(t)$ for all $t \in T$. Then we have $u_2^*\theta^*(t)u_1 = \phi^*(t)$ for all $t \in T$. Now, for each $x, y \in T$, $\omega_1(xy^*) = \phi(x)\phi(y)^* = u_1^*\theta(x)u_2u_2^*\theta^*(t)u_1 = u_1^*\theta(x)\theta(y)^*u_1 = u_1^*\rho_1(xy^*)u_1$ and similarly $\omega_2(xy^*) = u_2^*\rho_2(xy^*)u_2$. Therefore ω_i and ρ_i are unitarily equivalent via the unitary u_i, $i = 1, 2$. Thus,

$$\omega = \begin{bmatrix} \omega_1 & \phi \\ \phi^* & \omega_2 \end{bmatrix} = \begin{bmatrix} u_1^*\rho_1 u_1 & u_1^*\theta u_2 \\ u_2^*\theta^* u_1 & u_2^*\rho_2 u_2 \end{bmatrix} = \begin{bmatrix} u_1^* & 0 \\ 0 & u_2^* \end{bmatrix} \begin{bmatrix} \rho_1 & \theta \\ \theta^* & \rho_2 \end{bmatrix} \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}.$$

Hence ρ is uniatrily equivalent to ω by the unitary $U = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$. □

Fuller, Hartz and Lupini [33] introduced the notion of rectangular extreme points. Suppose that X is an operator space, and $\phi : X \to B(H, K)$ is a completely contractive linear map. A rectangular operator convex combination is an expression $\phi = \alpha_1^*\phi_1\beta_1 + \alpha_2^*\phi_2\beta_2 + \cdots + \alpha_n^*\phi_n\beta_n$, where $\beta_i : H \to H_i$ and $\alpha_i : K \to K_i$

are linear maps, and $\phi_i : X \to B(H_i, K_i)$ *are completely contractive linear maps for* $i = 1, 2, \cdots, n$ *such that* $\alpha_1^*\alpha_1 + \cdots + \alpha_n^*\alpha_n = 1$*, and* $\beta_1^*\beta_1 + \cdots + \beta_n^*\beta_n = 1$*. Such a rectangular convex combination is proper if* α_i, β_i *are surjective, and trivial if* $\alpha_i^*\alpha_i = \lambda_i I$*,* $\beta_i^*\beta_i = \lambda_i I$ *, and* $\alpha_i^*\phi_i\beta_i = \lambda_i\phi$ *for some* $\lambda_i \in [0, 1]$*. A completely contractive linear map* $\phi : X \to B(H, K)$ *is a rectanagular operator extreme point if any proper rectangular convex combination is trivial. The set of all rectangular operator states from* X *to* $B(H, K)$ *is denoted by* $CB(X, B(H, K))$*.*

We show that a boundary representations for an operator space is actually a rectangular operator extreme points of the rectangular operator convex sets $CB(X, B(H, K))$*. For that, we recall the connections between rectangular operator extreme points on* X *and pure CP-maps on* $\mathcal{S}(X)$*.*

Proposition 4.3.5. *[33, Proposition 1.12] Suppose that* $\phi : X \to B(H, K)$ *is a completely contarctive map and* $\mathcal{S}(\phi) : \mathcal{S}(X) \to B(K \oplus H)$ *is the associated unital completely positive map defined on the Paulsen system. The following assertions are equivalent:*

1. $\mathcal{S}(\phi)$ *is a pure completely positive map*
2. $\mathcal{S}(\phi)$ *is an operator extreme point*
3. ϕ *is a rectangular operator extreme point.*

Definition 4.3.3. *Let* X *be an operator space generating a TRO* T*. Let* $\phi : T \to B(H, K)$ *be a representation. We say that* ϕ *is a finite representation for* X *if for every* *isometries* $u : H \to H$ *and* $v : K \to K$*, the condition* $v^*\phi(x)u = \phi(x)$*, for all* $x \in X$ *implies that* u *and* v *are unitaries.*

The following theorem characterize the boundary representations of TRO's for operator spaces.

Theorem 4.3.2. *Let* X *be an operator space generating a TRO* T*. Let* $\phi : T \to B(H, K)$ *be an irreducible representation of* T*. Then* ϕ *is a boundary representation for* X *if and only if the following conditions are satisfied:*

(i) $\phi_{|X}$ *is a rectangular operator extreme point.*

(ii) ϕ *is a finite representation for* X*.*

(iii) X *separates* ϕ*.*

Proof. Assume that $\phi : T \to B(H, K)$ be a boundary representation for X. Let $\omega : C^*(\mathcal{S}(X)) \to B(K \oplus H)$ be a representation of $\mathcal{S}(X)$ such that $\omega_{|_{\mathcal{S}(X)}} = \mathcal{S}(\phi)$. By Theorem 4.1.3, ω is a boundary representation for $S(X)$. Using [4, Theorem 2.4.5], we have $\mathcal{S}(\phi)$ is a pure UCP map, ω is a finite representation for $\mathcal{S}(X)$ and $\mathcal{S}(X)$ separates ω. Thus, Proposition 4.3.5 implies that $\phi_{|X}$ is a rectangular operator extreme point, Proposition 4.3.1 implies that ϕ is a finite representation for X and Proposition 4.3.3 implies that X separates ϕ.

Conversely, assume that all the three conditions are satisfied. Using, Proposition 4.3.5, Proposition 4.3.2 and Proposition 4.3.4 we have, $\mathcal{S}(\phi)$ is a pure UCP map, ω

is a finite representation for $\mathcal{S}(X)$ and $\mathcal{S}(X)$ separates ω. Thus [4, Theorem 2.4.5] implies that ω is a boundary representation for $\mathcal{S}(X)$. By [33, Proposition 1.8], ϕ is a boundary representation for X. □

Remark 4.3.1. *The statement* (i) *of Theorem 4.3.2 says that a boundary representation for an operator space is an extreme point(rectangular operator extreme point) of the convex set* $CB(X, B(H, K))$. *Thus the boundary representations are actually 'on the boundary' of the set* $CB(X, B(H, K))$.

4.4 Rectangular hyperrigidity

In this section, we introduce the notion of rectangular hyperrigidity in the context of operator spaces in TRO's. Rectangular hyperrigidity is the generalization of Arveson's [10] notion of hyperrigidity in the context of operator systems in C^**-algebras. We define rectangular hyperrigidity as follows:*

Definition 4.4.1. *A finite or countably infinite set* G *of generators of a TRO* T *is said to be rectangular hyperrigid if for every faithful representation from* T *to* $B(H, K)$ *and every sequence of completely contractive (CC) maps* $\phi_n : B(H, K) \to B(H, K)$ *with* $\|\phi_n\|_{cb} = 1$, $n = 1, 2 \cdots$,

$$\lim_{n\to\infty} \|\phi_n(g) - g\| = 0, \ \forall \ g \in G \implies \lim_{n\to\infty} \|\phi_n(t) - t\| = 0, \ \forall \ t \in T. \quad (4.1)$$

As in Arveson's [10] notion of hyperrigity, we have lightened the notion of rectangular hyperrigidity by identifying T with image $\pi(T)$, where $\pi : T \to B(H, K)$ faithful nondegenerate representation. Significantly, rectangular hyperrigid set of operators implies not only that equation 4.1 should hold for sequences of CC maps ϕ_n with $\|\phi_n\|_{cb} = 1$, but also that the property should persist for every other faithful representation of T.

Proposition 4.4.1. *Let T be a TRO and G a generating subset of T. Then G is rectangular hyperrigid if and only if linear span of G is rectangular hyperrigid.*

Proof. The proof follows directly from the definition of rectangular hyperrigidity. □

Proposition 4.4.2. *Let A be a C^*-algebra and S be an operator system in A such that $A = C^*(S)$. If S is rectangular hyperrigid, then S is hyperrigid.*

Proof. The proof follows from the fact that (see [58, Proposition 3.6]), every UCP map is completely bounded with CB norm 1. □

In definition 4.4.1, suppose T is a C^*-algebra, $H = K$ then by [58, Proposition 2.11] and [58, Proposition 3.6] the both notions rectangular hyperrigidity and hyperrigidity coincides. Thus, the rectangular hyperrigidity is a generalized notion of hyperrigidity adapted in the context of TROs.

Now, we prove a characterization of rectangular hyperrigid operator spaces which leads to study the operator space analogue of Saskin's theorem ([70], [12, Theorem

4]) relating retangular hyperrigity and boundary representations for operator spaces.

Theorem 4.4.1. *For every separable operator space X that generates a TRO T, the following are equivalent:*

(i) *X is rectangular hyperrigid.*

(ii) *For every nondegenerate representation $\pi : T \to B(H_1, K_1)$ on seperable Hilbert spaces and every sequence $\phi_n : T \to B(H_1, K_1)$ of CC maps with $\|\phi_n\|_{cb} = 1$, $n = 1, 2, ...$*

$$\lim_{n\to\infty} \|\phi_n(x) - \pi(x)\| = 0, \ \forall\, x \in X \implies \lim_{n\to\infty} \|\phi_n(t) - \pi(t)\| = 0, \ \forall\, t \in T.$$

(iii) *For every nondegenerate representation $\pi : T \to B(H_1, K_1)$ on seperable Hilbert spaces, $\pi_{|X}$ has the unique extension property.*

(iv) *For every TRO T_1, every triple morphism of TRO's $\theta : T \to T_1$ with $\|\theta\|_{cb} = 1$ and every completely contractive map $\phi : T_1 \to T_1$ with $\|\phi\|_{cb} = 1$,*

$$\phi(x) = x, \ \forall\, x \in \theta(X) \implies \phi(t) = t, \ \forall\, t \in \theta(T).$$

The spirit and the line of argument in the proof of the above the theorem are the same as those by Arveson [10, Theorem 2.1], where we can replace operator systems, UCP maps and representations of C^*-algebras by operator spaces, CC maps and triple *morphism of TROs. Further, we need to use Haagerup-Paulsen-Wittstock [58, Theorem 8.2] extension theorem in place of Arveson extension theorem [4, Theorem 1.2.3].*

Example 4.4.1. *Let H be an infinite dimensional Hilbert space and V be the unilateral right shift operator on H. Then the operator space $S = span\{I, V, V^*\}$ is not rectangular hyperrigid. To see this, take $\phi_n : B(H) \to B(H)$ as $\phi_n = V^* I_S(\cdot) V$ for each $n = 1, 2 \cdots$. Then ϕ_n is a completely contractive linear map with $\|\phi_n\|_{cb} = 1$ and ϕ_n is identity on S. Hence $\lim\limits_{n\to\infty} \|\phi_n(s) - s\| = 0 \; \forall s \in S$ but $\lim\limits_{n\to\infty} \|\phi_n(VV^*) - VV^*\| = \|I - VV^*\| = 1$. Note that the arguments in this example carries over to any isometry V which is not a unitary.*

We deduce the following necessary conditions for rectangular hyperrigidity:

Corollary 4.4.1. *Let X be a separable operator space generating a TRO T. If X is rectangular hyperrigid then every irreducible representation of T is a boundary representation for X.*

Proof. The assertion is an immediate consequence of condition (ii) of Theorem 4.4.1.

□

Problem 4.4.1. *If every irreducible representation of TRO T is a boundary representation for a separable operator space $X \subseteq T$, then X is rectangular hyperrigid.*

Proposition 4.4.3. *Let X be an operator space generating TRO T. Let $\pi_i : T \to B(H_i, K_i)$ be a non-degenerate representation such that $\pi_{i|_X}$ has the unique extension* *property for $i = 1, 2, ..., n$. Then the direct sum of rectangular operator states*

$$\oplus_{i=1}^{n} \pi_{i|_X} : X \to B(\oplus_{i=1}^{n} H_i, \oplus_{i=1}^{n} K_i)$$

has the unique extension property.

Proof. Assume that $\pi_{i|_X} : T \to B(H_i, K_i)$ has unique extension property for X, $i = 1, 2, \cdots n$. By Corollary 4.1.1, we have $\mathcal{S}(\pi_{i|_X}) : \mathcal{S}(X) \to B(K_i \oplus H_i)$ has unique extension property for $\mathcal{S}(X)$. Using [10, Proposition 4.4], $\oplus_{i=1}^{n} \mathcal{S}(\pi_{i|_X})$ has unique extension property for $\mathcal{S}(X)$. Note that $\oplus_{i=1}^{n} \mathcal{S}(\pi_{i|_X}) = \mathcal{S}(\oplus_{i=1}^{n} \pi_{i|_X})$. Therefore, by [33, Proposition 1.8] we have $\oplus_{i=1}^{n} \pi_{i|_X}$ has unique extension property for X.

□

Here, we settle the Problem 4.4.1 when TRO is finite dimensional. Thus, we have a finite dimensional version of the classical Saskin's theorem.

Theorem 4.4.2. *Let X be an operator space whose generating TRO T is finite dimensional, such that every irreducible representation of T is a boundary representation for X. Then X is rectangular hyperrigid.*

Proof. Using item (iii) of Theorem 4.4.1, it is enough to prove that for every non-degenerate representation $\pi : T \to B(H, K)$, the rectangular operator state $\pi_{|_X}$ has the unique extension property. Since T finite dimensional, [14, Theorem 3.1.7] implies that every nondegenerate representation of a finite dimensional TRO is the finite direct sum of irreducible repersentations. By our assumption every irreducible representation restricted to X has unique extension property. By Proposition 4.4.3 finite direct sum of irreducible representation restricted to X has unique extension property. Therefore every nondegenerate representation restricted to X has the unique extension property. □

Now, we explore relations between rectangular hyperrigity of an operator space and hyperrigidity of the corresponding Paulsen system.

Theorem 4.4.3. *Let X be a separable operator space generating a TRO T. Paulsen system $\mathcal{S}(X)$ is hyperrigid in C^*-algebra $C^*(\mathcal{S}(X))$ if and only if X is rectangular hyperrigid in TRO T.*

Proof. Assume that Paulsen system $\mathcal{S}(X)$ is hyperrigid in C^*-algebra.
Let $\phi_n : B(H,K) \to B(H,K)$ be CC maps with $\|\phi_n\|_{cb} = 1$, $n = 1, 2 \cdots$, such that

$$\lim_{n\to\infty} \|\phi_n(x) - x\| = 0 \ \ , \forall\, x \in X.$$

Then the corresponding maps $\mathcal{S}(\phi_n) : B(K \oplus H) \to B(K \oplus H)$, $n = 1, 2 \cdots$ are

UCP maps. For all $x, y \in X$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}
\left\| \mathcal{S}(\phi_n)(\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix}) - \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \right\| &= \left\| \begin{bmatrix} \lambda & \phi_n(x) \\ \phi_n(y)^* & \mu \end{bmatrix} - \begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \right\| \\
&= \left\| \begin{bmatrix} 0 & \phi_n(x) - x \\ \phi_n(y)^* - y^* & 0 \end{bmatrix} \right\| \\
&\leq \|\phi_n(x) - x\| + \|\phi_n(y)^* - y^*\|.
\end{aligned}$$

Since $\mathcal{S}(X)$ is hyperrigid in $C^*(\mathcal{S}(X))$, we conclude that for every $t \in T$

$$\lim_{n\to\infty} \left\| \mathcal{S}(\phi_n)(\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}) - \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \right\| = 0.$$

Thus,

$$\lim_{n\to\infty} \|\phi_n(t) - t\| = 0 \; \forall \, t \in T.$$

Conversely, suppose X is rectangular hyperrigid. By item (iii) of Theorem 4.4.1 every nondegenerate representation of T restricted to X has the unique extension propery. From [13, Proposition 3.1.2 and Equation 3.1], we have a one to one correspondence between the representations of TRO T and its linking algebra. Using Corollary 4.1.1 we get, every nondegenerate representation of the C^*-algebra $C^*(\mathcal{S}(X))$ restricted to $\mathcal{S}(X)$ has the unique extension property. Thus, by [10, Theorem 2.1], $\mathcal{S}(X)$ is hyperrigid. $\square$

The following remark gives some partial answers to the Problem 4.4.1 with extra assumptions.

Remark 4.4.1. *Suppose that T is a TRO in $B(H,K)$ and every irreducible representation $\psi : T \to B(H,K)$ is a boundary representation for X. Then by Theorem 4.1.3, each $\mathcal{S}(\psi)$ is a boundary representation of $C^*(\mathcal{S}(X))$ for $\mathcal{S}(X)$. Suppose either $C^*(\mathcal{S}(X))$ has countable spectrum [10, Theorem 5.1] or $C^*(\mathcal{S}(X))$ is Type I C^*-algebra with $C^*(\mathcal{S}(X))''$ is the codomain for UCP maps on $C^*(\mathcal{S}(X))$ [46, Corollary 3.3]. Then $\mathcal{S}(X)$ is hyperrigid. Therefore by Theorem 4.4.3, X is rectangular hyperrigid in T.*

Chapter 5

Boundary representations for unbounded operators

In this chapter, we initiate a study of non-commutative Choquet boundary for certain spaces of unbounded operators. Like C-algebras in the case of bounded operators on Hilbert spaces, the $*$-algebras associated with unbounded operators are called locally C*-algebras. The notion of locally C^*-algebras was introduced by Atushi Inoue [40]. In the literature, locally C^*-algebra have been studied by several authors under different names like pro-C^*-algebras, O^*-algebras, LCM^*-algebras, and multinormed C^*-algebras.*

Like the theory of non-commutative Choquet boundary for operator systems in C^*-algebras, we develop the theory of non-commutative Choquet boundary for local oper-

ator systems in locally C-algebras. We define a suitable notion of boundary representations for local operator systems in locally C*-algebras and called as local boundary representations. We prove that local boundary representations provide an intrinsic invariant for a nice class of local operator systems. An appropriate analog of purity of local CP-maps on local operator systems is used to characterize local boundary representations for local operator systems in Frechet locally C*-algebras.*

5.1 Locally C*-algebras

Let $\mathcal{A}$ *be a unital* $*$*-algebra with unit* $1_{\mathcal{A}}$*. A seminorm* p *on* $\mathcal{A}$ *is said to be sub-multiplicative, if* $p(1_{\mathcal{A}}) = 1$ *and* $p(ab) \leq p(a)p(b)$ *for every* $a, b \in \mathcal{A}$*. A sub-multiplicative seminorm* p *satisfies the condition* $p(a^*a) = p(a)^2$ *for every* $a \in \mathcal{A}$*, is called a* C^**-seminorm. Let* $(\Lambda, \leq)$ *be a directed poset. A family of seminorms* $\mathcal{P} = \{p_\alpha : \alpha \in \Lambda\}$ *on* $\mathcal{A}$ *is called an upward filtered family, if* $\alpha \leq \beta$ *in* Λ*, then* $p_\alpha(a) \leq p_\beta(a)$ *for every* $a \in \mathcal{A}$*.*

Definition 5.1.1. *A locally* C^**-algebra* $\mathcal{A}$ *is a* $*$*-algebra together with an upward filtered family of* C^**-seminorms* $\mathcal{P}$ *on* $\mathcal{A}$ *such that* $\mathcal{A}$ *is complete with respect to the locally convex topology generated by the family* $\mathcal{P}$*.*

Example 5.1.1. *Let* $\mathcal{A} = C(\mathbb{R}^n)$ *be the set of all complex valued continuous functions on* $\mathbb{R}^n$*. Then* $\mathcal{A}$ *is a* $*$*-algebra with respect to point wise multiplication and involution* $f \to \overline{f}$*. For each* $n \in \mathbb{N}$*, let* $K_n = \{h \in \mathbb{R}^n : \|h\| \leq n\}$ *be the closed disk centered at* *origin and radius n in $\mathbb{R}^n$. Define $p_n : \mathcal{A} \to \mathbb{R}$ by*

$$p_n(f) = \sup_{h \in K_n} \|f(h)\| \ , \forall f \in \mathcal{A}.$$

Then $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ is an upward filtered family of C^*-seminorms. It is easy to see that $\mathcal{A}$ is complete with respect to the locally convex topology generated by the family $\mathcal{P}$. Let $\{f_\lambda : \lambda \in \Lambda\}$ be a net which is Cauchy in $\mathcal{A}$. Then the restriction of the net $\{f_\lambda : \lambda \in \Lambda\}$ to K_n is Cauchy with respect to the norm p_n on $C(K_n)$. As K_n is compact, $\{f_\lambda | K_n : \lambda \in \Lambda\}$ converges to some function $f_n \in C(K_n)$. Since $K_n \subseteq K_{n+1}$, $f_n(h) = f_{n+1}(h)\ \forall h \in K_n$, $\forall n \in \mathbb{N}$. Thus, we can define a unique continuous function $f : \mathbb{R}^n \to \mathbb{C}$ such that $f|K_n = f_n$ and the net $\{f_\lambda\}$ converges to f. Hence $\mathcal{A}$ is a locally C^*-algebra with respect to $\mathcal{P}$.

Remark 5.1.1. *The Example 5.1.1 gives an example of a commutative locally C*-algebra. A concrete non-commutative example of a locally C^*-algebra is given later, in Example 5.1.4.*

Throughout this chapter, $\mathcal{A}$ always denotes a locally C^*-algebra with a prescribed family of C^*seminorms $\{p_\alpha : \alpha \in \Lambda\}$.

Let $I_\alpha = \{a \in \mathcal{A} : p_\alpha(a) = 0\}$ and $\mathcal{A}_\alpha$ be the quotient C^*-algebra $\mathcal{A}/I_\alpha$ with the C^*-norm induced by p_α. Denote the cannonical quotient $*$-homomorphism from $\mathcal{A}$ to

$\mathcal{A}_\alpha$ by π_α. Note that, for $\alpha \leq \beta$ in Λ, there is a cannonical $*$-homomorphism

$$\pi_{\alpha\beta} : \mathcal{A}_\beta \rightarrow \mathcal{A}_\alpha \text{ ,where } \pi_{\alpha\beta}(a + I_\beta) = a + I_\alpha$$

and that satisfies $\pi_{\alpha\beta}\pi_\beta = \pi_\alpha$. Then we can identify $\mathcal{A}$ as the the inverse limit of the projective system $\{\mathcal{A}_\alpha, \pi_{\alpha,\beta} : \alpha, \beta \in \Lambda\}$ of C^*-algebras [61].

5.1.1 The space $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$

Let H be a complex Hilbert space and $\mathcal{D}$ be a dense subspace of H.

Definition 5.1.2. *A quantized domain in H is a triple $\{H, \mathcal{E}, \mathcal{D}\}$, where $\mathcal{E} = \{H_l : l \in \Omega\}$ is an upward filtered family of closed subspaces of H such that the union space $\mathcal{D} = \bigcup_{l \in \Omega} H_l$ is dense in H.*

In short, we say $\mathcal{E}$ is a quantized domain in H with its union space D. A quantized doamin $\mathcal{E}$ is called a quantized Frechet domain if $\mathcal{E}$ is a countable family.

Example 5.1.2. *Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_1, e_2 \cdots\}$. Let $H_n = span\{e_1, e_2, \cdots e_n\}$ and $\mathcal{D} = \bigcup_{n=1}^{\infty} H_n$. Then $\mathcal{E} = \{H_n : n \in \mathbb{N}\}$ is a quantized domain in H with its union space $\mathcal{D}$.*

Example 5.1.3. *Consider the complex Hilbert space $H = L^2(\mathbb{R})$. Let $\mathcal{D}$ be the dense subspace of H,*

$$\mathcal{D} = \{f \in L^2(\mathbb{R}) : \text{ supp(f) is compact } \}.$$

Define $H_n = \{f \in L^2(\mathbb{R}) : \text{supp(f)} \subseteq [-n, n]\}$. *Then each* H_n *is a closed linear subspace of* H *and* $\mathcal{D} = \bigcup_{n=1}^{\infty} H_n$. *Thus* $\mathcal{E} = \{H_n : n \in \mathbb{N}\}$ *is a quantized domain in* H *with its union space* $\mathcal{D}$.

Corresponding to a quantized domain $\mathcal{E} = \{H_l : l \in \Omega\}$ *we can associate an upward filtered family* $\mathscr{P} = \{P_l : l \in \Omega\}$ *of projections in* $B(H)$ *where* P_l *is the orthogonal projection of* H *onto the closed subspace* H_l.

Let us denote $L(\mathcal{D})$ *by the set of all linear operators on the linear subspace* $\mathcal{D}$. *The set of all noncommutative continuous functions on a quantized domain* $\mathcal{E}$ *is defined as*

$$\mathcal{C}_{\mathcal{D}}(\mathcal{E}) = \{T \in L(\mathcal{D}) : TP_l = P_l T P_l \in B(H), \text{ for all } l \in \Omega\}.$$

Note that $\mathcal{C}_{\mathcal{D}}(\mathcal{E})$ *is an algebra and if* $T \in L(\mathcal{D})$, *then*

$$T \in \mathcal{C}_{\mathcal{D}}(\mathcal{E}) \text{ if and only if } T(H_l) \subseteq H_l \text{ and } T|_{H_l} \in B(H_l) \text{ for all } l \in \Omega.$$

The $*$*-algebra of all noncommutative continuous functions on a quantized domain* $\mathcal{E}$ *is defined as*

$$\mathcal{C}^*_{\mathcal{E}}(\mathcal{D}) = \{T \in \mathcal{C}_{\mathcal{D}}(\mathcal{E}) : P_l T \subseteq T P_l, \text{ for all } l \in \Omega\}.$$

Note that $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is a unital subalgebra of* $\mathcal{C}_{\mathcal{D}}(\mathcal{E})$. *The details of the adjoint of operators in* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is given in [26, Proposition 3.1]. For* $T \in L(\mathcal{D})$, *it is easy to see that* $T \in$

$\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *if and only if for all* $l \in \Omega$

$$T(H_l) \subseteq H_l,\ T|_{H_l} \in B(H_l) \text{ and } T(H_l^{\perp} \cap \mathcal{D}) \subseteq H_l^{\perp} \cap \mathcal{D}.$$

Example 5.1.4. *Let* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *as above. Define* $q_l : \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}) \to \mathbb{R}$ *by*

$$q_l(T) = \|T|_{H_l}\| \text{ for all } T \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}).$$

Then $\mathcal{Q} = \{q_l : l \in \Omega\}$ *is an upward filtered family of* C^**-seminorms on* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$*. Also,* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is complete with respect to the locally convex topology generated by the family* $\mathcal{Q}$*. Hence* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is a locally* C^**-algebra.*

5.1.2 Local CP-maps and Local CC-maps

Anar Dosiev [26] introduced the notions of local hermitian and local positivity in locally C^**-algebras.*

Definition 5.1.3. *An element* $a \in \mathcal{A}$ *is called local hermitian if* $a = a^* + x$ *for some* $x \in \mathcal{A}$ *such that* $p_\alpha(x) = 0$ *for some* $\alpha \in \Lambda$ *and an element* $a \in \mathcal{A}$ *is called local positive if* $a = b^*b + x$ *for some* $b, x \in \mathcal{A}$ *such that* $p_\alpha(x) = 0$ *for some* $\alpha \in \Lambda$*. In this case, we call* a *is* α*-hermition (and* α*-positive, respectively).*

We use $a \geq_\alpha 0$ *to denote* a *is* α*-positive. A direct computation shows that* $a \geq_\alpha 0$ *in* $\mathcal{A}$ *if and only if the* $\pi_\alpha(a) \geq 0$ *in the* C^**-algebra* $\mathcal{A}_\alpha$*.*

Let $\mathcal{A}$ be a locally C^*-algebra. For a linear subspace S of $\mathcal{A}$ denote $S^* = \{x^* : x \in S\}$. We say S is self adjoint if $S = S^*$.

Definition 5.1.4. *A local operator system in $\mathcal{A}$ is a unital self adjoint linear subspace of $\mathcal{A}$.*

Every operator system in a C^*-algebra is a local operator system.

Example 5.1.5. *We give some examples for local operator systems:*

(i) *Consider the locally C^*-algebra $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ and let $T \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. Then*

$$S = \{\lambda_1 I + \lambda_2 T + \lambda_3 T^* : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}\}$$

is a local operator system and it is called the local operator system generated by the element T.

(ii) *Consider the locally C^*-algebra $C(\mathbb{R})$ of all complex valued continuous functions on the real line $\mathbb{R}$. Then $S = \{f \in C(\mathbb{R}) : f \text{ is real valued }\}$ is a local operator system in $C(\mathbb{R})$ with unit constant function 1.*

(iii) *Let H be a separable infinite dimensional Hilbert space with O.N.B. $\{e_n : n \in \mathbb{N}\}$ and consider the quantized domain $\mathcal{E}$ as in Example 5.1.2. An element T of $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ can be represented as infinite matrix $T = [t_{ij}]$ with respect to the basis*

$\{e_n : n \in \mathbb{N}\}$. *Then the set*

$$S = span\{E_{ij} : |i - j| \leq 1\}$$

is a local operator system in $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. *Note that the elements of* S *are tridiagonal matrices.*

An element a *in a local operator system* S *is local positive if* a *is local positive in* $\mathcal{A}$. *Consider another locally* C^**-algebra* $\mathcal{B}$ *with the associated family of seminorms* $\{q_l : l \in \Omega\}$. *Let* S_1 *and* S_2 *be local operator systems in* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively.*

Definition 5.1.5. *Let* $\phi : S_1 \to S_2$ *be a linear map. Then* ϕ *is said to be*

1. *local positive if for each* $l \in \Omega$ *there corresponds* $\alpha \in \Lambda$ *such that* $\phi(a) \geq_l 0$ *whenever* $a \geq_\alpha 0$ *in* S_1.
2. *local bounded if for each* $l \in \Omega$ *there exists an* $\alpha \in \Lambda$ *and* $C_{l\alpha} > 0$ *such that* $q_l(\phi(a)) \leq C_{l\alpha} p_\alpha(a)$ *for all* $a \in S_1$.
3. *local contractive if it's local bounded and* $C_{l\alpha}$ *can be chosen to be 1 in the definition of local bounded.*

It's important to note that a local positive map is a positive map.

Proposition 5.1.1. *[41] Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be locally* C^**-algebras and let* $\phi : \mathcal{A} \to \mathcal{B}$ *be a linear map. If* ϕ *is local positive, then* ϕ *is positive.*

For $n \in \mathbb{N}$, let $M_n(\mathcal{A})$ denotes the set of all $n \times n$ matrices over $\mathcal{A}$. Naturally $M_n(\mathcal{A})$ is a locally C^*-algebra with the defining family of seminorms $\{p_\alpha^n : \alpha \in \Lambda\}$, where $p_\alpha^n([a_{ij}]) = \|\pi_\alpha^{(n)}([a_{ij}])\|_\alpha$ for $[a_{ij}]$ in $M_n(\mathcal{A})$. We use $\phi^{(n)}$ to denote the n-amplification of the map ϕ, that is,

$$\phi^{(n)} : M_n(S_1) \to M_n(S_2) \text{ defined by } \phi^{(n)}([a_{ij}]) = [\phi(a_{ij})]$$

for $[a_{ij}]$ in $M_n(S_1)$.

Definition 5.1.6. *(Local CP-map). Let $\phi : S_1 \to S_2$ be a linear map. Then ϕ is said to be local completely positive if for each $l \in \Omega$, there exists $\alpha \in \Lambda$ such that*

$$\phi^{(n)}([a_{ij}]) \geq_l 0 \text{ in } M_n(S_2) \text{ whenever } [a_{ij}] \geq_\alpha 0 \text{ in } M_n(S_1).$$

Definition 5.1.7. *Let $\phi : S_1 \to S_2$ be a linear map. Then ϕ is said to be*

1. *local completely bounded(local CB-map) if for each $l \in \Omega$, there exists $\alpha \in \Lambda$ and $C_{l\alpha} > 0$ such that*

$$q_l^n([\phi(a_{ij})]) \leq C_{l\alpha} p_l^n([a_{ij}]), \text{ for every } n \in \mathbb{N}.$$

2. *local completely contractive(local CC-map) if it's local CB-map and $C_{l\alpha}$ can be chosen to be 1 in the definition of local CB-map.*

We use $\mathcal{CPCC}_{loc}(S,\, \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ to denotes the class of all local completely positive and local completely contractive maps(local CPCC-map) from a local operator system S to $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$.

Example 5.1.6. *Let $a \in \mathcal{A}$ be a fixed element of the locally C*-algebra $\mathcal{A}$. Then the map $\phi_a : \mathcal{A} \to \mathcal{A}$ defined by $\phi_a(x) = a^*xa$ is a local CP-map. To see this, let $\alpha \in \Lambda$. Consider $x \geq_\alpha 0$ in $\mathcal{A}$. Then $x = y^*y + b$, where $p_\alpha(b) = 0$. Note that*

$$\begin{aligned}\phi_a(x) &= \phi_a(y^*y + b) = \phi_a(y^*y) + \phi_a(b)\\ &= a^*(y^*y)a + \phi_a(b) = (ya)^*ya + \phi_a(b)\end{aligned}$$

where $p_\alpha(\phi_a(b)) = 0$ as $p_\alpha(\phi_a(b)) = p_\alpha(a^*ba) \leq p_\alpha(a^*)p_\alpha(b)p_\alpha(a) = 0$. That is $\phi_a(x) \geq_\alpha 0$. Hence ϕ_a is local positive. Now, let A be the diagonal matrix with all of its diagonal entries are a and $X = [x_{ij}] \geq_\alpha 0$ in $M_n(\mathcal{A})$. Then $X = Y^*Y + B$ with $p^{(n)}_\alpha(B) = 0$.

$$\begin{aligned}\phi^{(n)}_a([x_{ij}]) &= [\phi_a(x_{ij})]\\ &= [a^*x_{ij}a]\\ &= \begin{bmatrix} a^*x_{11}a & \cdots & a^*x_{1n}a \\ a^*x_{21}a & \cdots & a^*x_{2n}a \\ \vdots & \ddots & \vdots \\ a^*x_{n1}a & \cdots & a^*x_{nn}a \end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} a^* & 0 & \cdots & 0 \\ 0 & a^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a^* \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ x_{21} & \cdots & x_{2n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix}$$

$$= A^*XA$$

$$= A^*(Y^*Y + B)A$$

$$= (YA)^*YA + A^*BA$$

where $p_\alpha^n(A^*BA) = 0$ *as* $p_\alpha^n(A^*BA) \leq p_\alpha^n(A^*)p_\alpha^n(B)p_\alpha^n(A) = 0$. *Since* α *is independent of* n, ϕ_a *is a local CP-map.*

Example 5.1.7. *[11] Let* H *be a separable infinite dimensional Hilbert space with an orthonormal basis* $\{e_n : n \in \mathbb{N}\}$. *For* $n \in \mathbb{N}$, *let* $H_n = span\{e_1, e_2, \cdots, e_n\}$. *Then* H_n *is a closed subspace of* H *and* $D = \bigcup_{n=1}^{\infty} H_n$ *is dense in* H. *Consider the operators on* H *as infinite matrices with respect to the basis* $\{e_n : n \in \mathbb{N}\}$. *Then the elements of* $\mathcal{C}_\mathcal{E}^*(\mathcal{D})$ *are 'block diagonal' operators which are possibly unbounded. Let* $A \in B(H)$ *such that*

$$0 \leq A \leq I.$$

Write $A = [a_{ij}]$. *Define* $\phi_A : B(H) \to B(H)$ *by*

$$\phi_A(T) = [a_{ij}t_{ij}] \text{ for } T = [t_{ij}] \in B(H).$$

This ϕ_A is called the Schur product map. It is well known that the map ϕ_A can be written as

$$\phi_A(T) = V^*(A \otimes T)V$$

where V is the isometry $V : H \to H \otimes H$ given by $V(e_n) = e_n \otimes e_n$, $\forall n \in \mathbb{N}$ with adjoint $V^*(h_1 \otimes h_2) = \sum_{i=1}^{\infty} \langle e_i \otimes e_i, h_1 \otimes h_2 \rangle e_i$ for all $h_1, h_2 \in H$. Note that ϕ_A is the composition of the two CPCC-maps, namely $T \to A \otimes T$ and $T \to V^*TV$. Thus ϕ_A is also a CPCC-map. then extend the definition of ϕ_A to the whole of $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ as follows. Define $\psi_A : \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}) \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ by

$$\psi_A(T)|_{H_n} = \phi_A(P_n T|_{H_n}) \text{ for all } T \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}), n \in \mathbb{N}$$

where P_n is the projection corresponding to H_n. As ψ_A is the composition of the CPCC-map ϕ_A and the local CPCC-map $T \to P_n T|_{H_n}$, it's also a local CPCC-map.

A local CPCC-map satisfies Kadison-Schwarz inequality.

Lemma 5.1.1. *[26] Let $\phi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a local CC-map. If ϕ is a local CP-map then $\phi(a)^*\phi(a) \leq \phi(a^*a)$ on $\mathcal{D}$ for all $a \in \mathcal{A}$. If $\phi(a)^*\phi(a) = \phi(a^*a)$ then $\phi(ba) = \phi(b)\phi(a)$ for all $b \in \mathcal{A}$.*

5.1.3 Representations

By a representation of a locally C-algebra we mean a unital local contractive $*$-homomorphism $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ for some quantized domain $\mathcal{E}$.*

A locally convex version of Gelfand-Naimark-Segal theorem was proved by A.Dosiev [26].

Theorem 5.1.1. *[26, Theorem 7.2] For a locally C^*-algebra $\mathcal{A}$, there exist a Hilbert space H, a quantized domain $\{H; \mathcal{E}; \mathcal{D}\}$ and a local isometrical $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$.*

Remark 5.1.2. *The Theorem 5.1.1 tell us that the elements of a locally C^*-algebra can be treated as an unbounded operator on a suitable quantized domain.*

A locally convex version and an unbounded version of the celebrated Stinespring's dilation theorem is also appeared in the work of A. Dosiev [26].

Theorem 5.1.2. *[26, Theorem 5.1] Let $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. Then there exists a Hilbert space H^ϕ and a quantized domain $\mathcal{E}^\phi = \{H^\phi_\alpha : \alpha \in \Lambda\}$ in H^ϕ with its union space $\mathcal{D}^\phi$, a contraction $V_\phi : H \to H^\phi$, and a unital local contractive $*$-homomorphism $\pi_\phi : \mathcal{A} \to C^*_{\mathcal{E}^\phi}(\mathcal{D}^\phi)$ such that*

$$\phi(a) \subseteq V^*_\phi \pi_\phi(a) V_\phi \text{ and } V_\phi(H_\alpha) \subseteq H^\phi_\alpha$$

for every $a \in \mathcal{A}$ and $l \in \Lambda$. Moreover, if $\phi(1_{\mathcal{A}}) = 1_{\mathcal{D}}$, then V_ϕ is an isometry.

Remark 5.1.3. *If $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a local contractive $*$- homomorphism, then π is a local CP-map. Also, any map of the form $\phi(a) \subseteq V^*\pi(a)V$ is a local CP-map on $\mathcal{A}$, where V is an isometry on H such that $V(H_l) \subseteq H_l$ for every $l \in \Lambda$. Hence the Theorem 5.1.2 characterizes local CP-maps from locally C^*-algebras into $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ in terms of local contractive $*$-homomorphisms.*

Any triple $(\pi_\phi, V_\phi, \{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\})$ that satisfies the conditions of the Theorem 5.1.2 is called a Stinespring representation for ϕ.

The minimality of the Stinespring representation was introduced and studied recently by Bhat and et al in [11].

Definition 5.1.8. *[11] A Stinespring representation $(\pi_\phi, V_\phi, \{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\})$ of ϕ is said to be minimal, if $H^\phi_l = [\pi_\phi V_\phi H_l]$, for every $l \in \Lambda$.*

Given any Stinespring represesentation of a map $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$, one can reduce it to minimal Stinespring representation.

An important property of minimal Stinespring representation for a map $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ is the following;

Lemma 5.1.2. *[11] Let $(\pi_\phi, V_\phi, \{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\})$ be a minimal Stinespring representa-*

tion for a map $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$*. Then*

$$[\pi_\phi(\mathcal{A})V_\phi(H_\alpha^\perp \cap \mathcal{D})] = (H_\alpha^\phi)^\perp, \text{ for every } \alpha \in \Lambda$$

Any two minimal Stinespring representations are unitarily equivalent in the following sense.

Theorem 5.1.3. *[11] Let* $(\{H_1; \mathcal{E}_1; \mathcal{D}_1\}, \pi_1, V_1)$ *and* $(\{H_2; \mathcal{E}_2; \mathcal{D}_2\}, \pi_2, V_2)$ *be two minimal Stinespring representations of the map* $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$*. Then there exists a unitary* $U : H_1 \to H_2$ *such that*

$$UV_1 = V_2 \text{ and } U\pi_1(a) = \pi_2(a)U \text{ for all } a \in \mathcal{A}.$$

5.2 Local positive linear maps

In this section, we prove an analog of the Arveson extension theorem for local CC-maps on linear subspaces of $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *for a quantized Frechet domain* $\mathcal{E}$*. This result is crucial in establishing an important theorem in this chapter.*

Let S *be a local operator system in the locally* C^**-algebra* $\mathcal{A}$*. A linear functional* $f : S \to \mathbb{C}$ *is an* α*-contractive linear functional if* $|f(a)| \leq p_\alpha(a)$ *for all* $a \in S$*. Note that, by Hahn-Banach extension theorem, there is an* α*-contractive linear map* $\tilde{f} : \mathcal{A} \to \mathbb{C}$ *such that* $\tilde{f}|_S = f$ *and* $|\tilde{f}(a)| \leq p_\alpha(a)$ *for all* $a \in \mathcal{A}$*.*

For $a \in \mathcal{A}$ we define the α-spectrum of a to be the spectrum of $\pi_\alpha(a)$ in the C^*-algebra $\mathcal{A}_\alpha$. We use $\sigma_\alpha(a)$ to denote the α-spectrum of a.

Lemma 5.2.1. *Let S be a local operator system in a locally C^*-algebra $\mathcal{A}$ and let $f : S \to \mathbb{C}$ be a unital α-contractive linear functional. Let $\tilde{f}$ be a Hahn-Banach extension of f to $\mathcal{A}$. If $a = x^*x + b \in S$ is an α-positive element of $\mathcal{A}$, then $0 \leq \tilde{f}(x^*x) \leq r_\alpha$, where r_α is the spectral radius of $\pi_\alpha(a)$.*

Proof. Assume that $\tilde{f}(x^*x) \notin [0, r_\alpha]$. Since a closed interval in the real line is the intersection of all closed disks containing it in the complex plane, there exists a closed disk $D_r(\mu)$ centered at $\mu \in \mathbb{C}$ and radius r such that $|\tilde{f}(x^*x) - \mu| > r$ and $[0, r_\alpha] \subseteq D_r(\mu)$. Then $\sigma_\alpha(x^*x - \mu 1) \subseteq D_r(0)$ as $\sigma_\alpha(x^*x) \subseteq [0, r_\alpha] \subseteq D_r(\mu)$. Since $\pi_\alpha(x^*x)$ is a positive element of $\mathcal{A}_\alpha$, $\pi_\alpha(x^*x - \mu 1)$ is a normal element of $\mathcal{A}_\alpha$. The spectral radius and norm are same for normal elements of a C^*-algebra gives us $\|\pi_\alpha(x^*x - \mu 1)\|_\alpha \leq r$. Now using the fact $\tilde{f}$ is a unital α-contraction, we have

$$
\begin{aligned}
|\tilde{f}(x^*x) - \mu| = |\tilde{f}(x^*x - \mu 1)| &\leq p_\alpha(x^*x - \mu 1) \\
&= \|\pi_\alpha(x^*x - \mu 1)\|_\alpha \leq r.
\end{aligned}
$$

This is a contradiction. Hence $\tilde{f}(x^*x) \in [0, r_\alpha]$. □

Theorem 5.2.1. *Let S be a local operator system in a locally C^*-algebra $\mathcal{A}$ and let $\mathcal{E}$ be a quantized domain with its union space $\mathcal{D}$. Let $\phi : S \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a unital local* *contractive map. Then ϕ is a local positive map.*

Proof. Fix $l \in \Omega$. Since ϕ is local contractive, there exists $\alpha \in \Lambda$ such that $\|\phi(a)\|_l \leq p_\alpha(a)$ for every $a \in S$. Let $a \in S$ and $a = x^*x + b$ where $x, b \in \mathcal{A}$ and $p_\alpha(b) = 0$ for some $\alpha \in \Lambda$.

We will show that $\phi(a)|_{H_l}$ is a positive operator on H_l. Let $h \in H_l$ with $\|h\| = 1$. Define $f_h : S \to \mathbb{C}$ by $f_h(y) = \langle \phi(y)|_{H_l} h, h\rangle$. Then $f_h(1) = 1$ and

$$|f_h(y)| \leq \|\phi(y)\|_l \leq p_\alpha(y).$$

Therefore, the linear functional f_h is a unital α-contraction. Let $\tilde{f}_h : \mathcal{A} \to \mathbb{C}$ be an α-contractive Hahn-Banach extension of f_h. Then

$$\langle \phi(a)|_{H_l} h, h\rangle = f_h(a) = \tilde{f}_h(a) = \tilde{f}_h(x^*x) + \tilde{f}_h(b).$$

Note that, $\tilde{f}_h(b) = 0$ as $\tilde{f}_h$ is an α-contraction and $p_\alpha(b) = 0$. Using Lemma 5.2.1 we conclude that $\tilde{f}_h(x^*x) = \langle \phi(a)|_{H_l} h, h\rangle$ is positive. Therefore, $\phi(a)$ is local positive and that completes the proof. □

Corollary 5.2.1. *Let S be a local operator system in a locally C^*-algebra $\mathcal{A}$ and let $\mathcal{E}$ be a quantized domain with its union space $\mathcal{D}$. Let $\phi : S \to C^*_{\mathcal{E}}(\mathcal{D})$ be a unital local CC-map. Then ϕ is a local CP-map.*

Proof. Let ϕ be a local CC-map. Then for each $n \in \mathbb{N}$, the amplification $\phi^{(n)}$ is a local contractive map. By Theorem 5.2.1, $\phi^{(n)}$ is a local positive map for each $n \in \mathbb{N}$. Thus, ϕ is a local CP-map. □

We can use Theorem 5.2.1 to establish the following result.

Theorem 5.2.2. *Let S be a local operator system in a locally C^*-algebra $\mathcal{A}$ and let $\mathcal{E}$ be a quantized domain with its union space $\mathcal{D}$. Let $\phi : S \to C^*_{\mathcal{E}}(\mathcal{D})$ be a unital linear map. Then ϕ is a local CC-map if and only if ϕ is a local CP-map.*

Proof. Let ϕ be a local CC-map. Fix $l \in \Omega$. There exists a $\alpha \in \Lambda$ such that $\|\phi^{(n)}([a_{ij}])\|_l \leq p_\alpha^{(n)}([a_{ij}])$ for all $[a_{ij}] \in M_n(S)$,$n \in \mathbb{N}$. From the proof of Theorem 5.2.1 we have $\phi^{(n)}([a_{ij}]) \geq_l 0$ whenver $[a_{ij}] \geq_\alpha 0$. Thus ϕ is a local CP-map.

Conversely, assume that ϕ is a local CP-map. Fix $l \in \Omega$. There exists a $\alpha \in \Lambda$ such that $\phi^{(n)}(A) \geq_l 0$ whenever $A \geq_\alpha 0$ in $A \in M_n(S)$ and $n \in \mathbb{N}$. Let $A \in M_n(S)$ such that $p_\alpha^{(n)}(A) \leq 1$. Then

$$\begin{bmatrix} 1_n & A \\ A^* & 1_n \end{bmatrix} \geq_\alpha 0 \text{ in } M_{2n}(S).$$

Applying the map $\phi^{(2n)}$, we have

$$\begin{bmatrix} I_n & \phi^{(n)}(A) \\ \phi^{(n)}(A^*) & I_n \end{bmatrix} \geq_l 0 \text{ in } M_{2n}(C^*_{\mathcal{E}}(\mathcal{D})).$$

Thus

$$\begin{bmatrix} I_n & \phi^{(n)}(A) \\ \phi^{(n)}(A^*) & I_n \end{bmatrix}\Bigg|_{H_l^n \oplus H_l^n} \geq 0 \text{ in } B(H_l^n \oplus H_l^n).$$

Equivalently $\|\phi^n(A)|_{H_l^n}\| \leq 1$. Hence $\|\phi^n(A)\|_l \leq p_\alpha^{(n)}(A)$ for every $A \in M_n(S)$. That is, ϕ is a local CC-map. ☐

The following lemma tells us that local positive mappings are self adjoint.

Lemma 5.2.2. *[26] Let S_1 and S_2 be local operator systems and let $\phi : S_1 \to S_2$ be a local positive mapping. Then $\phi(x^*) = \phi(x)^*$ for every $x \in S_1$. In particular, $\phi^{(n)}(x^*) = \phi^{(n)}(x)^*$ for every $x \in M_n(S_1)$, whenever ϕ is a local CP-map.*

Theorem 5.2.3. *Let $\mathcal{A}$ be a unital locally C^*-algebra and let M be a unital subspace of $\mathcal{A}$. If $\phi : M \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ is a unital local contraction, then there is a local positive extension $\tilde{\phi}$ of ϕ to $M + M^*$ given by $\tilde{\phi}(x + y^*) = \phi(x) + \phi(y)^*$. Moreover, $\tilde{\phi}$ is the only local positive extension of ϕ to $M + M^*$.*

Proof. First, we will show that the map $\tilde{\phi}$ is well-defined. Let

$$M_* = \{a \in M : a^* \in M\}.$$

Clearly, M_* is a local operator system in $\mathcal{A}$. Also, the map ϕ is a unital local contractive map on M_*. Using Theorem 5.2.1 we have ϕ is a local positive map. Then ϕ is self adjoint on M_*, thanks to Lemma 5.2.2. To see $\tilde{\phi}$ is well defined, consider $a_1, a_2, b_1, b_2 \in M$ with $a_1 + b_1^* = a_2 + b_2^*$. Equivalently, $a_1 - a_2 = (b_2 - b_1)^*$. Thus $b_2 - b_1 \in M_*$. Then using the fact that ϕ is self adjoint on M_*, we have

$$\begin{aligned}
\phi(a_1 - a_2) &= \phi((b_2 - b_1)^*) \\
&= [\phi(b_2 - b_1)]^* \\
&= \phi(b_2)^* - \phi(b_1)^* \\
\phi(a_1) + \phi(b_1)^* &= \phi(a_2) + \phi(b_2)^*.
\end{aligned}$$

Hence $\tilde{\phi}(a_1 + b_1^*) = \tilde{\phi}(a_2 + b_2^*)$. That is, $\tilde{\phi}$ is well-defined.

To see $\tilde{\phi}$ is local positive; fix $l \in \Omega$. By local contractivity of ϕ, there exists an $\alpha \in \Lambda$ such that $\|\phi(a)\|_l \leq p_\alpha(a)$ for all $a \in \mathcal{A}$. Let $a + b^* \in M + M^*$ be an α-positive element. We will show that $\tilde{\phi}(a + b^*)$ is local positive by showing that $\tilde{\phi}(a + b^*)|_{H_l}$ is a positive operator on H_l. Let $h \in H_l$ with $\|h\| = 1$. Define $f : M \to \mathbb{C}$ by $f(y) = \langle \phi(y)h, h\rangle$. Then $|f(y)| \leq \|\phi(y)\|_l \leq p_\alpha(y)$ for every $y \in M$. Using Hahn-Banach extension theorem, f extends to $f_1 : M + M^* \to \mathbb{C}$ with $|f_1(y)| \leq p_\alpha(y)$ for every $y \in M + M^*$. By Theorem 5.2.1 we have that f_1 is local positive. Also, $0 \leq f_1(a+b^*) = f_1(a) + \overline{f_1(b)} = f(a) + \overline{f(b)} = \langle \phi(a)h, h\rangle + \overline{\langle \phi(b)h, h\rangle} = \langle \tilde{\phi}(a+b^*)h, h\rangle$. Hence $\tilde{\phi}$ is local positive.

To show $\tilde{\phi}$ is unique; let $\psi : M + M^* \to C^*_{\mathcal{E}}(\mathcal{D})$ be a local positive extension of ϕ.

The map ψ is self adjoint by Lemma 5.2.2. Then the following computation

$$\psi(a+b^*) = \psi(a)+\psi(b^*) = \psi(a)+\psi(b)^*$$
$$= \phi(a)+\phi(b)^* = \tilde{\phi}(a+b^*)$$

shows that $\psi = \tilde{\phi}$. □

Let $\mathcal{F}$ be a quantized Frechet domain with its union space $\mathcal{O}$. A. Dosiev proved the analog of Arveson's extension theorem for unital local CP-maps from local operator systems into $\mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$.

Theorem 5.2.4. *[26] Let $\mathcal{F}$ be a quantized Frechet domain with its union space $\mathcal{O}$ and let S be a local operator system in the locally C^*-algebra $\mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$. Then the following are equivalent:*

(i) S is an injective local operator system;

(ii) there is a morphism-projection $\mathcal{C}^*_{\mathcal{F}}(\mathcal{O}) \to \mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$ onto S.

In particular, $\mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$ is an injective local operator system.

Using Theorem 5.2.3, we deduce an analog of Arvesion extension theorem for local CC-maps on subspaces of locally C^*-algebras. A locally C^*-algebra $\mathcal{A}$ is called Frechet locally C^*-algebra if there is a local isometrical $*$-homomorphism $\mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ for some quantized Frechet domain $\mathcal{E}$ with its union space $\mathcal{D}$.

Theorem 5.2.5. *Let $\mathcal{F}$ be a quantized Frechet domain and let $\mathcal{A}$ be a Frechet locally C^*-algebra. Let M be a unital linear subspace of $\mathcal{A}$ and $\phi : M \to \mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$ be a unital local CC-map. Then ϕ has a local CP-extension to $\mathcal{A}$.*

Proof. Note that $M_n(M+M^*) = M_n(M)+M_n(M^*)$ for all n. Since ϕ is a local CC-map, by repeated application of Theorem 5.2.3 there is a local CP-map $\tilde{\phi} : M+M^* \to \mathcal{C}^*_{\mathcal{F}}(\mathcal{O})$. Then by Arveson-Dosiev extension theorem 5.2.4 $\tilde{\phi}$ extended to a local CP-map on $\mathcal{A}$. □

5.3 Irreducible representations and pure local CP-maps

By a representation of a locally C^*-algebra $\mathcal{A}$ we always mean a local contractive $*$-homomorphism from $\mathcal{A}$ into $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ for some quantized domain $\mathcal{E}$.

Definition 5.3.1. *Let $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a representation. The commutant of $\pi(\mathcal{A})$ is denoted by $\pi(\mathcal{A})'$ and is defined as*

$$\pi(\mathcal{A})' = \{T \in B(H) : T\pi(a) \subseteq \pi(a)T, \text{ for all } a \in \mathcal{A}\}$$

Definition 5.3.2. *A representation $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ is said to be irreducible if*

$$\pi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}) = \mathbb{C}I_{\mathcal{D}}$$

The following result is crucial in our discussions.

Theorem 5.3.1. *Let $\mathcal{E}$ be a quantized Frechet domain. Let $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ and $(\pi, V, \{H'; \mathcal{E}'; \mathcal{D}'\})$ be a Stinespring representation of the map ϕ. If π is irreducible, then $(\pi, V, \{H'; \mathcal{E}'; \mathcal{D}'\})$ is a minimal Stinespring representation for the map ϕ.*

Proof. If possible assume that there exists an $l_1 \in \mathbb{N}$ such that $[\pi(\mathcal{A})VH_{l_1}] \neq H'_{l_1}$. Since $V(H_l) \subseteq H'_l$ and H'_l is invariant for $\pi(a)$, for every $a \in \mathcal{A}$, we must have

$$[\pi(\mathcal{A})VH_{l_1}] \subsetneq H'_{l_1}.$$

Let $l_0 = \min\{l \in \mathbb{N} : \pi(\mathcal{A})VH_l] \neq H'_l\}$. Take P to be the orthognal projection of H' onto the closed subspace $[\pi(\mathcal{A})VH_{l_0}]$. We claim that $P \in \pi(A)' \cap \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$. First, we prove that $P \in \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$.

To see $P(H'_l) \subseteq H'_l$; let $l \in \mathbb{N}$. If $l \geq l_0$, then as $\mathcal{E}'$ is an upward filtered family and P is a projection we must have $P(H'_l) \subseteq [\pi(\mathcal{A})VH_{l_0}] \subsetneq H'_{l_0} \subseteq H'_l$. If $l < l_0$, then the choice of l_0 gives us $H'_l = [\pi(\mathcal{A})VH_l]$. Then, to show $P(H'_l) \subseteq H'_l$ it is enough to show that $P(\pi(\mathcal{A})VH_l) \subseteq H'_l$. Since $l < l_0$ and P is a projection with range $[\pi(\mathcal{A})VH_{l_0}]$, we have $H_l \subseteq H_{l_0}$. Thus,

$$\pi(\mathcal{A})VH_l \subseteq \pi(\mathcal{A})VH_{l_0}$$

$$\implies \quad P(\pi(\mathcal{A})VH_l) = \pi(\mathcal{A})VH_l \subseteq H_l'.$$

Hence $P(H_l') \subseteq H_l'$ for every $l \in \mathbb{N}$.

Note that, as $P(H_l') \subseteq H_l'$ and P is a projection we have $P|_{H_l'} \in B(H_l')$.

Now, we show that $P(H_l'^{\perp} \cap \mathcal{D}') \subseteq H_l'^{\perp} \cap \mathcal{D}'$. For $x \in H_l'^{\perp} \cap \mathcal{D}'$ and $y \in H_l'$ we need to show that $\langle Px, y\rangle = 0$. If $l < l_0$, then we have $H_l' = [\pi(\mathcal{A})VH_l]$. Since $H_l' = [\pi(\mathcal{A})VH_l] \subseteq [\pi(\mathcal{A})VH_{l_0}]$, we have $Py = y$ for every $y \in H_l'$. Then it follows that

$$\langle Px, y\rangle = \langle x, Py\rangle = \langle x, y\rangle = 0.$$

If $l \geq l_0$, then $[\pi(\mathcal{A})VH_{l_0}] \subsetneq H_l'$. Thus $H_l'^{\perp} \cap \mathcal{D}' \subseteq [\pi(\mathcal{A})VH_{l_0}]^{\perp}$. It follows that $Px = 0$ for all $x \in H_l'^{\perp} \cap \mathcal{D}'$. Therefore $\langle Px, y\rangle = 0$ for all $y \in H_l'$. Hence $P \in \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$.

To see $P \in \pi(\mathcal{A})'$, let $a \in \mathcal{A}$. First, we observe that $P\pi(a)h' = \pi(a)h'$ whenever $h' \in [\pi(\mathcal{A})VH_{l_0}]$. As the restriction of $\pi(a)$ to H_{l_0}' is a bounded operator on H_{l_0}', it is enough to consider h' in the dense subspace span$(\pi(\mathcal{A})VH_{l_0})$. Let $h' = \sum_{i=1}^{n} \pi(a_i)Vh_i$ for some $a_i \in \mathcal{A}$, $h_i \in H_{l_0}$ and $n \in \mathbb{N}$, $i = 1, 2, \cdots n$. Then,

$$\begin{aligned} \pi(a)h' &= \pi(a)(\sum_{i=1}^{n} \pi(a_i)Vh_i) \\ &= \sum_{i=1}^{n} \pi(aa_i)Vh_i \in [\pi(\mathcal{A})VH_{l_0}]. \end{aligned}$$

It follows that $P\pi(a)h' = \pi(a)h'$ whenever $h' \in [\pi(\mathcal{A})VH_{l_0}]$.

Now, consider $h' \in \mathcal{D}'$. Write $h' = h_1' + h_2'$ where $h_1' \in [\pi(\mathcal{A})VH_{l_0}]$ and $h_2' \in [\pi(\mathcal{A})VH_{l_0}]^{\perp} \cap \mathcal{D}'$. It follows that $P(h_2') = 0$ and $P\pi(a)h_1' = \pi(a)h_1'$. Then

$$\begin{aligned}
\|P\pi(a)h' - \pi(a)Ph'\|^2 &= \|P\pi(a)(h_1' + h_2') - \pi(a)P(h_1' + h_2')\|^2 \\
&= \|P\pi(a)h_1' + P\pi(a)h_2' - \pi(a)Ph_1' + \pi(a)Ph_2'\|^2 \\
&= \|P\pi(a)h_2'\|^2 \\
&= \langle P\pi(a)h_2', P\pi(a)h_2'\rangle \\
&= \langle \pi(a^*)P\pi(a)h_2', h_2'\rangle.
\end{aligned}$$

Let $h_3' = P\pi(a)h_2' \in [\pi(\mathcal{A})VH_{l_0}]$. Then

$$\pi(a^*)h_3' = \pi(a^*)(\sum_{i=1}^{n} \pi(a_i)Vh_i) = \sum_{i=1}^{n} \pi(a^*a_i)Vh_i \in [\pi(\mathcal{A})VH_{l_0}].$$

But $h_2' \in [\pi(\mathcal{A})VH_{l_0}]^{\perp}$ will imply that $\langle \pi(a^*)h_3', h_2'\rangle = 0$. Hence $P\pi(a)h' = \pi(a)Ph'$ for every $a \in \mathcal{A}$ and $h' \in \mathcal{D}'$. Hence $P \in \pi(A)' \cap \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$.

But $P \in \pi(A)' \cap \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$ is a contradiction as π is irreducible, and $P \neq 0$ and $P \neq I_H$. Hence π is a minimal Stinespring representation for ϕ. □

Remark 5.3.1. *It is well known that a representation θ of a C^*-algebra $\mathcal{C}$ is irreducible if and only if the commutant of $\theta(\mathcal{C})$ is trivial. If we take $\mathcal{A}$ to be a C^*-algebra and $\mathcal{E} = \{H\}$ in Definition 5.3.2, then Definition 5.3.2 coincides with the usual definition* *of irreducible representations of C^*-algebra. Also, our definition of irreducibility is motivated by the commutant considered to establish a Radon-Nikodym type theorem for local CP-maps in [11].*

5.3.1 Pure maps on local operator systems

We introduce the notion of pure local CP-maps on local operator system and study its connection with boundary representations for local operator systems. For this, we use the convexity structure of the set $\mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$.

Proposition 5.3.1. *For a local operator system S, the set $\mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ is a linear convex set.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{CPCC}_{\text{loc}}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ and $0 < t < 1$. Fix $l \in \Omega$. There exist $\alpha_r, \beta_r \in \Lambda$, $r = 1, 2$, such that

$$\phi_r^{(n)}([a_{ij}]) \geq_l 0 \text{ whenever } [a_{ij}] \geq_{\alpha_r} 0 \text{ and}$$

$$\|\phi_r^{(n)}([a_{ij}])\|_l \leq p^n_{\beta_r}([a_{ij}]) \text{ for every } n \in \mathbb{N}.$$

Replace ϕ_1 and ϕ_2 by $t\phi_1$ and $(1-t)\phi_2$ respectively. Then, for $\alpha = \max\{\alpha_1, \alpha_2\}$, we have

$$t\phi_1^{(n)}([a_{ij}]) + (1-t)\phi_2^{(n)}([a_{ij}]) \geq_l 0 \text{ whenever } [a_{ij}] \geq_\alpha 0.$$

Thus, $t\phi_1 + (1-t)\phi_2$ is a local CP-map. To see its local CC, take $\beta = \max\{\beta_1, \beta_2\}$. Then for every $[a_{ij}] \in M_n(S)$,

$$\begin{aligned}\|t\phi_1^{(n)}([a_{ij}]) + (1-t)\|\phi_2^{(n)}([a_{ij}])\|_l &\leq \|t\phi_1^{(n)}([a_{ij}])\|_l + \|(1-t)\|\phi_2^{(n)}([a_{ij}])\|_l \\ &\leq tp^n_{\beta_1}([a_{ij}]) + (1-t)p^n_{\beta_2}([a_{ij}]) \\ &\leq p^n_\beta([a_{ij}]).\end{aligned}$$

. □

Definition 5.3.3. *A map $\phi \in \mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ is called pure if for any map $\psi \in \mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ such that $\phi - \psi \in \mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$, then there is a scalar $t \in [0,1]$ such that $\psi = t\phi$.*

Remark 5.3.2. *A recent pre-print [42] also defines the notion of purity along similar lines.*

Now, we establish the connection between pure maps and irreducible representations in the unbounded setting. First, let us recall a couple of results which are crucial in the proof of main theorem the of this section.

The following is a Radon-Nikodym type theorem in the unbounded setting.

Theorem 5.3.2. *[11] Let $\phi, \psi \in \mathcal{CPCC}_{loc}(A,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. Then $\psi \leq \phi$ if and only if there* *exists a unique* $T \in \pi_\phi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}^\phi}(\mathcal{D}^\phi)$ *such that* $0 \leq T \leq I$ *and*

$$\psi(a) \subseteq V_\phi^* T \pi_\phi(a) V_\phi \text{ for all } a \in \mathcal{A},$$

where $(\{H^\phi; \mathcal{E}^\phi; \mathcal{D}^\phi\}, \pi_\phi, V_\phi)$ *is the Stinespring representation for the map* ϕ.

Corollary 5.3.1. *[11] Let* $\phi \in \mathcal{CPCC}_{loc}(A,\ \mathcal{C}^*_\mathcal{E}(\mathcal{D}))$. *There is a bijective correspondence between the sets* $\{T \in \pi_\phi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}^\phi}(\mathcal{D}^\phi) : 0 \leq T \leq I\}$ *and* $\{\psi \in \mathcal{CPCC}_{loc}(A,\ \mathcal{C}^*_\mathcal{E}(\mathcal{D})) : 0 \leq \psi \leq \phi\}$ *given by* $T \to \phi_T$, *where* $\phi_T(a) = V_\phi^* T \pi_\phi(a) V_\phi$ *for every* $a \in \mathcal{A}$.

Theorem 5.3.3. *A map* $\phi \in \mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_\mathcal{E}(\mathcal{D}))$ *is pure if and only if* ϕ *is of the form* $\phi(a) \subseteq V^*\pi(a)V$ *for all* $a \in \mathcal{A}$, *where* π *is an irreducible representation of* $\mathcal{A}$ *on some quantized domain* $\mathcal{E}'$ *with its union space* $\mathcal{D}'$ *and* $V \in L(\mathcal{D}, \mathcal{D}')$, $V \neq 0$ *and* $V(H_l) \subseteq H'_l$ *for all* $l \in \Omega$.

Proof. Let $\phi \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_\mathcal{E}(\mathcal{D}))$ be pure. Using Theorem 5.1.2 we have a unital representation $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}'}(\mathcal{D}')$ for some quantized domain $\mathcal{E}'$ with its union space $\mathcal{D}'$ such that $\phi(a) \subseteq V^*\pi(a)V$ where $V \in L(\mathcal{D}, \mathcal{D}')$ and $V(H_l) \subseteq H'_l$ for all $l \in \Omega$. Clearly $V \neq 0$. Now, let $T \in \pi(\mathcal{A})' \cap \mathcal{C}^*_\mathcal{E}(\mathcal{D})$ with $0 \leq T \leq I$. Taking $\psi(.) = V^*T\pi(.)V|_\mathcal{D}$ in Theorem 5.3.2 we have $\psi \leq \phi$. As ϕ is pure it follows that $\psi = t\phi$. Applying Corollary 5.3.1, $T = tI$. Hence π is irreducible.

Conversely, let π be an irreducible representation of $\mathcal{A}$ on some quantized domain $\mathcal{E}'$ with its union space $\mathcal{D}'$ and V be a non zero operator in $L(\mathcal{D}, \mathcal{D}')$ such that $V(H_l) \subseteq H'_l$ for all $l \in \Omega$. To show that $\phi(.) \subseteq V^*\pi(.)V$ is pure, consider $\psi \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A}, \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ with $\psi \leq \phi$. As π is irreducible by Theorem 5.3.1 $(\pi, V, \{H', \mathcal{E}', \mathcal{D}'\})$ is a minimal Stinespring representation for ϕ. Now, applying Corollary 5.3.1, there exists a unique $T \in \pi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ such that $0 \leq T \leq I$ and $\psi(a) \subseteq V^*T\pi(a)V$ for all $a \in \mathcal{A}$. Since π is irreducible, $T = tI$. It follows that $\psi = t\phi$ and hence ϕ is pure. □

Proposition 5.3.2. *Let S_1 and S_2 be local operator systems in a locally C^*-algebra $\mathcal{A}$ such that $S_1 \subseteq S_2$. Let $\phi : S_2 \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a unital local CP-map such that its a linear extreme point of $\mathcal{CPCC}_{loc}(S_2, \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. If $\phi|_{S_1}$ is pure, then ϕ is a pure.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{CPCC}_{\text{loc}}(S_2, \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ such that $\phi = \phi_1 + \phi_2$. Since $\phi|_{S_1}$ is pure, there exists $t \in (0, 1)$ such that $\phi_1|_{S_1} = t\phi|_{S_1}$ and $\phi_2|_{S_1} = (1 - t)\phi|_{S_1}$. The maps $\frac{1}{t}\phi_1$ and $\frac{1}{1-t}\phi_2$ are unital local CP-map on S_2. By Theorem 5.2.2 both the maps are local CC-maps. It follows that $\frac{1}{t}\phi_1, \frac{1}{1-t}\phi_2 \in \mathcal{CPCC}_{\text{loc}}(S_2, \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. Then the expression $\phi = t\frac{1}{t}\phi_1 + (1 - t)\frac{1}{1-t}\phi_2$ and the assumption ϕ is linear extreme implies that ϕ is pure. □

5.4 Local Boundary representations

In this section, we introduce the notion of local boundary representations for locally C^*-algebras and establish its connection with pure local CP-maps.

Definition 5.4.1. *Let S be a linear subspace of a locally C^*-algebra $\mathcal{A}$ such that S generates $\mathcal{A}$. A representation $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ is said to have local unique extension property for S if $\pi|_S$ has a unique local completely positive extension to $\mathcal{A}$, namely π itself.*

Remark 5.4.1. *Let $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a representation of $\mathcal{A}$. Then $\pi|_S$ has just one multiplicative local CP-extension to $\mathcal{A}$, namely π itself, but in general, there may exist other local CP-extensions of $\pi|_S$.*

Example 5.4.1. *For a self adjoint operator $T \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$, let $S = span\{I, T, T^2\}$ and $\mathcal{B}$ be the locally C^*-algebra generated by S in $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. We show that the identity representation $I_{\mathcal{B}}$ of $\mathcal{B}$ has local unique extension property. Let $\phi : \mathcal{B} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a local completely positive map such that $\phi(x) = x$ for all $x \in S$. Consider a minimal Stinespring representation $(\pi, V, \{H'; \mathcal{E}'; \mathcal{D}'\})$ of ϕ. To prove $\phi = I_{\mathcal{B}}$ on $\mathcal{B}$ it is enough to show that V is a unitary. Note that V is an isometry as ϕ is unital. We claim that $V(\mathcal{D})$ is invariant for $\pi(\mathcal{B})$. Then by minimality $H' = [\pi(\mathcal{B})V(\mathcal{D})] \subseteq [V(\mathcal{D})] \subseteq H'$ will imply V is a unitary. Now, to see the claim let us first show that $\pi(T)V(\mathcal{D}) \subseteq V(\mathcal{D})$. For that, we show that $\pi(T)V(H_l) \subseteq V(H_l)$ for every $l \in \Omega$. Let $l \in \Omega$ and $g \in H'_l$,*

$$
\begin{aligned}
\|(I - VV^*)\pi(T)VV^*g\|^2 &= \langle (I - VV^*)\pi(T)VV^*g,\ (I - VV^*)\pi(T)VV^*g\rangle \\
&= \langle VV^*\pi(T)(I - VV^*)\pi(T)VV^*g,\ g\rangle \\
&= \langle VV^*\pi(T)\pi(T)VV^*g - VV^*\pi(T)VV^*\pi(T)VV^*g,\ g\rangle \\
&= \langle V\phi(T^2)V^*g - V\phi(T)\phi(T)V^*g,\ g\rangle
\end{aligned}
$$

$$= \langle VT^2V^*g - VT^2V^*g,\ g\rangle$$

$$= 0.$$

Thus $(I - VV^*)\pi(T)VV^* = 0$ *on* H_l'*. Since* T *is self adjoint,* $VV^*\pi(T)(I - VV^*) = 0$ *on* H_l'*. These two observations and the facts* $\pi(T)_{|_{H_l'}} \in B(H_l')$*,* $V_{|_{H_l}}$ *is an isometry and* $\pi(T)V(H_l) \subseteq H_l'$ *will give* $\pi(T)V(H_l) \subseteq V(H_l)$*. As* l *is arbitrary, it follows that* $\pi(T)V(\mathcal{D}) \subseteq V(\mathcal{D})$*. To show* $\pi(\mathcal{B})V(\mathcal{D}) \subseteq V(\mathcal{D})$*, let* $T_0 \in \mathcal{B}$ *and* $\mathcal{B}_T = span\{I, T, T^2, T^3, \cdots\}$*. Then* $T_0 = \lim T_\lambda$*, where* $T_\lambda \in \mathcal{B}_T$*. For* $h \in H_l$*,*

$$\begin{aligned}\|\pi(T_\lambda)Vh - \pi(T_0)Vh\|_{H_l'} &= \|\pi(T_\lambda - T_0)Vh\|_{H_l'} \\ &\leq p_\alpha(T_\lambda - T_0)\|h\|_{H_l},\end{aligned}$$

where α *corresponds to* l *in the local contractivity of* π*. As* $\{T_\lambda\}$ *converges to* T_0*, we have* $p_\alpha(T_\lambda - T_0) \to 0$ *and hence* $\{\pi(T_\lambda)Vh\}$ *converges to* $\pi(T_0)Vh$ *in* H_l'*. Therefore,*

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh].$$

As $\pi(T)$ *leaves* $V(H_l)$ *invariant, so is every element of* $\mathcal{B}_T$*. Then using the fact that* VH_l *is a closed subspace (as* V *is an isometry and* H_l *is a closed subspace),*

$$\pi(T_0)Vh \in [\pi(T_\lambda)Vh] \subseteq [\pi(T_\lambda)V(H_l)] \subseteq [V(H_l)] = V(H_l).$$

Therefore $\pi(\mathcal{B})V(H_l) \subseteq V(H_l)$ *for every* l *and hence* $\pi(\mathcal{B})V(\mathcal{D}) \subseteq V(\mathcal{D})$.

Example 5.4.2. *Let* K *be an infinite dimensional separable complex Hilbert space with a complete orthonormal basis* $\{e_n : n \in \mathbb{N}\}$. *Consider* $K_n = span\{e_1, e_2, \cdots e_n\}$ *and* $H_n = K \oplus K_n$. *Then* $\mathcal{E} = \{H_n : n \in \mathbb{N}\}$ *is a quantized domain in the Hilbert space* $H = K \oplus K$ *with union space* $\mathcal{D} = \cup\{H_n : n \in \mathbb{N}\}$. *Define* $V : H \to H$ *to be the map* $V_0 \oplus 1_K$ *where* $V_0 : K \to K$ *be the unilateral right shift operator and* 1_K *be the identity operator on* K. *Note that* V *is an isometry but not a unitary. Also,* $V(K \oplus K_n) \subseteq K \oplus K_n$ *and*

$$V((K \oplus K_n)^{\perp}) = V(0 \oplus K_n^{\perp}) = 0 \oplus K_n^{\perp} = (K \oplus K_n)^{\perp}.$$

Therefore $V|_{\mathcal{D}} \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$.

Consider the local operator system $S = span\{1_{\mathcal{D}}, V|_{\mathcal{D}}, V^*|_{\mathcal{D}}\}$ *in* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *and let* $\mathcal{B}$ *the locally* C^**-algebra generated by* S *in* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. *We claim that the inclusion map from* S *to* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *have two distinct local CP-extension to* $\mathcal{B}$. *Obviously the inclusion representation* $I_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is a local CP-extension of the inclusion map on* S. *Define* $\psi : \mathcal{B} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *by* $\psi(a) = V^* I_{\mathcal{B}}(a) V|_{\mathcal{D}}$ *for all* $a \in \mathcal{B}$. *Clearly* ψ *is a unital local CP-map on* $\mathcal{B}$. *For all scalars* c_1, c_2 *and* c_3 *we have*

$$\begin{aligned}\psi(c_1 1 + c_2 V + c_3 V^*|_{\mathcal{D}}) &= V^*(c_1 1 + c_2 V + c_3 V^*) V|_{\mathcal{D}} \\ &= c_1 1 + c_2 V + c_3 V^*|_{\mathcal{D}}.\end{aligned}$$

Therefore $\psi|_S = I_{\mathcal{B}}|_S$*. Now the element* $VV^*|_{\mathcal{D}} \in \mathcal{B}$*. But*

$$\psi(VV^*|_{\mathcal{D}}) = V^*(VV^*|_{\mathcal{D}})V|_{\mathcal{D}} = I_{\mathcal{D}} \neq VV^*|_{\mathcal{D}}.$$

That is $\psi \neq I_{\mathcal{B}}$ *on* $\mathcal{B}$*. Therefore, the irreducible representation* $I_{\mathcal{B}}$ *doesn't have local unique extension property for* S*.*

Definition 5.4.2. *Let* S *be a linear subspace of a local* C^**-algebra* $\mathcal{A}$ *such that* S *generates* $\mathcal{A}$*. An irreducible representation* $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *is called a local boundary representation for* $\mathcal{S}$ *if* π *has local unique extension property for* S*.*

Remark 5.4.2. *The Definition 5.4.1 and Definition 5.4.2 are meaningful for local operator systems in arbitrary locally* C^**-algebras. But the Arveson's extension theorem in the context of locally* C^**-algebras is available only for* $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ *for quantized Frechet domain* $\mathcal{E}$ *and thus we restrict our studies to the context of Frechet locally C*-algebras.*

Now, we show that the local boundary representations are intrinsic invariants for local operator systems. Let $\mathcal{A}_1$ *be a locally* C^**-algebra and* $\mathcal{A}_2 = C^*_{\mathcal{E}_2}(\mathcal{D}_2)$ *be the locally* C^**-algebras of all non-commutative continuous functions on a quantized Frechet domain* $\mathcal{E}_2$ *with its union space* $\mathcal{D}_2$*.*

Theorem 5.4.1. *Let* S_1 *and* S_2 *be unital linear subspaces of* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *respectively. Let* $\phi : S_1 \to S_2$ *be a unital surjective local completely isometric linear map. Then for every local boundary representation* π_1 *of* $\mathcal{A}_1$ *there exists a local boundary representation* π_2 *of* $\mathcal{A}_2$ *such that* $\pi_2 \circ \phi(a) = \pi_1(a)\ \forall\ a \in S_1$*.*

Proof. By Theorem 5.2.5 we can extend ϕ to a local CP-map $\tilde{\phi} : \mathcal{A}_1 \to \mathcal{A}_2$. Consider the map $\psi : S_2 \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ given by $(\psi \circ \phi)(a) = \pi_1(a)$. Clearly ψ is a unital local CC-map. Again by Theorem 5.2.5 there exists a local CP-extension of ψ, say π_2, where $\pi_2 : \mathcal{A}_2 \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ such that $(\pi_2 \circ \phi)(a) = \pi_1(a)$ for every $a \in S_1$. Since π_1 is a boundary representation, $(\pi_2 \circ \phi)(a) = \pi_1(a)$ for every $a \in \mathcal{A}_1$. Note that the locally C^*-algebra generated by $\tilde{\phi}(\mathcal{A}_1)$ is equal to $\mathcal{A}_2$ and π_2 is continuous for the respective topologies. Thus, to prove π_2 is an algebra homomorphism it's enough to prove that $\pi_2(xy) = \pi_2(x)\pi_2(y)$ for every $x \in \tilde{\phi}(\mathcal{A}_1)$ and for all $y \in \mathcal{A}_2$. But in view of Lemma 5.1.1, it's enough to prove that

$$\pi_2(x)^*\pi_2(x) = \pi_2(x^*x) \ \ \forall x \in \tilde{\phi}(\mathcal{A}_1).$$

Let $a \in \mathcal{A}_1$. Then using the fact that a local positive map is positive (Proposition 5.1.1) and Lemma 5.1.1 we have, on $\mathcal{D}$,

$$\begin{aligned}
\pi_2(\tilde{\phi}(a))^*\pi_2(\tilde{\phi}(a)) &\leq \pi_2(\tilde{\phi}(a)^*\tilde{\phi}(a)) = \pi_2(\tilde{\phi}(a^*)\tilde{\phi}(a)) \\
&\leq \pi_2(\tilde{\phi}(a^*a)) \\
&= \pi_1(a^*a) \\
&= \pi_1(a^*)\pi_1(a) \\
&= \pi_2(\tilde{\phi}(a))^*\pi_2(\tilde{\phi}(a)).
\end{aligned}$$

Therefore $\pi_2(\tilde{\phi}(a)^*\tilde{\phi}(a)) = \pi_2(\tilde{\phi}(a))^*\pi_2(\tilde{\phi}(a))$ on $\mathcal{D}$. Thus π_2 is a representation of $\mathcal{A}_2$. In fact we proved that any local CP-extension of $\psi = \pi_2|_{S_2}$ to $\mathcal{A}_2$ is multiplicative on $\mathcal{A}_2$. Equivalently, π_2 has local unique extension property for S_2.

Now, note that $\pi_1(\mathcal{A}_1) \subseteq (\pi_2 \circ \tilde{\phi})(\mathcal{A}_1) \subseteq \pi_2(\mathcal{A}_2)$. Thus, for commutants we have $\pi_2(\mathcal{A}_2)' \subseteq \pi_1(\mathcal{A}_1)'$. Then the irreducibility of π_2 follows from the irreducibility of π_1. This completes the proof. □

Corollary 5.4.1. *Let S_1 and S_2 be local operator systems of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let $\phi : S_1 \to S_2$ be a unital invertible local CP-map such that ϕ^{-1} is also a local CP-map. Then for every local boundary representation π_1 of $\mathcal{A}_1$ there exists a local boundary representation π_2 of $\mathcal{A}_2$ such that $\pi_2 \circ \phi(a) = \pi_1(a) \; \forall \; a \in S_1$.*

Remark 5.4.3. *We expect the above theorem and consequently the corollary to be true for any Frechet locally C^*-algebras in place of $\mathcal{A}_2 = C^*_{\mathcal{E}_2}(\mathcal{D}_2)$.*

5.4.1 Characterisation of boundary representations

The following theorem shows that the restriction of a local boundary representation to the local operator system is a pure map.

Theorem 5.4.2. *Let S be a local operator system in a Frechet local C^*-algebra $\mathcal{A}$ such that S generates $\mathcal{A}$. Let $\mathcal{E}$ be a quantized Frechet domain with its union space $\mathcal{D}$, and $\pi : \mathcal{A} \to C^*_{\mathcal{E}}(\mathcal{D})$ be a boundary representation for S. Then $\pi|_S$ is a pure map on S.*

Proof. Let $\pi_1, \pi_2 \in \mathcal{CPCC}_{\text{loc}}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ such that $\pi|_S = \pi_1 + \pi_2$. Then by Arveson-Dosiev extension theorem (Theorem 5.2.4), each π_i extends to a local CPCC map on $\mathcal{A}$, call it $\tilde{\pi}_i$, $i = 1, 2$. We will show that $\tilde{\pi}_1 + \tilde{\pi}_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. For that, fix $l \in \mathbb{N}$. Then there exists α_i and β_i such that $\tilde{\pi}_i(a) \geq_l 0$ whenever $a \geq_{\alpha_i} 0$ in $\mathcal{A}$ and $\|\tilde{\pi}_i(b)\|_l \leq p_{\beta_i}(b)$ for every $b \in \mathcal{A}$. Take $\alpha = \max\{\alpha_1, \alpha_2\}$ and $\beta = \max\{\beta_1, \beta_2\}$. Using the fact that the family of semi-norms $\{p_n\}_{n\in\mathbb{N}}$ is an upward filtered family, we have $\tilde{\pi}_i(a) \geq_l 0$ whenever $a \geq_\alpha 0$ in $\mathcal{A}$ and $\|\tilde{\pi}_i(b)\|_l \leq p_\beta(b)$ for every $b \in \mathcal{A}$. Therefore, $\tilde{\pi}_1 + \tilde{\pi}_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$.

Now, since $\tilde{\pi}_1 + \tilde{\pi}_2|_S = \pi_1 + \pi_2 = \pi|_S$ and π is a boundary representation for S, we must have $\pi(a) = \tilde{\pi}_1(a) + \tilde{\pi}_2(a)$ for every $a \in \mathcal{A}$. The irreducibility of π and the Theorem 5.3.3 implies that π is a pure map. Thus, for each i, there exist $t_i \in [0, 1]$ such that $\tilde{\pi}_i(a) = t_i\pi(a)$ for every $a \in \mathcal{A}$. It follows that $\pi_i = t_i\pi|_S$. Hence $\pi|_S$ is a pure map on S. □

Now, we show that certain irreducible representations of $\mathcal{A}$ that are pure CPCC-maps on S are local boundary representations. For this, we need to introduce a couple of new notions. Let S be a local operator system in a local C^*-algebra $\mathcal{A}$ such that $\mathcal{A}$ is generated by S, and let $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a representation of $\mathcal{A}$. We say that π is a finite representation for S if for every isometry $V \in B(H)$ with $V(H_l) \subseteq H_l$ for every $l \in \Lambda$, the condition $\pi(x) \subseteq V^*\pi(x)V$ for every $x \in S$ implies V is a unitary. We say that the local operator system S separates the irreducible representation π if for any *irreducible representation ρ of $\mathcal{A}$ on some quantized domain $\mathcal{E}'$ with its union space $\mathcal{D}' = \bigcup_{l \in \Lambda} H'_l$ and an isometry V in $B(H, H')$ that satisfies $V(\mathcal{H}_l) \subseteq \mathcal{H}'_l$ for every $l \in \Lambda$ such that $\pi(x) \subseteq V^* \rho(x) V$ for all $x \in S$ implies that π and ρ are unitarily equivalent representations of $\mathcal{A}$.*

Theorem 5.4.3. *Let S be a local operator system in a local C^*-algebra $\mathcal{A}$ such that $\mathcal{A}$ is generated by S. Then an irreducible representation $\pi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ is a local boundary representations for S if and only if the following conditions hold;*

(i) $\pi|_S$ *is a pure map on* S

(ii) *Every local CP-extension of* $\pi|_S$ *to* $\mathcal{A}$ *is a linear extreme point of* $\mathcal{CPCC}_{loc}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$

(iii) π *is a finite representation for* S

(iv) S *separates* π.

Proof. Let π be an irreducible representation of $\mathcal{A}$. Assume that π is a local boundary representation for S. Then the statement (i) follows by Theorem 5.4.2.

(ii): Since π is a local boundary representation, there is only one local CP-extension of $\pi|_S$ to $\mathcal{A}$, namely π itself. Let $\phi_1, \phi_2 \in \mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ such that $\pi = \phi_1 + \phi_2$. Then $\pi|_S = \phi_1|_S + \phi_2|_S$. But $\pi|_S$ is pure by statement (i). Thus $\phi_1|_S = t\pi|_S$ and $\phi_2|_S = (1-t)\pi|_S$ for some $t \in [0,1]$. If $0 < t < 1$, then $\pi|_S = \frac{1}{t}\phi_1|_S$ and $\pi|_S = \frac{1}{1-t}\phi_2|_S$. Now the maps $\frac{1}{t}\phi_1$ and $\frac{1}{1-t}\phi_2$ on $\mathcal{A}$ are unital local CP-extensions of $\pi|_S$. But π is a boundary representation for S would imply that $\pi = \frac{1}{t}\phi_1$ and $\pi = \frac{1}{1-t}\phi_2$ on $\mathcal{A}$. That is, π is a linear extreme point of $\mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$.

(iii): Consider an isometry V on H such that $\pi(x) \subseteq V^*\pi(x)V$ for every $x \in S$ and $V(H_l) \subseteq H_l$ for every $l \in \Lambda$. Then $\phi(a) := V^*\pi(a)V|_{\mathcal{D}}$ for all $a \in \mathcal{A}$ is a unital local CP-extension of $\pi|_S$. As π is a local boundary representation we must have $\pi(a) = V^*\pi(a)V|_{\mathcal{D}}$ for all $a \in \mathcal{A}$. We claim that $V \in \pi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. Clearly V is bounded and $V(H_l) \subseteq H_l\ \forall\ l$. Let $x \in H_l^{\perp} \cap \mathcal{D}$. Since π is irreducible, by Theorem 5.3.1 $(\pi, V, \{H, \mathcal{E}, \mathcal{D}\})$ is a minimal Stinespring for π. Then by Lemma 5.1.2, $Vx = \pi(1)Vx \in H_l^{\perp}$. It follows that $Vx \in H_l^{\perp} \cap \mathcal{D}$ as $V(H_l) \subseteq H_l$. Thus $V(H_l^{\perp} \cap \mathcal{D}) \subseteq H_l^{\perp} \cap \mathcal{D}$ and hence $V \in \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. To see $V \in \pi(\mathcal{A})'$; first note that $dom(V\pi(a)) = \mathcal{D} \subseteq dom(\pi(a)V)$ for all $a \in \mathcal{A}$. Let $h \in \mathcal{D}$ and $a \in \mathcal{A}$.

$$
\begin{aligned}
&\|V\pi(a)h - \pi(a)Vh\|^2 \\
&=\langle V\pi(a)h - \pi(a)Vh, V\pi(a)h - \pi(a)Vh\rangle \\
&=\|V\pi(a)h\|^2 - \langle \pi(a)Vh, V\pi(a)h\rangle - \langle V\pi(a)h, \pi(a)Vh\rangle + \|\pi(a)Vh\|^2 \\
&=\|\pi(a)h\|^2 - \langle V^*\pi(a)Vh, \pi(a)h\rangle - \langle \pi(a)h, V^*\pi(a)Vh\rangle + \|\pi(a)Vh\|^2 \\
&=\|\pi(a)h\|^2 - \langle \pi(a)h, \pi(a)h\rangle - \langle \pi(a)h, \pi(a)h\rangle + \|\pi(a)Vh\|^2 \\
&=\|\pi(a)Vh\|^2 - \|\pi(a)h\|^2 = \langle \pi(a)Vh, \pi(a)Vh\rangle - \langle \pi(a)h, \pi(a)h\rangle \\
&=\langle V^*\pi(a^*)\pi(a)Vh, h\rangle - \langle \pi(a)^*\pi(a)h, h\rangle \\
&=\langle \pi(a^*a)h, h\rangle - \langle \pi(a^*a)h, h\rangle = 0.
\end{aligned}
$$

Therefore $V\pi(a) \subseteq \pi(a)V$ for every $a \in \mathcal{A}$ and hence $V \in \pi(\mathcal{A})' \cap \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$. By the irreducibility of π implies $V = \lambda I_H$, $\lambda \in \mathbb{C}$. Thus, the isometry V is a unitary. Hence π is a finite representation for S.

(iv): Assume that ρ is an irreducible representation of $\mathcal{A}$ on some quantized domain $\mathcal{E}'$ with its union space $\mathcal{D}' = \bigcup\limits_{l\in\Lambda} H'_l$ and an isometry V in $B(H, H')$ that satisfies $V(\mathcal{H}_l) \subseteq \mathcal{H}'_l$ for every $l \in \Lambda$ such that $\pi(x) \subseteq V^*\rho(x)V$ for all $x \in S$. As π is a local boundary representation for S, it follows that $\pi(a) \subseteq V^*\rho(a)V$ for all $a \in \mathcal{A}$. Here π and ρ are irreducible representations of $\mathcal{A}$. By Theorem 5.3.1 the Stinespring representations $(\pi, I_H, \{H, \mathcal{E}, \mathcal{D}\})$ and $(\rho, V, \{H', \mathcal{E}', \mathcal{D}'\})$ are minimal for π. Then Theorem 5.1.3 will imply that π and ρ are unitarily equivalent. Hence S separate π.

Conversely assume that the irreducible representation π satisfies all the four conditions. Let $\phi : \mathcal{A} \to \mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ be a local CP-map such that $\phi(a) = \pi(a)$ for every $a \in S$. By condition (ii), ϕ is a linear extreme point of $\mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. Then statement (i) and Proposition 5.3.2 will imply that ϕ is a pure map in $\mathcal{CPCC}_{\text{loc}}(\mathcal{A},\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$. If $\{\omega; V; \{K, \mathcal{F}, \mathcal{O}\}\}$ is a minimal Stinespring representation for ϕ, then by Theorem 5.3.3 ω is irreducible. Also,

$$\pi(a) = \phi(a) = V^*\omega(a)V|_{\mathcal{D}} \text{ for all } a \in S.$$

As π separates S, π and ω are unitarily equivalent. Let $U : K \to H$ be a unitary such that $U(\mathcal{O}) \subseteq \mathcal{D}$ and

$$\omega(a) = U^*\pi(a)U|_{\mathcal{D}} \text{ for all } a \in S.$$

Then

$$\pi(a) = V^*U^*\pi(a)UV|_{\mathcal{D}} \text{ for all } a \in S.$$

Since π is a finite representation and UV is an isometry on H, we have UV is a unitary. Thus $V = U^*(UV)$ is also a unitary. Therefore $\phi(a) = V^*\pi(a)V|_D$ on $\mathcal{A}$ is a representation of $\mathcal{A}$ which coincides with π on S. Therefore $\phi(a) = \pi(a)$ for all $a \in \mathcal{A}$ and hence π is a local boundary representation for S. □

Chapter 6

Conclusions and Recommendations

In this thesis, we studied the notions of boundary representations and hyperrigidity for operator spaces, a weaker notion of boundary representations and hyperrigidity for operator systems, and boundary representations for spaces of unbounded operators.

We study the amplifications of a weaker notion of boundary representations and hyperrigidity for operator systems in C^*-algebras called the weak boundary representations. We proved that a representation π of the C^*-algebra $\mathcal{A} = C^*(S)$ is a weak boundary representations for operator system S if and only if all of its the amplifications $\pi^{(n)}$ are weak boundary representations for the operator systems $M_n(S)$. Also, we deduced that an operator system S is quasi hyperrigid in the generated C^*-algebra $C^*(S)$ if and only if the tensor product operator system $M_n(\mathbb{C}) \otimes S$ is quasi hyperrigid in the tensor product C^*-algebra $M_n(\mathbb{C}) \otimes C^*(S)$, for all $n \geq 2$. This actually gives *a partial answer to the following question. Two operator systems S_1 and S_2 are quasi hyperrigid in their generated C^*-algebras if and only if the tensor product $S_1 \otimes S_2$ is quasi hyperrigid in its generated C^*-algebra ?. Our result shows that the answer to the preceding question is affirmative if the C^*-algebra generated by one of the operator system is $M_n(\mathbb{C})$. The general case is still unknown.*

We studied the concept of boundary representations for operator spaces. We deduced that boundary representations of operator spaces are in one to one correspondence with the boundary representations of the associated Paulsen system. We introduced weak boundary representation for operator spaces and proved that a weak boundary representation for an operator space induces a weak boundary representation for the corresponding Paulsen system and vice versa. We established a characterisation theorem for boundary representations; a map ϕ is a boundary representation for an operator space X if and only if ϕ is a rectangular operator extreme point for X, ϕ is a finite representation for X and X separates ϕ. This result justifies the terminology 'boundary' in the sense that boundary representations are extreme points of certain convex sets. Also, we initiated and studied the concept of hyperrigidity for operator spaces. We prove that if an operator space is rectangular hyperrigid in the TRO generated by the operator space, then every irreducible representation of the TRO is a boundary representation for the operator space. In the case of operator systems in C^*-algebras, the converse of the preceding statement, that is; if every irreducible representation is a boundary representation, then the operator system is hyperrigid; is *called the Arveson's hyperrigidity conjecture. Along the same line we posed the rectangular hyperrigidity conjecture and we established that the conjecture true in the the finite dimensional setting of $M_{nm}(\mathbb{C})$. The rectangular hyperrigidity conjecture can also be seen as a non commutative counter part of the classical Saskin's theorem to the setting of operator spaces and TROs. Validity of the conjecture in the general case of infinite dimensional setting is still unknown. An equivalence between rectangular hyperrigidity of an operator space and hyperrigidity of the corresponding Paulsen system is also established and hence the hyperrigidity conjecture is equivalent to rectangular hyperrigidity conjecture.*

We initiated a study of non-commutative Choquet boundary in the setting of spaces unbounded operators and in a more general setting of locally convex topological spaces. We used the notion of locally C^*-algebra to study non-commutative Choquet boundary to these new settings. An analogue of Arveson's extension theorem is proved for local CC-maps on unital subspaces of locally C^*-algebras. The notion of irreducible representations for locally C^*-algebras is introduced and it's used to establish a characterisation of purity of local completely positive maps. We proved that a local completely positive map is pure if and only if its minimal Stinespring representation is irreducible. Several examples for the new notions are provided. The relevance of local boundary representations are shown. The local boundary representations are intrinsic in variants for local operator systems. That is; let S_1 and S_2 be unital linear subspaces of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. Let $\phi : S_1 \to S_2$ be a unital surjective local completely isometric *linear map. Then for every local boundary representation π_1 of $\mathcal{A}_1$ there exists a local boundary representation π_2 of $\mathcal{A}_2$ such that $\pi_2 \circ \phi(a) = \pi_1(a) \ \forall \ a \in S_1$. Also, we characterised boundary representations in terms of purity and linear extreme points of certain convex sets.*

Here we will indicate a few problems for future research.

Arveson's hyperrigidity conjecture [10] says that, if every irreducible representation of C^*-algebra $A = C^*(S)$ is a boundary representation for a separable operator system S, then S is hyperrigid. The hyperrigidity conjecture is shown to be true only for certain C^*-algebras. Proving this conjecture for the genera C^*-algebra is an important problem.

In Corollary 3.3.1, we proved that all the amplification of a weak boundary representation for an operator system are all weak boundary representations for the corresponding amplified operator systems. From this result, it's immediate that the tensor product of weak boundary representation with the identity representation of $M_n(\mathbb{C})$ is also weak boundary representation. This inspires to pose a problem; Is the tensor product of weak boundary representations for operator systems is a weak boundary representation for the tensor product of operator systems ?. As the only irreducible representation of $M_n(\mathbb{C})$ is the identity representation(upto unitarily equivalence), the only weak boundary representation of $M_n(\mathbb{C})$ is the identity. Then Corollary 3.3.1 answers the problem mentioned above in the particular case where $\mathcal{A}_2 = M_n(\mathbb{C})$.

The general case has to be studied for tensor products of C^*-algebras, where one can consider different tensor product such as spatial, minimal or maximal tensor products.

The Problem 3.3.1 says that, for operator systems S_1 and S_2 and the generated C^*-algebras $\mathcal{A}_i = C^*(S_i)$, $i = 1, 2$ is it true that $S_1 \otimes S_2$ is quasi hyperrigid in $\mathcal{A}_1 \otimes_\gamma \mathcal{A}_2$ if and only if S_i is quasi hyperrigid in $\mathcal{A}_i$ for each $i = 1, 2$? We are able to solve this problem for the case $\mathcal{A}_2 = M_n(\mathbb{C})$, and the general case needs to be investigated.

In Theorem 4.3.2, we characterised boundary representations for operator spaces using rectangular operator extreme(which is an analogue of C^*-extreme points). It is interesting to check whether there exists any relation between boundary representations of operator spaces with the extreme points of suitable rectangular matrix convex sets.

We solved the Problem 4.4.1 for the finite dimensional case in Theorem 4.4.2. Also, we mentioned in Remark 4.4.1 that the conjecture is true with an additional assumption on the C^*-algebra generated by the Paulsen system associated to an operator space. The Problem 4.4.1 is still open for the general case.

In Theorem 5.2.5 we proved that a unital local CC-map on a unital subspace M of a Frechet locally C^*-algebra has local CP-extension to the local operator system $M + M^*$. This is analogue of Arveson' extension theorem for local CC-maps on local operator spaces in Frechet locally C^*-algebras. It will be interesting to investigate the case where the assumption that the locally C^*-algebra is Frechet is not assumed.

Davidson and Kennedy [24] proved the existence of boundary representations in *the case C^*-algebras. The existence of local boundary representations in the case of locally C^*-algebras is yet to be studied.*

In Theorem 5.4.1, we showed that local boundary representations are intrinsic invariants for local operator systems that generates Frechet locally C^*-algebras. We expect that this result is true for general local operator systems..

A possible version of hyperrigidity to the setting of spaces of unbounded operators is yet to be studied by introducing a suitable topology for the space of local CP-maps. Investigating the connections between local boundary representations and hyperrigidity, and establishing a possible analogue of Saskin's theorem is an important future work.

In Proposition 5.3.1 we showed that, for a local operator system S, the set $\mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ of all local CPCC-maps from S to $\mathcal{C}^*_{\mathcal{E}}(\mathcal{D})$ is a linear convex set. Exploring the non commutative convexity structure, specifically the C^*-convexity of the set $\mathcal{CPCC}_{loc}(S,\ \mathcal{C}^*_{\mathcal{E}}(\mathcal{D}))$ is an interesting topic of further research.

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