Ph.D. THESIS MATHEMATICS

TRANSIT IN GRAPHS

Thesis submitted to the

University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

under the Faculty of Science

by

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MAY 2023

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CERTIFICATE

I hereby certify that the thesis entitled "TRANSIT IN GRAPHS" is a bonafide work carried out by **Smt. Reshmi K M,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**TRANSIT IN GRAPHS**" is based on the original work done by me under the supervision of **Dr. Raji Pilakkat**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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ACKNOWLEDGEMENT

At the outset, I would like to express my sincere appreciation to Dr Raji Pilakkat, Professor, Department of Mathematics, University of Calicut, my PhD advisor and mentor. I am grateful to her for the time, effort, understanding and patience rendered to me. I express my gratitude to her for allowing me to research under her guidance. Her insightful comments and vast knowledge helped me develop as a researcher. She was always approachable and welcoming.

At various stages of my research, I was supported and encouraged by numerous individuals in the Department of Mathematics at Calicut University. I extend my thanks to Dr. Anil Kumar V., Head of the Department for helping me. I also thank Dr Preethi Kuttipulackal, Dr Sini P., Dr Prasad T, Dr Mubeena T, the Librarian and all non-teaching staff of the Department.

As a former HoD and my teacher, Professor Ramachandran P T will always be remembered.

I am happy to express my thanks to the research scholars, the M.Phil and M.Sc Students in this department for their help and cooperation.

I cannot forget to thank my colleagues at Govt Engineering College, Kozhikode and Govt Arts and Science College, Kozhikode. Gratitude goes out to all the friends who stood beside me and were an inspiration to me. There must be "an extraordinary thanks" to Dr Shikhi M, my friend and colleague for her unconditional help and support.

I owe a lot to my parents, Mr. Radhakrishnan Nair and Smt. Rajalakshmi E K for their love, care, prayers and sacrifices during these years.

And finally, I'm beyond grateful to Sreejith Kurungott, my life partner and my kids Tharun and Pranav for understanding me. I also thank them for the love, support, encouragement and entertainment provided during this difficult period in the most positive way.

Reshmi K M

Contents

List of symbols

G	A simple finite graph
$E(G)$	Edge set of G
$V(G)$	Vertex set of G
$T(v)$	Transit of v
$TI(G)$	Transit Index of G
$N[v]$	Closed neighbouhood of v
$\langle N[v] \rangle$	Subgraph induced by $N[v]$
$d(v)$	Degree of the vertex v in G
$diam(G)$	Diameter of the graph G
$\overline{G}$	Complement of G
$W(G)$	Wiener Index of G
$d_G(u,v)$	distance between u and v in G
$T_{k,d}$	k regular Dendrimer
$L(G)$	Line Graph of G
G^k	k- th Power of G
$\|S\|$	Cardinality of the set S

List of Symbols

K_n	Complete graph on n vertices
P_n	Path on n vertices
C_n	Cycle on n vertices
$K_{p,q}$	Complete bipartite graph with $p+q$ vertices
$S_n = K_{1,n}$	Star Graph
$B_{n,n}$	Bistar graph on $2n+2$ vertices
$K_{n_1,n_2,\ldots,n_m}$	Complete m-partite graph
$L_{m,n}$	Lollipop graph
W_{n+1}	Wheel graph
$B(n,m)$	Bow graph
F_n	Friendship graph
$H+K$	Join of the graphs H and K
$H\Box K$	Cartesian product of the graphs H and K
G^h	G fibre
$\sigma_G(g,h)$	Number of geodesics connecting g and h in G.
$\sigma_G(g,h/u)$	Stress of u.
L_n	Ladder graph
B_m	m-book graph
$H \circ K$	Corona of the graphs H and K
$Msp(v)$	majorised shortest paths through v
$\mathcal{M}_v$	Collction of all $Msp(v)$
$\mathcal{M}_G$	$\cup_{v\in V}\mathcal{M}_v$
τ	transit decomposition
τ_{min}	Transit decomposition of minimum cardinality
θ	Transit decomposition number

θ_a	$\lvert\mathcal{M}_G\rvert$
$S(G)$	Subdivision graph of G
$T(m,n)$	Tadpole graph
$\simeq_T$	Transit isomorphism
$T_H(v)$	Transit of v in H
$T(v,H)$	Transit of v induced by a subgraph H
$T(v,S)$	Transit of v induced by a subset S of V
$T(v,S/T)$	Transit of v induced by a subset S against T
$T_G(v,s)$	Transit of v induced by s in G
$[T]$	Transit induced matrix.
MON	motor octane number
(G,w)	weighted graph G

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Introduction

Graph Theory has evolved a lot after its inception in 1735. Though this branch of mathematics, in its initial stage were trailing around some puzzles, it has advanced to an important branch of mathematics having applications in numerous fields. Graph theory basically studies the relationships between 'things'. The techniques employed in graph theory are widely used in many disciplines including Chemistry, Biology, Physics and Computer science. Be it in a social network or a biological network or a chemical structure or an electrical circuit, the underlying networks are simply graphs.

The notion of graphs was well employed in chemistry. Molecules were modelled as graphs, which gave insight into their chemical and physical properties. Some of the physical properties of chemical components depend purely on their structure. Such physical properties of alkanes were well examined and predicted using their molecular graphs. Alkanes are organic compounds comprising carbon and hydrogen atoms. As the hydrogen atom does not give any information about the molecule, the molecular graph of alkanes is formed by replacing carbon atoms with vertices and carbon-carbon bonds with edges. Research conducted on chemical graphs shows a significant correlation between the chemical/physical properties of compounds and the topological indices of their chemical graphs.

Topological indices are graph invariant. Balaban index [1] [2], Hosoya index [11], Wiener index [31] etc are a few to name. As these indices turned out to be efficient tools to study and predict the physical and chemical properties of chemical compounds, chemical graph theory evolved as a branch of mathematical chemistry. Since then a lot of molecular descriptors have been defined and studied.

Topological index being a graph invariant, finds application in other branches aswell. In computer networks and social networks they are used as parameters for behavioural prediction, network analysis, network modelling and allied areas. Centrality measures are also graph invariants, which are studied in the context of social and computer networks. In this thesis titled 'Transit in Graphs', a graph invariant called "Transit Index" is introduced. Analysis of transit index in alkanes showed that the correlation between MON, a physical property of alkanes and the index is significant. This encouraged the theoretical study of transit index and concepts derived from it.

The thesis is organised into 9 chapters, excluding this introductory chapter. A preview of the thesis is presented here.

Preview

Chapter 1 details the basic concepts and definitions that are assumed in the thesis. They are already defined and studied. Most of the concepts are very familiar to graph theorists. The first section is about the basic terminologies and the second section deals with graph operations. A brief introduction to chemical graph theory and centrality measures are given in the third and fourth sections.

In **Chapter 2** the transit of a vertex in a graph and the transit index of a graph is defined. The transit of a vertex in a graph is the sum of the length of all the shortest paths passing through it. The sum of the transit of every vertex in the graph is termed as transit index. In the first section, some general results regarding these concepts are found. Some bounds for the transit of a vertex are also attained. Given the adjacency matrix of a graph, an effective method of computing the transit index is devised. The relationship between the transit index of a tree and its wiener index is established. The next section deals with the transit index of trees. An expression for the transit index of a path is derived. It is established that the path has the maximum transit index among trees of the same order. In the subsequent sections of the chapter, the transit index is computed for various graph classes.

In **Chapter 3** binary products of graphs are examined. Binary operation fuses two graphs to form a single graph. This chapter investigates how individual graph knowledge helps in the computation of transit of vertices/ transit index, in graph products. The focus is mainly on the Join of graphs, the Cartesian product and the Corona product.

Section 1 of **Chapter 4** defines terms related to the transit of a vertex and the transit index of a graph. Transit equivalent class, transit dominant class and transit null graph are among them. Section 2 deals with the majorised shortest paths, which facilitate the computation of the transit index of a graph.

This section ends with an algorithm for identifying majorised shortest paths of a graph. Next section pertain to transit decomposition, which utilizes the notion of majorised shortest paths in a graph. Transit decomposition number is defined and they are computed for a few graphs.

Chapter 5 discusses subdivision graphs. A subdivision graph is obtained by subdividing every edge. Transit index and transit decomposition are explored in subdivision graphs.

Graph isomorphism is a phenomenon in which the same graph appears in different forms. In **Chapter 6** we look at graphs displaying similar transit decomposition. Such graphs are termed transit isomorphic. This concept is investigated in line graphs. We also identify certain graph classes that are transit isomorphic to each other.

A graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can reduce a graph to a simpler graph while keeping certain structures intact. **Chapter 7** projects convex amalgamation of graphs. Certain supplementary definitions related to the transit of a vertex are made. Some significant properties of transit of a vertex induced by another can be viewed at a glance from the transit induced matrix. It is an $n \times n$ matrix $T = (t_{ij})$, where t_{ij} is the transit of v_i induced by v_j. Second section considers subgraph amalgamation.

The transit index being graph invariant finds application in chemical graph theory. The transit of a vertex can be treated as a centrality measure in networks. **Chapter 8** depicts the significance of the concepts introduced in the thesis. It was established that the transit index of alkanes has a strong correlation with the physical property MON (motor octane number). Transit decomposition also gives an insight into this physical property. In transportation networks, the concept of the transit of a vertex can be a functional tool.

Areas for future studies are suggested in the concluding chapter. Endnotes include publications, presentations, and a bibliography.

Chapter 1

Preliminaries

As a prelude to the subsequent chapters, this chapter introduces the terms and basic concepts used in this thesis. We adopt the basic definitions and notations from 'Graph Theory' [15], written by J.A. Bondy and U.S.R. Murty. The chapter is divided into four sections. Basic terminologies in graph theory are discussed in the first section. In the second section, we will give a brief overview of various graph operations. Third section is about chemical graph theory, while fourth section gives light to centrality measures.

1.1 Basic terminologies

A **graph** G consists of an ordered pair $(V(G), E(G))$ where $V(G)$ is the set of vertices and $E(G)$, disjoint from $V(G)$, the set of edges, along with an incidence function ρ_G that associates with each edge of $E(G)$ an unordered pair of (not necessarily distinct) vertices of $V(G)$. If e is an edge and u and v are vertices such that $\rho_G(e) = \{u, v\}$, then e is said to join u and v, and the vertices u and

v are called the ends of e. A graph F is defined to be a **subgraph** of a graph G if $V(F) \subseteq V(G), E(F) \subseteq E(G)$ and ρ_F is the restriction of ρ_G to $E(F)$. We then say that G is a subgraph of F or that F is a **supergraph** of G, and write $G \supseteq F$ or $F \subseteq G$, respectively.

The numbers of vertices and edges in G are called the **order**, denoted as $|G|$ and **size** of G, respectively. An edge is said to be **incident** with the end vertices, and vice versa.

A **finite graph** has its vertex set and edge set to be finite sets. An edge with identical ends is called a **loop**. If the end vertices of two or more edges are the same, they are called **parallel edges**. A graph with no loops or parallel edges is known as a **simple graph**.

Two vertices are said to be adjacent if they are incident with a common edge and two edges incident with a common vertex is termed as **adjacent edges**. Two adjacent vertices that are distinct are termed as **neighbours**. The **set of neighbours/ neighborhood** of a vertex v in a graph G is denoted by $N_G(v)$. $N_G[v] = N_G(v) \cup \{v\}$ is called the **closed neighborhood** of v. The number of vertices in $N_G(v)$ is called the **degree** of v. Vertices of degree 1 are called **pendant** vertices. An edge of a graph with one of its vertices as a pendant vertex is termed as **pendant edge**. Vertices with degree more than one is referred to as **internal vertex.** A subset S of the set of vertices of a graph G in which any two distinct vertices are adjacent is called a **clique** in G. A **simplicial vertex** has its closed neighbourhood to be a clique.

A linear sequence of distinct vertices arranged in such a manner that two consecutive vertices are adjacent is called a **path**, denoted by P_n. A **cycle** on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. A graph G is **connected** if there is a path between every pair of vertices. The **length** of a path or a cycle is the number of its edges. An **acyclic graph** is one that does not contain a cycle. A **tree** is a connected acyclic graph. A path or cycle of length k is called a **k-path or k-cycle**, respectively; the path or cycle is odd or even according to the parity of k. The length of a shortest path is called the **distance** between x and y and denoted $d_G(x, y)$. The **diameter** d of a graph G is the greatest distance between two vertices of G.

A graph is said to be a **spanning subgraph** of a graph G, if its vertex set is the entire vertex set of G. G and H are said to be **isomorphic**, denoted by $G \simeq H$, whenever there exist bijections $\theta : V(G) \longrightarrow V(H)$ and $\phi : E(G) \longrightarrow E(H)$ with the property $\rho_G(e) = uv$ if and only if $\rho_H(\phi(e)) = \theta(u)\theta(v)$. This pair of mappings is termed as an isomorphism between G and H.

If e is an edge of G, a graph on $m - 1$ edges obtained by deleting e from G but leaving the vertices and the remaining edges intact is the operation of **edge deletion**. The resulting graph is denoted by $G - e$. Similarly, if v is a vertex of G, the graph on $n - 1$ vertices obtained by deleting from G the vertex v together with all the edges incident with v is the **vertex deletion** operation.

A **cut-vertex** is a vertex whose removal will disconnect the graph. A **block** is a maximal connected subgraph of a given graph G that has no cut vertex. A graph is a **block graph** if every block is a clique.

A simple graph in which every cycle of length greater than three has a chord is called **chordal graph**. An edge e of a connected graph G is called a **bridge** in G if $G-e$ is disconnected. A **matching** in a graph is a set of pairwise nonadjacent edges. In metric graph theory, a **convex subgraph** of an undirected graph G is a subgraph that includes every shortest path in G between two of its vertices. A **geodetic graph** is an undirected graph such that there exists a unique shortest path between each two vertices.

An **adjacency matrix** of a simple graph is an $n \times n$ square matrix $A = (a_{ij})$ where $a_{i,j} = \begin{cases} 1 & , \quad v_i \text{ adjacent to } v_j \\ 0 & , \qquad\qquad otherwise \end{cases}$

1.2 Graph operations

Let G be a simple graph. A simple graph whose vertex set is V and whose edges are the pairs of nonadjacent vertices of G is called the **complement** $\overline{G}$ of G. The **line graph** $L(G)$ is the graph derived from the graph G by replacing every edge by a vertex and any two vertices of $L(G)$ are connected by an edge whenever the corresponding edges of G are incident with the same vertex of G. A **subdivision of an edge** e is to delete e, add a new vertex x, and join x to the ends of e (this amounts to replacing e by a path of length two).

The **union** of simple graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The **join** of two graphs G and H is a graph formed from disjoint copies of G and H by connecting every vertex of G to every vertex of H.

The **cartesian product** of simple graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$

such that either $u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in$ E(H) and $u_1 = u_2$. The **corona product** $G \circ H$, is obtained by taking one copy of G and $|V(G)|$ copies of H; and by joining each vertex of the $i - th$ copy of H to the $i - th$ vertex of G, where $1 \leq i \leq |V(G)|$. Suppose T is a subgraph of both G and H. Fix a copy of T contained in G and another in H. The **amalgamation** of G and H along T is the graph $G \vee_T H$ obtained by identifying the fixed copies of T. For any finite collection of graphs G_i, each with a fixed isomorphic subgraph T as common, the subgraph amalgamation is the graph obtained by taking the union of all the G_i and identifying the fixed subgraphs T.

1.3 Chemical graph theory

In **chemical graph theory**, the molecular structure of a compound is presented by a graph, where the atoms are represented by vertices and bonds are represented by edges. Usually the vertices corresponding to hydrogens are removed. What results is known as the **molecular graph. Graph invariants** are properties of graphs that are invariant under graph isomorphisms. A **topological graph index**, also called a **molecular descriptor**, is a mathematical formula that can be applied to any graph which models some molecular structure. **Quantitative Structure-Activity Relationship (QSAR)** is a branch of computer aided drug discovery (CADD) that relates chemical structures to biological activity. In general, QSAR can be separated into two major components: a quantitative description of molecular structure (descriptor) and a mathematical model that uses these multidimensional descriptors as input to predict activity.

Alkane are a series of compounds that contain carbon and hydrogen atoms with single covalent bonds. An alkane consists of hydrogen and carbon atoms arranged in a tree structure in which all the carbon–carbon bonds are single. Alkanes have the general chemical formula C_nH_{2n+2}. **Octane** is a hydrocarbon and an alkane with the chemical formula C_8H_{18}. Octane has 18 structural isomers that differ by the amount and location of branching in the carbon chain. An **octane rating**, or **octane number**, is a standard measure of a fuel's ability to withstand compression in an internal combustion engine without detonating. The higher the octane number, the more compression the fuel can withstand before detonating or knocking. **Motor Octane Number (MON)** is a type of octane rating.

1.4 Centrality measures

Centrality measures assigns numbers or rankings to vertices within a graph corresponding to their network position. Applications include identifying the most influential person(s) in a social network, key infrastructure vertices in the Internet or urban networks, super-spreaders of disease and brain networks. Centrality concepts were first developed in social network analysis. **Networks** are simply graphs. Any electric circuit or network can be converted into its equivalent graph by replacing the passive elements and voltage sources with vertices and the connection with edges. Some of the commonly used centrality measures are as follows.

The **distance function** $d_G(v)$ of a vertex $v \in V(G)$ is defined as $d_G(v) =$

$\sum_{u \in V(G)} d(v,u)$, the sum of the distances between v and all other vertices. The **centroid** of a graph G is the set of vertices minimizing d. A shortest path from u to v is also called a $u-v$ **geodesic**. The number of shortest $u-v$ paths is denoted by $\sigma(\mathbf{u},\mathbf{v})$ and the number of shortest $u-v$ path with 'a' as an internal vertex is denoted by $\sigma(\mathbf{u},\mathbf{v}/\mathbf{a})$. The number of geodesics passing through a vertex a is called its **stress** , denoted by $\sigma_{\mathbf{G}}(\mathbf{a})$ or simply $\sigma(\mathbf{a})$ if there is no confusion . The maximum distance from a vertex to any other vertex is the **eccentricity** of it. The **center** of a graph is the set of all vertices of minimum eccentricity. The **radius** of G, rad(G), is the minimum eccentricity.

Throughout the thesis, G denotes a finite, simple, connected and undirected graph with vertex set V and edge set E. Also $|V| = n$ **and** $|E| = m$.

Chapter 2

Transit index of a graph

Topological indices and centrality measures are graph invariant. Numerous studies have been carried out in these areas. The first notable topological index was the wiener index, named after Harry Wiener, a pioneer in chemical graph theory. It is defined as the sum of the lengths of the shortest paths between all pairs of vertices in the chemical graph representing the non-hydrogen atoms in the molecule. Wiener made fundamental contributions to the study of topological indices and established a correlation between the Wiener index and boiling points (hence viscosity and surface tension) of the paraffin. He could establish relationships with many chemical properties of alkanes with the wiener index.

Centrality measures are a vital tool for understanding graphs. Each measure has its definition of importance. Some are based on the degree of a vertex while others take the closeness to other vertices as the score of significance.

In his paper [27], Shimbel introduced the concept of the stress of a vertex. It is the number of shortest paths on which a vertex lies. This was further modified to produce measures of centrality. It found applications in social networking, for analyzing communication dynamics.

Keeping in mind the above two concepts, we introduce a new index, called the transit index of a graph. It considers the distances in the graph as well as the degree of vertices.

In computing the stress of a vertex, we only take into account the number of shortest paths through it; the length of the paths is not considered. Be it in data transmission or in the measure of closeness, the length of the paths also matters. Hence, in the computation of transit we account for the number of shortest paths as well as their length. Clearly, the definition shows that the index is graph invariant since it depends only on its structure.

2.1 Transit Index

Definition 2.1.1. Let $G(V, E)$ be a graph and let $v \in V$. Then we define the transit of a vertex v in G denoted by $T_G(v)$ or simply $T(v)$ as "the sum of the lengths of all shortest paths with v as an internal vertex " and the transit index of G denoted by $TI(G)$ as

$$TI(G) = \sum_{v \in V} T(v)$$

Remark 2.1.1. If there exists a shortest path in G with v as an internal vertex, then $T(v) > 0$. In particular if v has two non-adjacent neighbours, then $T(v) > 0$.

Theorem 2.1.1. For a vertex $v \in V,\ T(v) = 0$ iff v is a simplicial vertex.

Proof. Let $v \in V$ with $T(v) = 0$. Consider the degree $d(v)$ of v. If $d(v) = 0, 1$, then we are done. Let $d(v) > 1$. Let $V_k = \{v_1, v_2, \ldots, v_k\}$ be the neighbours of v. Let us suppose that v is not a simplicial vertex. i.e. $\langle N[v] \rangle$ is not a clique. Without loss of generality, let us assume that v_r and v_s are two non-adjacent neighbours of v. In this case v_r, v, v_s forms a shortest path through v. A contradiction to the assumption.
Conversely let $N[v] = V_k \cup \{v\}$ forms a clique.
If $d(v) = 0, 1$, then there are no paths passing through v and hence $T(v) = 0$. Let $d(v) > 1$. If possible, let $T(v) \neq 0$. Then there exist a shortest path $P : u, \ldots, u_1, v, v_1, \ldots, w$ passing through v. Since u_1, v_1 are neighbours of v,they are adjacent. Hence the $u{-}w$ path $P{-}\{v\}$ obtained from P by deleting the vertex v from P forms a $u - w$ path with length smaller than P, a contradiction. □

Example 2.1.1. Consider the graph in Figure 2.14. There are 6 vertices and 6 edges. We find the transit of each vertex and hence the Transit index of the graph G. 1,3,5 and 6 are simplicial vertices. Hence by Proposition 2.1.1 their transits are zero. The shortest paths in G with 2 as an internal vertex are 123, 124, 1245, 1246, 324, 3245, 3246. Hence $T(2) = 18$. Also 4 is an internal vertex of the paths 1245, 1246, 3245, 3246, 245, 246. Hence $T(4) = 16$. Thus $TI(G) = 34$

An immediate upper bound for transit of a vertex of a graph of order n and diameter d is $\binom{n-1}{2} d$ and an obvious lower bound is zero. These bounds are sharp for Star graphs.

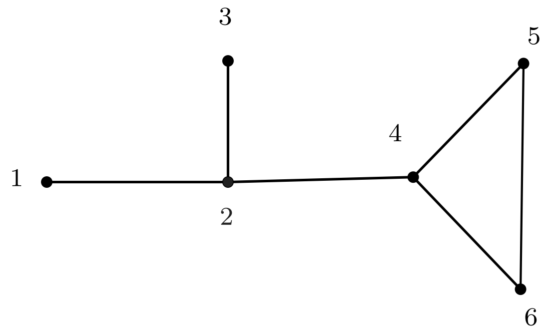


Figure 2.1: Graph G

Theorem 2.1.2. Let G be a graph with order n and diameter d. Then for any vertex $v \in V$, $0 \leq T(v) \leq \binom{n-1}{2}d$.

Proof. Let $v \in V$. The maximum length of the shortest path through any vertex is d. If all the shortest path in G between the $n-1$ vertices passes through v, the transit of v, $T(v) = \binom{n-1}{2}d$. For any pendant vertex $T(v) = 0$. Hence the proof. $\square$

Theorem 2.1.3. Let $d_G(v) \geq 2$. Let $e(v)$ denote the number of edges in the induced subgraph $\langle N(v) \rangle$. Then, $T(v) \geq d_G(v)(d_G(v) - 1) - 2e(v)$.

Proof. If u_1, u_2 are non-adjacent vertices in $N(v)$, then $u_1 v u_2$ is a shortest path of length 2 in G with v as an internal vertex. Considering every non adjacent vertices of $N(v)$, the result follows. $\square$

Theorem 2.1.4. If G is a graph with diameter 2 and let $d_G(v) \geq 2$. Then, $T(v) = d_G(v)(d_G(v) - 1) - 2e(v)$.

For each $u \in N(v)$, define $A_u = \{xy \in E(G)/x \in N(v) \;\&\; d(u, y) = 3\}$ and

$\alpha = \sum_{u \in N(v)} |A_u|$

Theorem 2.1.5. Let G be a graph with diameter 3. Then for $v \in V(G)$, $T(v) = 2[\binom{d_G(v)}{2} - e(v)] + 3\alpha$.

Proof. In G, the shortest paths passing through v are either of length 2 or of length 3. As shown in Theorem 2.1.3, the contribution to $T(v)$ is $2[\binom{d_G(v)}{2} - e(v)]$. The paths of length 3 are of the form $uvxy$ where $u, x \in N(v)$ and such that $xy \in E(G)$ with $d(u, y) = 3$. Hence the number of paths passing through v and originating at u is $|A_u|$. Thus contribution of all paths of length 3 will be 3α. Hence the result. $\square$

Matrices serve as a tool for representing, studying and manipulating graphs. The adjacency matrix and its operations give a compact way of handling them. In the succeeding section power of the adjacency matrix is used in the computation of the transit index of a graph.

Transit index and adjacency matrix

Let A denote the adjacency matrix of G. Then using the Lemma 2.1.1, we can find the number and length of shortest paths in G. This enables us to compute the transit index of G.

Lemma 2.1.1. [17] The number of walks of length l in G, joining v_i to v_j is the entry in position (i, j) of the matrix A^l

Consider the matrices $A, A^2, \ldots, A^d$, where d is the diameter of G. Denote the ijth element of A^k by $(a_{ij})^k$. For $i \neq j$, define $m_{ij} = (a_{ij})^k$, where k is the minimum for which $(a_{ij})^k \neq 0; \quad 1 \leq k \leq d$. Then the number of shortest paths between the vertices i and j of G is m_{ij}. If $m_{ij} = (a_{ij})^l, 1 \leq l \leq d$, then $d(i,j) = l$. Using these notations and ideas we write down the expression for transit index of a graph G.

Theorem 2.1.6. Let $A = (a_{ij})$ be the adjacency matrix. Then $TI(G) = \sum_{i<j} m_{ij} d(i,j)[d(i,j)-1]$, $m_{ij} = (a_{ij})^k$, where k is the minimum for which $(a_{ij})^k \neq 0; \quad 1 \leq k \leq d$

Proof. Let i, j be two vertices of G. Suppose there are l distinct shortest paths of length d connecting them. Then each of these path contributes towards the transit of every internal vertex of it. There are $d-1$ internal vertices for each path. Hence the contribution of the vertex pair (i,j) towards the transit index of G is $l \times d \times (d-1)$. Considering every unordered pair of vertices, we have the result. □

Corollary 2.1.1. In G, let s_i denote the number of shortest path of length d_i, then $TI(G) = \sum_i s_i d_i(d_i - 1)$

As two vertices in different components of a graph do not contribute anything to transit, only connected graphs are considered throughout the thesis. Next, we consider graphs with connected complements.

Lemma 2.1.2. [33] Let G be a connected graph with the connected complement. If $diam(G) > 3$, then $diam(\overline{G}) = 2$.

Theorem 2.1.7. Let $G(V,E)$ be a connected graph with connected complement. If $diam(G) > 3$, then $TI(\overline{G}) = 2 \sum_{uv \in E} |V \setminus N[u] \cup N[v]|$

Proof. By Lemma 2.1.2 $diam(\overline{G}) = 2$. For every edge uv in G, u and v are non adjacent vertices of $\overline{G}$. Hence there are m pairs of vertices at a distance 2 in $\overline{G}$, while other pairs are adjacent. The number of paths connecting u and v in $\overline{G}$ is equal to the number of vertices that are nonadjacent to both u and v in G. Hence the result. □

Next, we establish a relation connecting the transit index and the weiner index of trees. The relation we arrive at is not true for a general graph G, as there may be more than one shortest path connecting a pair of vertices.

Relation of transit index with weiner index

The Wiener index $W(G)$ of G, also known as the "path number" or "Wiener number" is a graph index defined for a graph. The length of the shortest path between every pair of vertices in a graph is added together to form it. ie, $W(G) = \frac{1}{2} \sum\limits_{(u,v)\in V\times V} d(u,v)$.

Theorem 2.1.8. Let T be a tree of order n. Then $TI(T) = \sum\limits_{\{u,v\}\subset V} (d(u,v) - 1)d(u,v) = \sum d^2(u,v) - W(T)$, where the summation is taken over all unordered pair of vertices in T.

Proof. T being a tree, every pair of vertices is connected by a unique path. Consider the path P, between vertices u and v in T. The length of P contributes to the transit of every internal vertex of P. There are $d(u,v)-1$, internal vertices. Hence the contribution of P towards the transit of T is $(d(u,v)-1)d(u,v)$. Thus considering every shortest path in T, we have $TI(T) =$

$$\sum_{\{u,v\}\subset V}(d(u,v)-1)d(u,v)=\sum d^2(u,v)-\sum d(u,v)=\sum d^2(u,v)-W(T) \quad \square$$

Trees form an important class of graphs. They were first studied by Cayley(1857). Trees are simple, undirected, connected, acyclic graphs. Many diverse fields find application in them, such as networking, social analysis, chemical graphs, electrical circuits, and so on. A tree T on n vertices has $n-1$ edges. Next we carry out our study in trees.

2.2 Trees

A rooted tree is a tree with a vertex labelled as its root. This root is usually taken as a point of reference. In most of the figures the root is placed at the top and other vertices are listed below as branches. The next theorem deals with transit of a root in a tree.

Theorem 2.2.1. Let T be a tree and v be a root of T. Let $\{T_j\}$ be the branches of v, with vertex set $V_j = \{u_{ij}, i = 1, 2, \ldots, n_j\}$ and edge set $E_j = \{e_{ij}, i = 1, 2, \ldots, n_j\}, j = 1, 2, \ldots, k$. Let $d(e_{ij})$ denote the number of vertices below e_{ij}. Then the transit of v is

$$T(v)=\sum_j\left[(n-1-n_j)\sum_i d(e_{ij})\right]=\sum_j\left[(n-1-n_j)\sum_i d(v,u_{ij})\right],$$

where $n_j = |V_j|$.

Proof. To find the transit of a vertex, we find the contribution of each edge to $T(v)$. In every shortest path passing through v, the edges will be used by vertices lying below it to travel to all the vertices in other branches. The contribution of

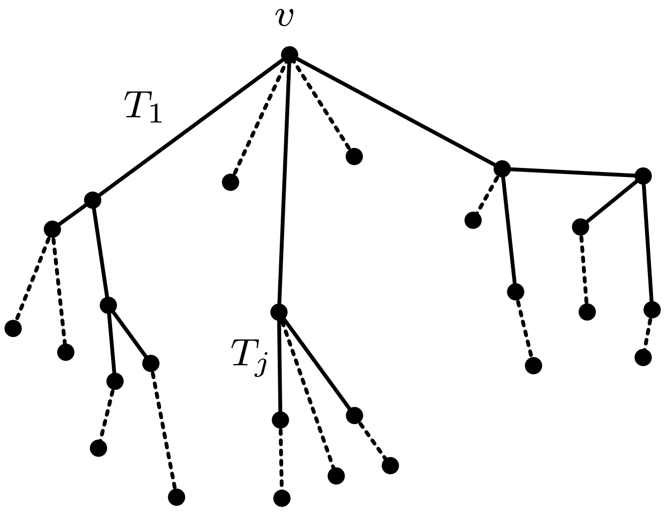


Figure 2.2: Tree T, rooted at v

an edge u_{ij} of T_j is $d(e_{ij})(n-1-n_j)$. Hence contribution of the whole branch T_j will be$(n-1-n_j)\sum_i d(e_{ij})$

$$\therefore T(v) = \sum_j \left[(n-1-n_j)\sum_i d(e_{ij}) \right]$$

Since the contribution of the edges in T_j can also be computed as $\sum_i d(v, u_{ij})$, we have

$$T(v) = \sum_j \left[(n-1-n_j)\sum_i d(v, u_{ij}) \right]$$

□

Corollary 2.2.1. In a tree, $T(v) = \sum_j (n-1-n_j)\frac{n_j(n_j+1)}{2}$, when v has all its branches as paths.

Algorithm 2.2.1. Algorithm for finding transit of a root in a tree.

Require: Tree T, root v, branches $T_1, T_2, \ldots, T_k$

Ensure: Transit of the vertex v, $T(v)$

1. $n_j = |V(T_j)|$

2. Compute $d(e_{ij})$ = number of vertices lying below e_{ij} in T_j

3. Find $d_j = \sum_i d(e_{ij})$

4. Compute $D_j = (n - 1 - n_j)d_j$

5. Determine $T(v) = \sum_{j=1}^{k} D_j$

Illustration Consider the tree with 13 vertices in the Figure 2.3, rooted at v. The branches are T_1, T_2, T_3 with $n_1 = 3, n_2 = 4, n_3 = 5$

For $j = 1, d(e_{11}) = 3, d(e_{21}) = 1, d(e_{31}) = 1, D_1 = (13 - 1 - 3)(3 + 1 + 1) = 45;$

For $j = 2, d(e_{12}) = 4, d(e_{22}) = 1, d(e_{32}) = 2, d(e_{42}) = 1, D_2 = 64;$

For $j = 3, d(e_{13}) = 5, d(e_{23}) = 2, d(e_{33}) = 1, d(e_{43}) = 2, d(e_{53}) = 1, D_3 = 77;$

Thus $T(v) = 186$

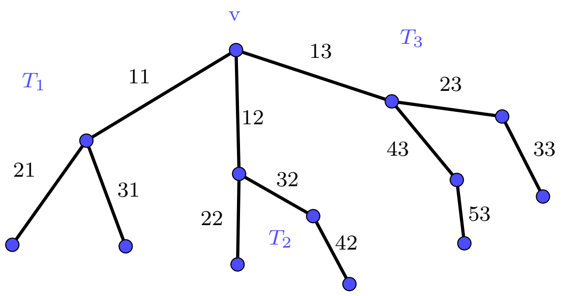


Figure 2.3: Tree T rooted at v

Dendrimers

Dendrimers[9] are highly ordered, branched polymeric molecules used in nanomedicine research. They are used as delivery or carrier systems for drugs and genes. Dendrimers are typically symmetrical about their cores. A regular dendrimer $T_{k,d}$ is a tree with a central vertex v. The radius of $T_{k,d}$ is k, the distance between each pendant vertex and v. Theorem 2.2.1 can be used to derive an expression for transit of v in $T_{k,d}$

Theorem 2.2.2. In $T_{k,d}$, $T(v) = \frac{d}{d-2}[(d-1)^k - 1]\sum_{i=1}^{k} i(d-1)^i$.

Proof. In $T_{k,d}$ all vertices other than the pendant vertices have degree d. We proceed as in Theorem 2.2.1. There are d identical branches, say $T_j, 1 \leq j \leq d$ in $T_{k,d}$ by considering it as a tree rooted at v. Hence n_j is a constant. Let the vertices on each branch at a distance i from v be denoted by u_i. Then, $d(u_i, v) = i$. In a branch, the number of vertices at a distance i from v (i.e. vertices of the type u_i) is $(d-1)^{i-1}$. Thus for the branch T_j, $\sum d(v, u_{ij}) = \sum_{i=1}^{k} i(d-1)^{i-1}$
Therefore, $n_j = \sum_{i=1}^{k}(d-1)^{i-1} = \frac{(d-1)^k - 1}{d-2}$ and $n = d\ n_j + 1$.

$$T(v) = \sum_j \left[(n - 1 - n_j)\sum_i d(v, u_{ij})\right] = \frac{d}{d-2}[(d-1)^k - 1]\sum_{i=1}^{k} i(d-1)^i$$

□

Path

Paths are fundamental concepts of graph theory. A path is an example of a tree, and in fact the paths are exactly the trees in which no vertex has degree 3 or more. This section deals with Transit index of a Path graph.

Theorem 2.2.3. For a path P_n, transit index$=\frac{n(n+1)(n^2-3n+2)}{12}$

Proof. Let $P_n : v_1v_2\dots v_n$. Then v_k divides P_n into two paths say P_1 with $k-1$ vertices and P_2 with $n-k$ vertices. We will compute $T(v_k)$ by counting the

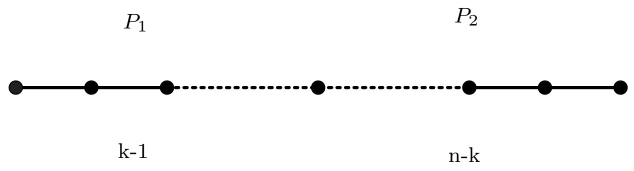


Figure 2.4: Path P_n

number of times each edge appears in the shortest path passing through v_k The edges in P_1 will be used $1.(n-k-1), 2.(n-k-1), 3.(n-k-1), \dots, k.(n-k-1)$ times respectively and the edges in P_2 will be used $1.(k-1), 2.(k-1), 3.(k-1), \dots, (n-k-1)(k-1)$ times. Hence

$$\begin{aligned}
T(v_k) &= 1.(n-k-1) + 2.(n-k-1) + 3.(n-k-1) + \dots + k.(n-k-1) \\
&\quad +1.(k-1) + 2.(k-1) + 3.(k-1) + \dots + (n-k-1)(k-1) \\
&= \frac{(k-1)k(n-k)}{2} + \frac{(n-k+1)(n-k)(k-1)}{2} \\
&= \frac{(k-1)(n-k)[k+n-k+1]}{2} \\
T(v_k) &= \frac{(n+1)(k-1)(n-k)}{2}
\end{aligned}$$

Hence, the transit index, $TI(P_n)$

$$\begin{aligned}
&= \sum_{k=1}^{n} T(v_k) \\
&= \sum_{k=1}^{n} \frac{(n+1)(n-k)(k-1)}{2}
\end{aligned}$$

$$= \frac{n(n+1)(n^2-3n+2)}{12}$$

□

The next proposition gives a reccursive formula for finding the transit of Path graphs.

Proposition 2.2.1. For $n = 1, 2, \ldots$, $TI(P_{n+1}) = TI(P_n) + \frac{n(n^2-1)}{3}$

Example 2.2.1. Using Theorem 2.2.3, $TI(P_2) = 0$, $TI(P_3) = 2$,$TI(P_4) = 10$, $TI(P_5) = 30$ and so on. It can be verified that $TI(P_5) = TI(P_4) + \frac{4(4^2-1)}{3}$

Next we discuss complement of a path graph. We know that $\overline{P_n}$ is connected only for $n \geq 4$. Also P_4 is the unique graph of order 4 with $P_4 \simeq \overline{P_4}$

Theorem 2.2.4. For $n \geq 4$, $TI(\overline{P_n}) = 2[n^2 - 4n + 2]$

Proof. Let $P_n : v_1 v_2 \ldots v_n$. In $\overline{P_n}$, the only non adjacent pairs of vertices are $(v_i, v_{i+1]})$, with $i = 1, 2, \ldots, n-1$. Also it is clear that, $diam(\overline{P_n})$ is 2. Now it remains to count the number of paths connecting v_i to v_{i+1} in $\overline{P_n}$. If $i \neq 1, n-1$, there will be $n-4$ paths and for $i = 1, n-1$ there are $n-3$ paths. Hence $TI(\overline{P_n}) = 2[n^2 - 4n + 2]$. □

From an existing graph new graphs can be formed by deletion or addition of vertices and edges. In the following two lemmas we investigate the effects of such modifications of a graph on the transit index of it.

Lemma 2.2.1. Let P_n be a path on n vertices. By adding a pendant vertex, the transit index is maximised if the vertex is added to either of the ends and minimised when it is added to the center vertex of P_n.

Proof. Consider the path P_n, with vertices $1, 2, \ldots, n$, with 1 and n being the pendant vertices. Let us attach a new vertex v to the kth vertex of P_n. Let I denote the increment in transit index due to this action. i.e. $I = TI(P_n + v) - TI(P_n)$. We will show that I is minimum when $k = \frac{n+1}{2}$

$$
\begin{array}{rcl}
I & = & \sum_{u \in V} \text{Increment in T(u)} \\
\text{For u=k} & \text{the increment is} & 2+3+\ldots+k+2+3+\ldots+n-k+1 \\
\text{For u=k-1} & \rightarrow & 3+4+\ldots+k \\
\vdots & & \\
\text{For u=2} & & k \\
\text{For u=1} & & 0 \\
\text{For u=k+1} & & 3+4+\ldots+n-k+1 \\
\text{For u=k+2} & & 4+5+\ldots+n-k+1 \\
\vdots & & \\
\text{For u=n-1} & & n-k+1 \\
\text{For u=n} & & 0 \\
\therefore I & = & (k-1)k(k+1)+(n-k)(n-k+1)(n-k+2)
\end{array}
$$

Now if we consider I as a real function of k on the closed interval $[1, n]$, its extrema are either at boundaries or when $\frac{dI}{dk} = 0$. $\frac{dI}{dk} = 0 \Longrightarrow k = \frac{n+1}{2}$. Hence extrema occurs at $k = 1, n, \frac{n+1}{2}$.

k	I(k)
1	(n-1)n(n+1)
n	(n-1)n(n+1)
$\frac{n+1}{2}$	$\frac{(n-1)(n+1)(n+3)}{8}$

Clearly maximum is for $k = 1, n$ and minimum for $k = \frac{n+1}{2}$ □

Lemma 2.2.2. Let e be an edge of G. If G and $G - e$ are connected, $TI(G) < TI(G - e)$.

Proof. Let G be a connected graph. Let $e = uv$ be such that $G - e$ is connected. By removing e, u and v becomes non adjacent. Since $G - e$ is connected there exist some shortest path P connecting u and v of length ≥ 2. This will increase the transit of every internal vertex of P in $G - e$. Hence the proof. □

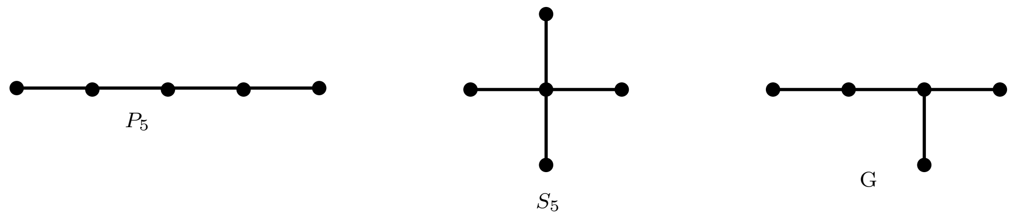
Among all connected graphs on n vertices, trees have the maximum transit index.

Corollary 2.2.2. If G is a connected graph and if T is a spanning tree of G, then TI(G)$\leq TI(T)$

Theorem 2.2.5. Among all trees on n vertices, the transit index is maximum for the path P_n.

Proof. Proof is by induction on n.

The result is trivially true for $n = 2, 3$ as there exist only one tree . For $n = 4$, there are only 2 non isomorphic trees. One is the path P_4 and other is the star S_4. We have $TI(P_4) = 10$ and $TI(S_4) = 6$. Hence true for $n = 4$. For $n = 5$, there are 3 non isomorphic trees, P_5, S_5 and G as shown in the figure. Here



$TI(P_5) = 30, TI(S_5) = 12, TI(G) = 14$. Hence true for $n = 5$ also. Assume that transit index is maximum for P_n among all trees with $\leq n$ vertices. Consider all trees on $n+1$ vertices. Let the transit index be maximum for some tree T, on $n+1$ vertices. We need to show that $T = P_{n+1}$. On the contrary assume $T \neq P_{n+1}$. Hence T has atleast 3 pendant vertices. Let v be any such pendant vertex . Removing it from T, we get $T' = T - \{v\}$, a tree on n vertices. By induction hypothesis and assumption we have the following relation.

$$TI(T') < TI(P_n) < TI(P_{n+1}) < TI(T) \tag{2.1}$$

We know that

$$TI(P_{n+1}) - TI(P_n) = \frac{n(n^2-1)}{3} \tag{2.2}$$

We calculate an upper bound for $TI(T) - TI(T')$ and show that it is less than $TI(P_{n+1}) - TI(P_n)$, which will contradict Equation 2.1 and prove $T = P_{n+1}$. Let the neighbour of v in T be u. Consider T' as a tree rooted at u. Let $T_1, T_2, \ldots, T_s$ be the branches with number of vertices $n_1, n_2, \ldots, n_s$ respectively. Then it is evident that $\sum_j n_j = n-1$, $1 < j \leq n-1$. We consider the increase in transit of

T' due to addition of the vertex v to it, forming T. Increase in $T(u)$ is due to the path connecting v to different vertices in the branches T_j. The maximum length of the path connecting v to vertices of T_j is $n_j + 1$. So the maximum possible increment in $T(u)$ will be

$$< (n-1)^2 + (n-1) \tag{2.3}$$

Consider vertices other than u of T. Fix a branch T_j and consider the n_j vertices of it. The increase in transit of these vertices depends only on the length of paths connecting vertices of T_j and v through u. By induction hypothesis the maximum increase in transit will happen when these $n_j(\leq n-1)$ vertices lie on a path. The maximum of the sum of length of paths connecting vertices on T_j to v and passing through these vertices is $\frac{1}{6}(2n_j^3 - 3n_j^2 - 5n_j + 6)$. Thus the maximum increment in transit due to all branches is

$$\sum_j \frac{1}{6}(2n_j^3 - 3n_j^2 - 5n_j + 6) < \frac{1}{6}[2(n-1)^3 - 3(n-1)^2 - 5(n-1) + 6(n-1)] \tag{2.4}$$

An upper bound for $TI(T) - TI(T')$ is got by adding Equations 2.3 and [2.4]. ie,$TI(T) - TI(T') < \frac{1}{6}[2n^3 - 3n^2 + 7n - 6] < \frac{n(n^2-1)}{3} = TI(P_{n+1}) - TI(P_n)$, a contradiction to Equation 2.1. $\square$

Corollary 2.2.3. For any tree T of order n, $TI(S_n) \leq TI(T) \leq TI(P_n)$

Corollary 2.2.4. For a connected graph G on n vertices, $0 \leq TI(G) \leq n(n+1)\frac{(n^2-3n+2)}{12}$. The bounds are attained by K_n and P_n respectively.

Next we compute transit index of certain graphs that are derived from path graph. A **comet** (Figure 2.5) is formed by appending multiple pendant edges to one end of a path.

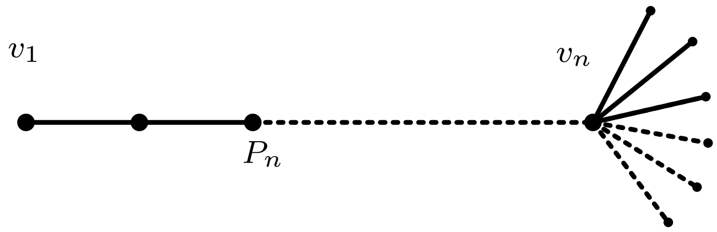


Figure 2.5: Comet

Theorem 2.2.6. Let G be the graph got by appending m pendant edges to one end of P_n. Then $TI(G) = TI(P_n) + \frac{mn(n^2-1)}{3} + m(m-1)$

Proof. In the graph G, transit is zero for the newly added vertices. Hence $TI(G) = TI(P_n) + I$, where I is the increase in transit of vertices of P_n due to the newly appended edges. Let v_k be any vertex of P_n. Then the increase in $T(v_k)$ is due to the paths connecting the vertices on the left of it to the newly added vertices. For the end vertex of P_n, the newly appended edges add 2 for each pair of newly added vertices. This increase can be computed as

$= nm + (n-1)m + \ldots + (n-k+2)m + m(m-1) = \frac{m}{2}[k(2n+3) - k^2 - (2n+2) + m(m-1)$.

$\therefore I = \sum_{1}^{n} \frac{m}{2}[k(2n+3) - k^2 - (2n+2)] + m(m-1) = \frac{mn(n^2-1)}{3} + m(m-1)$.

Hence the theorem. □

Remark 2.2.1. Applying the recursive formula for a path, $T(P_{n+1}) = T(P_n) + \frac{n(n^2-1)}{3}$, the Transit of a comet G can be expressed as, $TI(G) = mT(P_{n+1}) - (m-1)T(P_n) + m(m-1)$.

The **triangular snake graph** (Figure 2.6) can be viewed as the graph formed by replacing every edge of P_n by a triangle, thus adding $n-1$ vertices and $2(n-1)$ edges.

Theorem 2.2.7. If G is the triangular snake graph on $2n-1$ vertices, $TI(G) = TI(P_n) + \frac{(n-2)(n-1)n(n+1)}{4}$.

Proof. Let $v_1, v_2, \ldots, v_n$ denote the vertices of the path P_n. The newly added vertices are named as $u_1, u_2, \ldots, u_{n-1}$. For every u_i, $N[u_i]$ is a clique. Hence $T(u_i) = 0, \forall i$. Also $N[v_1], N[v_n]$ are cliques. $\therefore T(v_1) = T(v_n) = 0$. Hence we

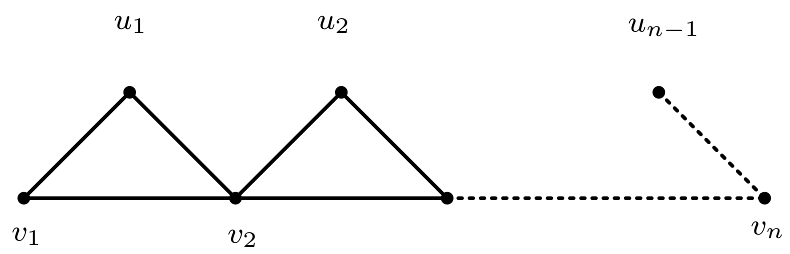


Figure 2.6: Snake graph

need to compute only the transit of v_i for $i \neq 1, n$. The transit of these vertices are due to path connecting v_i among themselves, path connecting u_i among themselves and paths connecting v_i to u_i. i.e. $TI(G) = TI(P_n) + I$, where I denote the increase in transit of v_i due to the addition of u_i. Consider v_k . The increase in its transit is due to

1. Paths connecting u_i to $u_j, i < k, j > k$

2. Paths connecting u_i to $v_j, i < k, j > k$

3. Paths connecting v_i to $u_j, i < k, j > k$

It can be seen that the increase in all the three cases are the same and equal to

$A = (2+3+\ldots+n-k+1)+(3+4+\ldots+n-k+2)+\ldots(k+(k+1)+\ldots+(n-1)$

Hence increase in transit of v_k is $3A$. If we take $a = 2+3+\ldots+n-k+1$, $A = a+(a+n-k)+(a+2(n-k))+\ldots$ Hence increase in transit of v_k is $= 3[a(k-1)+\frac{(n-k)(k-2)(k-1)}{2}] = \frac{3}{2}(n-k)(k-1)(n+1)$. $= \frac{3}{2}(n+1)[(n+1)k-k^2-n]$ Hence $I = \sum_{k=1}^{n} \frac{3}{2}(n+1)[(n+1)k-k^2-n] = \frac{(n-2)(n-1)n(n+1)}{4}$. Hence the proof. □

2.3 Cycle

Theorem 2.3.1. Let C_n be a cycle with n even. Then

i) $TI(C_n) = \frac{n^2(n^2-4)}{24}$

ii) $TI(C_{n+1}) = \frac{n(n^2-4)(n+1)}{24}$

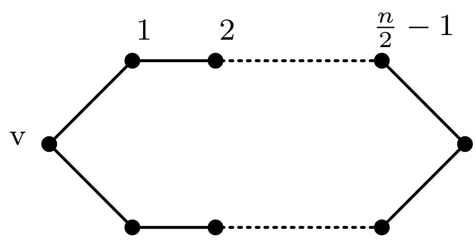


Figure 2.7: Cycle

Proof. (i)Consider the vertex v in the figure. The maximum length of the shortest path passing through v is of length $\frac{n}{2}$. The sum of the lengths of the shortest path originating

$$\begin{aligned}
\text{from } 1 \quad & is \quad 2+3+\ldots+\frac{n}{2} \\
\text{from } 2 \quad & is \quad 3+4+\ldots+\frac{n}{2} \\
& \vdots \\
\text{from } \frac{n}{2}-1 \quad & is \quad \frac{n}{2} \\
\text{Hence, } T(v) \quad & = \quad (\frac{n}{2}-1)\frac{n}{2}+(\frac{n}{2}-2)(\frac{n}{2}-1)+\ldots+2.1+1.0 \\
& = \quad \sum_{k=1}^{\frac{n}{2}}(k-1)k = \frac{(n^2-4)n}{24}
\end{aligned}$$

Due to symmetry $T(v)$ is the same for every $v \in V$.

$$\therefore TI(C_n) = \frac{n^2(n^2-4)}{24}, \text{n is even}$$

(ii) Consider C_{n+1}, with n even. The maximum length of the shortest path passing through any vertex v remains to be $\frac{n}{2}$. hence as in the case of even cycle $T(v) = \frac{(n^2-4)n}{24}$ $\therefore TI(C_{n+1}) = (n+1)T(v) = (n+1)\frac{(n^2-4)n}{24}$ □

The **wheel graph**, W_{n+1} is a graph obtained from C_n, $n \geq 3$ by adding a new vertex and by making it adjacent to all other vertices of C_n. Figure 2.8 represents a wheel graph.

Theorem 2.3.2. $TI(W_{n+1}) = n(n-1), n > 3$ and for $n = 3, TI(W_{3+1}) = 0$

Proof. Let $n > 3$. In W_{n+1}, the diameter is 2. Hence no shortest path is of length more than 2. Let v be any vertex on the outer circle C_n. The only shortest path passing through it is between its adjacent vertices. $\therefore T(v) = 2$, for $v \in C_n$. Consider the center vertex c. To find its transit we consider the contribution of

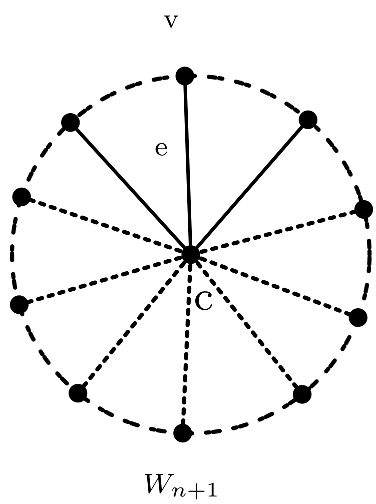


Figure 2.8: Wheel graph

each edge to it. Every edge on C_n contributes 0 to $T(c)$. Consider the edges of the type e, as shown in the Figure 2.8, which are the spokes of the wheel. e will be used only by v to travel to every vertices other than its adjacent ones. Hence the contribution is $(n-3)$. $\therefore T(c) = n(n-3)$. i.e.$TI(W_{n+1}) = 2n+n(n-3) = n(n-1)$ For $n = 3$, we get $W_{3+1} = K_4$. $\therefore$ its transit is zero. □

A **shell graph** is a cycle C_n with $(n-3)$ chords sharing a common end vertex called the apex. A Shell graph on n vertices are denoted as $C(n, n-3)$.

Theorem 2.3.3. For a shell graph on n vertices, $G = C(n, n-3)$, $TI(G) = n(n-3), n \neq 4$. For $C(4,1), TI(G) = 4$

Proof. Consider $G = C(n, n-3), n \neq 4$. Let a denote the apex vertex in G. The other vertices are denoted by $v_1, v_2, \ldots, v_{n-1}$. Shortest paths through v_i is the one connecting v_{i-1} to $v_{i+1}, (i \neq 1, n-1)$ of length 2. Hence $T(v_i) = 2$. The apex vertex a is adjacent to every v_i. Hence $T(a) = \frac{(n-1)(n-2)}{2} - (n-2) = (n-2)(n-3)$. Thus $TI(G) = n(n-3)$. For $n = 4$, $TI(G)$ can be computed as 4 from the graph. □

The **bow graph**, $B(n,m)$ is a double shell in which each shell has order n and m with a common apex vertex. Thus $|V| = n + m - 1$. For $n \geq 3$, $B(n,n)$ is called a uniform bow graph.

Theorem 2.3.4. $TI(B(,n,m)) = n(n-3)+m(m-3)+2mn$ and $TI(B(,n,n)) = 2n(2n-3)$.

Proof. The computation of transit in $B(n,m)$ is similar to the shell graph. Every shortest path in the bow graph will be of length 2. Let a be the vertex common to both the shell. Then the paths of length 2 connecting vertices from one shell to other will also account to $T(a)$. Thus $T(a) = (n-2)(n-3)+(m-2)(m-3)+2(n-1)(m-1)$. Considering vertices u_i in the first shell, $\sum_i T(u_i) = 2(n-3)$ and for vertices v_i in the second shell, $\sum_i T(v_i) = 2(m-3)$. Thus, $TI(B(n,m)) = n(n-3) + m(m-3) + 2mn$.

Taking $n = m$ in the result for bow graph we get the transit index for uniform bow graph. □

The **Friendship graph** (Figure 2.9), F_n is constructed by coalescence of n copies of the cycle C_3 of length 3, with a common vertex.

Theorem 2.3.5. $TI(F_n) = 4n(n-1)$.

Proof. In F_n, the diameter is 2. For every vertex v other than the coalescence vertex, $N[v]$ is a clique. Hence $T(v) = 0$. Hence $TI(F_n) = TI(c)$ The edges of

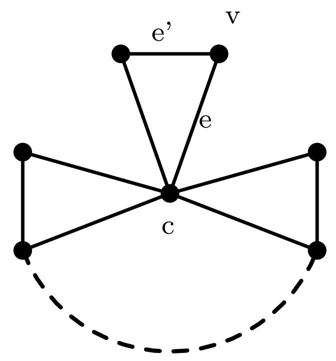


Figure 2.9: Friendship graph

the type e' does not contribute to $T(c)$. Hence we count the number of times the edges of the type e in the Figure 2.9 is used. The edge e will be used by the vertex v to travel to all vertices other than its adjacent ones. Hence contribution of e is $2(n-1)$. There are $2n$ such edges. $\therefore T(c) = 4n(n-1)$. i.e. $TI(F_n) = 4n(n-1)$. $\square$

2.4 Star Graphs

Theorem 2.4.1. For a star graph $S_n = K_{1,n}$, $TI(S_n) = (n-1)(n-2)$

Proof. In star graph $S_n = K_{1,n}$, $n-1$ vertices are pendant vertices. For each such vertices $T(v) = 0$. There are $\binom{n-1}{2}$ shortest path of length 2 passing through the center vertex. Hence $TI(S_n) = 2.\binom{n-1}{2} + 0$. i.e $TI(S_n) = (n-1)(n-2)$ $\square$

Bistar $B_{n,n}$ is the graph obtained by joining the centre (apex) vertices of

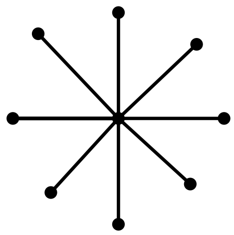


Figure 2.10: Star Graph

two copies of $K_{1,n}$ by an edge.

Theorem 2.4.2. If G is the bistar $B(n,n)$, TI(G)=$2n(4n+1)$

Proof. In G, let u, v denote the apex vertices. Every vertex other than u, v have transit zero. Also $T(u) = T(v)$. The shortest paths through u are of 3 types. Paths of length two connecting pendant vertices of the same star, Paths of length 2 connecting pendant vertices of the first star to v and paths of length three connecting pendant vertices of first star to the pendant vertices of the second star. Hence the result. □

Theorem 2.4.3. If G is obtained by joining the apex vertices of $K_{1,n}$ and $K_{1,m}$ by an edge, TI(G)=$m(m+1)+n(n+1)+6mn$

2.5 Complete Graphs

For every vertex v in a complete graph K_n, $\langle N[v]\rangle = K_n$, a clique. Hence $T(v) = 0, \forall v \in V$. So complete graphs are graphs having transit index zero.

Next we investigate the Transit index of certain graphs that are constructed from complete graphs.

Theorem 2.5.1. For $n \geq 3$, deleting an edge from K_n, increases the transit index by $2(n-2)$.

Proof. The deletion of the edge $e = uv$, makes u and v non-adjacent. Hence every other vertex will be an internal vertex of the shortest path between u and v of length 2. Thus $TI(K_n - e) = 2(n-2)$ $\square$

Theorem 2.5.2. Let G be the graph obtained by attaching a vertex to one of the vertices of a complete graph. Then $TI(G) = 2(n-1)$

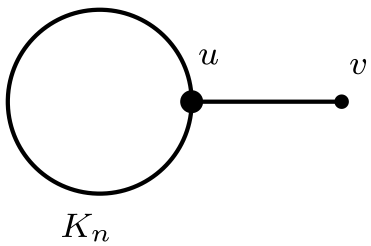


Figure 2.11: Graph G

Proof. Let the new vertex be v and the vertex to which it is attached be u. Then for every vertex in G (Figure 2.11) other than u, $N[v_i]$ is a clique. Hence transit is zero. There are $n-1$ paths of length 2 connecting v to vertices of $K_n - \{u\}$, passing through v.

$\therefore TI(G) = 2(n-1)$ $\square$

Theorem 2.5.3. Let G be the graph formed by attaching a vertex each to every vertex of K_n. (Also called the corona product $K_n \circ K_1$, Refer Section 3.3). Then $TI(G) = 5n(n-1)$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of K_n and $u_1, u_2, \ldots, u_n$, be the vertices attached to $v_1, v_2, \ldots, v_n$ respectively. Since u_i's are pendant vertices $T(u_i) = 0, \forall i$. The shortest path passing through v_i are either $u_i v_j$ paths or $u_i u_j$ paths of length 2 and 3 respectively. Hence $T(v_i) = 2(n-1) + 3(n-1)$. Thus $TI(G) = 5n(n-1)$ □

Theorem 2.5.4. Let G be the graph formed by merging a vertex of K_n and K_m. Then $TIG) = 2(n-1)(m-1)$

Proof. Let v be the coalescence vertex . For every vertex u of G other than v, $T(u) = 0$, as $N[u]$ is a clique. The shortest paths passing through v are those connecting the $n-1$ vertices of K_n with $m-1$ vertices of K_m, each of length 2. Hence $TI(G) = T(v) = 2(n-1)(m-1)$ □

Theorem 2.5.5. Let G be the graph formed by merging a vertex of K_n with a vertex of C_m. Then $TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}$, if m is even and $TI(G) = TI(C_m) + \frac{(n-1)(m-1)(m+1)(m+3)}{12}$, if m is odd.

Proof. Let us denote the coalescence vertex by v and $P_1 : v_1 \ldots v_{\frac{m}{2}+1}$, as shown in Figure 2.12

Case 1: m even

Clearly, $TI(G) = TI(C_m) + TI(K_n) + I$, where I denote the increment in transit due to merging of graphs. The transit for vertices in K_n remains zero, except for

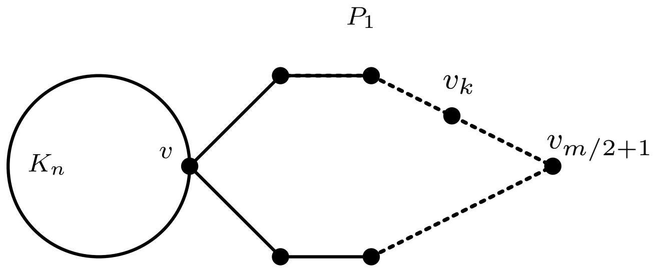


Figure 2.12: Graph G

v. The vertex at the distance $\frac{m}{2}$ from v on C_m has no increment. Let v_k denote the kth vertex on P_1 the shortest path, v_1 being v. Then increment for v_k is due to the shortest paths from vertices on its right to vertices of K_n including v. This can be computed as

$= [(k+1)+(k+2)+\ldots+\frac{m}{2}+1](n-1) = \left[(\frac{m}{2}+1)(\frac{m}{2}+2)-k-k^2\right]\frac{(n-1)}{2}$

Now due to similar positions we have $T(v_k) = T(v_{m-k+2})$. Hence we have $I =$

$=2\sum_{1}^{\frac{m}{2}}\left[(\frac{m}{2}+1)(\frac{m}{2}+2)-k-k^2\right]\frac{(n-1)}{2} = \frac{(n-1)(m+4)(m+2)m}{12}$.

$\therefore TI(G) = TI(C_m) + \frac{(n-1)(m+4)(m+2)m}{12}$

Case 2: m odd

Let v_k denote the kth vertex on $P_1 : v_1 \ldots v_{\frac{m+1}{2}}$, v_1 being v. Then increment for v_k is due to the shortest paths from vertices on its right to vertices of K_n including v. This can be computed as

$I = (k+1)+(k+2)+\ldots+\frac{m+1}{2} = \frac{(m+1)}{2}\frac{(m+3)}{4} - \frac{k}{2} - \frac{k^2}{2}$. In this case also, $T(v_k) = T(v_{m-k+2})$.

Hence $TI(G) = 2\sum_{1}^{\frac{m-1}{2}}\left[\frac{(m+1)}{2}\frac{(m+3)}{4} - \frac{k}{2} - \frac{k^2}{2}\right] = \frac{(n-1)(m-1)(m+1)(m+3)}{12}$

$\therefore TI(G) = TI(C_m) + \frac{(n-1)(m-1)(m+1)(m+3)}{12}$. □

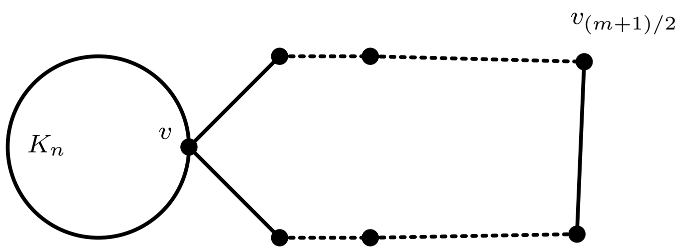


Figure 2.13: Graph G

The (m,n) **lollipop graph** is the graph obtained by joining a complete graph K_m to a path graph P_n with a bridge, represented as $L_{m,n}$. Lollipop graphs are known to be geodetic. Let v denote the vertex on K_m connected to the path P_n and v_k be the vertex on P_n at the kth position with v_n denoting the pendant vertex. In the next theorem we give expressions for transit of vertices in a Lollipop graph.

Theorem 2.5.6. In $L_{m,n}$, $T(v) = \frac{(m-1)(n+3)n}{2}$ and $T(v_k) = \frac{(n-k)}{2}[(n+2)k+(m-1)(n+k+3)]$

Proof. The transit of v is due the shortest paths connecting vertices of P_n to vertices of K_m. Hence $T(v) = [2+3+\ldots+(n+1)](m-1)=\frac{(m-1)(n+3)n}{2}$. Now considering v_k the transit is due to vertices of P_n and due to those on K_m. In Theorem 2.2.3 we have the contribution due to vertices on P_n as $\frac{(n+1)(k-1)(n-k)}{2}$. To this we add the contribution due to vertices on K_m to get $\frac{(n-k)}{2}[(n+2)k+(m-1)(n+k+3)]$ □

The $n-$ **barbell graph** is a special type of undirected graph consisting of two non-overlapping $n-$vertex cliques together with a single edge that has an end vertex in each clique.

Theorem 2.5.7. If G is the $n-$barbell graph, TI(G)=2$(n-1)(3n-1)$

Proof. In G, let the two cliques be connected by the edge uv, where u is a vertex in the first clique and v is a vertex of second clique. Clearly, every vertex of G other than u, v have transit zero. Also u and v being structurally identical have the same transit. The shortest paths through u are those of length 2 connecting vertices of first clique to v and due to the paths of length 3 connecting vertices of first clique to the second. Hence $T(u) = 2(n-1) + 3(n-1)^2$. Hence the proof. □

2.6 Complete Bipartite Graphs

Theorem 2.6.1. Let $G = K_{p,q}$ where $V = V_1 \cup V_2$ the bi-partition with $|V_1| = p, |V_2| = q$. Then $TI(G) = pq[p+q-2]$

Proof. Let $v \in V_1$. Then the number of shortest path through v is $\binom{q}{2}$, of length 2. If $v \in V_2$, then $T(v) = 2C(p, 2)$. Hence

$$\begin{aligned} TI(G) &= \sum_{v \in V} T(v) \\ &= \sum_{v \in V_1} T(v) + \sum_{v \in V_2} T(v) \\ &= 2[\frac{pq(q-1)}{2}] + 2[\frac{pq(p-1)}{2}] \\ &= pq[p+q-2] \end{aligned}$$

□

Theorem 2.6.2. Let G be the complete s-partite graph. Then $TI(G) = \sum_{i=1}^{s} 2n_i \left[\sum_{j\neq i} \binom{n_j}{2}\right]$

Proof. Let $V_1, V_2, \ldots, V_s$ be the partition of the vertex set V. Then no two vertices in V_i are adjacent to each other. But every vertex in $V_j, j \neq i$ is adjacent to vertices of V_i. The shortest paths passing through v_i are those connecting vertices of V_j to itself, of length 2. Hence $T(v_i) = 2\sum_{j\neq i} \binom{n_j}{2}$ $\therefore TI(G) = \sum_{v_i\in V_1} T(v_i) + \sum_{v_i\in V_2} T(v_i) + \cdots + \sum_{v_i\in V_s} T(v_i) = \sum_{i=1}^{s} 2n_i \left[\sum_{j\neq i} \binom{n_j}{2}\right]$ □

Cocktail party graph is the complete n-partite graph, $K_{2,2,\ldots,2}$

Corollary 2.6.1. If G is the cocktail party graph $TI(G) = 4n(n-1)$

Proof. In the Theorem 2.6.2, take $n_i = 2, \forall i$ and $s = n$ with $|G| = 2n$. □

Crown graph is the unique $n-1$ regular graph with $2n$ vertices, obtained from the complete bipartite graph $K_{n,n}$ by deleting a perfect matching. Or it is the graph with vertices as two sets $\{u_i\}$ and $\{v_i\}$, with an edge from u_i to v_j whenever $i \neq j$. Figure 2.14 depicts a crown graph.

Theorem 2.6.3. For the Crown graph G, $TI(G) = 2n(n^2-1)$.

Proof. Let the bipartition be V, U, with $V = \{v_1, v_2, \ldots, v_n\}$ and $U = \{u_1, u_2, \ldots, u_n\}$. Consider a vertex of V, say v_k. Note that $d(u_i, v_i) = 3$ and $d(u_i, v_j) = 2, i \neq j$. The shortest path through v_k are those connecting v_i to $v_j, i \neq j$ of length 2 and those connecting v_i to u_i of length 3. Hence $T(v_k) = 2\binom{n-1}{2} + 3(n-1) = n^2 - 1$. Transit for every vertex are equal. $\therefore TI(G) = 2n(n^2-1)$. □

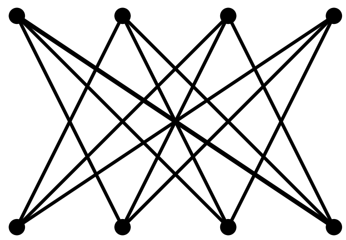


Figure 2.14: Crown graph

In this chapter, we obtained expressions for the transit index for some of the most common graph classes. Also, we could establish that, among all trees, the path has the maximum transit index. This shows that branching reduces the transit index of a graph.

Chapter 3

Binary Products of Graphs

The study of topological indices and centrality measures in graph products has attracted the interest of many researchers. In graph theory, a graph product is a binary operation on graphs. Specifically, it pulls two graphs G_1 and G_2 together into a new graph H. In this chapter we investigate how the knowledge of individual graphs helps in the computation of transit of vertices/ transit index, in graph products. Our focus is mainly on Join of graphs, Cartesian and Corona products.

3.1 Join

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. Let $|V_1| = n_1, |V_2| = n_2, |E_1| = m_1$ and $|E_2| = m_2$.

Theorem 3.1.1. $TI(G_1 + G_2) = 2\left[[\binom{m_1}{2} - m_1]n_2 + [\binom{m_2}{2} - m_2]n_1 + D\right]$, where D denotes the total number of shortest paths in G_1 and G_2 of length 2.

Proof. In G every shortest path is of length ≤ 2. Every pair of nonadjacent vertices in G_1 will contribute a transit 2 to each of the vertices in G_2 and vice versa. In G_1, number of adjacent vertices is m_1 and in G_2 will be m_2. In G_1 and G_2 there may be vertices at a distance 2 to each other. They will remain at the same distance in $G_1 + G_2$ also. Thus we arrive at the result. $\square$

Inductively, Theorem 4.2.5 can be extended to finitely many graphs as follows.

Theorem 3.1.2. Let $G_1, G_2, \ldots, G_s$ be graphs with $n_1, n_2, \ldots, n_s$ number of vertices and $m_1, m_2, \ldots, m_s$ number of edges and $G = G_1 + G_2 + \ldots + G_s$. If d_i denote the number of shortest paths of length 2 in G_i, then $TI(G) = 2\sum_{i=1}^{s}\left[\binom{n_i}{2} - m_i\right]\sum_{i\neq j} n_j + 2\sum_{i=1}^{s} d_i$

Examples

Many well known graph classes can be viewed as join of graphs. We discuss the transit of a few of them here.

s-partite graph: The complete s-partite graph can be viewed as join of s graphs, $G = \overline{K_{n_1}} + \overline{K_{n_2}} + \ldots + \overline{K_{n_s}}$. Here $d_i = 0, \forall i$. Hence $TI(G) = 2\sum_{i=1}^{s}\left[\binom{n_i}{2} - m_i\right]$

Wheel graph : $W_{n+1} = C_n + K_1$. Here, $D = n, n_1 = n, n_2 = 1, m_1 = n, m_2 = 0$. Thus $TI(W_{n+1}) = \left[\binom{n}{2} - n + 0 + n\right] = n(n-1)$.

Star graph : $S_{n+1} = \overline{K_n} + K_1$. Here $D = 0, n_1 = n, n_2 = 1, m_1 = 0, m_2 = 0$. Thus $TI(S_{n+1}) = n(n-1)$.

n-dipyramidal graph : $G = C_n + K_2$. Here, $D = n, n_1 = n, n_2 = 2, m =$

$n-1$. Then, $TI(G) = 2(n^2 - n + 6)$.

Cone graph : The cone graph $C(r,s) = C_r + \overline{K_s}$. It can easily be verified that $D = r, n_1 = r, n_2 = s, m_1 = r, m_2 = 0$. Thus $TI(C(r,s)) = s\,[r(r-3) - s + 1] + 2r$.

Windmill graph. This graph is the join of m copies of K_{n-1} with K_1. Theorem 3.1.2 cannot be applied here directly. But the computation is almost similar. Windmill graphs are geodetic and every shortest path is of length 2, which connects a vertex in K_{n-1} to other $(m-1)$ copies of K_{n-1}. Here, $TI(G) = m(m-1)(n-1)^2$.

3.2 Cartesian Product

Cartesian products of graphs have been one of the most studied types of graph products since the 1960s. Graph cartesian products are applicable in many fields, such as coding theory, network design, chemical graph theory, etc. It has been extensively studied from a variety of perspectives. Networks like grids and hypercubes are Cartesian products of graphs. In this section we study the parameter Transit index/ Transit of a vertex in Cartesian product of graphs.

Definition 3.2.1. [32] The Cartesian product of two graphs G and H, denoted by $G \Box H$ is a graph with vertex set, $V(G \Box H) = V(G) \times V(H)$, that is the set $\{(g,h)/g \in V(G), h \in V(H)\}$. The edge set of $G \Box H$ consists of all pairs $[(g_1,h_1),(g_2,h_2)]$ of vertices with $g_1g_2 \in E(G)$ and $h_1 = h_2$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$.

Thus, for each edge g_1g_2 of G and each edge h_1h_2 of H, there are four edges in $G\Box H$, namely $(g_1,h_1)(g_2,h_1),(g_1,h_2)(g_2,h_2),(g_1,h_1)(g_1,h_2), and (g_2,h_1)(g_2,h_2)$. For any $h \in V(H)$, the subgraph of $G\Box H$ induced by $V(G)\times\{h\}$ is called G-fibre or G-layer, denoted by G^h. H-fibre or H-layer is also defined in a similar manner. They are isomorphic to G and H respectively. $G\Box H$ contains $|H|$ copies of G and $|G|$ copies of H.

Proposition 3.2.1. [32] If (g,h) and (g',h') are vertices of Cartesian product $G\Box H$, then $d_{G\Box H}((g,h),(g',h')) = d_G(g,g') + d_H(g',h')$.

Proposition 3.2.2. [28] If $u = (g,h)$ and $v = (g',h')$ are vertices in $G\Box H$, then the number of shortest $u-v$ paths in $G\Box H$ is $\sigma_{G\Box H}((g,h),(g',h')) =$
$\sigma_G(g,g') \times \sigma_H(h,h') \times \binom{d_G(g,g')+d_H(g',h')}{d_G(g,g')}$

In the following proposition we give a general formula for computing transit of any vertex in $G\Box H$, with respect to parameters of G and H.

Theorem 3.2.1. Let G and H be two simple graphs that are connected. Let $a = (a_1,a_2)$ be any vertex of $G\Box H$. Then, $T(a) = \sum_{(u,v)} \sigma_G(u_1,v_1)\times\sigma_H(u_2,v_2)\times \binom{(d_G(u_1,a_1)+d_H(u_2,a_2)}{d_G(u_1,a_1)} \times \binom{(d_G(a_1,v_1)+d_H(a_2,v_2)}{d_G(a_1,v_1)} \times [d_G(u_1,v_1) + d_H(u_2,v_2)]$, where the summation is taken over all $u-v$ geodesic, with 'a' as an internal vertex.

Proof. By definition of transit of a vertex, $T(a)$ is the sum of the lengths of all geodesics with 'a' as an internal vertex. ie, $T(a) = \sum \sigma_{G\Box H}(u,v/a)d_{G\Box H}(u,v)$. Let $u = (u_1,u_2)$ and $v = (v_1,v_2)$. We have $\sigma(u,v/a) = \sigma(u,a)\times\sigma(a,v)$. Now by Proposition 3.2.2 , $\sigma_{G\Box H}(u,a) = \sigma_G(u_1,a_1)\times\sigma_H(u_2,a_2)\times\binom{d_G(u_1,a_1)+d_H(u_2,a_2)}{d_G(u_1,a_1)}$ and $\sigma_{G\Box H}(a,v) = \sigma_G(a_1,v_1)\times\sigma_H(a_2,v_2)\times\binom{d_G(a_1,v_1)+d_H(a_2,v_2)}{d_G(a_1,v_1)}$. Now by Proposition 3.2.1 $d_{G\Box H}(u,v) = d_G(u_1,v_1) + d_H(u_2,v_2)$. Thus, $T(a) = \sigma_G(u_1,a_1)\times$

$\sigma_H(u_2, a_2) \times \binom{d_G(u_1,a_1)+d_H(u_2,a_2)}{d_G(u_1,a_1)} \times \sigma_G(a_1, v_1) \times \sigma_H(a_2, v_2) \times \binom{d_G(a_1,v_1)+d_H(a_2,v_2)}{d_G(a_1,v_1)} \times [d_G(u_1, v_1) + d_H(u_2, v_2)]. = \sum_{(u,v)} \sigma_G(u_1, v_1) \times \sigma_H(u_2, v_2) \times \binom{d_G(u_1,a_1)+d_H(u_2,a_2)}{d_G(u_1,a_1)} \times \binom{d_G(a_1,v_1)+d_H(a_2,v_2)}{d_G(a_1,v_1)} \times [d_G(u_1, v_1) + d_H(u_2, v_2)]$ □

In the following sections we use Proposition 3.2.1 to compute transit of vertices in various graphs that are Cartesian products. We begin with the grid graph.

Grid Graph

A two-dimensional grid graph, also known as a rectangular grid graph or two-dimensional lattice graph, is the graph Cartesian product $P_m \Box P_n$ of path graphs on m and n vertices. Let us denote the vertices of P_n by $1, 2, \ldots, n$ and that of P_m by $1, 2, \ldots, m$, so that the distance between any two vertices u_1, u_2 on P_n or P_m will be $|u_1 - u_2|$. A grid graph $P_m \Box P_n$ has mn vertices and $2mn - m - n$ edges. If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two vertices of $P_m \Box P_n$, then $d(u, v) = |u_1 - v_1| + |u_2 - v_2|$. In this case $\sigma(u, v) = \binom{d}{d_1}$, where $d_1 = d(u_1, v_1)$ or $d(u_2, v_2)$. Let $a = (a_1, a_2)$ be a vertex of $P_m \Box P_n$. Then the fibre $P_m^{a_2}$ and $P_n^{a_1}$ divides the grid into 4 parts as seen in the figure. Two vertices $u = (u_1, u_2), v = (v_1, v_2)$ of the graph contributes to the transit of 'a' only when the $u - v$ geodesic contains 'a'. This happens in the following two cases.

(1) u and v lies in diametrically opposite parts. Let us denote the contribution to the transit, $T(a)$ in this case as T_D.

(2) u and v lie on the fibres $P_m^{a_2}$, $P_n^{a_1}$ with 'a' in between. Let the contribution to the transit, $T(a)$ in this case be T_F

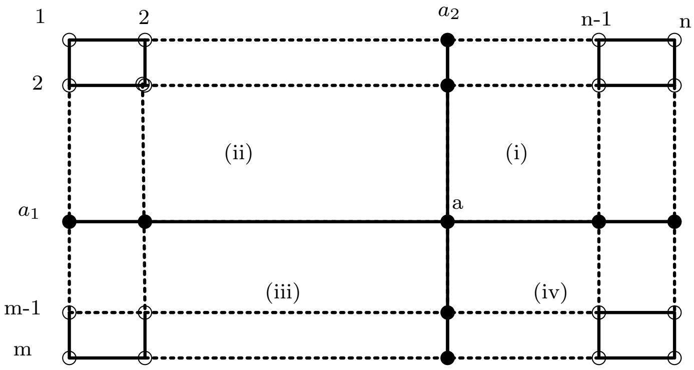


Figure 3.1: $P_m \square P_n$

Then, $T(a) = \sum_{(u,v)} T_D + \sum_{(u,v)} T_F$. An expression for T_D and T_F is found in Theorem 3.2.2

Theorem 3.2.2. For the vertex $a = (a_1, a_2)$ in $P_m \square P_n$, 1) $T_D = \binom{|a_1-u_1|+|u_2-a_2|}{|a_1-u_1|} \times |v_1 - a_1| + |a_2 - v_2||v_1 - a_1|[|v_1 - u_1| + |u_2 - v_2|]$
2) $T_F = d_{G\square H}(u, v)$

Proof. 1)As u and v are in diametrically opposite parts, it could be eaisly verified that, $\sigma_{G\square H}(u, a) = \binom{|a_1-u_1|+|u_2-a_2|}{|a_1-u_1|}$ $\sigma_{G\square H}(a, v) = \binom{|v_1-a_1|+|a_2-v_2|}{|v_1-a_1|}$ and $d_{G\square H}(u, v) = |u_1 - v_1| + |u_2 - v_2|$. The result follows.
2)In this case there exist only a single $u - v$ geodesic. Hence the contribution of the pair (u, v) will be the length of the $u - v$ geodesic. Hence the proof. □

Ladder Graph

The m-ladder graph shown in Figure 3.2 can be defined as $L_m = P_m \square P_2$, where P_m is a path graph. It is therefore equivalent to the $m \times 2$ grid graph. The ladder graph is named for its resemblance to a ladder consisting of two rails and n rungs between them.

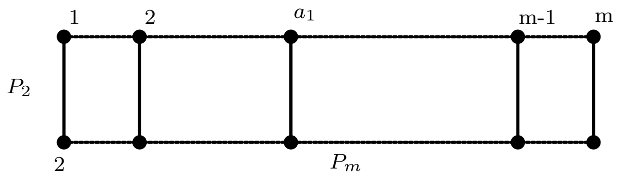


Figure 3.2: Ladder Graph L_m

Theorem 3.2.3. For any vertex $a = (a_1, a_2)$ of L_m,

$$T(a) = \frac{(m - a_1 + 1)(m + 2 - a_1)(a_1 - 1)(a_1 + 4m + 6 - a_1^2)}{12} + \frac{a_1(a_1 + 1)(m - a_1)(a_1 + 3m + 5)}{12} + \frac{(m + 1)(a_1 - 1)(m - a_1)}{2}$$

Proof. Since the two rails of the graph are identical, with out loss of generality we assume 'a' lies on the first rail(the fibre P_m^1). Let $a = (a_1, 1)$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two vertices of the graph L_m other than 'a'. A $u - v$ geodesic will contribute to $T(a)$ only in the following three cases.

Case(i): $1 \leq u_1 < a_1 < v_1 \leq m$ and $u_2 = v_2 = 1$. In this case we can view 'a' as an internal vertex of the path P_m. Hence the contribution to transit $T(a) = \frac{(m+1)(a_1-1)(m-a_1)}{2}$

Case(ii): $1 \leq u_1 < a_1 \leq v_1 \leq m$ and $u_2 = 1, v_2 = 2$. Here $d_{L_m}(u, v) = v_1 - u_1 + 1$ and $\sigma_{L_m}(u, v/a) = v_1 - a_1 + 1$. Hence the contribution of the pair (u, v) will be $[v_1 - a_1 + 1][v_1 - a_1 + 1]$. The total contribution in this case to $T(a_1, a_2)$ will be

$$= \sum_{v_1=a_1}^{m} \sum_{u_1=1}^{a_1-1} [v_1 - u_1 + 1][v_1 - a_1 + 1]$$

$$= \frac{(m - a_1 + 1)(m + 2 - a_1)(a_1 - 1)(a_1 + 4m + 6 - a_1^2)}{12}.$$

Case(iii): $1 \leq v_1 \leq a_1 < u_1 \leq m$ and $u_2 = 2, v_2 = 1$. In this case it can be verified that, $d_{L_m}(u,v) = u_1 - v_1 + 1$ and $\sigma_{L_m}(u, v/a) = a_1 - v_1 + 1$. The total contribution to $T(a_1, a_2)$ in this case will be

$$= \sum_{u_1=a_1+1}^{m} \sum_{v_1=1}^{a_1} [u_1 - v_1 + 1]a_1 - v_1 + 1]$$

$$= \frac{a_1(a_1+1)m - a_1)(a_1 + 3m + 5)}{12}$$

Considering every pair of vertices (u,v), with 'a' as an internal vertex, the result follows. □

Rook Graph

In graph theory, a rook graph is a graph that represents all legal moves of the rook chess piece on a chessboard. Each vertex of a rook graph represents a square on a chessboard, and each edge represents a legal move from one square to another. The $m \times n$ rook graph is the graph Cartesian product $K_m \square K_n$ of complete graphs.

Theorem 3.2.4. The transit index of $m \times n$ rook graph is given by, $TI(K_m \square K_n) = 2mn(m-1)(n-1)$.

Proof. In $K_m \square K_n$, every geodesic is of length 2. Let 'a' be any vertex of $K_m \square K_n$ and $u - a - v$ be a geodesic. Then it could be eaisly verified that $d(u,v) = 2$ and $\sigma_{K_m \square K_n}(u, v/a) = 1$. Hence $T(a) = \sum_{(u,v)} 2$. There are $(m-1)(n-1)$ pairs of non adjacent vertices with uav as a geodesic. So we get $T(a) = 2(m-1)(n-1)$. Hence the proof. □

Note: If the shortest paths passing through two vertices are same in number and length, we call them **transit identical**.

Stacked Prism Graph

A stacked (or generalized) prism graph is a simple graph given by the graph cartesian product $P_m \square C_n$ for positive integers m,n with $m \geq 3$. It can therefore be viewed as formed by connecting m concentric cycle graphs C_n along spokes. Therefore it has mn vertices and $n(2m-1)$ edges. The term "web graph" is sometimes used to refer a stacked prism graph.

We know that in a cycle, every vertex is transit identical. Earlier in Chapter 2 we have proved that transit of a vertex in C_{2k+1} and C_{2k} are the same. Hence we will consider only one case here, ie, $n = 2k+1$. We attempt to find the transit of an arbitrary vertex of the graph, $C_n \square P_m$. For this consider (r,k) lying on the fibre P_m^k. We find the contribution to $T(r,k)$ due to geodesic connecting vertices of fibre C_n^p to C_n^q, denoted by T_{pq}. Then $T(r,k) = \sum_{pq} T_{pq}$. The following theorem gives an expression for T_{pq}.

Theorem 3.2.5. 1)For $p < r < q$,

$$T_{pq} = 2 \times \left[\sum_{i=0}^{k-1} \binom{r-p+k-i}{k-i} [q-p+k-i] \right.$$

$$+(q-r+1) \sum_{i=1}^{k} \binom{r-p+k-i}{k-i} [q-p+k+1-i] +$$

$$\left. +\ldots+ \binom{q-r+k}{k} \sum_{i=k}^{k} \binom{r-p+k-i}{k-i} [q-p+2k-i] \right] + (q-p)$$

2) For $p = r < q$ and $p < r = q$,

$$T_{pq} = 2 \times \left[\sum_{i=0}^{k-1} \binom{r-p+k-i}{k-i} [q-p+k-i] \right.$$

$$+ (q-r+1) \sum_{i=1}^{k} \binom{r-p+k-i}{k-i} [q-p+k+1-i] + \dots$$

$$\left. + \dots + \binom{q-r+k-1}{k-1} \sum_{i=k-1}^{k-1} \binom{r-p+k-i}{k-i} [q-p+2k-1-i] \right]$$

Proof. For convenience we name the vertices of C_n as $0, 1, 2, \dots, 2k-1, 2k$ and that of P_m ad $0, 1, 2, \dots, m-1$. Let $0 \leq p < q < r \leq m-1$, as shown in the Figure 3.3. Let $u = (u_1, p) \in C_n^p$ and $v = (v_1, q) \in C_n^q$. Not every $u-v$ geodesic contribute to the transit of (r, k). The possible cases are when $0 \leq u_1 \leq k$ and $k \leq v_1 \leq 2k$ OR $0 \leq v_1 \leq k$ and $k \leq u_1 \leq 2k$.

1))For $p < r < q$, the computations for $0 \leq u_1 \leq k$ and $k \leq v_1 \leq 2k$ are tabulated. The computation for $0 \leq v_1 \leq k$ and $k \leq u_1 \leq 2k$ is similar and gives the same value. Both the cases added together gives the value for T_{pq}

2)The values obtained for $p = r < q$ and $p < r = q$ will be the same. This can also be computed from the same table by omitting the last column corresponding to the vertex(p, k).

(u, v)	$(p, 0)$	$(p, 1)$	$\dots$
(q, k)	$\binom{r-p+k}{r-p}\binom{q-r}{0}[q-p+k]$	$\binom{r-p+k-1}{k-1}\binom{q-r}{0}[q-p+k-1]$	$\dots$
$(q, k+1)$	0	$\binom{r-p+k-1}{k-1}\binom{q-r+1}{1}[q-p+k]$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(q, 2k)$	0	0	0

(u,v)	$\dots$	(p,k)
(q,k)	$\dots$	$\binom{r-p+k-k}{k-k}\binom{q-r}{0}[q-p+k-k]$
$(q,k+1)$	$\dots$	$\binom{r-p+k-k}{k-k}\binom{q-r+1}{1}[q-p+k-k+1]$
$\vdots$	$\vdots$	$\vdots$
$(q,2k)$	0	$\binom{r-p+k-k}{k-k}\binom{q-r+k}{k}[q-p+k-k+k]$

□

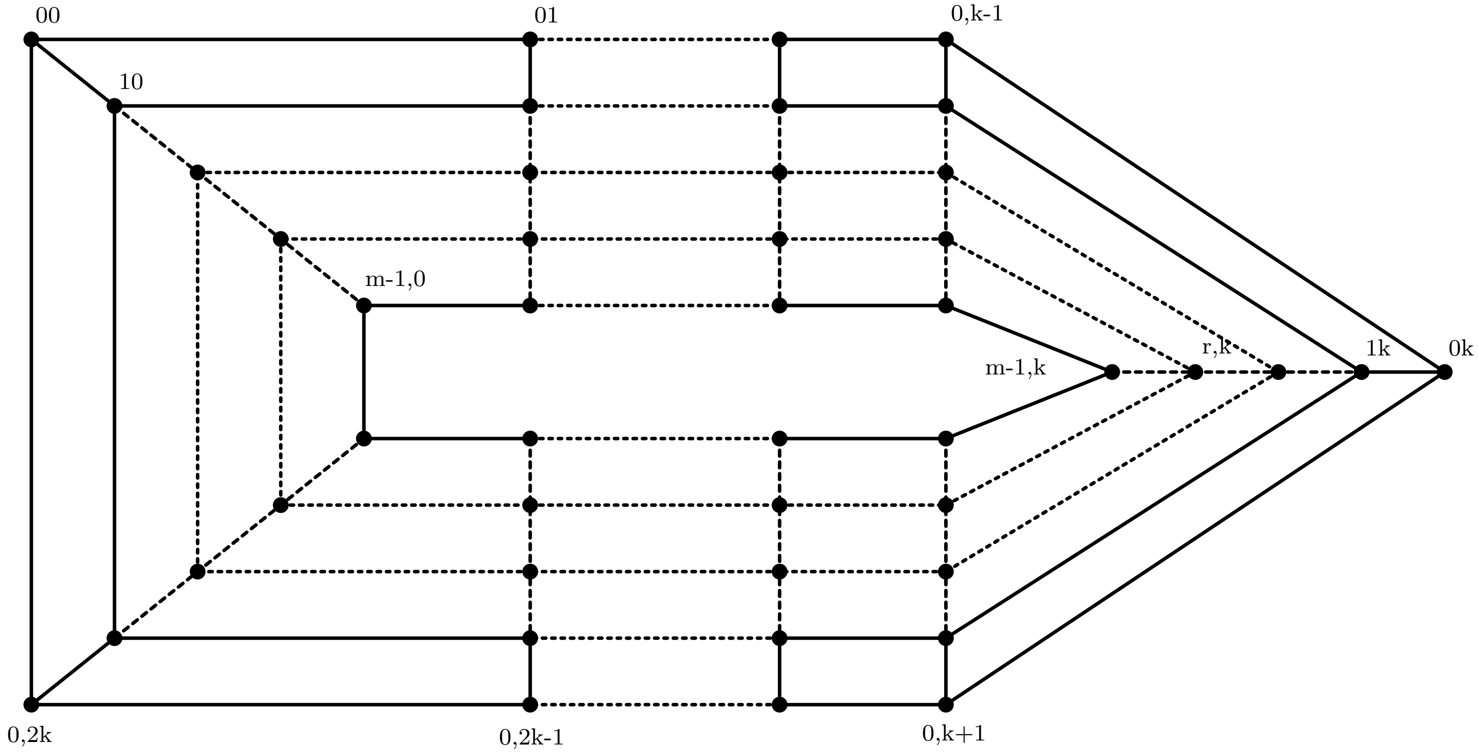


Figure 3.3: $P_m \Box C_n$

Prism Graph

A stacked prism graph $P_m \Box C_n$ with $m = 2$ is called a prism graph. A prism graph is sometimes called a circular ladder graph. Prism graphs are both planar and polyhedral. An n-prism graph has 2n vertices and 3n edges.

Theorem 3.2.6. If a is any vertex of the n-prism graph, $T(a) = \frac{k(k+1)(3k^2+15k+6)}{12}$, for both $n = 2k$ and $n = 2k + 1$.

Proof. We have observed that the transit of a vertex is same for C_{2k} and C_{2k+1}, as the length and number of geodesic through a vertex are the same for both graphs. The same argument holds in the case of n-prism also. So we prove the result for C_{2k}.

Let u be any vertex of the graph $P_m \Box C_n$. Without loss of generality we can assume $a = (0, k)$. (All vertices being transit identical.) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two vertices of the graph different from 'a'. A $u - v$ geodesic will contribute to $T(a)$ in the following three cases.

(i) Both u and v lie on the fibre C_n^0.

The contribution of such (u, v) pairs is same as the transit of a vertex in C_n, which is $\frac{(m^2-4)m}{24}$.

(ii) u and v lie on different copies of C_n. The contribution of the pairs (u, v), when $1 \leq u_2 \leq k - 1$ and $k \leq v_2 \leq 2k$ is tabulated in Table 3.1.

Using the values from table the contribution to $T(a)$ in this case is

$$\left[\frac{1}{2}\sum_{j=1}^{k-1} j(j+1)^2]\right] - (k+1) = \frac{k(k+1)(3k^2+11k+10)}{24} - (k+1)$$

. The values are similar when $1 \leq v_2 \leq k$ and $k + 1 \leq u_2 \leq 2k$. Besides those tabulated there are two paths of length $k + 1$ connecting $(0, 0)$ to $(1, k)$.

u	(0,1)		(0,2)			$\ldots$	$(0,k-1)$			
Length of $u-v$ geodesic	k	k+1	k-1	k	k+1	$\ldots$	2	3	$\ldots$	k+1
Number of $u-v$ geodesic	1	2	1	2	3	$\ldots$	1	2	$\ldots$	k

Table 3.1:

Thus altogether $T(a) = \frac{k(k+1)(3k^2+15k+6)}{12}$. Hence the proof. $\square$

Book Graph

The $m-$book graph is defined as the graph Cartesian product $B_m = S_{m+1}\Box P_2$, where S_{m+1} is a star graph and P_2 is the path graph on two vertices.

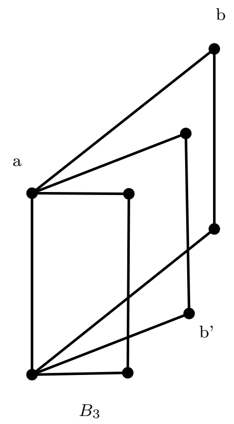


Figure 3.4: $S_4\Box P_2$

Theorem 3.2.7. The transit index of the $m-$book graph is, $TI(S_{m+1}\Box P_2) = 4m(m-1)$

Proof. Consider the Figure 3.4. As far as transit is concerned there are two type of vertices, a and b. B_m has m copies of C_4. The vertex a is common to all these m cycles. Hence each of them contribute 2 towards the transit of a. Every pair of vertices of the type b and b' will contribute 6 towards $T(a)$ and there are $m(m-1)$ such pairs. Hence $T(a) = 6m(m-1) + 2m$. Clearly $T(b) = 2$. Hence $TI(B_m) = 4m(3m-1)$ □

3.3 Corona Product

Definition 3.3.1. [7] Let G_1 and G_2 be two graphs. The corona product $G_1 \circ G_2$, is obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2; and by joining each vertex of the i-th copy of G_2 to the i-th vertex of G_1, where $1 \leq i \leq |V(G_1)|$

Whenever we consider $G_1 \circ G_2$, we use the following notations.

1. G_2^i the ith copy of G_2 in $G_1 \circ G_2$
2. $V(G_1) = \{u_1, u_2, \ldots, u_{n_1}\}, |E(G_1)| = m_1$
3. $V(G_2^i) = \{v_1^i, v_2^i, \ldots, v_{n_2}^i\}, |E(G_2^i)| = m_2, \forall i$

Lemma 3.3.1. [26] Let G_1 and G_2 be two arbitrary graphs. Then,

$$d_{G_1 \circ G_2}(u_i, u_p) = d_{G_1}(u_i, u_p), 0 \leq i, p \leq n_1$$

$$d_{G_1 \circ G_2}(u_i, v_q^p) = d_{G_1}(u_i, u_p) + 1, 0 \leq i, p \leq n_1, 0 \leq q \leq n_2$$

$$d_{G_1 \circ G_2}(v_j^i, v_q^p) = \begin{cases} d_{G_1}(u_i, u_p) + 2, & i \neq p \\ 1, & if \quad i = p, \quad v_j v_q \in E(G_2) \\ 2, & if \quad i = p, \quad v_j v_q \notin E(G_2) \end{cases}$$

Theorem 3.3.1. 1. $T_{G_2}(a) = 0$ iff $T_{G_1 \circ G_2}(a) = 0$, $a \in G_2$

2. $T_{G_2}(a) = T_{G_1 \circ G_2}(a)$ iff every shortest path in G_2 with 'a' as an internal vertex is of length 2, $a \in G_2$.

Proof. 1)Let $T_{G_2}(a) = 0 \implies \langle N_{G_2}[a] \rangle$ is a clique. In $G_1 \circ G_2$, every vertex in G_2^i will be joined to u_i of G_1. Hence $\langle N_{G_1 \circ G_2}[a] \rangle$ is also a clique, and thereby $T_{G_1 \circ G_2}(a) = 0$. Conversely, let $T_{G_1 \circ G_2}(a) = 0$. $\implies \langle N_{G_1 \circ G_2}[a] \rangle$ is also clique. $\implies \langle N_{G_1 \circ G_2}[a] \rangle - \{u_i\} = \langle N_{G_2}[a] \rangle$ is also a clique. The result follows.

2)$T_{G_2}(a) = T_{G_1 \circ G_2}(a)$. Since every vertex of G_2^i are connected by an edge to u_i in $G_1 \circ G_2$, the geodesics through 'a' in $G_1 \circ G_2$ are geodesics through 'a' in G_2 also. In $G_1 \circ G_2$, every vertex of G_2^i are at a maximum distance of 2 from each other. Conversely, let every shortest path in G_2 with 'a' as an internal vertex be of length 2. In $G_1 \circ G_2$, every geodesic of length 2 will remain a geodesic. Hence the proof. $\square$

Theorem 3.3.2. 1) $\sigma_{G_1 \circ G_2}(u_p) = (n_2+1)\Big[(n_2+1)\sigma_{G_1}(u_p) + n_2 \sum_{p \neq k=1}^{n_1} \sigma_{G_1}(u_p, u_k)\Big]$, $u_p \in V(G_1)$

2) $\sigma_{G_1 \circ G_2}(v_k^i) =$ Number of geodesic of length 2 with v_k as an internal vertex in G_2

Proof. 1) Let u_p be any vertex of G_1. Every geodesic in G_1 with u_p as an internal vertex will be counted in $\sigma_{G_1 \circ G_2}(u_p)$. Geodesics connecting $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$ will have $u_l - u_k$ geodesic as its part. Let P_1 be one of the $u_l - u_k$ geodesic with u_p as an internal vertex. Then P_1 will be part of some geodesics connecting vertices of $G_2^k \cup \{u_k\}$ to vertices of $G_2^l \cup \{u_l\}$. There will be n_2^2 geodesics that connects G_2^k to G_2^l, n_2 geodesics that connects G_2^k to u_l, n_2 geodesics that connects G_2^l to u_k, with P_1 as its part. Hence for the pair of vertices (u_k, u_l), there will be, $({n_2}^2 + 2n_2 + 1)\sigma_{G_1}(u_p)$ geodesics with u_p as an internal vertex. The geodesics connecting vertices of G_2^p to other vertices of $G_1 \circ G_2$ will have $u_p - u_k$ geodesic as a part for some k. If P_2 is one of the $u_p - u_k$ geodesic, it will be part of - n_2^2

geodesics that connects G_2^k to G_2^p and n_2 geodesics that connects G_2^p to u_k. Hence for every $u_p - u_k$ geodesic in G_1 there will be $\sigma_{G_1}(u_p, u_k)[n_2(n_2+1)]$ geodesics in $G_1 \circ G_2$ with u_p as an internal vertex. Considering every pair $u_k - u_p$ the result follows.

2) Since every vertex of G_2^i are joined to u_i, the maximum distance between vertices of G_2^i is 2. Hence the proof. $\square$

Next we find an expression for the transit of a vertex, u_p in $G_1 \circ G_2$, where G_1 and G_2 are arbitrary. Let (u_k, u_l) be a pair of vertices in G_1 such that $u_k - u_l$ geodesic has u_p as an internal vertex. Let $T_{kl}(u_p)$ denote the contribution to transit of u_p, due to geodesic connecting vertices of $G_2^k \cup \{u_k\}$ to $G_2^l \cup \{u_l\}$. Also we denote the contribution of vertices in G_2^p to $T(u_p)$ by $T_p(u_p)$.

Lemma 3.3.2. For arbitrary graphs G_1 and G_2,

$$T_{kl}(u_p) = \sigma_{G_1}(u_k, u_l/u_p)\Big[(n_2+1)^2 d(u_k, u_l) + 2n_2(n_2+1)\Big]$$

Proof. The following table give the length and number of geodesics through u_p

Vertices connected	Length	Number
G_2^k to G_2^l	$2 + d(u_k, u_l)$	$n_2^2\ \sigma(u_k, u_l/u_p)$
u_k to G_2^l	$1 + d(u_k, u_l)$	$n_2\ \sigma(u_k, u_l/u_p)$
G_2^k to u_l	$1 + d(u_k, u_l)$	$n_2\ \sigma(u_k, u_l/u_p)$
u_k to u_l	$d(u_k, u_l)$	$\sigma(u_k, u_l/u_p)$

The result follows. $\square$

Lemma 3.3.3.

$$T_p(u_p) = \sum_{p \neq k=1}^{n_1} \Big[\sigma_{G_1}(u_p, u_k)\big[n_2(n_2+1)d(u_p, u_k) + n_2(1+2n_2)\big]\Big] + 2\left[\binom{n_2}{2} - m_2\right]$$

Proof. The following table gives the contribution of geodesics through u_p to $T_p(u_p)$

Vertices connected	Length	Number
u_k to G_2^p	$1 + d(u_k, u_p)$	$n_2\ \sigma(u_k, u_p)$
G_2^k to G_2^p	$2 + d(u_k, u_p)$	$n_2^2\ \sigma(u_k, u_p)$
G_2^p to G_2^p	2	$\binom{n_2}{2} - m_2$

Considering every vertex $u_k, k \neq p$, the result follows. □

Theorem 3.3.3.

$$T(u_p) = T_p(u_p) + \sum_{kl} T_{kl}(u_p)$$

Proof. Geodesics through u_p are either considered in $T_p(u_p)$ or in $T_{kl}(u_p)$. Hence the result is evident. □

In the remaining sections we consider G_2 as arbitrary, while G_1 is replaced by various graph classes like $P_n, C_n, K_n, K_{m,n}$ and S_{n+1}

Path Graph

Let P_n be the path graph with vertices $1, 2, \ldots, n$. We give an expression for transit of k using Theorem 3.3.3 in $P_n \circ G_2$.

Theorem 3.3.4.

$$T(k) = \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1) + 4n_2\Big] +$$

$$n_2(n_2+1)\Big[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}\Big] + n_2(2n_2+1)(n-1) + 2\Big[\binom{n_2}{2} - m_2\Big]$$

Proof. Let $1 \leq l < k < m \leq n$. Since G_1 is a path, we have $\sigma_{G_1}(l, m/k) = 1$. Hence $T_{lm}(k) = (n_2+1)^2(m-l) + 2n_2(n_2+1)$

$$\therefore \sum_{l,m} T_{lm}(k) = (n_2+1)^2 \sum_{l=1}^{k-1} \sum_{m=k-1}^{n} (m-l) + \sum_{l=1}^{k-1} \sum_{m=k-1}^{n} 2n_2(n_2+1)$$
$$= (n_2+1)^2 T_{G_1}(k) + (k-1)(n-k-1)2n_2(n_2+1)$$
$$= (n_2+1)^2 \frac{(n+1)(k-1)(n-k)}{2} + (k-1)(n-k-1)2n_2(n_2+1)$$
$$= \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1) + 4n_2\Big]$$

In a similar manner we compute $T_k(k)$

$$T_k(k) = \sum_{k \neq i=1}^{n} \Big[(d(k,i)+1)n_2 + (d(k,i)+2)n_2^2\Big] + 2\Big[\binom{n_2}{2} - m_2\Big]$$
$$= (n_2 + n_2^2)\left[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}\right] + n_2(2n_2+1)(n-1) + 2\Big[\binom{n_2}{2} - m_2\Big]$$

The result follows. $\square$

Examples

In the following examples we compute transit for the vertices in various corona products of P_n. From Theorem 3.3.4, we have

$$T(k) = \frac{(k-1)(n_2+1)(n-k)}{2}\Big[(n_2+1)(n+1) + 4n_2\Big] + n_2(n_2+1)\Big[\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2}\Big] + n_2(2n_2+1)(n-1) + 2\Big[\binom{n_2}{2} - m_2\Big]$$

$= T_1 + T_2$, say where $T_2 = 2\Big[\binom{n_2}{2} - m_2\Big]$

(a) $G_2 = P_m$. Here $n_2 = m, m_2 = m-1$. Hence $T(k) = T_1 + (m-1)(m-2)$. For pendant vertices of P_m^i, the transit is 0 and 2 for others, $\forall i$.

(b) $G_2 = C_m$. Here $n_2 = m_2 = m$. For every vertex in C_m^i, there exist only one geodesic of length 2 through it. Here $T(k) = T_1 + m(m-3)$ and $T(v_k^i) = 2$, $\forall k, i$.

(c) $G_2 = K_m$. Then $n_2 = m, \quad m_2 = \binom{m}{2}$. Thus, $T(k) = T_1$ and $T(v_k^i) = 0$, $\forall k, i$.

(d) $G_2 = S_m$. Here $n_2 = m, \quad m_2 = m-1$ Hence $T(k) = T_1 + (m-1)(m-2)$. $T(v_k^i) = 0$, for pendant vertices and for central vertex of S_m, $T(v_k^i) = (m-1)(m-2)$

(e) $G_2 = K_{l_1,l_2}$. $n_2 = m = l_1 + l_2$ and $m = l_1 l_2$. $T(k) = T_1 + (m-1)m - 2l_1 l_2$ and $T(v_k^i) = T_{K_{l_1,l_2}}(v_k)$

3.3.1 Cycle

In this section we consider G_1 to be a cycle. We have already seen that transit of vertices in cycles of order $2n$ and $2n+1$ are the same. Hence we consider $G_1 = C_{2n_1+1}$. We represent the vertices by $0, 1, \ldots, 2n_1$. Also every vertex in the cycle being transit identical, it is enough we compute the transit for n_1.

Theorem 3.3.5. If a is any vertex of the cycle C_{2n} or C_{2n+1}, its transit in the corona product $C_{2n} \circ G_2$ or $C_{2n+1} \circ G_2$ is given by $T(a) = \frac{(n_2+1)(n_1-1)n_1}{3}[n_2 n_1 + 4n_2 + n_1 + 1] + n_2 n_1\big[(n_2+1)(n_1+1) + 2(1+2n_2)\big] + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. For any k, l we know that $\sigma_{G_1}(k, l/n_1) = 1$.

$$
\begin{aligned}
T_{k,l}(n_1) &= \Big[(n_2+1)^2 d(k,l) + 2n_2(n_2+1)\Big] \\
\therefore \sum_{k,l} T_{k,l}(n_1) &= (n_2+1)^2 \sum_{k,l} d(k,l) + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= (n_2+1)^2 T_{G_1}(n_1) + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= (n_2+1)^2 \frac{(n^2-1)n}{24} + \frac{n_1(n_1-1)}{2} 2n_2(n_2+1) \\
&= \frac{(n_2+1)(n_1-1)n_1}{3}[n_2n_1 + 4n_2 + n_1 + 1]
\end{aligned}
$$

Next we compute $T_{n_1}(n_1)$

$$
\begin{aligned}
T_{n_1}(n_1) = \sum_{n_1 \neq i = 0}^{2n_1} \Big[n_2(n_2+1)d(n,i) + n_2(1+2n_2)\Big] + 2\left[\binom{n_2}{2} - m_2\right], \\
\sigma_{G_1}(n_1, i) \text{ being } 1 \\
= n_2n_1\big[(n_2+1)(n_1+1) + 2(1+2n_2)\big] + 2\left[\binom{n_2}{2} - m_2\right]
\end{aligned}
$$

And $T_{G_1 \circ G_2}(n_1) = \sum_{k,l} T_{k,l}(n_1) + T_{n_1}(n_1)$. Hence the proof. □

3.3.2 Star Graph

Let $G_1 = S_{n+1}$. In a star there are n pendant vertices and one central vertex. All pendant vertices are transit identical. Hence we need to compute transit of one of the pendant vertex and the central vertex in $S_{n+1} \circ G_2$. Let us name the vertices as $1, 2, \ldots, n+1$, where $n+1$ is the central vertex.

Theorem 3.3.6. In $S_{n+1} \circ G_2$, $T(n+1) = n\big[(n-1)(n_2+1)(2n_2+1) + n_2(3n_2+2)\big] + 2\left[\binom{n_2}{2} - m_2\right]$ and $T(i) = n_2\big[(n_2+1)(2n-1) + n(2n_2+1)\big] + 2\left[\binom{n_2}{2} - m_2\right], i \neq n+1$

Proof. Consider the vertex $n+1$. We have $\sigma(k, l/(n+1)) = 1$ and $d(k,l) = 2$

$$\text{Thus } T_{k,l}(n+1) = 2(n_2+1)(2n_2+1)$$

$$\therefore \sum_{k,l} T_{k,l}(n+1) = \binom{n}{2} 2(n_2+1)(2n_2+1)$$
$$= n(n-1)(n_2+1)(2n_2+1)$$

While computing $T_{n+1}(n+1)$, we see that $\sigma_{S_{n+1}}(n+1,i) = 1$ and $d(n+1,i) = 1, \forall i$. Thus we get $T_{n+1}(n+1) = nn_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$, which completes the computation for $T(n+1)$. Now consider the vertex $i \neq n+1$. It can easily be verified that $\sigma(k, l/i) = 0, \forall k, l$. Hence $\sum_{k,l} T_{k,l}(i) = 0$. For a fixed i, $\sigma(i,k) = 1, \forall k$ and $d(i, n+1) = 1$ and $d(i,k) = 2, k \neq n+1 \therefore T_i(i) = n_2\big[4nn_2 + 3n - n_2 - 1\big] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the proof. □

3.3.3 Complete Graph and Bipartite Graph

Theorem 3.3.7. In the corona product $K_n \circ G_2$, the transit of any vertex of K_n is $(n-1)n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. Since every vertex of K_n is transit identical, we consider one of them. Let u_i be any vertex of K_n. $\sigma(u_k, u_l/u_i) = 0 \Longrightarrow \sum T_{k,l}(u_i) = 0$. Again $\sigma(u_i, u_k) = 1, \forall k \neq i$ and $d(u_i, u_k) = 1$. $\therefore T_i(u_i) = \sum_{k \neq i} \big[n_2(n_2+1) + n_2(2n_2+1)\big] + 2\left[\binom{n_2}{2} - m_2\right]$. Hence the result. □

Next we consider a complete bipartite graph K_{l_1,l_2} with bipartition V_1, V_2. Let $V_1 = a_1, a_2, \ldots, a_{l_1}$ and $V_2 = b_1, b_2, \ldots, b_{l_2}$. Then all a_i are transit identical. Similarly all b_i are also transit identical. Computation of $T(a_i)$ and $T(b_i)$ are similar. Hence we compute $T(a_i)$ only.

Theorem 3.3.8. In $K_{l_1,l_2} \circ G_2$, the transit of a_i, $T(a_i) = \binom{l_2}{2} 2(n_2+1)(2n_2+1) + l_2 n_2(4n_2+3)(l_1-1) + l_2 n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$

Proof. The shortest path in K_{l_1,l_2} through a_i are those connecting vertices of V_2. $\therefore T_{k,l}(a_i) = \sigma_{K_{l_1,l_2}}(b_k, b_l/a_i)\big[(n_2{+}1)^2 d(u_k,u_l){+}2n_2(n_2{+}1)\big]$. Thus $\sum_{k,l} T_{k,l}(a_i) = \binom{l_2}{2}2(n_2+1)(2n_2+1)$. While computing $T_i(a_i)$, we see that vertices in V_1 and V_2 behaves differently. Hence we split the summation as follows.

$$T_i(a_i) = \sum_{a_j} \sigma(a_i,a_j)\Big[n_2(n_2+1)d(a_i,a_j) + n_2(2n_2+1)\Big]$$
$$+\sum_{b_j} \sigma(a_i,b_j)\Big[n_2(n_2+1)d(a_i,b_j) + n_2(2n_2+1)\Big] + 2\left[\binom{n_2}{2} - m_2\right]$$
$$= l_2n_2(4n_2+3)(l_1-1) + l_2n_2(3n_2+2) + 2\left[\binom{n_2}{2} - m_2\right]$$

$\square$

Chapter 4

Transit Decomposition

When studying large networks, distinguishing vertices on the basis of their importance is a significant area of study. Centrality measures serve these purposes. The center and centroid of a graph are among them. Transit of a vertex can also be treated as one of the centrality measures. In this chapter we define a few concepts regarding the transit index of a graph and the transit of vertices in graphs. By using these concepts, we are able to perform computations more easily. Additionally, they provide a better understanding of the relevance and importance of the concept of transit.

4.1 Transit Identical

Definition 4.1.1. Two vertices of a graph are called **transit identical** if the shortest paths passing through it are same in number and length. In Figure 4.1, u and v are transit identical. Certain graphs like cycles have the property that every pair of vertices are transit identical, such graphs are called **transit uniform**

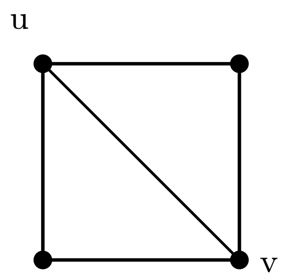


Figure 4.1: Transit Identical vertices

graphs.

If two vertices of a graph are structurally symmetric, then they are transit identical. But the converse is not always true. In Figure 4.2 a and b are structurally different but transit identical. This shows that, the shortest paths through a and b are same in number and length.

The example shown in Figure 4.3 illustrates vertices with equal transit but not transit identical. Eventhough a and b have transit 12, the number of shortest paths through them and their lengths are different. These examples show that transit identical vertices have their own status and hence can be used for characterisation of vertices in graphs.

Remark 4.1.1. Transit identical does not imply structural symmetry or equal value for transit.
Define a relation $\sim$ on $V(G)$ as follows.
For any two vertices a, b declare $a \sim b$ iff a and b are transit identical. The $\sim$ being an equivalence relation partitions $V(G)$ into disjoint classes, with each

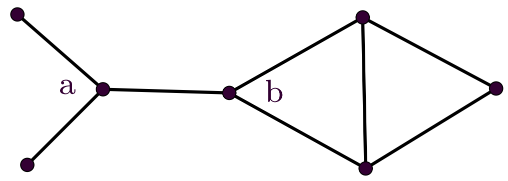


Figure 4.2: Transit identical & not structurally identical

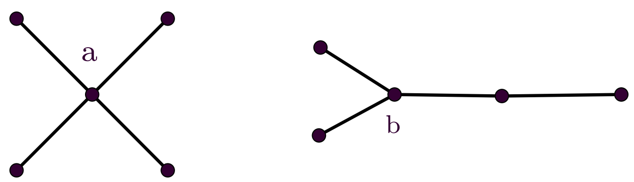


Figure 4.3: Transit equal but not transit identical

class having vertices that are transit identical.

Definition 4.1.2. Equivalence classes with respect to $\sim$, defined on $V(G)$ as above are called transit equivalence classes.

Based on the equivalence classes we make the following definitions.

Definition 4.1.1. The transit equivalence class of $V(G)$ with maximum transit is defined as transit dominant class of G and the class with transit zero as null class of G.

Example 4.1.1. In cycles all vertices are transit identical and hence there is only one equivqlence class. For paths P_n there are $\lceil \frac{n}{2} \rceil$ equivalence classes, for

$n > 2$. The transit dominant class has one element for n odd and two elements for n even. S_n, the star graph on $n > 2$, has only two equivalence classes, of which one is the null class with $n-1$ elements and other the dominant class with one element. Complete graph K_n has only the null class, while the complete bipartite graph $K_{m,n}$ has two equivalence classes, neither of which is a null class, provided $m, n > 1$.

Note:

In any graph G, the null class corresponds to the collection of all simplicial vertices of it.

4.2 Majorized Shortest Paths

As the computation of transit in large graphs is tedious, we introduce a new concept called majorised shortest paths that makes the task easier. Also, towards the end of the section, we provide an algorithm to sort out all the majorised shortest paths in a graph.

Definition 4.2.1. A path M through v is called a majorized shortest path through v, abbreviated as Msp(v), if it satisfies the following conditions.

1. M is a shortest path in G with v as an internal vertex.

2. There exist no path M' such that, M' is a shortest path in G with v as an internal vertex and M as a sub-path of it. We denote the collection of all $Msp(v)$ by $\mathcal{M}_v$ and $\bigcup\limits_{v \in V} \mathcal{M}_v$ by $\mathcal{M}_G$

Example 4.2.1. Consider Figure 4.4. Let $M_1 : 1234, M_2 : 1235, M_3 : 123$. Then M_1 and M_2 are $Msp(2)$, while M_3 is not a majorized shortest path.

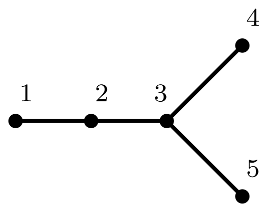


Figure 4.4: Majorized shortest path

$\mathcal{M}_G$ for various graphs

For a path P_n, $\mathcal{M}_G$ ={ P_n }. For a star S_n, $\mathcal{M}_G$ is the collection of all paths of length 2 connecting two pendant vertices. We know that there are $\binom{n-1}{2}$ such paths and their intersection is {c}, where c is the central vertex. $\therefore TI(G) = C(n-1,2) \times TI(P_3) = (n-1)(n-2)$

For a cycle C_n, $n > 3$, every majorized shortest path has length d which is equal to the diameter of C_n. For every vertex $v \in C_n$, $\mid \mathcal{M}_v \mid = d - 1$. Hence $\mid \mathcal{M}_G \mid$ = n.

Theorem 4.2.1. For a graph G, $\mathcal{M}_G$ is unique.

Proof. For a while suppose that, $\mathcal{M}_G$ is not unique. Let $\mathcal{M}_G$ and $\mathcal{M}'_G$ be two different collections of msp in G. Let $M_1 \in \mathcal{M}_G$ and $M_1 \notin \mathcal{M}'_G$. Then M_1 is msp for some v in G. Then M_1 is a shortest path in G and hence should be part of some M_1' in $\mathcal{M}'_G$. Being an msp itself, it cannot be part of any other msp. Hence, $M_1 \in \mathcal{M}_G$. Which proves the statement. □

Theorem 4.2.2. Let $e = uv$ be any edge of G. If e is not a part of any majorized shortest path in G, then $e \in C_3$

Proof. Let us assume that e is not a part of any majorized shortest path in G. Let $v_1 \neq u$ be a neighbour of v in G. Then the shortest path from v_1 to u is of length ≤ 2. If it is 2, the path v_1vu will be a part of the majorized shortest path through v. Hence $d(u, v_1) = 1$, showing $e = uv$ is part of C_3. □

Theorem 4.2.3. For any vertex v of a tree T, $Msp(v)$ connects two of its pendant vertices, $\forall v \in V$. Conversely every path connecting two pendant vertices forms a msp for every internal vertex of it.

Proof. Suppose M be a $Msp(v)$, $v \in T$. Let $M : v_1v_2 \dots v \dots v_k$. Suppose if possible one of the end vertex of M be a non pendant vertex of T. Without loss of generality let us assume v_1 is not a pendant vertex. Then $d(v_1) > 1$. Let u be a neighbor of v_1 other than v_2. Then the $u - v$ path $uv_1v_2 \dots v \dots v_k$ is a shortest path in T with v as an internal vertex and with M as a subpath of it. This is a contradiction.

Conversely, let M be a path connecting two pendant vertices, say u_1 and u_2 of T. Let v be an internal vertex of M. We need to show that $M \in \mathcal{M}_v$. Assume $M \notin \mathcal{M}_v$ .Then either (i)M is not a shortest path in T or (ii)M is a subpath of some M' with v as an internal vertex. Since T is a tree, $u_1 - u_2$ path is unique and hence M is a shortest path. So (i) does not hold. Again u_1, u_2 are pendant vertices proves (ii) wrong. Hence the proof. □

Corollary 4.2.1. For any tree T with p pendant vertices, $|\mathcal{M}_T| = \binom{p}{2}$, where T is a tree and p the number of pendant vertices of T.

Lemma 4.2.1. Consider the graph $G(V, E)$. Let $v \in V$ and $\mathcal{M}_v$ be the collection of all majorized paths in G with v as an internal vertex. If $\mathcal{M}_v$ =$\{M_1, M_2\}$, then $T(v) = T_{M_1}(v) + T_{M_2}(v) - T_{M_1 \cap M_2}(v)$

Proof. Given $\{M_1, M_2\} = Msp(v)$ .Let $\mathcal{S}$ be the collection of all shortest paths in G with v as an internal vertex. Then $T(v)$ = sum of lengths of paths in $\mathcal{S}$ Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the collection of all subpaths of M_1, M_2 with v as an internal vertex, respectively. Then $T_{M_i}(v)$ is the sum of the lengths of the paths in $\mathcal{S}_i$. Consider $M_1 \cap M_2$. Either $M_1 \cap M_2 = \{v\}$ or $M_1 \cap M_2$ is a subpath of M_1 and M_2 with v as an internal vertex. In the first case $T(v) = T_{M_1}(v) + T_{M_2}(v)$. In the second case, let $\mathcal{S}'$ be the collection of subpaths of $M_1 \cap M_2$ with v as an internal vertex. Then $\mathcal{S}' \subset \mathcal{S}_1$ and $\mathcal{S}' \subset \mathcal{S}_2$. Hence the proof. $\square$

Let $G(V, E)$ and $\mathcal{M}_v$ be the collection of all majorized path in G with v as an internal vertex. If $\mathcal{M}_v$ =$\{M_i, i = 1, 2, \ldots k\}$, then the Lemma 4.2.1 could be extended by applying the inclusion-exclusion principle in set theory as follows.

Lemma 4.2.2. Let $G(V, E)$ be a graph and $v \in V$. If $\mathcal{M}_v = \{M_1, M_2, \ldots, M_k\}$ then, $T(v) = T_{M_1}(v) + \ldots + T_{M_k}(v) - T_{M_1 \cap M_2}(v) - \ldots - T_{M_{k-1} \cap M_k}(v) + \ldots + (-1)^{k+1} T_{M_1 \cap M_2 \cap M_3 \ldots \cap M_k}(v)$
$=\sum_{i=1}^{k} T_{M_i}(v) - \sum_{i \neq j} T_{M_i \cap M_j} + \ldots + (-1)^{k+1} T_{M_1 \cap M_2 \cap M_3 \ldots \cap M_k}(v)$

Theorem 4.2.4. Let $G(V, E)$ and $\mathcal{M}_G$ be the collection of all majorized paths in G. If $\mathcal{M}_G$ =$\{M_1, M_2\}$, then $TI(G) = TI(M_1) + TI(M_2) - TI(M_1 \cap M_2)$

Proof. $TI(G) = \displaystyle\sum_{v \in V} T(v) = \sum_{v \in V} [T_{M_1}(v) + T_{M_2}(v) - T_{M_1 \cap M_2}(v)] = \sum_{v \in M_1} T(v) + \sum_{v \in M_2} T(v) - \sum_{v \in M_1 \cap M_2} T(v) = TI(M_1) + TI(M_2) - TI(M_1 \cap M_2)$ $\square$

The result in Theorem 4.2.4 could be extended by applying the inclusion-exclusion principle in set theory as follows.

Theorem 4.2.5. Let $G(V, E)$ be a graph and $\mathcal{M}_G$ be the collection of all majorized path in G. If $\mathcal{M}_G = \{M_i, i = 1, 2, \ldots k\}$, $TI(G) = TI(M_1) + \ldots + TI(M_k) - TI(M_1 \cap M_2) + \ldots - TI(M_{k-1} \cap M_k) + \ldots + (-1)^{k+1} TI(M_1 \cap M_2 \cap M_3 \ldots \cap M_k)$.

Thus knowing the majorized shortest paths of a graph, one could compute the transit index of a graph in an easy way. Following is an algorithm based on the Theorem 4.2.5

Algorithm to find msp in G

From the collection of all shortest paths connecting vertices in G, an algorithm can be derived to find the majorized shortest paths in G.
Dijkstra's algorithm (SPF algorithm) is an algorithm for finding the shortest paths between vertices in a graph. Using SPF one can find the set of all shortest paths between vertices in a graph. Now applying the following algorithm to the set of all shortest paths in G we get the set of all majorized shortest path in a graph G.
Algorithm Require: Graph G and $S = \{M_1, M_2, \ldots, M_s\}$, the collection of all shortest paths in G, obtained from Dijkstra's algorithm(or any other)
Ensure: Collection of all majorised paths in G

1. Perform 2-5 for $i = 1, 2, \ldots, s-1$
2. Fix M_i
3. Find $M_i \cap M_{i+j}$, for $j = 1, 2, \ldots, s-i$
4. If $M_i \cap M_{i+j} = M_i$, $S = S \setminus \{M_i\}$
5. If $M_i \cap M_{i+j} \neq M_i$, $S = S$

6. On completion S will be the collection of all majorised paths in G.

4.3 Transit Decomposition

Definition 4.3.1. A decomposition of a graph G into a collection of subgraphs $\tau = \{T_1, T_2, \ldots, T_r\}$, where each T_i is either a chordless cycle of G with atleast two of its subpaths in $\mathcal{M}_G$ or a majorized shortest path of G such that, $TI(G) = \sum_i TI(T_i) - \sum_{i \neq j} TI(T_i \cap T_j) + \ldots + (-1)^{r+1} TI(T_1 \cap T_2 \cap \ldots \cap T_r)$ is called a Transit Decomposition of G. We denote a transit decomposition of minimum cardinality by τ_{min}.

The minimum cardinality of a transit decomposition of G is called the Transit decomposition number, denoted by $\theta(G)$ or simply θ if there is no confusion. Clearly $\mathcal{M}_G$ is a transit decomposition of G. We denote $|\mathcal{M}_G|$ by $\theta_a(G)$ or simply θ_a.

Example 4.3.1. Consider the graph G in the Figure 4.5.

Let $M_1 : 1234$; $M_2 : 1254$; $M_3 : 345$; $M_4 : 325$; $C_1 : 23452$.

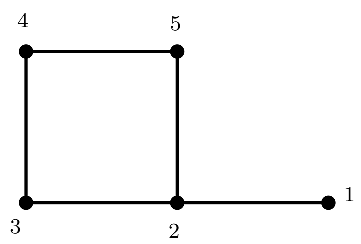


Figure 4.5: Graph G

Here $\mathcal{M}_G = \{M_1, M_2, M_3, M_4\}$, $\tau_{min} = \{M_1, M_2, C_1\}$, the transit decomposition of minimal cardinality. Hence $\theta = 3$, while $\theta_a = 4$.

Remark 4.3.1. If G is chordal, $\theta = \theta_a$.

Remark 4.3.2. If e is an edge of G that does not belong to any clique in G, it will be a part of some $T_i \in \tau$

Theorem 4.3.1. For a graph G, τ_{min} is unique.

Proof. On the contrary, suppose that $\tau_1 \neq \tau_2$ be two transit decompositions of G with minimum cardinality. Let $T \in \tau_1$ and $T \notin \tau_2$. As a first case, let us assume that T is a path. Then there exist some $T_i \in \tau_2$, say T' such that T is a subpath of it. Since T is a majorised shortest path(msp) in G, T' cannot be a msp of G. Hence T' must be a chordless cycle of G, with atleast two of its subpaths in $\mathcal{M}_G$. Then by the definition of τ_{min} we see that $T' \in \tau_1$.

Now if T is a chordless cycle in G, clearly $T \in \tau_2$. This proves our claim. □

Theorem 4.3.2. Let C_m be a chordless cycle in G, with m even. If one of the shortest path in C_m is in $\mathcal{M}_G$, then $C_m \in \tau_{min}$

Proof. Let $P : c_1c_2\ldots c_r$ be the shortest subpath of C_m which belongs to $\mathcal{M}_G$ .Then c_1 and c_r are vertices of degree two and $d(c_1, c_r) = \lfloor \frac{m}{2} \rfloor$. Consider $P' = C_m \setminus P$. Then P' is also a msp in G. Thus P, P' are two subpaths of G in $\mathcal{M}_G$. Hence $C_m \in \tau_{min}$. □

Transit decomposition number for various graphs

In this section we compute the transit decomposition number for some well known graphs and some graph constructions.

If G is a tree , $\tau_{min} = \mathcal{M}_G$ and $\theta = \theta_a = \binom{p}{2}$, where p is the number of pendant vertices. If G is a cycle, $\theta = 1$ and $\theta_a = n, n > 3$. If G is a path, $\theta = \theta_a = 1$. If G is a complete graph, $\theta = \theta_a = 0$.

Bipartite graph

Theorem 4.3.3. Let $G = K_{p,q}$, the bipartite graph. Then $\theta = \frac{p(p-1)q(q-1)}{4}$ and $\theta_a = \frac{pq(p+q-2)}{2}$.

Proof. In the case of a complete bipartite graph, every shortest path is of length ≤ 2. Hence every shortest path is a majorized shortest path. $\therefore \theta_a = \sum_1^q c(p,2) + \sum_1^p \binom{q}{2} = \frac{pq(p+q-2)}{2}$. The chordless cycles in $K_{p,q}$ is of length 4. Also every shortest path is part of some chordless cycle. Hence $\theta =$ the number of cycles in $K_{p,q}$ of girth 4 $= \binom{p}{2} \times \binom{q}{2} = \frac{p(p-1)q(q-1)}{4}$ □

Wheel graph

Theorem 4.3.4. Let $G = W_n$, $n > 4$, be the wheel graph. Then $\theta = \theta_a = \frac{(n-1)(n-4)}{2}$

Proof. In the wheel graph every chordless cycle is C_3. Hence $\theta = \theta_a$. Note that the diameter of the graph is 2. Hence every majorized shortest path is of length ≤ 2. Since P_2 is not a majorized shortest path(msp), all msp in G are isomorphic to P_3. On the cycle of the wheel, starting with every vertex there are 2 msp. Hence on a total (n-1) paths.

Other msp are those, that starts and ends on the cycle of the wheel and passes through the center. With each vertex on the cycle we can associate (n-4) such paths. Hence on a total $\frac{(n-1)(n-4)}{2}$ paths. Thus $\theta = \theta_a = \frac{(n-1)(n-4)}{2}$ □

Bow graph

Theorem 4.3.5. If $G = B(n, m)$, the bow graph, $\theta = \theta_a = \frac{n(n-3)}{2} + \frac{m(m-3)}{2} + (m-1)(n-1)$. For a uniform bow graph, $B(n, n)$, $\theta = \theta_a = 2n^2 - 5n + 1$

Proof. The majorised shortest paths in the bow graph and uniform bow graph are isomorphic to P_3. Also, the cycles are C_3. Hence $\theta = \theta_a =$ the count of paths of length 2. Hence the result. □

A 1-factorization of a complete graph is a partition of the edge set into perfect matchings. A decomposition of this kind exists only when the number of vertices is even, say 2n. In the next theorem, the graph considered is the one got by deleting a one factor from K_{2n}.

Theorem 4.3.6. If $G = K_{2n} - I$, $\theta_a(G) = 2n(n-1)$ and $\theta(G) = \frac{n(n-1)}{2}, n > 2$, where I is the one factor of K_{2n}

Proof. In $K_{2n} - I$, there will be n pair of vertices which are non adjacent. For a vertex v, $d(v) = n - 1$. Note that every vertex of G are transit identical. There will be $n - 1$ number of $msp(v)$ of length 2. Hence $\theta_a(G) = 2n(n-1)$.
Of the n pair of non adjacent vertices taking 2 pair at a time we get a chordless cycle of length 4. Hence $\theta(G) = \frac{n(n-1)}{2}$. □

In the next two theorems we consider unicyclic graphs.

Theorem 4.3.7. Let G be a unicyclic graph, with cycle $C_r, r > 3$. If the number of vertex of C_r with $d(v) > 2$ is one, then

1. $\theta(G) = \binom{p}{2} + 2p + 1$

2. $\theta_a(G) = \frac{1}{2}(p^2 + 3p + 2r - 4)$, where p is the number of pendant vertices of G.

Proof. 1. When forming τ_{min} we first include C_r, as there will be atleast two subpaths of it as msp. Corresponding to every pendant vertex, there will be 2 majorized shortest paths connecting it to vertices of the cycle C_r. Thus including 2p paths to τ_{min}. Again there are $\binom{p}{2}$ majorized shortest path connecting pendant vertices among themselves. Hence $\theta(G) = \binom{p}{2} + 2p + 1$.

2. Here $\tau_{min} = \mathcal{M}_G$. In the previous case if we exclude C_r and include every majorized shortest path of vertices of C_r, which are r in number, we get $\mathcal{M}_G$. Note that of these r msp, two of them forms a part of msp connecting pendant vertices to vertices of C_r. Hence we get $\theta_a(G) = \frac{1}{2}(p^2+3p+2r-4)$.

□

Theorem 4.3.8. Let G be a unicyclic graph with cycle $C_r, r > 3$. Let u and v be two vertices at a distance $\lfloor \frac{r}{2} \rfloor$ to each other with $d(v_1), d(v_2) > 2$, and all other vertices be of degree 2 each. Let T_1 be a tree with p_1 pendant edges and T_2 be a tree with p_2 pendant edges rooted at u and v respectively . Then

$$\theta(G) = \begin{cases} \binom{p_1+p_2}{2} + p_1 + p_2 + 1 & , \text{ when r is odd} \\ 2p_1p_2 + \binom{p_1}{2} + \binom{p_2}{2} + 1 & , \text{ when r is even} \end{cases}$$

Proof. Clearly the cycle $C_r \in \tau_{min}$.

Case 1

Let u_1, u_2 and v_1, v_2 be the vertices of C_r adjacent to u and v respectievly. When r is odd, the msp connecting pendant vertices of T_1 to T_2 is unique. Hence they will be p_1p_2 in number. Either of v_1, v_2 (also u_1, u_2) lie on such msp. Without loss of generality let us assume that u_1 and v_1 lie on these msp. There will be p_1 number of msp connecting pendant vertices of T_1 to v_2 and p_2 number of msp connecting pendant vertices of T_2 to u_2. Hence we get $\theta(G) = \binom{p_1+p_2}{2}+p_1+p_2+1$.

Case 2

When r is even $\lfloor \frac{r}{2} \rfloor = \frac{r}{2}$. Hence u and v are diametrically opposite vertices of C_r. For every pendant vertex of T_1 there are 2 msp connecting it to a vertex of T_2. Altogether there are $2 \times p_1 \times p_2$ number of msp. The number of msp connecting pendant vertices of T_1 among themselves is $\binom{p_1}{2}$ and the case of T_2 is $\binom{p_2}{2}$. Thus $\theta(G) = 2p_1p_2 + \binom{p_1}{2} + \binom{p_2}{2}) + 1$. $\square$

Chapter 5

Subdivision Graphs

Complex networks in computer networking or large molecular graphs of chemical substances can be constructed from simple graphs using different graph operations. One such is the process of subdivision of a graph. Subdivision graphs exhibit various structural properties that make them interesting and useful for various networks. Thus, the study of topological descriptors in subdivision graphs plays an influential role in understanding complex graph structure. In this chapter we study transit index and transit decomposition of subdivision graphs.

Definition 5.0.1. [18] The edge subdivision operation for an edge $uv \in E$ is the deletion of uv from G and the addition of two edges uw and wv along with the new vertex w.

Definition 5.0.2. [18] A graph which has been derived from G by a sequence of edge subdivision operations is called a subdivision of G.

5.1 Subdivision of a Graph

In this section we analyse the transit decomposition of a subdivision graph $S(G)$ and compare it with that of G.

Theorem 5.1.1. Let G be a graph with no odd cycles. Let τ_{min} be a transit decomposition of G of minimum cardinality. τ' denote the collection of all sub division of paths/cycles in τ_{min}. Then τ' is a transit decomposition of $S(G)$, the subdivision graph of G.

Proof. Let $\tau_{min} = \{T_1, T_2, \ldots, T_r\}$ and $\tau' = \{T'_1, T'_2, \ldots, T'_r\}$. Since τ_{min} is of minimum cardinality, every chordless cycle of G belongs to τ_{min}. Also note that every path in $S(G)$ have subdivision vertices in alternate position.

Claim: If $M' : v_1, v_2, \ldots, v_k$ is a shortest path in in $S(G)$, then M' is a subpath of some $T'_i \in \tau'$. Here 3 cases arise. In each case we will show the claim is true.

Case 1

Both v_1 and v_k are in G. The path got by deleting the subdivision vertices from M' will be a path connecting v_1 to v_k in G and will be a shortest path. Hence it will be part of some path/cycle, say T_i in τ_{min}. Clearly M' will be part of $T'_i \in \tau'$.

Case 2

Either of v_1 or v_k is in G. Without loss of generality let us assume v_1 is in G and v_k is a subdivision vertex. Clearly the path $v_1, v_3, \ldots, v_{k-1}$ is a shortest path in G. Suppose $w \neq v_{k-1}$ is a neighbour of v_k. Since G has no odd cycles it is clear that v_{k-3} is not a neighbour of w. We claim that the path $M : v_1, v_3, \ldots, v_{k-1}, w$ is a shortest path in G, which will prove the theorem for Case 2. On the contrary let us assume that M is not the shortest path from v_1 to w. Then it is evident that some(atleast v_{k-3}, v_{k-1} and w) or all of the vertices in M are part of a cycle. Let us assume that the vertices $v, \ldots, v_{k-1}, w$ are part of a cycle. Then the $v - v_{k-1}$ path and $v - w$ path are of same length, a contradiction to the fact that G has no odd cycles. Hence our claim.

Case 3

Both v_1 and v_k are subdivision vertices. Hence $d(v_1) = d(v_k) = 2$. Let $u \neq v_2$ and $w \neq v_{k-1}$ be the neighbours of v_1 and v_k respectively. If $uv_2v_4 \ldots w$ is a shortest path of G, we are done. If the edges uv_2 and $v_{k-1}w$ are not part of a cycle, there is nothing to prove. Since G contains cycles, some or all of the vertices of the path $uv_2v_4 \ldots w$ may lie on same or different cycles. Hence there can be more than one path connecting u to w. Due to our assumption that M' is the shortest path and due to the fact that G contains only even cycles, the path $uv_2v_4 \ldots w$ is a shortest path. Hence the proof. □

Theorem 5.1.2. Let G be not a cycle and let τ be a transit decomposition of G. If τ' denotes the collection of all subdivision of edges of τ, τ' will be a transit decomposition of $S(G)$, the sub division graph of G, only if every edge of G is part of some majorized path in τ.

Proof. Suppose on the contrary, let $e = uv$ be not a part of any majorized path in τ. Clearly e belongs to some cycle, say C. Let the sub division of e be uwv. Let w' be any vertex of G that is not in C. Then the shortest path connecting w to w' in $S(G)$ will not be a sub path of any element in τ'. Which proves τ' is not a transit decomposition of $S(G)$ □

Tadpole graph

The tadpole graph $T_{m,n}$ is a special type of graph consisting of a cycle on $m(\geq 3)$ vertices and a path on n vertices, connected by a bridge, say e.

Corollary 5.1.1. Let G denote the tadpole graph $T_{2m,n}$. Then the transit decomposition for $S(G), \tau_{S(G)} = \{T_1', T_2', T_3'\}$, where $T_1', T_2' \simeq P_{2m+2n+1}$ and $T_3' \simeq C_{4m}$.

Proof. The graph G contains no odd cycles. Hence if τ_{min} is a transit decomposition of G with minimum cardinality, the transit decomposition for $S(G), \tau_{S(G)}$ is got by subdividing every edge of paths/cycles in τ_{min}. Since C_{2m} is a cycle in G, $C_{2m} \in \tau_{min}$. Let $e = uv$ be the bridge in G, with u as a vertex of the cycle C_{2m}. Let u' be vertex diametrically opposite to u. The paths connecting the pendant vertex of G to u' are of length $n + m$ and form majorized paths of G. Hence $\tau_{min} = \{T_1, T_2, T_3\}$ where $T_1 \simeq C_{2m}$ and $T_1, T_2 \simeq P_{n+m+1}$. Thus by Theorem 5.1.1, the result follows. □

Remark 5.1.1. Consider the tadpole graph $T_{2m+1,n}$. This graph has an odd cycle and hence the Theorem 5.1.1 does not hold good here. To find the transit index of its subdivision graph we form the transit decomposition, $\tau_{S(G)}$. It is evident that here $S(G) = T_{4m+2,2n}$. $\therefore \tau_{S(G)} = \{T_1, T_2, T_3\}$, where $T_1 \simeq C_{4m+2}, T_2 \simeq T_3 \simeq P_{2m+2n+2}$. Also note that $T_1 \cap T_2 \simeq T_1 \cap T_3 \simeq P_{2m+2}$ and $T_2 \cap T_3 \simeq P_{2n+1}$

Trees

Theorem 5.1.3. Let G be a tree. Let S denote the graph got by subdividing every edge of G. Then $\mathcal{M}_S$ is got by subdividing paths of $\mathcal{M}_G$.

Proof. Let $M : v_1v_2 \dots v \dots v_{k-1}v_k \in\mathcal{M}_G$. Let $M' : v_1u_1v_2u_2 \dots v \dots v_{k-1}u_{k-1}v_k$ be the subdivision of M.

Claim 1 : M' is a shortest path connecting v_1 to v_k in S.

Suppose if possible let M' is not a shortest path connecting v_1 to v_k in S. Then there exist some path $N' : v_1n_1n_2 \dots n_sv_k$, where $s + 2 < 2k - 1$. Clearly $n_1, n_3, \dots, n_s$ are subdivision vertices. Hence the path $N : v_1n_2n_4 \dots n_{s-1}v_k$ is a path in G connecting v_1 to v_k of length $\frac{s-1}{2} - 1$. But $\frac{s-1}{2} - 1 < k - 1$, a contradiction to the fact that M is a shortest path connecting v_1 to v_k. Hence the claim.

Claim 2 : There exist no path M'' in S such that M'' is a shortest path with v as an internal vertex and M' as a subpath of it.

Suppose on the contrary, let M'' be a shortest path in S with v as an internal vertex and M' as a subpath of it. Then M'' connects two pendant vertices of S. Let $M'' : z_1u_1 \dots M' \dots u_{s-1}z_s$. Then the path $M'' - \{u_i\}$ is a path in G with M as a subpath and v as an internal vertex. A contraction. Hence the claim. These two claims prove the theorem. □

Illustration

Consider the Figure 5.1. Let $M_1 : 1234, M_2 : 1235, M_3 : 435$ $\mathcal{M}_G = \{M_1, M_2, M_3\}$,

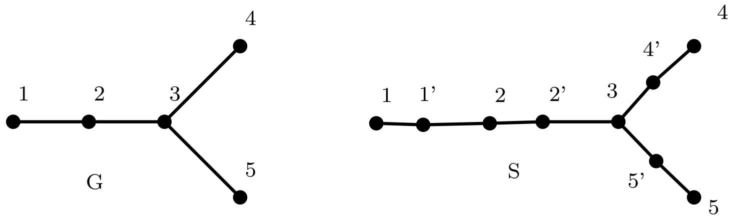


Figure 5.1: G and its subdivision S

where$M_1, M_2 \simeq P_4$ and $M_3 \simeq P_3$. Hence $\mathcal{M}_S = \{M_1', M_2', M_3'\}$, where $M_1', M_2' \simeq P_7$ and $M_3' \simeq P_5$. Here $M_1' \cap M_2' \simeq P_5, M_1' \cap M_3' \simeq P_3$ and $M_2' \cap M_3' \simeq P_3$. Thus $TI(S) = 2TI(P_7) + TI(P_5) - TI(P_5) - 2TI(P_3). = 2 \times 140 + 30 - 30 - 2 \times 2 = 276$

5.2 Transit Index of Subdivision Graphs

In this section we express the transit decomposition of $S(G)$ in terms of $TI(G)$ and $n = |V(G)|$.

Theorem 5.2.1. Let G be the graph got by subdividing every edge of the path P_n. Then $TI(G) = TI(P_n) + \frac{n(n-1)(15n^2-31n+14)}{12}$

Proof. Since G is got by subdividing every edge of P_n, $G \simeq P_{2n-1}$. Hence by Theorem 2.2.3, we get $TI(G) - TI(P_n) = \frac{n(n-1)(15n^2-31n+14)}{12}$ $\square$

A result similar to that of path P_n hold good in the case of cycles.

Theorem 5.2.2. Let G be the graph got by subdividing every edge of a cycle.

Then

$$TI(G) = \begin{cases} TI(C_n) + \frac{n(n-1)(5n^2+6n+1)}{8} & , \quad \text{n odd} \\ TI(C_n) + \frac{n^2(5n^2-4)}{8} & , \quad \text{n even} \end{cases}$$

Proof. For a cycle C_n its subdivision graph is the cycle C_{2n}. Now using Theorem 2.3.1, the result follows. □

Theorem 5.2.3. Let G be the graph got by the subdivision of a single edge $e = uv_1$ of the star graph S_n. Then $TI(G) = n^2 + 3n - 8 = TI(S_n) + 6n - 10$

Proof. In the graph G every vertex other then the central vertex u and the newly added vertex v have transit zero. The shortest paths through u are the one's

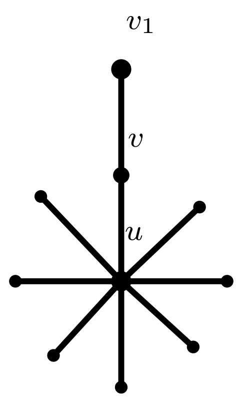


Figure 5.2: S(G)

connecting v to other $(n-2)$ vertices of star and the ones connecting v_1 to the $(n-2)$ vertices of star. i.e. $T(u) = (n-1)(n-2) + 3(n-2) = TI(S_n) + 3(n-2)$ The shortest path through v are those connecting v_1 to other $(n-2)$ vertices of star and connecting v_1 to u. i.e. $T(v) = 3(n-2)+2$. Hence $TI(G) = n^2+3n-8 = TI(S_n)+6n-10$ □

Theorem 5.2.4. Let G be the graph got by the subdivision of every edge of the star graph S_n. Then $TI(G) = (n-1)(13n-24) = TI(S_n) + 2(n-1)(6n-11)$

Proof. In G let the pendant vertices be $v_1, v_2, \ldots, v_{n-1}$, newly added vertices be $u_1, u_2, \ldots, u_{n-1}$ and the center vertex be u. $T(v_i) = 0, \forall i$. The shortest paths

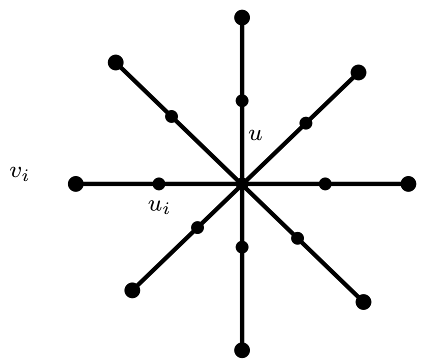


Figure 5.3: $S(S_n)$

through u are those

(1) connecting v_i to v_j of length 4, (2) connecting v_i to u_j of length 3,

(3) connecting u_i to u_j of length 2.

$\therefore T(u) = 4\frac{(n-1)(n-2)}{2} + 3(n-1)(n-2) + 2\frac{(n-1)(n-2)}{2} = (n-1)(6n-12)$

The shortest paths through u_i are

(1) connecting v_i to v_j of length 4, (2) connecting v_i to u_j of length 3,

(3) connecting v_i to u of length 2.

$\therefore T(u_i) = 4(n-2)+3(n-2)+2 = 7n-12$. This gives $TI(G) = (n-1)(6n-$

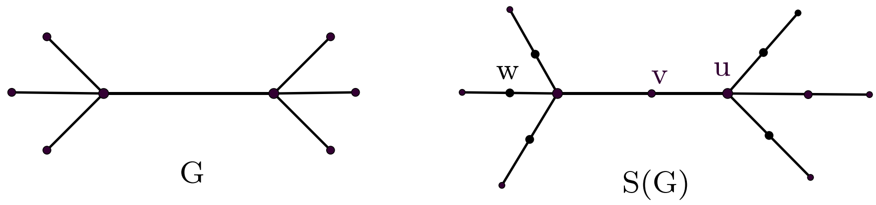


Figure 5.4: Bistar and its subdivision graph

$12)+(n-1)(7n-12) = (n-1)(13n-24). = (n-1)(n-2)+2(n-1)(6n-11). = TI(S_n) + 2(n-1)(6n-11).$ $\square$

Theorem 5.2.5. Let G be the bistar got by joining the apex vertex of two stars $K_{1,n}$ by an edge. If $S(G)$ denotes its subdivision graph, $TI(S(G)) = TI(G) + 124n^2 + 4n + 2$

Proof. Consider the Figure 5.4. All vertices other than the vertices of the type u, v, w have transit zero. It can be easily verified that $T(u) = 32n^2$, $T(v) = 32n^2 + 2n + 2$ and $T(w) = 18n + 2$. There are 2 vertices of type u and 2n vertices of the type w. This shows that $TI(S(G)) = 132n^2 + 6n + 2$. Also $TI(G)$ can be computed from the figure as $8n^2 + 2n$. Hence the result. $\square$

Theorem 5.2.6. Let G be the graph got by the subdividing a single edge $e = uv$ of the complete graph K_n. Then $TI(G) = 6n - 10$

Proof. Let the new vertex be w. After the subdivision, the distance between u and v becomes 2. Also the diameter of the graph is now 2. All the $n-2$ vertices of K_n other than u and v are adjacent to each other and at a distance 2 from w. Hence $T(u) = T(v) = 2(n-2)$. The only shortest path through w is the one connecting u to v. Hence $T(w) = 2$. For the remaining $(n-2)$

vertices the shortest path through it is the one connecting u to v, of length 2.
$\therefore TI(G) = 4(n-2) + 2 + 2(n-2) = 6n - 10$ □

Theorem 5.2.7. Let G be the graph got by sub dividing every edge of the complete graph K_n. Then $TI(G) = n(11n^2 - 40n + 37)$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of K_n and $u_1, u_2, \ldots, u_m$ be the sub division vertices, where $m = \binom{n}{2}$. In G, $d(v_i) = n-1$ and $d(u_i) = 2$. Note that $v_i's$ are transit identical and so are $u_i's$. We will calculate $T(v_i)$ and $T(u_i)$ separately and hence compute $TI(G)$.

1. Computing $T(v_i)$

Refer Figure 5.5. Let us fix v_1. It is adjacent to $n-1$ vertices of the type u_i. The shortest path connecting these $n-1$ vertices are of length 2 and passes through v_1. Hence contribute $\binom{C(n-1,2)}{n-1} 2 \times 2 = (n-1)(n-2)$ to the transit of v_1. The (n-1) vertices of the type u_i adjacent to v_1 travels through v_1 to reach other (n-2) vertices of type v_i, each of length 3. Hence add $(n-2)(n-1) \times 3$ to $T(v_1)$. For u_1 there are $m - (2n-3)$ vertices of the type u_i at a distance 4 from it. For each such vertex u_i there are two paths passing through v_1 of length 4. Hence contribute $4 \times 2 \times (m - 2n + 3)$ to the transit of v. $\therefore T(v) = (n-1)(n-2) + (n-2)(n-1) \times 3 + 4 \times 2 \times (m - 2n + 3). = 8n^2 - 32(n-1)$.

2. Computing $T(u_i)$.

Refer Figure 5.5. Now consider u_1. The path connecting v_1 to v_2 of length 2 passes through u_1. All the $(n-2)$ vertices of the type u_i adjacent to v_2 passes through u_1 to reach v_1. All these paths are of length 3. Hence add $2 \times 3 \times (n-2)$ to transit of u_1. $\therefore T(u_1) = 2 + 6(n-2) = 6n - 10$. $\therefore TI(G) = \sum_i T(v_i) + \sum_i T(u_i)$.
$= m(6n-10) + n(8n^2 - 32(n-1)) = n(11n^2 - 40n + 37)$. □

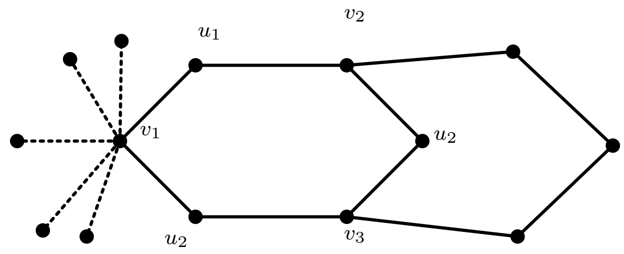


Figure 5.5: v_1 and adjacent vertices

Theorem 5.2.8. Let G be the graph got from W_{n+1}, the wheel graph, by sub dividing the cycle. Then $TI(G) = n[37n - 65] = TI(W_{n+1}) + n(36n - 64)$

Proof. Let $u_1, u_2, \ldots, u_n$ be the subdivision vertices of the edges $v_1v_2, v_2v_3, \ldots, v_nv_1$. We denote the central vertex of W_{n+1} by c. It can eaisly be verified that G has 3 transit equivalence classes. They are $\{u_1, u_2, \ldots, u_n\}$,{c} and $\{v_1, v_2, \ldots, v_n\}$. Hence we compute $T(u_1), T(c)$ and $T(v_1)$. The only paths through u_1 are those connecting v_1 to v_2, v_1 to u_2, u_1 to u_2 and u_n to u_2. Thus $T(u_1) = 12$ and $\sum\limits_i T(u_i) = 12n$. In the case of c, The paths are mainly of 3 types, connecting v_i to v_j, u_i to u_j and v_i to u_j. On considering all such paths we get $T(c) = n[11n - 25]$. Next we consider the paths through v_1. They are paths originating either from u_1 or from u_2. Considering them we get $\sum\limits_i T(v_i) = n(22n - 40)$.

We thus arrive at $TI(G) = n[37n - 65] = TI(W_{n+1}) + n(36n - 64)$.

□

Chapter 6

Transit isomorphism

The study of graphs with similar properties has attracted the attention of graph theorists forever. Graph isomorphism is one such concept. It is a phenomenon in which the same graph appears in more than one form. A graph of this type is called an isomorphic graph.

A discussion of transit decomposition is presented in Chapter 6 and shown to be unique for graphs. Based on this concept, we define transit isomorphism as a relation among all finite connected simple graphs. We know that Transit Index of a graph is a "graph invariant" and hence isomorphic graphs have same transit index. In this chapter we look upon graphs that are transit isomorphic even if they are not isomorphic. A few ways of constructing transit isomorphic graphs are also dealt with.

Definition 6.0.1. Let G_1 and G_2 be two graphs. We say that G_1 is transit isomorphic to G_2, if there exists a bijection, say Ψ from $\tau_{min}(G_1)$ to $\tau_{min}(G_2)$ such that $\Psi(H_i) = H_i'$ implies $H_i \simeq H_i'$. We write $G_1 \simeq_T G_2$

Two graphs G_1 and G_2 are transit isomorphic need not imply that G_1 and G_2 are isomorphic.

Consider the graphs in Figure 6.1. Clearly they are non isomorphic.

Here $\tau_{min}(G_1) = \{T_1, T_2, T_3\}$, where $T_1 = 1234, T_2 = 1265, T_3 = 234562$

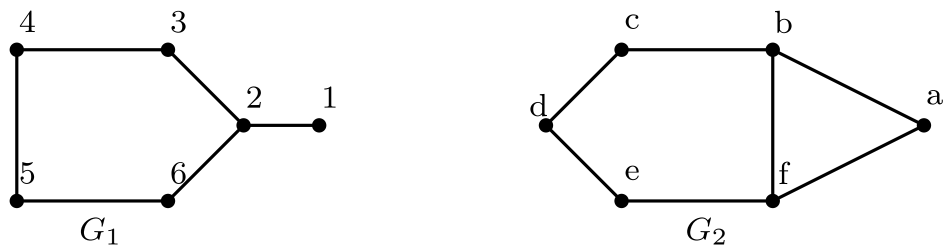


Figure 6.1: Transit isomorphic graphs

$\tau_{min}(G_2) = \{H_1, H_2, H_3\}$, where $H_1 = abcd, H_2 = afed, H_3 = bcdefb$. Define $\Psi : \tau_{min}(G_1) \to \tau_{min}(G_2)$ by $\Psi(T_i) = H_i, i = 1, 2, 3$. Then Ψ is a transit isomorphism.

Definition 6.0.1. Two graphs G_1 and G_2 are said to be strongly transit isomorphic if $G_1 \simeq_T G_2$ and $TI(G_1) = TI(G_2)$

The two graphs G_1 and G_2 in Figure 6.1 are strongly transit isomorphic. It can be verified that $TI(G_1) = TI(G_2) = 26$

The graphs G_1 and G_2 are transit isomorphic need not imply that $TI(G_1)$ and $TI(G_2)$ are equal. Consider the graphs in Figure 6.2. They are transit isomorphic, but their transit indices differ. Here $\tau_{min}(G_1) = \{T_1, T_2, T_3\}$, where $T_1 = 123457, T_2 = 123657, T_3 = 34563$. $\tau_{min}(G_2) = \{H_1, H_2, H_3\}$, where $H_1 = abcegh, H_2 = abdfgh, H_3 = cdfec$

Define $\Psi : \tau_{min}(G_1) \to \tau_{min}(G_2)$ by $\Psi(T_i) = H_i, i = 1, 2, 3$. Then Ψ is a transit isomorphism. But, $TI(G_1) = 142$ and $TI(G_2) = 148$

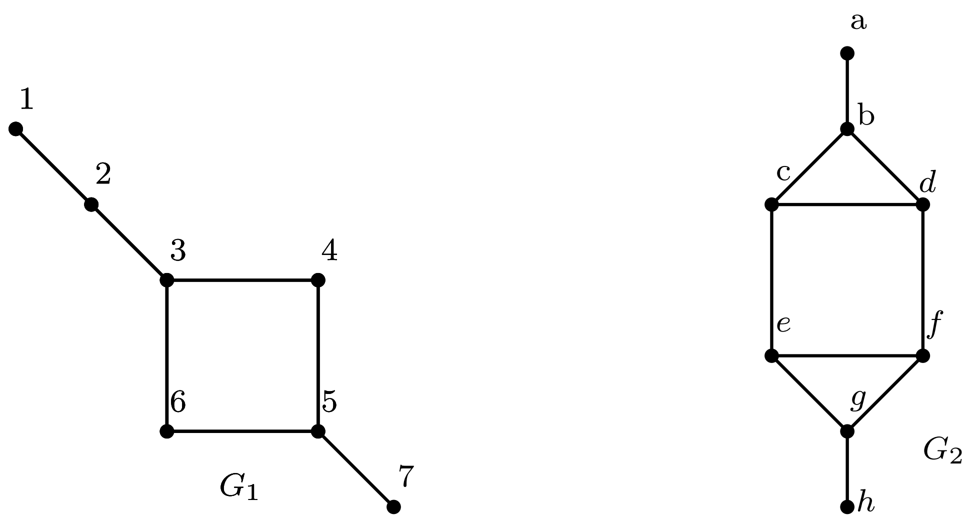


Figure 6.2: G_1 and G_2 with different transit indices

Remark 6.0.1. G_1 and G_2 are transit isomorphism does not imply that they have equal transit index or they are isomorphic. ie, Isomorphism $\Longrightarrow$ strong transit isomorphism $\Longrightarrow$ transit isomorphism, but the reverse relation need not always be true.

The relation $\simeq_T$ on the set of all simple connected finite graph is an equivalence relation. The disjoint classes, called the transit isomorphic classes due to $\simeq_T$ are denoted by $[G]_\psi$. Note that $[C_n]_\psi = \{C_n\}$ and $[P_n]_\psi = \{P_n\}$.

While discussing transit index of graph classes in Chapter 2, we have noticed that the diameter of Star and Wheel graph being 2, the majorized shortest paths are isomorphic to P_3. Also it can easily be verified that W_{n+1} and S_{n+1} have the same number, ie $n(n-1)/2$ of majorized shortest paths in their transit decomposition. Hence we have the following result.

Theorem 6.0.1. The wheel graph W_{n+1} is transit isomorphic to the star S_{n+1}

6.1 Graphs that are Transit Isomorphic to its Line Graph

A line graph $L(G)$, of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common. In this section we investigate the occurrence of transit isomorphism between a graph and its line graph.

Theorem 6.1.1. Let G be a unicyclic graph formed by identifying the center vertex of a star graph S_p with one of the vertices of the the cycle C_r, r odd. Then G and $L(G)$ are transit isomorphic.

Proof. We know that the line graph of a star is a complete graph and that of a cycle is isomorphic to the same cycle. In the case of the graph G under our consideration, $L(G)$ can be viewed as the graph got by identifying an edge of the complete graph K_{p+1} with any one of the edges of C_r. Let τ_1 and τ_2 denote the transit decompositions of G and $L(G)$ of minimum cardinality.

Let v be the vertex common to C_r and S_p in G. If e_v and e'_v are the edges of C_r incident to v in G, then they will form the end vertices of the edge common to K_{p+1} and C_r in $L(G)$. Let u_1 and u_2 be the vertices of C_r at a distance $\lfloor \frac{r}{2} \rfloor$ from v in G. Then $e = u_1u_2$ will be the vertex at a distance $\lfloor \frac{r}{2} \rfloor$ from e_v and e'_v in $L(G)$. Let $e_1 = vw_1, e_2 = vw_2, \ldots, e_{p-1} = vw_{p-1}$ be the pendant edges of G. Then $e_1, e_2, \ldots, e_{p-1}, e_v$ and e'_v will be the vertices of K_{p+1} in $L(G)$. The majorized shortest paths in τ_1 are those connecting w_i to u_1, say $T_{i,1}$ and those connecting w_i to u_2, say $T_{i,2}$ of length $\lfloor \frac{r}{2} \rfloor + 1$ each and $2p - 2$ in number. When we consider τ_2, the msp are those connecting the vertices $e_1, e_2, \ldots, e_{p-1}$ to e along e_v, say $H_{i,1}$ and along e'_v, say $H_{i,2}$. Again they are also of length $\lfloor \frac{r}{2} \rfloor + 1$ each and $2p - 2$ in number.

Clearly, there is only one chordless cycle in G and $L(G)$ and it is isomorphic to C_r (when $r > 3$). Also note that one of the edges of C_r is not a part of any msp of G. Thus C_r belongs to τ_{min} for both G and $L(G)$. Let it be T_1 and H_1, respectively. (When r=3, no cycles are there in τ_1 and τ_2.).Also $|\tau_1| = |\tau_2|$
Define $\Psi : \tau_1 \to \tau_2$ by $\Psi(T_1) = H_1, \Psi(T_{i,1}) = H_{i,1}$ and $\Psi(T_{i,2}) = H_{i,2}$ for $i = 1, 2, \ldots, p - 1$. Then Ψ is a transit isomorphism. $\square$

Illustration of Theorem 6.1.1

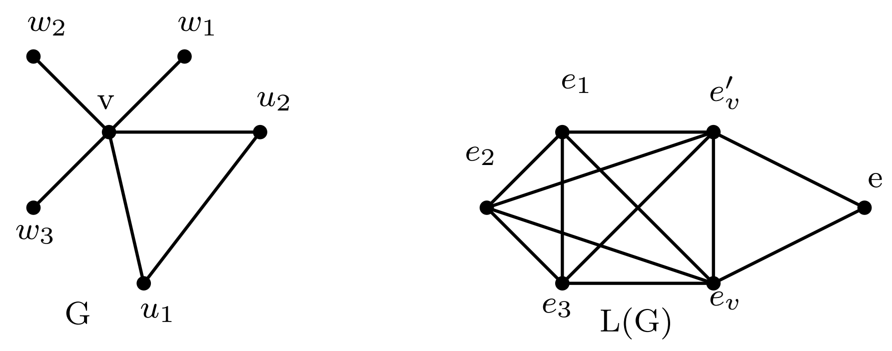


Figure 6.3: G and $L(G)$ when r=3 and p=4

Consider Figure 6.3. Here p=4 and r=3. In this case $|\tau_1| = |\tau_2| = 6$ and $T_{i,1}, T_{i,2}, H_{i,1}, H_{i,2}$ are all isomorphic to P_3

Remark 6.1.1. Note that in the Theorem 6.1.1 when r is even, G and $L(G)$ are not transit isomorphic. The cardinalities of the transit decompositions are equal but, the lengths of majorized paths in τ_1 will be one more than those in τ_2.

Theorem 6.1.2. Let G be the unicyclic graph formed by identifying pendant vertex of the path P_n with a vertex of the cycle C_m, m odd. Then G and $L(G)$ are transit isomorphic.

Proof. Consider the given graph G. Let us denote the common vertex by v. Name the edges of C_m that are incident to v by e_v and e'_v. Call the edge in P_n incident to v as e. Note that there are two vertices on C_m at a distance $\lfloor \frac{m}{2} \rfloor$ from v. Call them u_1 and u_2. Let $e_u = u_1u_2$. In $L(G)$, e_v , e'_v and e forms a clique or C_3. We can view $L(G)$ as the graph got by identifying two of the vertices of this C_3 with two adjacent vertices of C_m and the third vertex with one of the pendant vertex of the path P_{n-1}. Then e_u will be at a distance $\lfloor \frac{r}{2} \rfloor$ from e_v and e'_v.

Let τ_1 and τ_2 denote the transit decompositions of G and $L(G)$ of minimum cardinality. The majorized paths in τ_1 are the two paths T_1, T_2 connecting u_1 and u_2 respectively to the pendant vertex of G. They are of length $\lfloor \frac{m}{2} \rfloor + n - 1$. When we consider the majorized paths in τ_2, they are the ones connecting the pendant vertex (if $n \neq 2$) to e_u via e_v and e'_v respectively. They are of length $\lfloor \frac{m}{2} \rfloor + 1 + n - 2$. Call them H_1 and H_2.

In G and $L(G)$, C_m is a chordless cycle, with one of its edge not in any msp. Hence $C_m \in \tau_1, \tau_2$. We name them T and H. Define $\Psi : \tau_1 \to \tau_2$ by $\Psi(T_1) = H_1$, $\Psi(T_2) = H_2$ and $\Psi(T) = H$. Then Ψ is a transit isomorphism. $\square$

Remark 6.1.2. In the Theorem 6.1.2 when r is even, G and $L(G)$ are not transit isomorphic. The cardinalities of the transit decompositions are equal but, the lengths of majorized paths in τ_1 will be one more than those in τ_2.

Theorem 6.1.3. Let G be the bicyclic graph got by identifying a vertex of C_m with C_n, where m and n are of different parities. Then $G \simeq_T L(G)$

Proof. Without loss of generality, let us assume m is odd and n is even. Denote the vertex common to C_m and C_n by v. Let the vertex on C_n farthest from v be u and the edges incident to it be e_u and e'_u. Call the vertices on C_m farthest from v as v_1 and v_2. Let $e = v_1v_2$. $L(G)$ can be viewed as the graph got by identifying an edge e_1 of K_4 with one of the edges of C_m and another edge e_2, non adjacent to e_1 with an edge of C_n, as shown in the Figure 6.4. Let τ_1 and τ_2

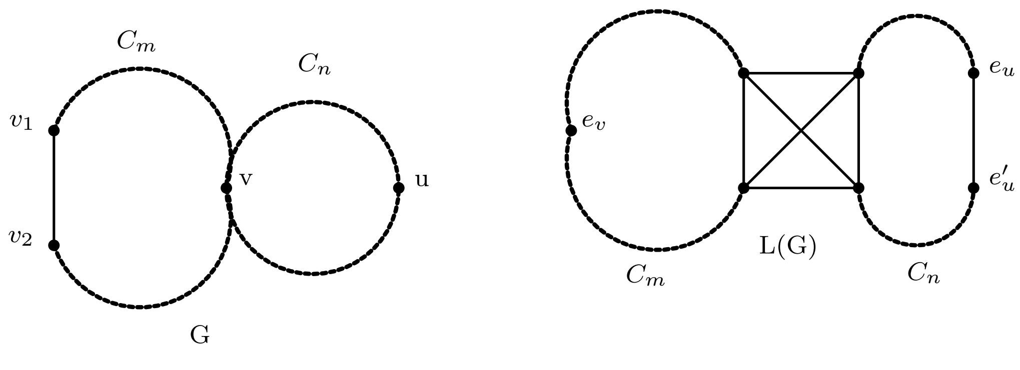


Figure 6.4: G and $L(G)$

denote the transit decompositions of G and $L(G)$ of minimum cardinality.

The msp in τ_1 are those connecting u to v_1 and u to v_2. They are 4 in number and all are of length$\lfloor \frac{m}{2} \rfloor + \frac{n}{2}$. Similarly, the msp in τ_2 are those connecting e_u and e'_u to e_v in $L(G)$. Again they are 4 in number and all are of length$\lfloor \frac{m}{2} \rfloor + 1 + \frac{n}{2} - 1$.

If $m \neq 3$, C_m and C_n will be in τ_1 and τ_2. (Otherwise only $C_n \in \tau_1, \tau_2$) Thus $|\tau_1| = |\tau_2|$. These facts can be used to define a bijection Ψ from $\tau_1 \to \tau_2$ as a transit isomorphism. $\square$

Remark 6.1.3. In the Theorem 6.1.3, all other options for n and m will give graphs for which G and $L(G)$ are not transit isomorphic. It can be verified that $|\tau_1| = |\tau_2|$, but the majorized paths in τ_1 and τ_2 do not agree in lengths.

Theorem 6.1.4. Let G be the bicyclic graph got by joining C_m with C_n by a bridge e, where m and n are of different parities. Then $G \simeq_T L(G)$

Proof. Without loss of generality, let us assume m is odd and n is even. Let $e = uv$ be the bridge with u on C_m and v on C_n. Let u_1 and u_2 be the vertices on C_m which are farthest from u, with $e_u = u_1u_2$. Let v_1 be the vertex on C_n farthest from v, with edges e_v and e'_v incident to v_1. Let us now consider $L(G)$. It will be a graph with 4 cycles, namely C_m, C_3, C_3, C_n. The two C_3's will share a common vertex, say w. The edge non incident to w in each of the C_3 will be part of C_m and C_n respectively, as depicted in Figure 6.5.

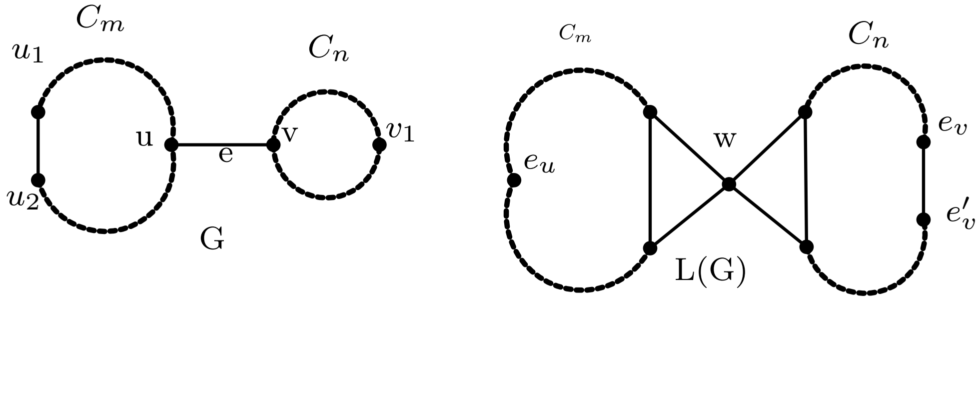


Figure 6.5: G and L(G)

Let τ_1 and τ_2 denote the transit decompositions of G and $L(G)$ of minimum cardinality. If $m \neq 3$, C_m and C_n will be in τ_1 and τ_2. (Otherwise only $C_n \in \tau_1, \tau_2$). The msp in τ_1 are those connecting v_1 to u_1 and v_1 to u_2. There are two shortest paths in each case, each of length $\frac{n}{2} + 1 + \lfloor \frac{m}{2} \rfloor$. When we look at $L(G)$ the majorized shortest paths are those connecting e_u to e_v and e_u to e_v'. Altogether they are 4 in number and of lengths $\lfloor \frac{m}{2} \rfloor + 2 + (\frac{n}{2} - 1)$. Thus we can define a bijection from $\tau_1 \to \tau_2$, which can have the properties of a transit isomorphism. Which will prove our claim. $\square$

Remark 6.1.4. In the Theorem 6.1.4, when n and m are of like parities, it can eaisly be shown that G and $L(G)$ are not transit isomorphic.

6.2 Construction of Transit Isomorphic Graphs

In this section we demonstrate a few methods of constructing graphs that are not isomorphic, but are transit isomorphic.

Construction 1

Graphs G and $L(G)$ in Figure 6.6 are not transit isomorphic. But by adding a pendant edge to the apex of one of the C_3 in $L(G)$ we can create a new graph G', which is transit isomorphic to G. The idea used here can be carried to a class of graphs and produce transit isomorphic graphs as follows.

In G, replace C_4 by any even cycle and the edges e_1 and e_2 by paths of any length. For this G, construct $L(G)$ and attach a pendant edge to the apex vertex

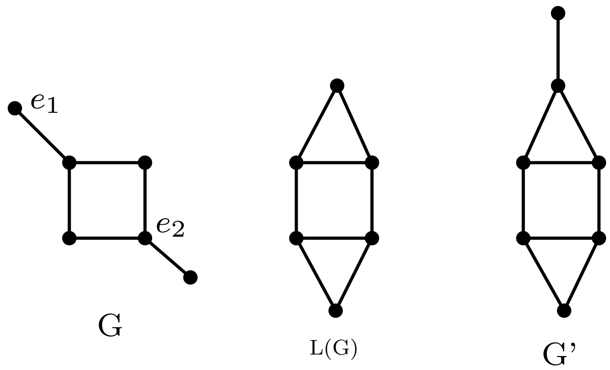


Figure 6.6: $G \simeq_T G'$

of C_3 (if paths are of length 2) or the pendant vertex (if paths are of length ≥ 3), as may be the case, in it to form G'. Then $G \simeq_T G'$

Construction 2

Consider G in Figure 6.7. It is formed by joining two opposite vertices of C_4 and by attaching a pendant edge to a vertex of degree 3. G' is the graph got by attaching two pendant edges to any one of the vertices of C_3. Clearly $G \simeq_T G'$.

The same construction can be done using any cycles C_n. Join a vertex of C_n to every non adjacent vertices of it, so that we have C_3 formed in every step. Now attach a pendant edge to the same vertex, thus increasing its degree to n. This forms G. (Actually G can be viewed as the shell graph $C(n, n-3)$ with a vertex attached to its apex vertex). To construct the transit isomorphic graph of G, add as many pendant edges to one of the vertex of C_3, so that its degree is n.

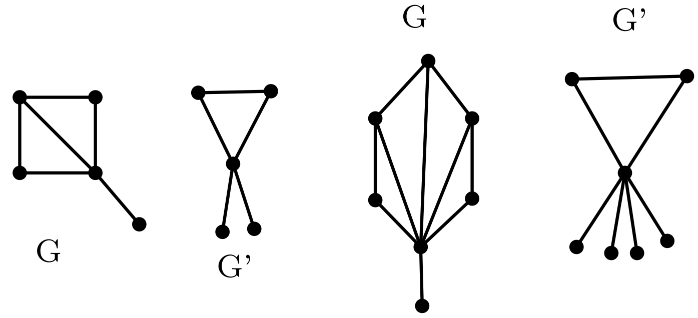


Figure 6.7: The case when $n = 2, 3$

Construction 3

Consider K_n, the complete graph on n vertices. We know that $TI(K_n) = 0$. Removal of a single edge from K_n will increase its transit index by $2(m-2)$. Every shortest path of it is of length 2. Hence τ_{min} will have $(m-2)$ isomorphic copies of P_3. If G is a graph whose τ_{min} contains only isomorphic copies of P_3, then $G' = K_m \setminus \{e\}$ will be transit isomorphic to G for suitable choice of m. The choice for m for some of the well known graphs are demonstrated below.

Star graph : $m = \frac{n^2-n+4}{2}$, **Friendship Graph :** $m = 2(n^2-n+1)$, **Shell graph :** $m = \frac{n^2-3n+4}{2}$, $\mathbf{K_{2n}} - \mathbf{I}$:(I is the one factor of K_{2n}), $m = n^2 - n + 2$

The same idea can be applied to join of any finite number of graphs. In such a binary product every shortest path is of length 2.

Chapter 7

Convex Amalgamation of Graphs

In this chapter, we look upon the transit on graph amalgamations. In graph theory, a graph amalgamation is a relationship between two graphs (one graph is an amalgamation of another). Amalgamations can provide a way to reduce a graph to a simpler graph while keeping certain structure intact. Hence this operation on graphs can be used to study properties of the original graph in an easier to understand context.

Definition 7.0.1. [5] In metric graph theory, a convex subgraph of an undirected graph G is a subgraph that includes every shortest path in G between two of its vertices.

Definition 7.0.2. Suppose T is a subgraph of both G and H. Fix a copy of T contained in G and another in H. The amalgamation of G and H along T is the graph $G \vee_T H$ obtained by identifying the fixed copies of T.

For any finite collection of graphs G_i, each with a fixed isomorphic subgraph T as common, the subgraph amalgamation is the graph obtained by taking the union of all the G_i and identifying the fixed subgraphs T.

7.1 Some definitions regarding Transit of a Vertex

Definition 7.1.1. Transit of a vertex in a subgraph: Let G be a graph and H a subgraph of G. Let $v \in H$. Then the transit of v in H denoted by $T_H(v) = \sum_{s,t \in H} \sigma_H(s, t/v)d(s, t)$, where $\sigma_H(s, t/v)$ is the number of shortest paths between s and t with v as an internal vertex and that lies entirely in H.

Definition 7.1.2. Transit of a vertex induced by a subgraph: Let G be a graph and H a subgraph of G. Let $v \in V(G)$. The transit of v induced by the subgraph H of G is $T(v, H) = \sum_{s,t \in V(H)} \sigma(s, t/v)d(s, t)$

Definition 7.1.3. Transit of a vertex induced by a subset: Let G be a graph and S a subset of G. Let $v \in V(G)$. The transit of v induced by the subset $S \subseteq V(G)$ is $T(v, S) = \sum_{s,t \in S} \sigma(s, t/v)d(s, t)$

Definition 7.1.4. Transit of a vertex induced by disjoint subsets: Let G be a graph and $S, T \subseteq V(G)$ with $S \cap T = \emptyset$. Then the transit of v induced by S against T is $T_G(v, S/T) = \sum_{s \in S, t \in T} \sigma(s, t/v)d(s, t)$

Definition 7.1.5. Transit of a vertex induced by another vertex: Let G be a graph and $s, t, v \in V(G)$. Then the transit of v induced by s is $T_G(v, s) = \sum_{s \neq t \in V} \sigma(s, t/v)d(s, t)$

Remark 7.1.1. 1. If x is a pendant vertex, $T_G(x, s) = 0$

2. Let $x \in V(G)$. Then $T(x) = \frac{1}{2} \sum_{x_i \in V} T(x, x_i)$

Results

1. Let P_n be a path on n vertices $x_1, x_2, \ldots, x_n$, then

$$T(x_i, x_j) = \frac{(n-i)(i-2j+n+1)}{2}, i > j$$

Let $i > j$, then $T(x_i, x_j) =$

$$\begin{aligned} &= (i-j+1) + (i-j+2) + \ldots + (n-j) \\ &= (i-j)(n-i) + \frac{(n-i)(n-i+1)}{2} \\ &= \frac{(n-i)(i-2j+n+1)}{2}, i > j \end{aligned}$$

2. If C_n is a cycle with $n+1$ vertices $x_0, x_1, \ldots, x_n$, n odd

$$T(x_0, x_i) = \begin{cases} \frac{(n+1-2i)(n+3+2i)}{8} & , \quad 0 < i < \frac{n+1}{2} \\ 0 & , \quad i = \frac{n+1}{2} \\ \frac{(2i-n-1)(3n-2i+5)}{8} & , \quad \frac{n+1}{2} < i \leq n \end{cases}$$

Let $0 < i < \frac{n+1}{2}$, then.

$$\begin{aligned} T(x_0, x_i) &= (i+1) + (i+2) + \ldots + \frac{n-1}{2} \\ &= \frac{(n+1-2i)(n+3+2i)}{8} \end{aligned}$$

When $i = \frac{n+1}{2}$, there are no geodesics with one end as x_i and with x_0 as internal vertex. Hence $T(x_0, x_i) = 0$. Let $\frac{n+1}{2} < i \leq n$, then

$$\begin{aligned} T(x_0, x_i) &= (n+1-i+1) + (n+1-i+2) + \ldots + \frac{n+1}{2} \\ &= \frac{(2i-n-1)(3n+5-2i)}{8} \end{aligned}$$

3. If C_n is a cycle with $n+1$ vertices $x_0, x_1, \ldots, x_n$, n even

$$T(x_0, x_i) = \begin{cases} \frac{(n-2i)(2i+2+n)}{8} & , \quad 0 < i < \frac{n}{2} \\ 0 & , \quad i = \frac{n}{2}, \frac{n}{2}+1 \\ \frac{(2i-n-2)(3n-2i+4)}{8} & , \quad \frac{n}{2}+1 < i \leq n \end{cases}$$

Let $0 < i < \frac{n}{2}$. Then,

$$\begin{aligned} T(x_0, x_i) &= (i+1) + (i+2) + \ldots + \frac{n}{2} \\ &= \frac{(n-2i)(2i+2+n)}{8} \end{aligned}$$

When $i = \frac{n}{2}, \frac{n}{2}+1$, there are no geodesics passing through x_0. Hence the transit induced by vertices $x_{\frac{n}{2}}, x_{\frac{n}{2}+1}$ on x_0 is zero. Now let $\frac{n}{2}+1 < i < n$. Then,

$$\begin{aligned} T(x_0, x_i) &= (n+1-i+1) + (n+1-i+2) + \ldots + \frac{n}{2} \\ &= \frac{(2i-n-2)(3n-2i+4)}{8} \end{aligned}$$

4. For a star S_n with central vertex x_0,

$$T(x_i, x_0) = 0; \quad T(x_0, x_i) = 2(n-2); \quad T(x_i, x_j) = 0, i, j \neq 0$$

5. For a wheel graph $W_n, n > 5$, with central vertex x_0,

$$T(x_i, x_0) = 0; \quad T(x_0, x_i) = 2(n-4); \quad T(x_i, x_j) = \begin{cases} 2, & d(x_i, x_j) = 1 \\ 0, & otherwise \end{cases}$$

Transit induced matrix

The transit of a vertex induced by another vertex can be viewed at a glance and expressed compactly with the help of an $n/timesn$ matrix, which we refer to as a transit induced matrix. The definition is as follows.

Definition 7.1.6. Let $G(V, E)$ and $V = \{v_1, v_2, \ldots, v_n\}$. We define the transit induced matrix of G as the $n \times n$ matrix T as follows. $T = (t_{ij})$, where t_{ij} is the transit of v_i induced by $v_j = T_G(v_i, v_j)$

We list a few properties of this matrix T.

Properties

1. The $n \times n$ matrix T, is not necessarily symmetric.

2. If v_i is a simplicial vertex of G, then the ith row of the matrix T is the zero vector.

3. Diagonal elements are always zero.

4. $|T| = 0$ if G has a simplicial vertex.

5. Zero will be a root of the charecteristic equation with multiplicity $\geq r$, where r is the number of simplicial vertices in G.

6. $[T] \equiv 0, \Longleftrightarrow G \simeq K_n$.

7. $t_{ij} = 0,\ \forall j$ and v_i is not a simplicial vertex implies that v_i lie on a cycle.

8. $\sum\limits_{i} t_{ij} = 2T(v_i)$.

9. $\sum\limits_{i}\sum\limits_{j} t_{ij} = 2TI(G)$.

Results

1. When G is the cycle C_n, the transit induced matrix T obtained will be a circulant matrix.

2. For the path graph P_n, $T = (t_{ij})$ has the following property. $t_{ij} = t_{n-i+1,n-j+1}$, $\forall i, j$

3.Let $G = F_n$, the friendship graph. T has $n - 1$ columns as zero. The column corresponding to the center vertex has all its elements as $2(n - 3)$.

4. For the star, $K_{1,n}$, T is similar to the transit induced matrix of the friendship graph. The column corresponding to the centeral vertex has all the elements to

be $2(n-1)$.

5. In the case of a wheel graph W_n with v_1 as the central vertex, $T = (t_{ij})$ has the following property. $t_{1n} = 2, t_{i,i+1} = 2; \forall i = 1, 2 \ldots, n-1$. The column corresponding to the central vertex has its elements as $2(n-3)$. In all other cases the entry is zero.

7.2 Transit in Subgraph Amalgamation

Theorem 7.2.1. Let G be the graph obtained by merging the graphs $\{G_i\}_{i=1}^{n}$ along n copies of isomorphic induced common convex subgraph H, where $H \subset G_i, \forall i$. Let $S_i = V(G_i - H)\forall i$. Then we have the following results.

$$T_G(x) = \sum_{i=1}^{n} T_{G_i}(x) - (n-1)T_H(x) + \sum_{i<j} T(x, S_i/S_j), x \in H$$

$$T_G(u) = T_{G_k}(u) + \sum_{i \neq k} T(u, S_k/S_i), u \in G_k - H$$

Proof. Consider the graph G got by merging $\{G_i\}_{i=1}^{n}$ along n copies of isomorphic induced common convex subgraph H, where each contain a copy of H as a subgraph. Now the subgraphs, $G_i - H$ and H forms a partition of G.

Let $x \in H$. The transit of x is due to the geodesics through x joining vertices in G_i themselves and due to vertices from different pairs of $G_i - H$. Since H is convex, in the sum $\sum_{i=1}^{n} T_{G_i}(x)$, the quantity $T_H(x)$ will be counted n times. Thus we get, $T_G(x) = \sum_{i=1}^{n} T_{G_i}(x) - (n-1)T_H(x) + \sum_{i<j} T(x, S_i/S_j)$

Now consider $u \in G_k - H$. The transit of u is either due to geodesics connecting vertices in G_k alone or due to those with one end in S_k and other end in G_k^c. Consider the geodesic $u' - u - u''$ through u. If $u' \in S_k$ and $u'' \in G_k$, the contribution of the path $u' - u - u''$ towards the transit of u will be accounted in $T_{G_k}(u)$. On the contrary, if $u' \in S_k$ and $u'' \in G_k^c$, the contribution of the path $u'uu''$ will be accounted in $T(u, S_k/S_i)$, for some $i \neq k$. Hence the result. $\square$

Algorithm for computing transit of a vertex in subgraph amalgamation

1 Algorithm for finding $T_G(x)$ when $x \in V(H)$.

Require: Graphs $G_1, G_2, \ldots, G_n$, convex common subgraph H, vertex $x \in V(H)$

Ensure: Transit of vertex x, $T_G(x)$

1. $S_i = V(G_i - H)$, for $i = 1, 2, \ldots, n$.
2. Compute transit of x in G_i, $T_{G_i}(x)$, for $i = 1, 2, \ldots, n$. Add all these transit values to get $T_{sum}(x)$
3. Compute transit of x in H, $T_H(x)$
4. Compute transit of x in S_i, $T_{S_i}(x)$ for $i = 1, 2, \ldots, n$
5. Find $T(x, S_i/S_j) = \sum_{s \in S_i, t \in S_j} \sigma_{st}(x) d(s,t)$, for all $i < j, i = 1, 2, \ldots, n$. Add all these transit to get $T_{subset}(x)$.
6. Determine $T_G(x) = T_{sum}(x) - (n-1) \times T_H(x) + T_{subset}(x)$

2 Algorithm for finding $T_G(x)$ when $x \in G_k - H$.

Require: Graphs $G_1, G_2, \ldots, G_n$, convex common subgraph H, vertex $x \in v(G_k - H)$

Ensure: Transit of vertex x, $T_G(x)$

1. $S_i = V(G_i - H)$, for $i = 1, 2, \ldots, n$.

2. Compute transit of x in G_k, $T_{G_k}(x)$.

3. Compute transit of x in S_i, $T_{S_i}(x)$ for $i = 1, 2, \ldots, n$

4. Find $T(x, S_i/S_k) = \sum_{s\in S_i, t\in S_k} \sigma_{st}(x)d(s,t)$, for all $i \neq k$. Add all these transit values to get $T_{sum}(x)$.

5. Determine $T_G(x) = T_{G_k}(x) + T_{sum}(x)$

Path amalgamation of graphs

In this section we consider the graph obtained by merging two cycles C_m and C_n along a common convex subgraph.

Corollary 7.2.1. Let G be the graph obtained by merging two cycles C_m and C_n along a common path $P_r = \{w_1, w_2, \ldots, w_r\}$, as common subgraph where $r < min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$ and $U = \{u_1, u_2, \ldots, u_{m-r}\}$ and $V = \{v_1, v_2, \ldots, v_{n-r}\}$, the vertex set of $C_m - P_r$ and $C_n - P_r$ respectively. Let $w_k \in P_r$ and $u_k \in U$. Then

$$T_G(w_k) = T_{C_m}(w_k) + T_{C_n}(w_k) - T_{P_r}(w_k) + T(w_k, U/V)$$

$$T_G(u_k) = T_U(u_k) + T(u_k, U/V)$$

Proof. Taking $G_1 = C_m, G_2 = C_n$ and $H = P_r$ in the theorem[7.2.1], we get the result. □

The transit of vertices in cycles and paths are already discussed and found. Hence we attempt to find the transit of $w_k(y_k)$ induced by U against V. ie, we find $T(w_k, U/V)$ and $T(y_k, U/V)$. We find these two quantities in the following proposition, considering various cases. Observe that every u_iv_j geodesics passing through w_k has P_r as a part of it, and hence the quantity $T(w_k, U/V)$ remains the same for $\forall k,\ 1 < k < r$. Also note that w_1 and w_k are at symmetric positions on the graph and hence have the same value for the transit induced by U against V. In a similar manner it can be understood that there are vertices in U occupying symmetric positions. Hence we need to compute $T(u_k, U/V)$ for all those k with $1 \leq k \leq \lceil \frac{m-r}{2} \rceil$.

Proposition 7.2.1. Let G be the graph obtained by merging two cycles C_m and C_n along a common path $P_r = \{w_1, w_2, \ldots, w_r\}$, as common subgraph where $r < min\{\lceil \frac{m}{2} \rceil, \lceil \frac{n}{2} \rceil\}$ and $U = \{u_1, u_2, \ldots, u_{m-r}\}$ and $V = \{v_1, v_2, \ldots, v_{n-r}\}$, the vertex set of $C_m - P_r$ and $C_n - P_r$ respectively. Let $w_k \in P_r$ and $u_k \in U$.

Case 1. When both m and n are even.

$T(w_k, U/V) = \sum_{i=1}^{\alpha} \sum_{j=\delta}^{n-r} d(u_iv_j) + \sum_{i=\beta}^{m-r} \sum_{j=1}^{\gamma} d(u_iv_j)$, where $1 < k < r$

$T(w_1, U/V) = T(w_k, U/V) + \sum_{i=1}^{\beta} \sum_{j=1}^{\gamma} d(u_iv_j) + \sum_{i=1}^{\alpha} \sum_{j=1}^{\delta} d(u_iv_j)$

$T(u_k, U/V) = \sum_{i=k+1}^{\alpha} \sum_{j=\delta}^{n-r} d(u_iv_j) + \sum_{\substack{i=k+1 \\ k>\alpha}}^{\beta} \sum_{j=1}^{\gamma} d(u_iv_j) +$
$\sum_{i=k+1}^{\alpha} \sum_{j=1}^{\delta} d(u_iv_j)$,where $u_k \in U$.

Case 2. When both m and n are odd.

$T(w_k, U/V) = \sum_{i=1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_iv_j) + \sum_{i=\beta_2}^{m-r} \sum_{j=1}^{\gamma_1} d(u_iv_j)$, where $1 < k < r$

$T(w_1, U/V) = T(w_k, U/V) + \sum_{i=\alpha_2}^{\beta_1} \sum_{j=1}^{\gamma_2} d(u_iv_j) + \sum_{i=1}^{\alpha_2} \sum_{j=1}^{\delta_1} d(u_iv_j)$

$T(u_k, U/V) = \sum_{i=k+1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_iv_j) + \sum_{\substack{i=k+1 \\ k>\alpha_2}}^{\beta_1} \sum_{j=1}^{\gamma_2} d(u_iv_j) +$
$\sum_{i=k+1}^{\alpha_2} \sum_{j=1}^{\delta_1} d(u_iv_j)$,where $u_k \in U$.

Case 3. When m is even and n is odd.

$T(w_k, U/V) = \sum_{i=1}^{\alpha} \sum_{j=\delta_2}^{n-r} d(u_iv_j) + \sum_{i=\beta}^{m-r} \sum_{j=1}^{\gamma_1} d(u_iv_j)$

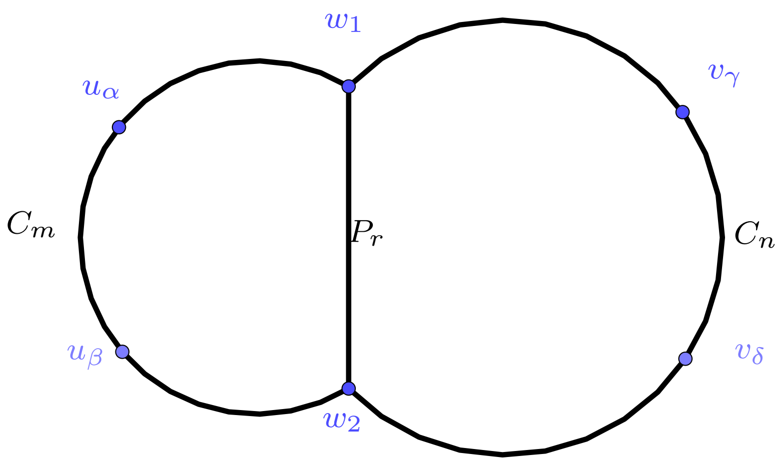


Figure 7.1: m and n even.

$$T(w_1, U/V) = T(w_k, U/V) + \sum_{i=\alpha+1}^{\beta} \sum_{j=1}^{\gamma_1} d(u_i v_j) + \sum_{i=1}^{\alpha} \sum_{j=1}^{\delta_1} d(u_i v_j)$$

$$T(u_k, U/V) = \sum_{i=k+1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{\substack{i=k+1 \\ k>\alpha}}^{\beta} \sum_{j=1}^{\gamma_2} d(u_i v_j) + \sum_{i=k+1}^{\alpha} \sum_{j=1}^{\delta_1} d(u_i v_j), \text{ where } u_k \in U.$$

Proof. In all of the cases we consider vertices w_1, $w_k(1 < k < r)$ and $u_k(1 \leq k < \lceil \frac{m-r}{2} \rceil)$.

Case 1. m and n are even.

Let u_α and u_β be the vertices eccentric to w_r and w_1 respectively, with respect to C_m. Considering the cycle C_n, let v_γ and v_δ be the vertices eccentric to w_r and w_1 respectively. Then it can be easily verified that $\alpha = \frac{m}{2} - r + 1$, $\beta = \frac{m}{2}$, $\gamma = \frac{n}{2} - r + 1$ and $\delta = \frac{n}{2}$. In computing $T(w_k, U/V), 1 < k < r$, we need to consider every geodesic with one end in U and other end in V, having P_r as a part of it. Those are paths of the following types.

(i)$u_i - v_j$ where $1 \leq i \leq \alpha$ and $\delta \leq j \leq n - r$

(ii)$u_i - v_j$ where $\beta \leq i \leq m - r$ and $1 \leq j \leq \gamma$. Hence $T(w_k, U/V) =$

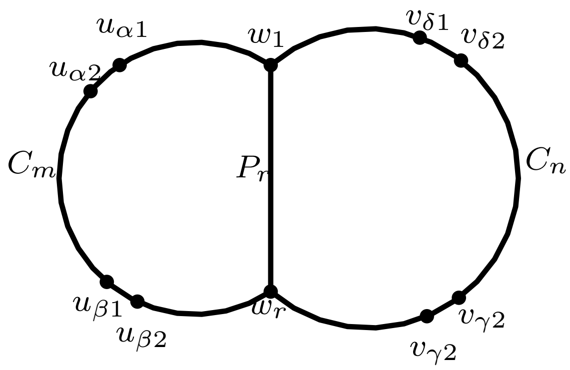


Figure 7.2: m and n odd

$\sum_{i=1}^{\alpha}\sum_{j=\delta}^{n-r} d(u_iv_j) + \sum_{i=\beta}^{m-r}\sum_{j=1}^{\gamma} d(u_iv_j)$

Next we consider the case of w_1. Every geodesic that contributed to $T(w_k, U/V), 1 < k < r$ also contributes to $T(w_1, U/V)$. Besides them the following paths also contribute.

(i)$u_i - v_j, \alpha \leq i \leq \beta$ and $1 \leq j \leq \gamma$. (ii)$u_i - v_j, 1 \leq i < \alpha$ and $1 \leq j \leq \delta$. Hence $T(w_1, U/V) = \sum_{i=1}^{\alpha}\sum_{j=\delta}^{n-r} d(u_iv_j) + \sum_{i=\beta}^{m-r}\sum_{j=1}^{\gamma} d(u_iv_j) + \sum_{i=\alpha+1}^{\beta}\sum_{j=1}^{\gamma} d(u_iv_j) + \sum_{i=1}^{\alpha}\sum_{j=1}^{\delta} d(u_iv_j)$

Next we consider $u_k \in U$ for $1 \leq k \leq \lceil \frac{m-r}{2} \rceil$. The paths to be considered here are those which starts at $u_t, t > k$ and passes through w_1 with u_k as an internal vertex. Considering such paths we arrive at the expression for $T(u_k, U/V)$

$$= \sum_{i=k+1}^{\alpha}\sum_{j=\delta}^{n-r} d(u_iv_j) + \sum_{\substack{i=k+1 \\ k>\alpha}}^{\beta}\sum_{j=1}^{\gamma} d(u_iv_j) + \sum_{i=k+1}^{\alpha}\sum_{j=1}^{\delta} d(u_iv_j)$$

Case 2. m and n are odd.

Since C_m and C_n are odd cycles there exists two vertices eccentric to w_1 and w_r. Let u_{α_1} and u_{α_2} be the vertices eccentric to w_r and u_{β_1} and u_{β_2} eccentric to w_1 respectively, with respect to the cycle C_m. Similarly let v_{γ_1} and v_{γ_2} be eccentric to w_r and v_{δ_1} and v_{δ_2} eccentric to w_1 respectively in C_n. On computation we get, $\alpha_1 = \lfloor \frac{m-r}{2} \rfloor + 1$, $\alpha_2 = \alpha_1 + 1$, $\beta_1 = \lfloor \frac{m}{2} \rfloor$ and $\beta_2 = \beta_1 + 1$. Also, $\gamma_1 = \lfloor \frac{n-r}{2} \rfloor + 1$, $\gamma_2 = \gamma_1 + 1$, $\delta_1 = \frac{n}{2}$ and $\delta_2 = \delta_1 + 1$.

The geodesics contributing to $T(w_k, U/V)$, $1 < k < r$ are the following.

(i) $u_i - v_j$ where $1 \le i \le \alpha_1$ and $\delta_2 \le j \le n-r$. (ii) $u_i - v_j$ where $\beta_2 \le i \le m-r$ and $1 \le j \le \gamma_1$. Hence $T(w_k, U/V) = \sum_{i=1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{i=\beta_2}^{m-r} \sum_{j=1}^{\gamma_1} d(u_i v_j)$.

Next we consider the case of w_1. As in case 1, it should be noted that $T(w_k, U/V)$, $1 < k < r$ is a part of $T(w_1, U/V)$.

The following paths also contribute to $T(w_1, U/V)$. (i)$u_i v_j, \alpha_2 \le i \le \beta_1$ and $1 \le j \le \gamma_2$. (ii)$u_i v_j, 1 \le i \le \alpha_2$ and $1 \le j \le \delta_1$. Hence $T(w_1, U/V) = \sum_{i=1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{i=\beta_2}^{m-r} \sum_{j=1}^{\gamma_1} d(u_i v_j) + \sum_{i=\alpha_2}^{\beta_1} \sum_{j=1}^{\gamma_2} d(u_i v_j) + \sum_{i=1}^{\alpha_2} \sum_{j=1}^{\delta_1} d(u_i v_j)$

If we proceed as in the case 1, we arrive at the expression for $T(u_k, U/V)$, for $u_k \in U$ as

$$\sum_{i=k+1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{\substack{i=k+1 \\ k>\alpha_2}}^{\beta_1} \sum_{j=1}^{\gamma_2} d(u_i v_j) + \sum_{i=k+1}^{\alpha_2} \sum_{j=1}^{\delta_1} d(u_i v_j)$$

Case 3. m is even and n is odd.

Since m is even, the vertices eccentric to w_r and w_1 in C_m is unique and let it be u_α and u_β respectively. In C_n the vertices eccentric to w_r and w_1 be $v_{\gamma_1}, v_{\gamma_2}$ and $v_{\delta_1}, v_{\delta_2}$ respectively. As in previous cases, these values can be computed as follows.

$\alpha = \frac{m}{2} - r + 1$, $\beta = \frac{m}{2}$, $\gamma_1 = \lfloor \frac{n-r}{2} \rfloor + 1$, $\gamma_2 = \gamma_1 + 1$, $\delta_1 = \frac{n}{2}$ and $\delta_2 = \delta_1 + 1$

Let $1 < k < r$. Then geodesics contributing to $T(w_k, U/V)$ are the following.

(i) $u_i - v_j$, $1 \le i \le \alpha$ and $\delta_2 \le j \le n-r$. (ii) $u_i - v_j$, $\beta \le i \le m-r$ and

$1 \leq j \leq \gamma_1$. Hence $T(w_k, U/V) = \sum_{i=1}^{\alpha} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{i=\beta}^{m-r} \sum_{j=1}^{\gamma_1} d(u_i v_j)$.

Now consider the case where $k = 1$. In addition to the geodesics considered in the case of $T(w_k, U/V)$, the following paths also contributes to $T(w_1, U/V)$ (i)$u_i - v_j$, $1 \leq i \leq \alpha$ and $1 \leq j \leq \delta_1$. (ii) $u_i - v_j$, $\alpha + 1 \leq i \leq \beta$ and $1 \leq j \leq \gamma_1$. Thus $T(w_1, U/V) = \sum_{i=1}^{\alpha} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{i=\beta}^{m-r} \sum_{j=1}^{\gamma_1} d(u_i v_j) + \sum_{i=\alpha+1}^{\beta} \sum_{j=1}^{\gamma_1} d(u_i v_j) + \sum_{i=1}^{\alpha} \sum_{j=1}^{\delta_1} d(u_i v_j)$

Finally we compute the transit of a vertex u_k, $1 \leq k \leq \lceil \frac{m-r}{2} \rceil$. As in the previous cases it can be estimated as

$$\sum_{i=k+1}^{\alpha_1} \sum_{j=\delta_2}^{n-r} d(u_i v_j) + \sum_{\substack{i=k+1 \\ k>\alpha}}^{\beta} \sum_{j=1}^{\gamma_2} d(u_i v_j) + \sum_{i=k+1}^{\alpha} \sum_{j=1}^{\delta_1} d(u_i v_j)$$

□

Chapter 8

Applications

This chapter discusses a few areas where the transit index of a graph and transit of a vertex may find applications. As acknowledged earlier, transit index is a graph invariant and transit of a vertex indicates a measure of its importance. In the following sections we briefly describe how these concepts may find application in chemical graph theory and networking problems.

8.1 Transit Index as a Molecular Descriptor

Chemical graph theory is an interdisciplinary field, where the molecular structure of a chemical compound is analyzed as a graph. Graph theoretical and computational techniques are employed in studying the mathematical properties of chemical compounds. The concept of chemical indices or molecular descriptor is one of the most crucial concepts in chemical graph theory. This associates a numerical value with a graph structure which is supposed to have some sort of correlation with chemical properties.

A molecular graph is a connected, undirected graph which admits a one-to-one correspondence with the structural formula of a chemical compound in which the vertices of the graph correspond to atoms of the molecule and edges of the graph correspond to chemical bonds between these atoms. The desirable properties of a molecular descriptor is discussed in [20] and [22]. Invariance with respect to labelling and numbering of the molecule atoms and an unambiguous algorithmically computable definition are among them. Moreover, good molecular descriptors SHOULD HAVE other important characteristics:

a. a structural interpretation.

b. a good correlation with at least one property.

c. no trivial correlation with other molecular descriptors.

d. gradual change in its values with gradual changes in the molecular structure.

e. not including in the definition experimental properties.

f. not restricted to a too small class of molecules.

g. preferably, some discrimination power among isomers.

h. preferably, not trivially including in the definition other molecular descriptors.

i. preferably, allowing reversible decoding (back from the descriptor value to the structure).

Of the above stated properties, most of them are met by the parameter 'Transit index' and hence it can be considered a molecular descriptor. On investigation we found that Transit index of molecular graphs of octane isomers bears a strong negative correlation with the chemical property- MON, the motor octane number, a measure of fuel performance .

Figure 8.1 depicts octane isomers. They are compounds of the form C_8H_{18}, with 8 carbon atoms and 18 hydrogen atoms. Octane has 18 structural isomers. Their chemical graphs are got by detaching hydrogen atoms. Transit of vertices of octane isomers are presented in Table 8.1

Octane Isomer	T(1)	T(2)	T(3)	T(4)	T(5)	T(6)	T(7)	T(8)
n-octane	0	27	45	54	54	45	27	0
2-methyl-heptane	0	0	42	50	51	43	26	0
3-methyl-heptane	0	23	51	0	48	41	25	0
4-methyl-heptane	0	24	39	54	0	39	24	0
3-ethyl-hexane	0	14	37	54	21	0	21	0
2,2-dimethyl-hexane	0	48	0	0	45	39	24	0
2,3-dimethyl-hexane	0	36	0	50	0	39	24	0
2,4-dimethyl-hexane	0	38	0	44	48	0	22	0
2,5-dimethyl-hexane	0	40	0	47	47	40	0	0
3,3-dimethyl-hexane	0	20	51	0	0	35	22	0
3,4-dimethyl-hexane	0	21	45	0	45	0	21	0
2-methyl-3-ethyl-pentane	0	20	50	20	0	34	0	0
3-methyl-3-ethyl-pentane	0	19	0	51	19	0	19	0
2,2,3-trimethyl-pentane	0	42	0	0	42	0	20	0
2,2,4-trimethyl-pentane	0	45	0	0	41	36	0	0
2,3,3-trimethyl-pentane	0	19	47	0	0	32	0	0
2,3,4-trimethyl-pentane	0	34	0	46	0	34	0	0
2,2,3,3-tetramethylbutane	0	39	0	0	39	0	0	0

Table 8.1: Transit of vertices in chemical graphs of Octane Isomers

Octane Isomer	TI(G)	MON
n-octane	252	-
2-methyl-heptane	212	23.8
3-methyl-heptane	188	35
4-methyl-heptane	180	39
3-ethyl-hexane	156	52.4
2,2-dimethyl-hexane	156	77.4
2,3-dimethyl-hexane	149	78.9
2,4-dimethyl-hexane	152	69.9
2,5-dimethyl-hexane	174	55.7
3,3-dimethyl-hexane	128	83.4
3,4-dimethyl-hexane	132	81.7
2-methyl-3-ethyl-pentane	124	88.1
3-methyl-3-ethyl-pentane	108	88.7
2,2,3-trimethyl-pentane	104	99.9
2,2,4-trimethyl-pentane	122	100
2,3,3-trimethyl-pentane	98	99.4
2,3,4-trimethyl-pentane	114	95.9
2,2,3,3-tetramethylbutane	78	-

Table 8.2: Transit Index and Motor Octane Number of Octane Isomers

In Table 8.2, transit index of octane isomers and MON are presented. Using the Table 8.2, the scatter plot between $TI(G)$ and MON is exhibited in Figure 8.2. The correlation coefficient obtained is **-0.9544**. This value indicates a strong correlation between Transit index of chemical graphs of octane isomers and their MON.

8.2 Transit Decomposition

In this section we give a summary of the study done in the structural graphs of octane isomers. As we know there are 18 octane isomers. Obviously, none of them are transit isomorphic. But a few of them shows a certain type of similarity in their transit decomposition. This similarity was also reflected in the values of their motor octane number. The significant observations are tabulated.

Observing Table 8.3, the 7 isomers may be grouped into three.

Group A:1,2 ; Group B:3,4 ; Group C:5,6,7

In each group the cardinality of transit decomposition is the same. ie, a bijection exists. The majorised shortest paths are agreeing in most of the cases and whenever they don't they differ in length with atmost one edge. This grouping makes sense when we observe the corresponding values of MON.

Hence we suppose that the study of transit decomposition in chemical structures may give meaningful information regarding their motor octane number.

Sl.No	Octane Isomer	$\|\tau_i\|$	Paths isomorphic to the majorised shortest paths in τ_i	MON
1	2,2 dimethyl hexane	6	$P_3, P_3, P_6, P_3, P_6, P_6$	77.4
2	2,3 dimethyl hexane	6	$P_3, P_4, P_6, P_4, P_6, P_5$	78.9
3	2 methyl,3 ethyl pentane	6	$P_5, P_5, P_5, P_5, P_5, P_5$	88.1
4	3 methyl,3 ethyl pentane	6	$P_4, P_5, P_6, P_4, P_5, P_4$	88.7
5	2,2,3 trimethyl pentane	10	$P_3, P_3, P_4, P_5, P_3, P_4, P_5, P_4, P_5, P_4$	99.9
6	2,2,4 trimethyl pentane	10	$P_3, P_3, P_5, P_5, P_3, P_5, P_5, P_5, P_5, P_3$	100
7	2,3,3 trimethyl pentane	10	$P_4, P_4, P_5, P_5, P_3, P_4, P_4, P_4, P_4, P_3$	99.4

Table 8.3: Majorized shortest paths

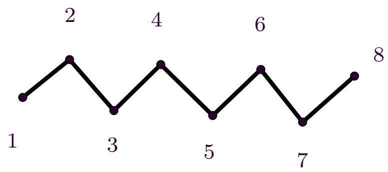
8.3 Transit of a Vertex as a Centrality Measure

Centrality measures are crucial tools for gaining a deeper understanding of networks. These measures calculate the importance of any given vertex in a network. Stress of a vertex, center of a graph, centroid of a graph, and betweenness centrality are some measures of centrality.

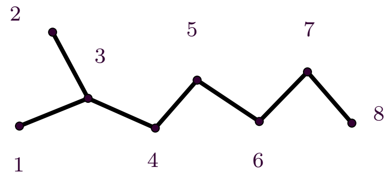
The transit of a vertex can be viewed as a measure of centrality. In a graph, the vertex/vertices with maximum transit are of particular importance. As the parameter transit depends both on the number and length of the geodesics passing through it, it has importance in communication as well as transportation networks.

Let us consider the example of a road transportation network. The underlying network is built by replacing roads with edges and cities lying on them with vertices. The vertex with the maximum transit may correspond to a city that is frequented by vehicles after long distance travel. These cities are therefore significant and deserve extra attention. For investment in facilitation centers, they should be considered first.

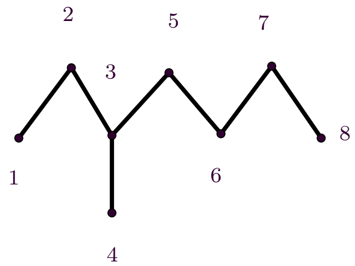
In electrical or communication networks, vertices with maximum transit should be considered for amplification or boosting as they may corresponds to junctions.



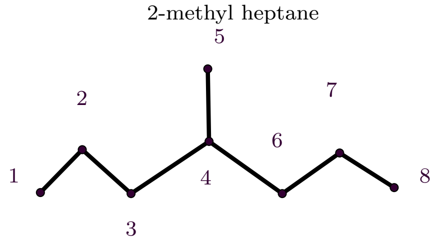
n-Octane



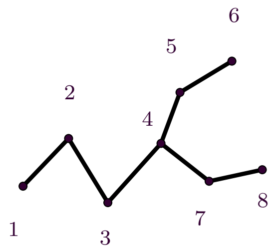
2-methyl heptane



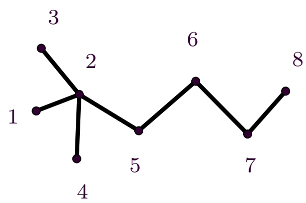
3-methyl heptane



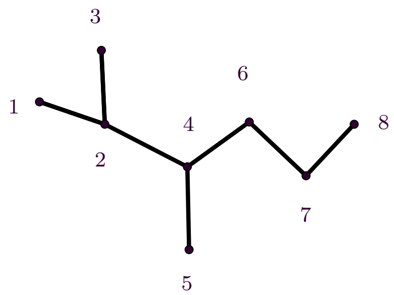
4-methyl heptane



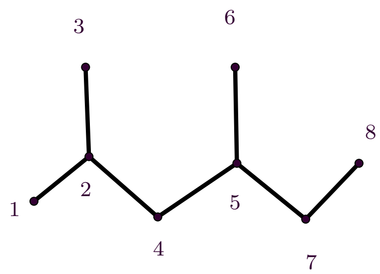
3-ethylhexane



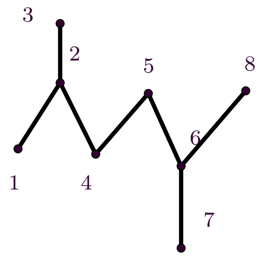
2,2-dimethy hexane



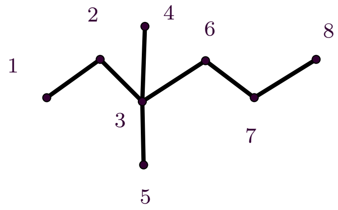
2,3-dimethyl hexane



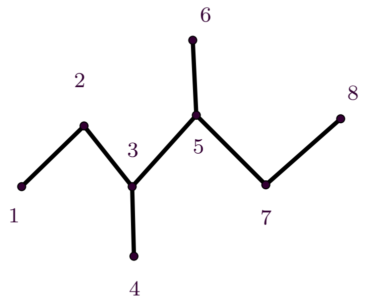
2,4-dimethyl hexane



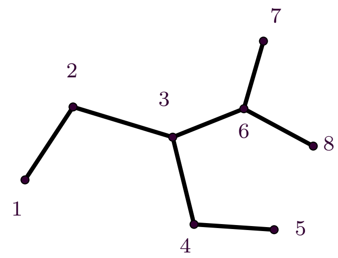
2,5-dimethyl hexane



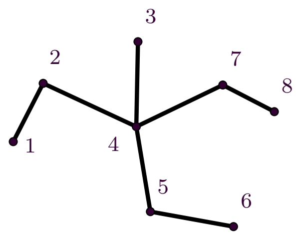
3,3-dimethyl hexane



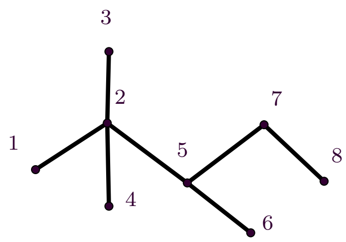
3,4-dimethyl hexane



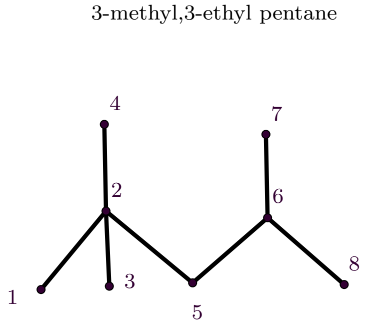
2-methyl, 3-ethyl-pentane



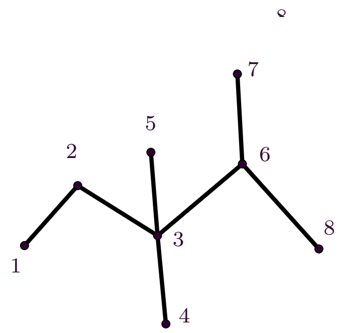
3-methyl,3-ethyl pentane



2,2,3-trimethyl pentane



2,2,4-trimethyl pentane



2,3,3-tri methyl pentane

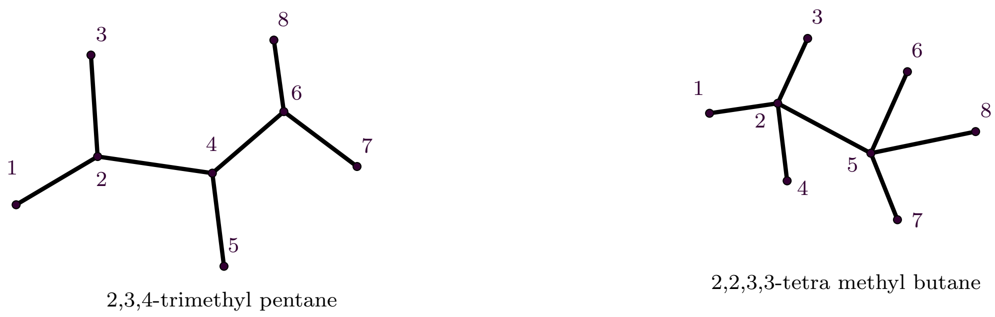


Figure 8.1: Chemical graphs of Octane Isomers

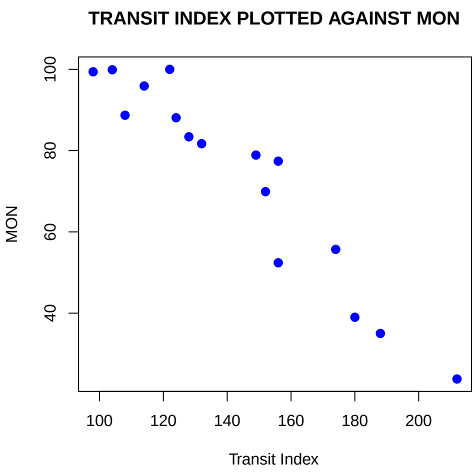


Figure 8.2: Scatter Plot

Chapter 9

Conclusion and Recommendations

This chapter is a summary of the thesis. It also discusses problems and areas for further study.

Summary

The transit of a vertex in a graph and the Transit Index of a graph are two major concepts dealt with in the thesis. These two concepts are introduced in Chapter 2. Some bounds are found for the transit of vertex in a graph. The index was studied in many graph classes. The concept of transit was explored in binary graph products, mainly Cartesian and Corona products.

A novel concept of majorised shortest paths and transit decomposition was introduced and studied in subdivision graphs. Isomorphic graphs are always inspiring; hence, a different kind of isomorphism called transit isomorphism was introduced and examined in line graphs. The behaviour of transit of vertices in convex amalgamated graphs was studied.

And finally, we could give some glimpses of applications for some of the concepts introduced.

Recommendations

In this section we discuss certain open problems. In Chapter 6, strongly transit isomorphic graphs were defined as follows.

Two graphs G and G' are said to be strongly transit isomorphic if $G \simeq_T G'$ and $TI(G) = TI(G')$

1.This concept of strongly transit isomorphic graphs is not much studied in this thesis and hence is an open area.

2. Conjecture: Let G and G' be transit isomorphic graphs. If there exist a one to one correspondence from $V(G)$ to $V(G')$, with corresponding vertices having the same transit, then $G \simeq G'$.

3. Transit isomorphic classes, can be investigated in detail.

4. Definition 4.1.1 dealing with transit dominant class and transit zero class is another area.

5. Transit induced matrix defined in Chapter 7 is another possible area of study.

In Chapter 8 we have seen that transit of a vertex can be considred as a centrality measure. In networks like transportation and similar ones it would be better if the edges have appropriate weights. Thus the concept of transit looks promising when extended to weighted graphs and strongly connected digraphs.

Let $G(V, E)$ be a graph. With each edge $e \in E$ of G, let there be associated a real number $w(e)$, called its weight. Then G, together with these weights on its edges, is called a weighted graph, and denoted (G, w). The length of a path P in (G, w) is the sum of the weights of its edges. The path P connecting two vertices with minimum length is the shortest path between them.

A directed graph, also called a digraph, is a graph in which the edges have a direction. A directed path in a digraph is a sequence of vertices in which there is a (directed) edge pointing from each vertex in the sequence to its successor in the sequence, with no repeated edges. A directed graph is strongly connected if there is a path between any two pair of vertices.
Now we extend the definition of transit of a vertex in a graph to vertices in weighted graph(strongly connected digraph) in the following manner.

Definition 9.0.1. Let (G, w) be a weighted graph(strongly connected digraph). The transit of a vertex $v \in V$ is the sum of the length of all shortest paths through v. The transit index of the graph G is the sum of the transit of all vertices in V.

1. Transit of a vertex and transit index of a graph can be explored in weighted and strongly connected digraphs.
2. Considering transit of a vertex as a centrality measure, study of its stability and continuity is a hopeful area of research.

Bibliography

[1] A.T. Balaban, *Distance connectivity index*, Chem. Phys. Lett. 89(1982) 399–404.

[2] A.T. Balaban, *Topological indices based on topological distances in molecular graphs*, Pure Appl. Chem. 55 (1983) 199–206.

[3] Andrey A. Dobrynin, Roger Entringer And Ivan Gutman; *Wiener Index of Trees: Theory and Applications*; Acta Applicandae Mathematicae 66: 211–249, 2001

[4] Anton Perdih, Franc Perdih; *Chemical Interpretation of Octane Number*; Acta Chim. Slov. 2006, 53, 306–315

[5] Bandelt, H.-J.; Chepoi, V. (2008), *Metric graph theory and geometry: a survey*, Contemporary Mathematics, 2008; Volume 453, DOI: https://doi.org/10.1090/conm/453

[6] Ernesto Estrada, Yaquelin Gutierrez; *The Balaban J Index in the multidimensional space of topological indices, Generalisations and QSPR Im-*

provements, MATDCY(44)155-167(2001)

[7] Frucht.R; Harary. F, *On the corona of two graphs*, Aeq. Math. 4, 322–325 (1970). https://doi.org/10.1007/BF01844162.

[8] Gajendra Pratap Singh, Aparajita Borah and Sangram Ray; *A review paper on corona product of graphs*; Advances and Applications in Mathematical Sciences, Volume 19, Issue 10, August 2020, Pages 1047-1054

[9] Gutman, I., Yeh, Y. N., Lee, S. L. and Chen, J. C.: *Wiener numbers of dendrimers*, Comm.Math. Chem. (MATCH) 30 (1994), 103–115.

[10] Gutman. I., Trinajstic.N., *Graph theory and molecular orbitals total π-electron energy of alternant hydrocarbons*, Chem. Phys. Lett., 17, (1972), 535-538.

[11] Hosoya, H. (1971). *Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons.* Bulletin of the Chemical Society of Japan, 44(9), 2332-2339.

[12] Harary. F; *Graph Theory* , Addison Wesley, 1969.

[13] Hendrik Timmerman; Todeschini, Roberto; Viviana Consonni; Raimund Mannhold; Hugo Kubinyi (2002). *Handbook of Molecular Descriptors.* Weinheim: Wiley-VCH. ISBN 3-527-29913-0.

[14] I. I. Jadav and G.V. Ghodasara, *Snakes related strongly graphs*, International Journal of Advanced Engineering Research and Science, Volume 3, Issue 9, 2016, pages 240-245.

[15] J.A. Bondy ,U.S.R. Murty; *Graph Theory* , Springer

[16] Mehdi Eliasia; Ghaffar Raeisib; Bijan Taerib, *Wiener index of some graph operations*, Discrete Applied Mathematics Volume 160, Issue 9,doi.org/10.1016/j.dam.2012.01.014

[17] Norman Biggs, *Algebraic Graph Theory*, Cambridge University Press

[18] O. Melnikov, V. Sarvanov, R. Tyshkevich, V. Yemelichev and I. Zverovich: *Exercises in Graph Theory*: Section 1.2

[19] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer, 2000.

[20] Randic M (1991) Generalized molecular descriptors. J Math Chem 7:155–168

[21] Randic, M. (1993), *Novel molecular descriptor for structure—property studies*, Chemical Physics Letters, 211 (10): 478–483

[22] Randic, M. (1996), *Molecular bonding profiles*, J. Math. Chem., 19, 375-392.

[23] Randic, M.; *Charecterisation of molecular branching*, J.Amer.Chem.Soc,97(23):6609-6615,1975

[24] Richard H, Imrich W, Klavzăr S, *Handbook of Product of Graphs*- 2011 - CRC press

[25] Schultz, H.P. (1989) *Topological Organic Chemistry. 1. Graph Theory and Topological Indices of Alkanes*. J. Chem. Inf. Comput. Sci, 29, 227-228. https://doi.org/10.1021/ci00063a012

[26] Sheeba Agnes.V, *Degree distance and Gutman index of corona product of graphs*, Transactions on combinatorics, Vol. 4 No.3(2015)

[27] Shimbel, Alfonso, *Structural parameters of communication networks*, Bulletin of Mathematical Bio- physics 15:501-507

[28] Sunil Kumar R. , Kannan Balakrishnan (2020), *Betweenness centrality in Cartesian product of graphs*, AKCE International Journal of Graphs and Combinatorics, 17:1, 571-583, DOI: 10.1016/j.akcej.2019.03.012

[29] Wagner.S; Wang.H, *Introduction to chemical graph theory*, Boca Raton, FL : CRC Press, Taylor & Francis Group, [2019] ©2019

[30] Weiner.H.; *Correlation of heats of isomerisation and differences in heats of vapourization of isomers*, J.Am.Chem.Soc,69:2636-2638,1947

[31] Wiener, H. (1947). *Structural determination of paraffin boiling points.* Journal of the American chemical society, 69(1), 17-20.

[32] Wilfried I, Sandi K, Douglas FR, *Topics in Graph Theory- Graphs and their Cartesian products*- AK Peters, 2008

[33] Xinhui An, Baoyindureng Wu; *The Wiener index of the kth power of a graph*, Applied Mathematics Letters 21 (2008) 436–440

[34] Y. Shanthakumari P; Siva Kota Reddy V; Lokesha P. S; *Hemavathi Topological aspects of Boron triangular nanotube and Boron-α Nanotube-II*; South east asian journal of mathematics and mathematical science; Volume : 16 , Issue : 3

APPENDIX I

List of publications

1. Reshmi K M; Raji Pilakkat, *Transit Index of a Graph and its correlation with MON of octane isomers*, Advances in Mathematics: Scientific Journal 2020, Vol. 9, No. 4

2. Reshmi K M; Raji Pilakkat, *Transit Index of various Graph Classes*, Malaya Journal of Matematik, Vol. 8, No. 2, 494-498, 2020

3. Reshmi K M; Raji Pilakkat, *Transit index by means of graph decomposition*, Malaya Journal of Matematik, doi :10.26637/MJM0804/0151

4. Reshmi K M; Raji Pilakkat, *Transit isomorphism and its study on octane isomers*, South East Asian J. of Mathematics and Mathematical Sciences Vol. 18, No. 3 (2022), pp. 255-264 DOI: 10.56827/SEAJMMS.2022.1803.21

5. Reshmi K M; Raji Pilakkat, *Transit Index of Subdivision Graphs*, Communications in Mathematics and Applications Vol. 12, No. 3, pp. 581–588,

2021

6. Reshmi K M; Raji Pilakkat, *Transit in Corona product of graphs*, Ratio Mathematica, Volume 47, 2023, DOI: 10.23755/rm.v39i0.860.

7. Reshmi K M; Raji Pilakkat, *Transit in Cartesian product of graphs.*(Communicated)

8. Reshmi K M; Raji Pilakkat, *Transit in Convex amalgamation of graphs.* (Communicated)

APPENDIX II

Paper Presentation

1. International Conference on Graph Connections; August 04 - 06, 2020, organised by Bishop Chulaparambil Memorial College, Kottayam

2. International Conference On Advances in Mathematics, Science and Technology (ICAMST-2020), 1-3 September 2020, Organized by Rajiv Gandhi University (A Central University) Rono Hills, Doimukh – 791112, Arunachal Pradesh, INDIA

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