CONTRIBUTIONS TO THE STUDY ON AGEING OF LIFE DISTRIBUTIONS BASED ON FAILURE RATE AND APPLICATIONS

Thesis submitted to the

UNIVERSITY OF CALICUT

for the award of the degree of

DOCTOR OF PHILOSOPHY

IN

STATISTICS

under the Faculty of Science

by

Kavya, P.

Under the Guidance of

Prof. (Dr.) M. Manoharan

Department of Statistics

University of Calicut

Kerala-673 635

July 2023

DEPARTMENT OF STATISTICS
UNIVERSITY OF CALICUT

Dr. M. MANOHARAN, M.Sc., Ph.D.
Senior Professor (Retired)

CALICUT UNIVERSITY (P. O.)
MALAPPURAM (District),
KERALA, INDIA PIN - 673 635
Mob : +91- 9447424043
Email : manumavila@gmail.com

……
….

Date : 26-10-2023

CERTIFICATE

*Certified that the corrections / suggestions recommended by the adjudicators of the Ph.D. Thesis entitled - "**CONTRIBUTIONS TO THE STUDY ON AGEING OF LIFE DISTRIBUTIONS BASED ON FAILURE RATE AND APPLICATIONS**" submitted by Smt. **Kavya P.**, Research Scholar of the Department under my supervision and guidance, have been incorporated in this copy of the thesis and that the contents of the thesis and soft copy are one and the same.*

Dr. M. Manoharan
Research Guide

CERTIFICATE

I hereby certify that the work presented in this thesis entitled 'Contributions to the study on ageing of life distributions based on failure rate and applications' is a bonafide work done by Mrs. Kavya P under my guidance for the award of the degree of Doctor of Philosophy in the Department of Statistics, University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree. Also certify that the contents of the thesis have been checked using anti-plagiarism data base and no unacceptable similarity was found through the software check.

University of Calicut

Date: 8/11/23

Dr. M. Manoharan

(Supervising Guide)

DEPARTMENT OF STATISTICS
No.
Date:.........
UNIVERSITY OF CALICUT

DECLARATION

I hereby declare that the work presented in this thesis entitled 'Contributions to the study on ageing of life distributions based on failure rate and applications' is based on the original work done by me under the guidance of Dr. M. Manoharan (Guide), Department of Statistics, University of Calicut, and has not been included in any other thesis submitted previously for the award of any degree.

University of Calicut

Date: 8/11/23

Kavya P

ACKNOWLEDGEMENT

This thesis would not have been possible without the guidance and the help of several individuals who in one way or another contributed and extended their valuable assistance in the preparation and completion of the study.

First, I would like to express my deepest gratitude to my Guide Dr. M Manoharan, Professor, Department of Statistics, University of Calicut, for his generous help, constructive criticism, scholarly guidance, valuable supervision, mental support and encouragement throughout the preparation of this thesis, without which this thesis would not have been materialized.

I would like to thank teachers Dr. K. Jayakumar, Head of the Department and Dr. C. Chandran for the technical help. I am indebted to the faculty members of the department Dr. Krishnarani S. D., and Dr. Dileep Kumar for sharing with me their knowledge and for providing the necessary facilities during the span of my study.

My sincere thanks are also due to Librarian Mrs. Suja and the non-teaching staff of the Department of Statistics, Calicut University for their help and co-operation. Also, I register my heartfelt thanks to my colleagues Vidya, Aparna, Praveen, Prasily, Fasna, Jeena, Arya, Jumna, Salima, Anjali, Jiji, Safwana, Shabeer, Anand, Greeshma, Rehna, Amjish, Nimisha for the co-operation and warmth I could enjoy from them. A special feeling of gratitude to Rajitha, Deepthi, Anakha, Nithin for their support and help provided during my thesis work.

Next I thank to my parents Karunakaran and Sreelekshmy and my sister Kavitha, brother-in-law Narayanan for the continuous encouragement, mental and financial support. I am grateful to my Cheriyamma and family for the support especially during the lock-down period. I am thankful to my in-laws; they made the last period of my research life smooth and tension-free. I am also grateful to my sweet little children Eva, Aarabhi and Ian for their understanding.

Finally my love and gratitude to my husband Sebastian Koothottil, without his motivation, encouragement, and mental support I wouldn't have been able to complete this research work.

Abstract of The Thesis

The thesis mainly focuses on the development of new lifetime distributions which are parsimonious and explores their ageing properties based on failure rate. The principle of parsimonious modeling of lifetimes has regained its importance recently, and the value of stochastic modeling in dealing with the inevitable uncertainty and risk is nowadays highly appreciated. In this thesis, we propose a transformation called Kavya-Manoharan (KM) transformation for obtaining a new class of parsimonious distributions and study its properties. Next, we construct three new lifetime models using exponential, Weibull and Lomax as the baseline distributions in the transformation, respectively known as KM-Exponential, KM-Weibull and KM-Lomax distributions, and investigate their ageing behaviour and other properties. The models introduced here have monotone failure rate functions. Comparing the proposed models with other models in the literature using a real-life data set, the newly introduced models show better fit to the data sets. We generalize the transformation and develop a new lifetime model using the exponential distribution as the baseline distribution, and the new model is called Generalized KM Exponential (GKME) distribution. The new model shows both monotone and non-monotone failure rate. With the help of real-life data sets, we show that the proposed model is more suitable compared to other distributions mentioned in this study. Estimation of stress-strength reliability for the newly proposed model is an other major work of the thesis, followed by a discussion on the asymptotic distribution of stress-strength reliability, simulation study and application. The thesis concludes by emphasizing the importance of ageing concepts in reliability theory and outlining future research directions.

സംഗ്രഹം

ഈ പ്രബന്ധം പ്രധാനമായും പാർസിമോണിയസ് ആയ പുതിയ ലൈഫ്ടൈം ഡിസ്ട്രിബ്യൂഷനുകൾ വികസപ്പിക്കുന്നതിൽ ശ്രദ്ധ കേന്ദ്രീകരിക്കുകയും പരാജയനിരക്കിന്റെ അടിസ്ഥാനത്തിൽ അവയുടെ പ്രായമാകൽ സ്വഭാവം പരിശോധിക്കുകയും ചെയ്യുന്നു. ലൈഫ്ടൈം വിതരണങ്ങളുടെ പാർസിമോണിയസ് മോഡലിംഗ് സമീപകാലത്ത് കൂടുതലായി ഗവേഷകശ്രദ്ധയാകർഷിച്ചുകൊണ്ടിരിക്കുന്ന ഒരു മേഖലയാണ്. ഈ പ്രബന്ധത്തിൽ ഞങ്ങൾ, ഒരു കൂട്ടം പാർസിമോണിയസ് വിതരണങ്ങൾ നിർമിക്കാനുതകുന്ന ഒരു പുതിയ ട്രാൻസ്ഫർമേഷൻ മുന്നോട്ടു വക്കുകയും, കാവ്യ-മനോഹരൻ (കെ-എം) എന്ന് പേരിട്ടിരിക്കുന്ന ഈ ട്രാൻസ്ഫർമേഷന്റെ സവിശേഷതകൾ ചർച്ച ചെയ്യുകയും ചെയ്യുന്നു. അടുത്തതായി, എക്സ്പോണൻഷ്യൽ, വെയ്ബുൾ, ലോമാക്സ് എന്നീ വിതരണങ്ങൾ ഈ പുതിയ ട്രാൻസ്ഫർമേഷനിൽ പ്രയോഗിക്കുന്നതിലൂടെ മൂന്നു പുതിയ ലൈഫ് ടൈം മോഡലുകൾ ഞങ്ങൾ നിർമിക്കുന്നു. ഇവയെ യഥാക്രമം കെ-എം എക്സ്പോണൻഷ്യൽ, കെ-എം വെയ്ബുൾ, കെ-എം ലോമാക്സ് എന്നിങ്ങനെ വിളിക്കാം. ഈ പുതിയ വിതരണങ്ങളുടെ പ്രായമാകൽ സവിശേഷതകളും മറ്റു സ്വഭാവങ്ങളും പ്രബന്ധത്തിൽ പരിശോധിക്കുന്നു. മറ്റു മോഡലുകളുമായി താരതമ്യപ്പെടുത്തുമ്പോൾ ഈ പുതിയ മോഡലുകൾക്ക്, ലഭ്യമായ ഡേറ്റസെറ്റുകളെ മെച്ചപ്പെട്ട രീതിയിൽ വിവരിക്കാൻ സാധിക്കുന്നു എന്നാണു കാണുന്നത്. . തുടർന്ന്, കെ-എം ട്രാൻസ്ഫർമേഷൻ സാമാന്യവൽക്കരികക്കുകയും, അതിൽ അടിസ്ഥാന വിതരണമായി എക്സ്പോണൻഷ്യൽ ഡിസ്ട്രിബ്യൂഷൻ ഉപയോഗിക്കുന്നതിലൂടെ ജനറലൈസ്ഡ് കെ-എം വിതരണം (ജികെഎംഇ) എന്ന മറ്റൊരു മോഡൽ കൂടി നിർമിക്കുകയും ചെയ്യുന്നു. ജികെഎംഇ വിതരണത്തിന്റെ സ്വഭാവസവിശേഷതകളും, മറ്റു മോഡലുകളുമായുള്ള താരതമ്യവും ഈ പ്രബന്ധത്തിൽ ചർച്ചാവിഷയമാകുന്നുണ്ട്. റിലയബിലിറ്റി സിദ്ധാന്തത്തിൽ പ്രായമാകൽ പഠനങ്ങളുടെ പ്രാധാന്യം ഊന്നിപ്പറഞ്ഞുകൊണ്ടും ഭാവി ഗവേഷണദിശകൾ സൂചിപ്പിച്ചുകൊണ്ടും പ്രബന്ധം ഉപസംഹരിച്ചിരിക്കുന്നു.

Contents

List of Figures

Chapter 1

Introduction

1.1 Reliability- The Basic Concepts

All of us quite often use the term “reliable” in various contexts such as a reliable friend, a reliable equipment, a reliable service centre, reliable news etc. As an abstract concept, it means something or someone we may depend upon. And when we purchase a computer or television or a transport vehicle or even a simple product such as an electric bulb or a battery we expect it to function properly for a reasonable period of time. The reliability of a unit or a system is defined as the probability that it will perform satisfactorily for a specified period of time without a major breakdown. From the beginning of twentieth century, the concept of reliability gets greater attention and many authors have studied and worked on it. In statistics, reliability is defined by Barlow and Proschan [1] as “reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered”. The time period is generally $[0,\ t]$. The adequate performance at time ‘t’ can be considered as the performance during $[0,\ t]$ provided no repair or replacement occurs. In engineering field reliability enjoys a special place because it is considered as one of the important characteristics of any device or equipment.

A reliability distribution or failure distribution represents an attempt to describe probabilistically the length of life of a material, a biological unit, a structure or a device. On the basis of actual observations lifetimes or failure times it is difficult to distinguish among the various nonsymmetric probability distributions. In order to discriminate among such probability distributions, it is necessary to appeal to a concept that permit us to base the differentiation on a physical consideration. Such a concept is based on the failure rate or hazard rate which has useful probabilistic interpretation.

In reliability analysis one mainly focuses on estimating the lifetime of the device, replacement policies, optimization of cost, time of destructive equipments etc. The methods and theories of reliability analysis are applied into various fields like small electronic devices to weapons, aerospace equipments, communication systems, forecasting of natural calamities like earthquake and volcano and so on. Lawless [2] shows the wide application of survival analysis in lifetime studies and medical treatment of human and other biological species.

Reliability of a product is a factor of great concern for producers and customers alike. Customers look for reliable products and the failure to guarantee the same puts the producers' business at risk. Reliability of equipments like weapons and aerospace devices is a matter of serious concern for both governments and societies. There is always a probability of failure during the ensured working period for any component. For example, lots of electronic manufacturers guarantee a minimum working period for their products, and some products breakdown before the guarantee period. The concept of failure facilitates the calculation of the reliability of a device. The concept of failure is used in connection to the situations where the functioning of a device is seriously affected or totally stopped due to the partial or total deprivation or change in the effects of the device.

The term reliability generally indicates the ability of a system to continue to perform its intended function. It is essentially the application of probability theory to the modeling of failures and prediction of success probability. The vast majority of the reliability analysis assumes that components and systems are in either of two states; functioning or failed. The reliability of such system is the probability that a specific function under given condition for a specific period of time performs without failure. The reliability of a fresh unit corresponding to a mission of duration t is, by definition,

$$\bar{F}(t) = P(T > t) = 1 - F(t) \tag{1.1}$$

where $F(.)$ is the life distribution of the unit. The hazard or failure rate $r(t)$ is defined by

$$r(t) = \frac{f(t)}{\bar{F}(t)} \tag{1.2}$$

when $f(t)$ (probability density function) exists and $\bar{F}(t) > 0$. Different types of lifetime distributions have been developed on the basis of failure patterns of the devices, for details see Marshall and Olkin [3].

1.2 Modes of failure and causes

A failure is the total or partial loss or change in the properties of a device in such a way that its functioning is seriously affected or totally stopped. The concept of failures and their details help in the evaluation of quantitative reliability of a device. In general, some components have well defined failures; others do not. Initial failure or infant mortality is a term used in connection to the situations where the item fails with high frequency in the beginning, when the component is installed. Infant mortality or initial failure is generally caused by manufacturing defects. Failures due to manufacturing defects are high at initial stages and gradually decrease and stabilize over a longer period of time.

Stable or constant failures due to chance can be observed on an item for a longer period. These types of failures are known as random failures and characterized by constant number of failures per unit time. Due to wear and tear with usage, the item gradually deteriorates and frequency of failures again increases. These types of failures are called as wear-out failures. At this stage failure rate seems to be very high due to deterioration, and the whole pattern of failures could be depicted by a bathtub curve.

1.3 Binary state system

The binary state system provides a foundation for the Mathematical and Statistical theory of reliability. The components in the system are assumed to be in one of two states functioning or failed. To indicate the state of the i^{th} component, we assign a binary indicator variable x_i to component i:

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed.} \end{cases}$$

for i= 1,2,...,n where n is the number of components in the system. Similarly, the binary variable Φ indicates the state of the system.

$$\Phi = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

We assume that the state of the system is determined completely by the states of the components, so that we may write $\Phi = \Phi(x)$, where x=$(x_1, x_2, ..., x_n)$. The function $\Phi(x)$ is called the structure function of the system. Clearly the structure function relates the state of the system with the states of the components.

A series structure functions if and only if each component functions. The structure function is given by

$$\Phi(x) = \prod_{i=1}^{n} x_i = min(x_1, x_2, ..., x_n)$$

A parallel structure function if and only if at least one component functions. The structure function is given by

$$\Phi(x) = \coprod_{i=1}^{n} x_i = max(x_1, x_2, ..., x_n)$$

where $\coprod_{i=1}^{n} = 1 - \prod_{i=1}^{n}(1 - x_i)$. The consideration of the fact that each component of a physical system serves some sort of useful function. In what way the functioning or failure state of a component contributes to the state of the whole system provides us the notion of component relevancy.

A system of components is coherent if its structure function Φ is increasing and each component is relevant. The ith component is irrelevant to the structure Φ of Φ is a constant in x_i. Otherwise the ith component is relevant to the structure.

A cut set K is a set of components that by failing causes the system to fail. A cut set is minimal if it cannot be reduced without losing its status as a cut set.

A path set S is a set of components that by functioning ensures that the system is functioning. A path set is minimal if it cannot be reduced without losing its status as a path set.

The reliability of a system is given by

$$P(\Phi(x) = 1) = h = E\Phi(x)$$

Under the assumption of independent components, we may represent system reliability as a function of component reliabilities. That is

$$h = h(p).$$

Accordingly the reliability function of series structure is h(p)=$\prod_{i=1}^{n} p_i$ and the reliability function of parallel structure is given by $h(p) = 1 - \prod_{i=1}^{n}(1 - p_i)$.

1.4 Mean residual life

Given that a unit is of age 't', then the remaining life is random. The expected value of this random residual life is called the mean residual life function (MRL). The mean residual life (MRL) has been used as far back as the third century A.D.(cf. Deevey [4] and Chiang [5]). The MRL has been studied by reliabilists, statisticians, survival analysts and others. A number of handy results have been derived in relation to it. The expected value of the random residual lifetime is called the MRL or mean remaining life. The MRL is often an important criterion for finding an optimal burn-in time for an item.

The MRL of a unit or a subject at age x is the average remaining life among those population members who have survived until time x. Then the mean residual life function is defined by

$$\mu(t) = E(X - t | X > t) = \int_0^t \bar{F}_t(x)dx$$

If F has a density f, we can then alternatively write

$$\mu(t) = \frac{\int_t^\infty x f(x)dx}{\bar{F}(t)}$$

Like failure rate function, the MRL describes a conditional concept of ageing; however, the MRL is more intuitive, especially in the health sciences.

In the context of industrial reliability studies of repair and replacement strategies, the MRL functions are often more pertinent than the failure rate function. In studies of human populations, demographers usually refers to the MRL by the names of life expectancy or expectation of life. Clearly, the MRL is of utmost importance to actuarial work associated with life insurance policies.

Note that there can be situations in which the MRL function exists and the failure rate function does not exist and vice versa. In theory and practice, however, we need both the MRL and the failure rate functions.

In cases where MRL and failure rate functions exist, they are related by,

$$\mu'(t) = \mu(t)r(t) - 1$$

for μ differentiable.

Knowledge of the MRL function completely determines the reliability function as follows

$$\bar{F}(t) = \begin{cases} \frac{\mu(0)}{\mu(t)}\ exp\ (\int_0^t \frac{du}{\mu(u)}) & \text{for } 0 < t < F^{-1}(1) \\ 0 & \text{for } t \geq F^{-1}(1) \end{cases}$$

where $F^{-1}(1) =^{def} sup\ (t|F(t) < 1)$.

Like the density function, the moment generating function or the characteristic function, the MRL completely determines the distribution via an inverse formula (e.g., see Cox [6], Kotz and Shanbhag [7] and Hall and Wellner [8]). The

MRL is used for both parametric and non-parametric modeling. It has a wide range of applications. In the biomedical setting researchers analyze survivorship studies by MRL. Morrison [9] mentions that IMRL (increasing mean residual life) distributions have been useful in modeling the social sciences for the life lengths of wars and strikes. Bhattacharjee [10] observes that MRL functions naturally come up in other areas such as optimal disposal of an asset, renewal theory, dynamic programming and branching processes.

1.5 Lifetime distributions

A lifetime (failure) distribution represents the probabilistic distribution of the length of life of a material, structure, device or an organism. On the basis of actual observations of times to failure it is difficult to distinguish among various non-symmetric probability functions. In such contexts, in order to discriminate among probability functions, it is necessary to appeal to a concept that permit us to base the differentiation on a physical considerations. Such a concept is based on the failure rate or hazard rate. This function has a useful probabilistic interpretation.

In reliability theory, we assume that the system and components are in either of the two states: functioning or failed. We next consider the life lengths of the system of components. In general, life lengths is random, and so we are led to a study of life distributions. To understand better which life distributions are important in reliability models, we consider first a notion of ageing. Ageing is conveniently studied in terms of the failure rate function. The ageing properties (increasing failure rate, decreasing failure rate, increasing failure rate average, decreasing failure rate average, new better than used etc) are discussed by several authors see Deshpande et al [11]. If we obtain a constant failure rate, then exponential distribution serves as a very useful model for reliability computation.

1.5.1 Notion of ageing

The notion of ageing plays an important role in reliability theory. Several classes of life distributions based on the notion of ageing have been studied during the past several years. The most well known classes are IFR (increasing failure rate), IFRA (increasing failure rate average), NBU (new better than used), NBUE (new better than used in expectation), DMRL (decreasing mean residual life) classes with their duals. The discrete versions and continuous versions of these classes have become common nature. These classes characterize either positive ageing or negative ageing. Positive ageing means adverse effect of age on the random residual life of the component or unit whereas negative ageing means beneficial effect. Concepts of ageing describe how a component or system improves or deteriorates with age.

Let X be a continuous non-negative random variable representing the lifetime of a unit. This unit may be a living organism, a mechanical component, a system of components etc. By lifetime we mean the period for which the unit is working satisfactorily. Age of the working unit is the time for which it is already working satisfactorily without failure.

Let F(x) be the cumulative distribution function of X, then the survival function of a fresh unit is $\bar{F}(x) = 1 - F(x) = P(X > x)$. Let X_t be the random variable representing the residual lifetime of a unit which has attained the age t. Then the distribution function and survival function of X_t are respectively $F_t(x)$ and $\bar{F}_t(x)$. It can be seen that

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

Note that this is the conditional probability that the unit survived upto time 't', will not fail before additional x unit of time. Further, $\bar{F}_0(x) = \bar{F}(x)$ is the survival function of a new unit.

By ageing we mean the phenomenon whereby an older system has shorter remaining lifetime in some statistical sense than a new or younger one. Obviously any study of the phenomenon of ageing is to be based on $\bar{F}_t(x)$ and functions related to it.

The conditional failure rate or hazard rate $r_F(t)$ at time 't' is defined as

$$r_F(t) = lim_{x \to 0} \frac{F(t+x) - F(t)}{x \ \bar{F}(t)}$$

so that

$$r_F(t) = \frac{f(t)}{\bar{F}(t)}$$

where f(t), the probability density function exists and $\bar{F}(t) > 0$.
We call $r_F(t)$ as the failure rate and write it simply as r(t). Some useful identities are

$$\int_0^x r(t) \ dt = -log \ \bar{F}(x)$$

and

$$\bar{F}(x) = exp \ (-R(x))$$

where $R(x) = \int_0^x r(t) \ dt$ is referred to as the cumulative hazard function.

Expressed by	F(t)	f(t)	$\bar{F}(t)$	r(t)
F(t)	-	$\int_0^t f(u)du$	$1 - \bar{F}(t)$	$1 - exp(-\int_0^t r(u)du$
f(t)	$\frac{dF(t)}{dt}$	-	$\frac{-d\bar{F}(t)}{dt}$	$r(t).exp(-\int_0^t r(u)du$
$\bar{F}(t)$	1-F(t)	$\int_t^\infty f(u)du$	-	$exp(-\int_0^t r(u)du$
r(t)	$\frac{\frac{dF(t)}{dt}}{1-F(t)}$	$\frac{f(t)}{\int_t^\infty f(u)du}$	$\frac{-dln\bar{F}(t)}{dt}$	-

Table 1.1: Relationship between the functions F(t), f(t), $\bar{F}(t)$, and r(t) (Lai and Xie [12])

Ageing is classified into three, No ageing, Positive ageing and Negative ageing.

No ageing is equivalent to the phenomenon that age has no effect on the residual survival time of a unit.

Now we consider a device which does not age stochastically, that is, probability distribution of the residual lifetime at age 't' of a unit does not depend on 't'. Hence

$$\bar{F}_t(x) = \bar{F}_0(x) \qquad \forall \;\; t, x > 0$$

This is equivalent to

$$\bar{F}(t + x) = \bar{F}(t)\bar{F}(x) \qquad \forall \;\; t, x > 0,$$

which is the Cauchy functional equation. It is well known as that among the continuous survival functions only the exponential survival function

$$\bar{G}(x) = e^{-\lambda x}, \quad x > 0, \;\; \lambda > 0$$

satisfies the above equation. This property of exponential lifetime is known as lack-of-memory property or no-ageing property in reliability theory.

Positive ageing describes the situation where the residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a component. This situation is common in reliability engineering as components tends to become worse with time due to increased wear and tear. On the other hand negative ageing has an opposite effect on the residual lifetime. Negative ageing is also known as beneficial ageing.

1.5.2 Basic reliability classes

Concept of ageing describes how a component or system improves or deteriorate with age. Many classes of life distributions are categorized or defined in the literature according to their ageing properties. An important aspect of such classification is that the exponential distribution is nearly always a member of each class. The notion of stochastic ageing plays an important role in any reliability analysis and many test statistics have been developed in the literature for testing exponentiality against different ageing alternatives.

The concept of increasing and decreasing failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR and DFR distributions. Other classes such as IFRA, NBU, NBUE, and DMRL have also been of much interest. The notion of harmonically new better than used in expectation (HNBUE) was introduced by Rolski [13] and studied by Klefsjo [(14), (15)].

We are now giving the formal definitions of basic reliability classes. Most of the reliability classes are defined in terms of the failure rate r(t), conditional survival function $\frac{\bar{F}(x+t)}{\bar{F}(t)}$, or the mean residual life $\mu(t)$. All these three functions provide probabilistic information on the residual lifetime and hence ageing classes may be formed according to the behavior of the ageing effect on a component.

F is said to be IFR if $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ is decreasing in $0 \leq t < \infty$ for each $x \geq 0$. It is decreasing failure rate(DFR) distribution if $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ is increasing in t or F is IFR (DFR) if and only if $-log\bar{F}(t)$ is convex (concave) when the density exists, IFR (DFR) is equivalent to $r(t) = \frac{f(t)}{\bar{F}(t)}$ being increasing (decreasing) in $t \geq 0$. F is said to be constant failure rate distribution if r(t) is constant or F is increasing as well as decreasing.

Assuming a failure rate decreasing during infant mortality phase, next constant during the so-called useful phase, and finally, increasing during the so-called wearout phase. In reliability literature such failure rate functions are said to have a bathtub shape. That is, F is said to be bathtub shaped failure rate distribution if r(t) is decreasing for $0 < t < t_0$, constant for $t_0 < t < t_1$, and increasing for

$t_1 < t < \infty$. Then F is said to be an inverse bathtub shaped failure rate distribution if r(t) is increasing in $0 < t < t_0$, constant for $t_0 < t < t_1$ and decreasing in $t_1 < t < \infty$.

There are many lifetime distributions that have been proposed in the literature, well studied by several researchers, and applied in a wide range of areas. We shall review some of them with respect their ageing properties.

Exponential Distribution:-

The exponential distribution is applied in a wide variety of statistical procedures. Currently among the most prominent applications are in the field of life testing. The exponential life distribution provides a good description of the life length of a unit which does not age with time. The density function is

$$f(t) = \lambda e^{-\lambda t}, \qquad \lambda > 0, t \geq 0$$

and

$$\bar{F} = e^{-\lambda t}, \qquad \forall \;\; t \geq 0$$

It has a constant failure rate, that is, $r(t) = \lambda, \;\; \forall \;\; t \geq 0$

A property of the exponential distribution which makes it especially important in reliability theory and application is that the remaining life of a used component is independent of its initial age (the memoryless property).

Gamma distribution:-

The density function of a standard two-parameter gamma distribution is

$$f(t) = (\frac{\lambda^{\alpha} t^{\alpha-1}}{\Gamma\alpha})e^{-\lambda t}, \qquad \alpha, \lambda > 0$$

The distribution function F(x) may be written as

$$F(t) = 1 - \sum_{i=0}^{\alpha-1} (\frac{(\lambda t)^i}{i!}) e^{-\lambda t}, \qquad for \;\; t \geq 0$$

It can be shown, by a change of variable, that

$$[r(t)]^{-1} = \int_0^{\infty} (1 + \frac{u}{t})^{\alpha-1} e^{-\lambda u} du$$

The distribution F(t) is DFR for $0 < \alpha \leq 1$ and IFR for $\alpha \geq 1$. For $\alpha = 1$, $F(x) = 1 - e^{-\lambda t}$, an exponential distribution which is both IFR and DFR.

Weibull distribution :-

The Weibull distribution is named after the Swedish Physicist Waloddi Weibull who in 1939 used it to represent the distribution of the breaking strength of material and in 1951 for variety of other applications. The survival function of two-parameter Weibull is

$$\bar{F}(t) = 1 - e^{(-\lambda t)^{\alpha}}, \qquad for \;\; t \geq 0, \;\; \alpha, \; \lambda > 0,$$

and the failure rate function, $r(t) = \lambda\alpha(\lambda t)^{\alpha} - 1$, for $t > 0$.

Thus the Weibull distribution F is IFR for $\alpha \geq 1$ and DFR for $0 < \alpha \leq 1$; for $\alpha = 1$, $F(t) = 1 - e^{-\lambda t}$, the exponential distribution. Note that r(t) is increases to ∞ for $\alpha > 1$ and decreases to zero for $\alpha < 1$ as $t \to \infty$. The parameter α is called shape parameter; as α increases, the failure rate function rises more steeply and the probability density becomes more peaked.

F is said to be IFRA if $-(\frac{1}{t})log\bar{F}(t)$ is increasing in $t \geq 0$. This is equivalent to $\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t)$, $0 < \alpha < 1$, $t \geq 0$. It is a decreasing failure rate average (DFRA) distribution if $-(\frac{1}{t})log\bar{F}(t)$ is decreasing in $t \geq 0$ or $\bar{F}(\alpha t) \leq \bar{F}^{\alpha}(t)$, $\forall \;\; 0 < \alpha < 1$.

Certain class of distribution such as new better than used (NBU), new worse than used (NWU), new better than used in expectation (NBUE) and new worse than used in expectation (NWUE) arise naturally in considering replacement policies.

F is said to be NBU if $\bar{F}_t(x) \leq \bar{F}(x)$, that is, $\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t)$ for $x, t \geq 0$. This means that a device of any particular age has a stochastically smaller remaining lifetime than does a new device. F is said to be NWE if $\bar{F}(x+t) \geq \bar{F}(x)\bar{F}(t)$ for all $x, t \geq 0$.

F is said to be NBUE if $\int_0^\infty \bar{F}_t(x)dx \leq \mu \quad$ for $t \geq 0$. This means that a device of any particular age has a smaller mean remaining lifetime than does a new device. F is said to be NWUE if $\int_0^\infty \bar{F}_t(x)dx \geq \mu \quad$ for all $t \geq 0$. This class of life distribution plays an important role in maintenance and replacement policies and in biostatistics applications.

F is said to be harmonically new better than used in expectation (HNBUE) if $\int_t^\infty \bar{F}(x)dx \leq \mu \ exp(-\frac{t}{\mu}) \quad$ for $t \geq 0$. Similarly, F is said to be harmonically new worse than used in expectation (HNWUE) if $\int_t^\infty \bar{F}(x)dx \geq \mu \ exp(-\frac{t}{\mu}) \quad$ for $t \geq 0$.

F is said to be decreasing mean residual life (DMRL) if the mean remaining life function $\mu(t) = \int_0^\infty \bar{F}_t(x)dx$ is decreasing in t, that is, $\mu(s) \geq \mu(t) \quad$ for $0 \leq s \leq t$. In other words, the older the device is, the smaller is its mean residual life. Similarly, F is said to be increasing mean residual life (IMRL) distribution if $\mu(s) \leq \mu(t) \quad$ for $0 \leq s \leq t$.

F is said to be Laplace class $\mathscr{L}$-distribution if for every $s \geq 0$, $\int_0^\infty e^{-st}\bar{F}(t)dt \geq \frac{\mu}{1+s}$. The expression $\frac{\mu}{1+s}$ can be written as for $\int_0^\infty e^{-st}\bar{G}(x)dx$, where $\bar{G}(x) = e^{\frac{-x}{\mu}}$. This means that the inequality is one between Laplace transforms of $\bar{F}$ and of an exponential survival function with the same mean as F (15).

F is said to be new better than used in failure rate (NBUFR) if $r(t) > r(0)$ for $t \geq 0$ (11). F is said to be new worse than used in failure rate(NWUFR) if $r(t) < r(0)$ for $t \geq 0$.

F is said to be new better than used in failure rate average (NBAFR or NBUFRA) if $r(0) \leq \frac{1}{t}\int_0^t r(x)dx \quad \forall\ t \geq 0$. Note that this is equivalent to $r(0) \leq \frac{-log\bar{F}(t)}{t}, \quad t \geq 0$. Similarly, F is said to be new worse than used in failure rate average (NWUFRA) if $r(0) \geq \frac{-log\bar{F}(t)}{t}, \quad t \geq 0$.

F is said to be new better than used in convex ordering (NBUC) if $\int_y^\infty \bar{F}(t|x)dt \leq \int_y^\infty \bar{F}(t)dt \quad \forall\ x, y \geq 0$.

These notion should be applied to complex systems. If we consider the time dynamics of such systems, we want to investigate how the reliability of the whole system changes in time if the components have one of the above mentioned ageing property.

1.5.3 Chain of implications of the ageing classes

The chain of possible implications of the ageing classes are given below.

$$
\begin{array}{ccccccccc}
IFR & \longrightarrow & IFRA & \longrightarrow & NBU & \longrightarrow & NBUFR & \longrightarrow & NBUFRA \\
\downarrow & & & & \downarrow & & & & \\
\downarrow & & & & NBUC & & & & \\
\downarrow & & & & \downarrow & & & & \\
DMRL & \longrightarrow & & & NBUE & \longrightarrow & HNBUE & \longrightarrow & \mathscr{L}
\end{array}
$$

If we reverse the inequalities and interchange increasing and decreasing, we obtain the classes DFR, DFRA, NWU, IMRL, NWUE, HNWUE, $\bar{\mathscr{L}}$, NWUFR, NWUFRA and NWUC. They satisfies the same chain of implications. These are sometimes referred to as the dual classes and their roles are to define negative ageing effects to a device.

Classes of life distributions having monotone residual variance are also studied in the literature. There are decreasing variance residual life (DVRL), net decreasing variance residual (NDVRL) and corresponding dual classes. It is observed that there is intimate connection between DVRL (IVRL) distributions. It is seen that no implications hold between DVRL and NBUE classes. One can present improved bounds on survival functions and moments of DVRL (IVRL) and NDVRL (NIVRL) classes compared to the known bounds for DMRL (IMRL) distributions. All these matters can be found in Launer [16] and Gupta et. al. [17].

Amongst the partial orders introduced in the literature some are used for directly comparing probability distributions with the exponential distribution to describe classes of distribution having certain positive ageing properties. Besides these direct comparisons, many partial orders are utilized for involving comparison between probability distributions of residual life times at different ages in order to describe positive ageing. Deshpande et. al. [18] distinguished between two types of positive ageing viz. 1) Younger better than older (YBO) type ageing wherein the effect of ageing is progressive and the unit deteriorates monotonically in some sense, with increasing age; and 2) New better than used (NBU) type ageing wherein the comparison is only between a new unit and a used unit. Deshpande et. al. [18] observed that the YBO and NBU comparisons in terms of some specific partial orders lead to the same class of distributions.

Apart from the classes of life distributions based on notions of ageing which are discussed in this work, the life distributions which are heuristic extension of IFR (DFR) and NBU (NWU) classes to random intervals are available in literature (see Singh and Deshpande [19]). The extended distributions are stochastic increasing failure rate (SIFR), and stochastic new better than used (SNBU). It is established by these authors that the following chain of implications hold between these classes.

$$
\begin{array}{ccc}
IFR & \longrightarrow & NBU \\
\downarrow & & \downarrow \\
SIFR & \longrightarrow & SNBU
\end{array}
$$

The closure of these ageing properties under Poisson shock model is also established.

1.5.4 Formal definitions of discrete life distribution classes

Most lifetime distributions are continuous and hence many continuous distributions have been studied and presented in the literature. However, in realistic situations it is possible to find discrete failure data, for example, a series of reports collected weekly or monthly displaying the number of failures of a device, with no specification for failure times. Another example which can be seen is when a device operating on demand and the worker staff observe the number of successful demands executed before failure.

We shall define the discrete version of the classes of continuous lifetime distributions. Following are the classes corresponding to positive ageing.

Let X be a discrete life length taking on integer values 0, 1, 2, ... The reliability function, denoted by $\bar{P}_k$, is defined as

$$\bar{P}_k = P(X > k), \quad k = 0, 1, 2, ...$$

Assuming $\bar{P}_0 = 1$.

Definition 1: The survival probability $\bar{P}_k$ is said to be discrete IFR if the discrete conditional survival function $\frac{\bar{P}_{k+1}}{\bar{P}_k}$ is decreasing in k=0, 1, 2, ... or the discrete failure rate function r(k), k=0, 1, 2, ... is increasing.

Definition 2: The survival probability $\bar{P}_k$ is discrete IFRA if $[\bar{P}_k]^{1/k}$ is decreasing in k=0, 1, 2,....

Definition 3: The survival probability $\bar{P}_k$ is said to be discrete NBU if $\bar{P}_{k+l} \leq \bar{P}_k \ \bar{P}_l$ for k, l=0, 1, 2, ... This means that a device of any particular age has a stochastically smaller remaining lifetime than does a new device.

Definition 4: The survival probability $\bar{P}_k$ is said to be discrete NBUE if P has finite mean and $\bar{P}_k \sum_{j=0}^{\infty} \bar{P}_j \geq \sum_{j=k}^{\infty} \bar{P}_j$ for k=0, 1, 2, ...

Definition 5: The survival probability is said to be DMRL if $\bar{P}_k$ has finite mean and $\frac{\sum_{j=k}^{\infty} \bar{P}_j}{\bar{P}_k}$ is decreasing in k=0, 1, 2, ...

Definition 6: The survival probability $\bar{P}_k$ is said to be discrete HNBUE if $\sum_{j=k}^{\infty} \bar{P}_j \leq \mu \ (1 - (1/\mu))^k$ for k=0, 1, 2, ...

By reversing the inequalities and changing decreasing to increasing we get the definitions corresponding to the dual classes. The implication between these classes of discrete distribution are the same as in the continuous case.

1.6 Review of literature and objectives

Ageing or lifetime of a system has an important role in the area of engineering and biology. Failure rate functions have served as the basis in the development of many lifetime models. Distributions which exhibit constant hazard rates got most of researchers'attention at early times. The exponential distribution is an example for a constant hazard rate function. The concept of non constant hazard rates has received increased attention with the realization that there existed such cases. The concept of non constant hazard rates and related ideas have been elaborated by Barlow and Proschan [20] and Shaked and Shanthikumar [21]. Koutras [22] and Ross et al [23] gives the real life applications of increasing failure rates. Barlow et al.[24], Barlow and Marshall [25], and Barlow and

Marshall [26] studied the properties and bounds for the distributions which show monotone hazard rate. The most popular distributions that have IFR are the gamma and Weibull distributions (these distributions also exhibit decreasing and constant failure rates). Gupta [27] and Gupta and Kundu [28] introduced the exponentiated exponential distribution and the weighted exponential distribution as alternatives to the gamma and Weibull distributions, respectively. Cancho et al. [29] introduced a new lifetime distribution with increasing failure rate, named as the Poisson-exponential (PE) distribution.

Probabilistic models of decreasing failure rate processes have studied by Cozzolino [30]. Proschan [31] gave the theoretical explanation of observed decreasing failure rate. Adamidis and Loukas [32] studied a decreasing failure rate distribution namely 'the exponential geometric distribution'. Tahmasbi and Rezaei [33] developed a DFR model by compounding exponential distribution with logarithmic series distribution and termed it as 'the exponential logarithmic distribution'. Chahkandi and Ganjali [34] introduced 'the exponential binomial distribution' as a decreasing failure rate family of distributions, which can be viewed as an alternative to both exponential geometric distribution and exponential logarithmic distribution. Kus [35] considered another class of decreasing failure rate distribution namely 'the exponential Poisson distribution (EPD)'.

After many years, researchers find that there are lot of situations where the hazard rate shows non-monotone failure patterns. The best example of such situation is the human life. Then they search different models that show non-monotone hazard rates and find that for different choices of parameters Weibull and gamma distributions' failure rate give non-monotone hazard rate. Many authors studied and developed non-monotone hazard rates by modifying existing models or generalizing the existing models. The non-monotone distributions serve as adequate models for the survival time of many industrial products.

Such failure rate curves are also known as the U-shaped or J-shaped curves. Many parametric families of bathtub curve failure rate distributions have been introduced in literature during past several years.

New bathtub curve fitting models were proposed one by one since the 1980s. Glaser [36] and Lawless [2] have been given many examples of bathtub curve failure rate life distributions. Hjorth [38] described bathtub curve failure rate distributions by mixtures of a set of increasing failure rate distribution for competing risk model. Lai et al. [37] discussed the bathtub curve failure rate distributions. Xie et al. [48] studied modified Weibull extension models with bathtub curve failure rate function useful in reliability related decision making and cost analysis. Xie et al. [40] investigated some models extending the traditional two-parameter Weibull distribution. Navarro and Hernandez [41] studied the shape of reliability functions by using the s-equilibrium distribution of a renewal process and also studied how to obtain distribution with bathtub curve failure rate using mixture of two positive truncated Normal distributions. Kundu [42] proposed two parameter exponentiated Exponential distribution and discussed several properties and different estimation procedures. Wondmagegnehu et al. [43] studied the failure rate of the mixture of an exponential distribution and a Weibull distribution. Block et al. [44] discussed the continuous mixture of whole families of distribution having a bathtub curve failure rate functions. Sarhan and Kundu [45] derived the generalized linear failure rate distribution and its properties. Extension of Weibull distributions to make it compatible with bathtub curve failure rate data are introduced by Mudholkar and Srivastava [46], Xie and Lai [47], and Xie et al. [48]. Chen [49] also introduced a two parameter bathtub curve failure rate model for survival data analysis. Wang [50] studied an additive model based on the Burr XII distribution for lifetime data with bathtub curve failure rate. Wang et al. [51] derived Weibull extension with bathtub curve failure rate function based on type-II censored samples.

Recently, Lemonte [53] proposed a new exponential type distribution with constant failure rate, increasing, decreasing, inverse bathtub and bathtub failure rate functions which can be used in modeling survival data in reliability problems and fatigue life studies. Zhang et al. [54] investigated the parameter estimation of 3-parameter Weibull related model with decreasing, increasing, bathtub and upside-down bathtub shaped failure rates. Parsa et al. [55] investigated the difference between the change points of failure rate and mean residual life functions of some generalized Gamma type distribution due to the capability of these distribution in modeling various bathtub curve failure rate functions. Wang et al. [56] discussed new finite interval lifetime distribution model for fitting bathtub curve failure rate curve. Shehla and Ali khan [57] studied reliability analysis using an exponential power model with bathtub curve failure rate function. Zeng et al. [58] derived two lifetime distributions, one with 4 parameters and the other with 5 parameters, for the modeling of bathtub curve failure rate data. Cordeiro et al. [59] have recently introduced and studied new bathtub shaped failure rate distributions like Beta extended Weibull family respectively. Maurya et al. [60] introduced new distribution which shows increasing, decreasing and bathtub curve failure rate functions. A new generalization of the Weibull-geometric distribution with bathtub failure rate have proposed by Nekoukhou and Bidram [61]. Recent books of Lai and Xie [12] and also Marshall and Olkin [3] give extensive discussions concerning mixtures of survival functions and also bathtub-shaped failure rates.

Shafiq and Viertl [62] proposed generalized estimators for the parameters and failure rates of bathtub curve failure rate distributions used to model fuzzy lifetime data. Cordeiro et al. [63] introduced new Lindley Weibull distribution which accommodates unimodal and bathtub shaped failure rates. Dey et al. [64]

introduced a new distribution alpha-power transformed Lomax distribution with decreasing and inverse bathtub curve failure rate distribution. Al-abbasi et al. [65] proposed a three parameter generalized Weibull uniform distribution that extends the Weibull distribution to have bathtub curve failure rate or decreasing failure rate property. Shoaee [66] investigated two bivariate models, viz., bivariate Chen distribution and bivariate Chen-Geometric distribution, that has bathtub curve failure rate or increasing failure rate functions. Ahsan et al. [67] studied the reliability analysis of gas-turbine engine with bathtub curve failure rate distribution. Chen and Gui [68] discussed the estimation problem of two parameters of a lifetime distributions with a bathtub curve failure rate functions based on adaptive progressive type-II censored data.

Abd-Elrahman [69] derived a new distribution with Increasing, decreasing and inverse bathtub curve failure rate functions. For the characterization of bathtub and other failure rate functions, see Glaser [36]. Bourguignon et al. [70] describe in detail the decreasing failure rate and inverse bathtub curve failure rate functions. Dimitrakopoulou et al [71], Sharma et al [72], Alkarni [73], Maurya et al [74], and Kavya and Manoharan [75] developed and studied inverse bathtub shaped hazard rate distributions.

The following are the specific objectives of the thesis. Inspired by our extensive review of the literature, we try to develop a new transformation for generating lifetime distributions. Our aim is to obtain a transformation such that the generated distributions are convenient to use in terms of mathematical calculation, simulation and data analysis. Next, we construct new lifetime models by applying existing distributions in the newly introduced transformation. The study of statistical properties is an important part of our work. In order to examine the flexibility and suitability of the proposed models, we compare the new models with other major lifetime distributions. It is also our objective to generalize the newly developed transformation to construct other lifetime distributions and examine their properties. Yet another objective is to estimate the stress-strength reliability for the proposed model.

1.7 An overview of the thesis

The thesis mainly focusses on the development of new lifetime distributions which are parsimonious. Chapter 1 reviews the existing literature and provides an introduction to our work. In chapter 2 we develop a new transformation which is parsimonious in parameters. We go on to discuss some of the properties of the transformation in the same chapter. Construction of three new lifetime distributions using the transformation and an examination of their properties are carried out in Chapter 3. Chapter 3 also includes a discussion of its flexibility and suitability. In Chapter 4, we generalize the transformation and develop new lifetime model using the exponential distribution as the baseline distribution. The chapter also includes the properties, simulation study and real life data analysis. Estimation of stress-strength reliability for the newly proposed model is the principal focus of Chapter 5, along with a discussion on the asymptotic distribution of stress-strength reliability, simulation study and application. In the final chapter we incorporate the conclusions and briefly outline potential lines of research for the future.

Chapter 2

Introduction of New Transformation For Lifetime Models

2.1 Introduction

The development of statistical distributions is an old but ever relevant topic in statistics[1]. After the monumental work by Pearson [76] for generating statistical distributions using the system of differential equation approach, several general methods have been developed for generating family of distributions. Burr [77] a method based on differential equation that can take on a wide variety of shapes for continuous distributions. Johnson's [78] translation method for generating system of frequency curves and the quantile method to generate distributions proposed by Hastings et al. [79] and Tukey [80] are other milestone set in this direction.

For the past several years, many authors have studied and introduced different methods to propose new distributions using the existing distributions as the

[1]This chapter is based on Kavya, P., Manoharan, M. (2021)

baseline distribution. For example, if $G(x)$ is some baseline distribution function, then a new distribution function is obtained as

$$F(x) = (G(x))^{\alpha},$$

where $\alpha > 0$ is a shape parameter. This is the well-known Lehmann family found useful in modeling failure time data as revealed in several research works, see, e.g., Gupta et al. [81]. Several useful distributions have been proposed in the literature using this idea see Nadarajah and Kotz [82] and Nadarajah [83].

Since 1980, methodologies of generating new distributions shifted to adding parameters to an existing distribution or combining existing distributions (see Lee et al. [84]). For instance, method of skew distributions by Azzalini ([85], [86]). By compounding a continuous distribution with a classic discrete distribution Adamidis and Loukas [32] have introduced exponential-geometric (EG), and Kus [87] has proposed exponential-Poisson distribution. Both these distributions show a decreasing failure rate. Generalized exponential-power series class of distributions given by Mahmoudi and Jafari [88] and Silva et al. [89] proposed a compound class of extended Weibull-power series distributions. Method of adding parameters to an existing distribution, the notable works were done by Mudholkar and Srivastava [46] and Marshall and Olkin [90]. Cordeiro et al. [91] proposed a new class of distributions by adding two new shape parameters. In most of the methods, the authors add new parameters to the existing models. Even though it gives greater flexibility to the model, the estimation of the parameters can become complicated.

Quadratic Rank Transmuted Map (QRTM) is a technique to generalize a baseline distribution, which is introduced by Shaw and Buckly [92]. Various generalizations have been introduced based on QRTM in the literature. Some of such generalized distributions are, transmuted extreme value distribution proposed by Aryal and Tsokos [93], transmuted log-logistic distribution given by Aryal [94], and transmuted inverse Weibull distribution introduced by Khan et al. [95].

Other well-known methods like beta-generated method proposed by Eugene et al [96], transformed-transformer method (T-X family) by Alzaatreh et al. [97], composite method by Cooray and Ananda [98], and inverse probability integral transformation method by Ferreira and Steel [99] are available in the literature.

Kumar et al. [100] introduced a method to get new lifetime distributions called DUS (Dinesh-Umesh-Sanjay) transformation. They also proposed another method by using the sine function (Kumar et al. [101]) to construct new life distributions. The main advantage of both of these methods is that the new distributions preserve the property of being parsimonious in the parameter. Recently, Maurya et al. [60] proposed a generalized DUS transformation for producing some useful lifetime distributions.

The readers can consult the survey article by Kotz and Vicari [102] which highlighted the milestone of the early development of statistical distributions. Further, Alexander et al. [103] mentioned the new techniques for building meaningful distributions are widely investigated, including the two-piece approach by Hansen [104], the pertubation approach of Azzalini and Capitanio [105], and compounding approach by Barreto-Souza et al. [106].

The principle of parsimonious modeling of lifetimes has regained its importance in recent times. The value of stochastic modeling in dealing with the inevitable uncertainty and risk is nowadays highly appreciated. However, several families of distributions/models used in stochastic modeling of lifetimes are often non-parsimonious, unnatural, theoretically unjustified, and sometimes unnecessary. In this chapter, we propose a transformation for obtaining a new class of distributions. The distributions obtained from this transformation are parsimonious in parameter.

2.2 KM Transformation

Our motivation was to introduce a new transformation without additional parameters to generate probability distributions and which fit data reasonably well. Based on our study of the relevant literature in the area, we were motivated to try different transformations. The one we describe below have many properties that make it suitable in a variety of situations. The success of the transformation, however, has to be decided based on how well it can fit real life data in different situations. The newly introduced transformation will now on be referred to as the Kavya-Manoharan (KM) transformation.

Let X be a random variable with cumulative distribution function (cdf) $G(x)$ and probability density function (pdf) $g(x)$ of some baseline distribution. Then the cdf $F(x)$ of new distribution is defined as,

$$F(x) = \frac{e}{e-1}[1 - e^{-G(x)}]. \tag{2.1}$$

The first derivative of $F(x)$ gives the pdf $f(x)$ of the new distribution. Then the pdf is

$$f(x) = \frac{e}{e-1} g(x) e^{-G(x)}, \quad -\infty < x < \infty \tag{2.2}$$

The new distribution function and the pdf satisfies all properties of a cdf and a pdf.

The survival function of the model is obtained as

$$\begin{aligned}\overline{F}(x) =& 1 - F(x) \\ =& \frac{e^{1-G(x)} - 1}{e - 1}\end{aligned} \tag{2.3}$$

The main objective of our study is to introduce a transformation that yields new lifetime models/distributions by using a given baseline distribution. We do not add any additional parameters to keep the model adjusted to the existing uncertainty and focus on modeling the lifetime with a procedure that supplies correct parsimonious results. The procedure transforms the response variable to achieve a model with interesting ageing properties as revealed through the hazard rates.

2.3 Hazard rate function

The hazard rate measures the propensity of an item to fail or die depending on the age it has reached. It is part of a wider branch of statistics called reliability theory, which is a set of methods for predicting the amount of time until a certain event occurs, such as the death or failure of an engineering system or component. The concept is applied to other branches of research under slightly different names, including duration analysis (economics), and event history analysis (sociology).

To find the hazard rate of the proposed model, substitute Equations (2.2) and (2.3) in (1.2), then the hazard rate function is obtained as

$$h(x) = \frac{g(x)e^{1-G(x)}}{e^{1-G(x)} - 1}. \tag{2.4}$$

2.4 Shapes of density and hazard rate functions

Here the shapes of the density and hazard functions are explained analytically. We first find the derivative of the density function with respect to x and equate it to zero.

$$f'(x) = \frac{e^{(1-G(x))}}{e-1}[g'(x) - (g(x))^2] = 0 \tag{2.5}$$

Here (2.5) may have more than one root. Suppose $x = x_0$ is a root of (2.5), then it corresponds to a local maximum if $f''(x) < 0$, a local minimum if $f''(x) > 0$, and a point of inflection if $f''(x) = 0$.

In similar way, the critical points of $h(x)$ are the roots of the equation

$$h'(x) = \frac{g'(x)e^{1-G(x)}}{e^{1-G(x)} - 1} - \frac{(g(x))^2 e^{1-G(x)}}{e^{1-G(x)} - 1} + \frac{(g(x))^2 e^{2(1-G(x))}}{(e^{1-G(x)} - 1)^2} = 0. \tag{2.6}$$

Here (2.6) may have more than one root. Suppose $x = x_0$ is a root of (2.6), then it corresponds to a local maximum if $h''(x) < 0$, a local minimum if $h''(x) > 0$, and a point of inflection if $h''(x) = 0$.

2.5 The reversed hazard rate function

The concept of reversed hazard rate of a random life is defined as the ratio between the life probability density to its distribution function. This concept plays a role in analyzing censored data and is applicable in such areas as Forensic Sciences. The reversed hazard rate of a random variable X is defined by

$$r_h(x) = \frac{f(x)}{F(x)}, \qquad F(x) > 0 \tag{2.7}$$

The reversed hazard function of the new distribution is,

$$r_h(x) = \frac{g(x)e^{-G(x)}}{1 - e^{-G(x)}} \tag{2.8}$$

2.6 Summary of the chapter

In this chapter, we have introduced a new transformation for lifetime models. The main advantage of the KM transformation is that there is no added parameters required. So, mathematical calculations, estimation of parameters, and simulation study of new lifetime models become convenient. The general equations for pdf, cdf, survival function, hazard rate and reversed hazard rate functions for the lifetime models are discussed. In the following chapters we gauge the success of the newly introduced transformation by applying it to different existing lifetime models and comparing the resultant models to real data.

Chapter 3

Development of New Lifetime Models Using KM Transformation

3.1 Introduction

Exponential, Weibull, and Lomax distributions have wide applications in reliability and survival analysis[1]. Using these distributions as the baseline models, many authors developed new distributions. Many types of generalized exponential distributions obtained by Khan and Jain [107]. Gupta and Kundu [108] introduced the three-parameter generalized exponential distribution and the monotonicity of the failure rates were studied. Gupta and Kundu [109] studied some properties of a new family of distributions, namely exponentiated exponential distribution, discussed in Gupta et al. [81]. Khan et al. [95] proposed the transmuted generalized exponential distribution using the QRTM method. Adamidis and Loukas [32] constructed a two parameter lifetime distribution by compounding the ex-

[1]This chapter is based on Kavya, P, Manoharan, M. (2021), and Manoharan, M., Kavya, P. (2022)

ponential distribution with the geometric distribution, called the exponential geometric (EG) distribution. Kus [87] obtained a compound of the exponential distribution with that of Poisson and it was named the exponential Poisson (EP) distribution. Tahmasbi and Rezaei [33] obtained the exponential logarithmic (EL) distribution by using the same compounding mechanism. The extended EG distribution was considered by Adamidis et al. [110].

Oguntunde et al. [111] studied the Weibull-exponential distribution, its properties and applications. Ahmad et al. [112] developed a new Weibull-X family of distributions along with its properties, characterizations and applications. Kharazmi [113] generalize the Weibull distribution to a new class referred to as the generalized weighted Weibull (GWW) distribution with one scale parameter and two shape parameters. It is investigated that the new model has increasing, decreasing and upside-down bathtub shaped hazard. Elbatal et al. [114], Kavya and Manoharan [75], and Ishaq and Abiodun [115] proposed new lifetime models using Weibull distribution.

The Lomax distribution has wide applications in many fields like economics, actuarial science, and so on. The Lomax distribution is also called Pareto Type II distribution. The distribution was introduced by Lomax [116] and it is a heavy-tailed distribution. It has also been useful in reliability and life testing problems in engineering and survival analysis as an alternative distribution [117, 118]. The Lomax distribution shows decreasing failure rate. Modified and extended versions of the Lomax distribution have been studied; examples include the weighted Lomax distribution [118], exponential Lomax distribution [119], exponentiated Lomax distribution [72], gamma Lomax distribution [121], transmuted Lomax distribution [97], Poisson Lomax distribution [123], McDonald Lomax distribution [53], Weibull Lomax distribution [124], power Lomax distribution [125], Kumaraswamy-Generalized Lomax distribution [126], Gompertz-Lomax distribution [127], and DUS-Lomax distribution [128]. Besides, estimation of the parameters of Lomax distribution under general progressive censoring has been considered by Al-Zahrani and Al-Sobhi [129].

The main objective of this chapter is to introduce three new lifetime models using KM transformation and study the analytical characteristics and the applicability of these models in reliability and survival analysis.

3.2 KM-Exponential model

The exponential distribution has been widely used in reliability and survival analysis because of its simplicity and analytical tractability. The exponential distribution's hazard function is a constant, but this need not be the case in real-life situations. So we try to obtain a new distribution with a non-constant hazard function by using the exponential distribution as the baseline distribution in our newly proposed transformation. Exponential distribution has pdf $g(x) = \lambda e^{-\lambda x}, \quad x > 0, \ \lambda > 0$. Now employing the KM transformation and using this in Equation (2.2), we get a new distribution. We call this the KM-Exponential (KME) distribution.

The pdf and cdf of the KME distribution are respectively,

$$f(x) = \frac{\lambda e^{-\lambda x} e^{e^{-\lambda x}}}{e-1}, \quad x > 0, \ \lambda > 0, \tag{3.1}$$

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-\lambda x})}], \quad x > 0, \ \lambda > 0, \tag{3.2}$$

and the hazard rate function is,

$$h(x) = \frac{\lambda e^{-\lambda x} e^{e^{-\lambda x}}}{e^{e^{-\lambda x}} - 1}, \quad x > 0, \ \lambda > 0. \tag{3.3}$$

The graphical representation gives a better understanding of the shapes of the

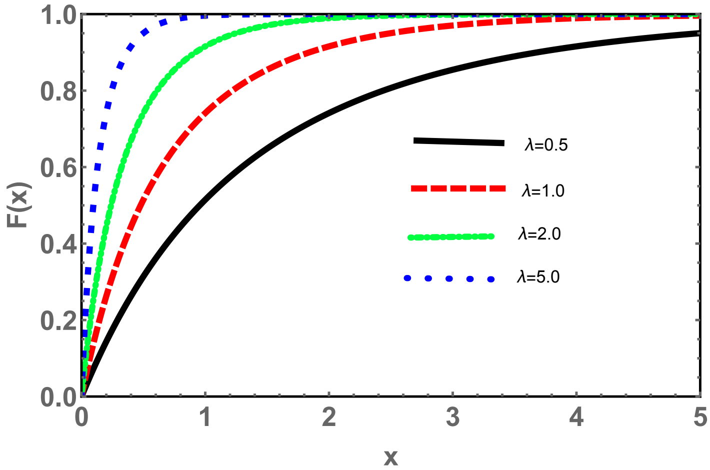
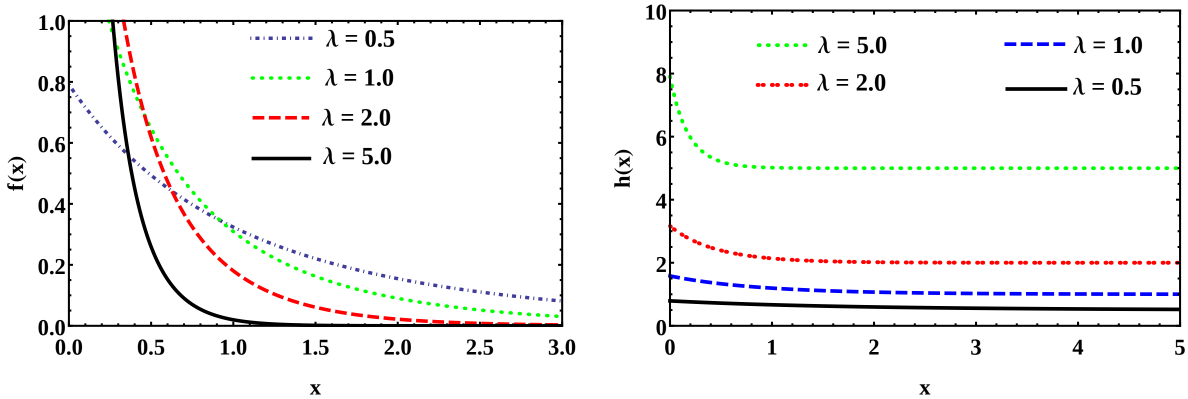


Figure 3.1: The cumulative distribution function plot of KME distribution

cdf, pdf and hazard rate. The graph is given in Fig. 3.1 and 3.2. Throughout this thesis, we use the software MATHEMATICA [130] for plotting the graphs. By examining the plots, we conclude that the hazard rate function of KME distribution exhibits decreasing failure rate property.



(a) Plot of pdf of the distribution (b) Hazard rate plot of the distribution

Figure 3.2: The pdf and hazard rate plot of KME distribution for different values of parameters.

3.2.1 Moments of KME distribution

The moments are a set of statistical parameters to measure a distribution. The r^{th} raw moment of the proposed distribution is

$$E(X^r) = \frac{\lambda}{e-1}\int_0^\infty x^r e^{-\lambda x} e^{e^{-\lambda x}} dx.$$

Expanding exponential term, we get,

$$\begin{aligned} E(X^r) &= \frac{\lambda}{e-1}\int_0^\infty x^r e^{-\lambda x}\sum_{m=0}^{\infty}\frac{(e^{-\lambda x})^m}{m!}dx, \\ &= \frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^\infty x^r e^{-(\lambda+\lambda m)x}dx, \end{aligned}$$

which reduces to,

$$E(X^r) = \frac{1}{\lambda^r(e-1)}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{r!}{(m+1)^{r+1}}. \tag{3.4}$$

Variance and other higher order central moments can be obtained from this expression. Put $r = 1$ in Equation (3.4), then the mean of the KME distribution is obtained as

$$E(X) = \frac{1}{\lambda(e-1)}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{(m+1)^2}.$$

The variance of the distribution is defined as

$$V(X) = E(X^2) - (E(X))^2$$

The variance of the proposed model is

$$V(X) = \frac{1}{\lambda^2(e-1)}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{2}{(m+1)^3} - \frac{1}{\lambda^2(e-1)^2}\left[\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{(m+1)^2}\right]^2.$$

3.2.2 Moment generating function

Let X be a random variable, then the moment generating function (mgf) is defined as,

$$M_X(t) = E(e^{tX}).$$

$$\begin{aligned} M_X(t) =& \frac{\lambda}{e-1}\int_0^\infty e^{tx}e^{-\lambda x}e^{e^{-\lambda x}}dx \\ =& \frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\int_0^\infty e^{-(\lambda+\lambda m-t)x}dx \\ =& \frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{(\lambda+\lambda m-t)} \quad for\ t<\lambda. \end{aligned}$$

We can also obtain the r^{th} raw moment from mgf by the equation

$$\mu_r^{'} = \frac{d^r M_X(t)}{dt^r}|_{t=0}.$$

3.2.3 Characteristic function

Here we have derived the characteristic function of the proposed model.

$$\Phi_X(t) =\frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{1}{(\lambda+\lambda m-it)},$$

where $i = \sqrt{-1}$.

3.2.4 Median

If M is the median of the distribution, then $\int_{-\infty}^{M} f(x)dx = 1/2$. Median of the new distribution is obtained as,

$$M = \frac{-1}{\lambda}log\left[1+log\left(\frac{e+1}{2e}\right)\right] \tag{3.5}$$

3.2.5 Mode

The mode is the value which occurs most frequently in a data set. The mode is only of interest for large data sets, as in small samples the mode strongly depends on random variations of the data. The relative position of the mode, the median, and the mean provides an indication of the skewness of a distribution. The pdf of the KME model is,

$$f(x) = \frac{\lambda}{e-1} e^{-\lambda x} e^{e^{-\lambda x}}$$

The mode of the KME (λ) can be obtained as a solution of the following nonlinear equation

$$\frac{d}{dx} f(x) = 0$$

Therefore,

$$f'(x) = \frac{-\lambda^2}{e-1} e^{-\lambda x} e^{e^{-\lambda x}} [1 + e^{-\lambda x}] = 0$$

It is not possible to obtain the explicit solution in the general case. It has to be obtained numerically.

3.2.6 Mean deviation

The mean deviation about mean is defined as

$$\begin{aligned}
\delta_1(x) &= \int_0^\infty |x - \mu| f(x) dx \\
&= \int_0^\mu (\mu - x) f(x) dx + \int_\mu^\infty (x - \mu) f(x) dx,
\end{aligned}$$

where μ is the mean. After simplification, we get,

$$\delta_1(x) = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} xf(x)dx,$$

where $F(.)$ is the proposed cdf, and

$$\begin{aligned}\int_{\mu}^{\infty} xf(x)dx =& \frac{\lambda}{e-1}\int_{\mu}^{\infty} xe^{-\lambda x}e^{e^{-\lambda x}}dx,\\ =& \frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{\mu}^{\infty} xe^{-(\lambda+\lambda m)x}dx.\end{aligned}$$

The complementary incomplete gamma function is defined as $\Gamma(n,x) = \int_x^{\infty} t^{n-1}e^{-t}dt$, which can also be written as $(n-1)!e^{-x}\sum_{k=0}^{n-1}\frac{x^k}{k!}$. Applying this result in the above expression, we get,

$$\delta_1(x) = 2\mu F(\mu) - 2\mu + \frac{2\lambda}{e-1}\sum_{m=0}^{\infty}\frac{e^{-(\lambda+\lambda m)\mu}(1+(\lambda+\lambda m)\mu)}{m!(\lambda+\lambda m)^2}.$$

Mean deviation about median is defined as

$$\begin{aligned}\delta_2(x) =& \int_0^{\infty} |x-M|f(x)dx,\\ =& \int_0^{M}(M-x)f(x)dx + \int_M^{\infty}(x-M)f(x)dx,\end{aligned}$$

where M denotes median. After simplification, we get

$$\delta_2(x) = -\mu + 2\int_M^{\infty} xf(x)dx.$$

Using complementary incomplete gamma function, we can write

$$\int_M^{\infty} xf(x)dx = \frac{\lambda}{e-1}\sum_{m=0}^{\infty}\frac{e^{-(\lambda+\lambda m)M}(1+(\lambda+\lambda m)M)}{m!(\lambda+\lambda m)^2}.$$

Therefore, mean deviation about median is

$$\delta_2(x) = -\mu + \frac{2\lambda}{e-1}\sum_{m=0}^{\infty}\frac{e^{-(\lambda+\lambda m)M}(1+(\lambda+\lambda m)M)}{m!(\lambda+\lambda m)^2}.$$

3.2.7 Quantile function

The quantile function is useful when generating random observations from a distribution. It can also be utilized in estimating measures of shapes (skewness and kurtosis) when the moments of the random variable do not exist. The p^{th} quantile function $Q(p)$ of a distribution is defined as

$$F(Q(p)) = p.$$

Here,

$$\begin{aligned}\frac{e}{e-1}[1-e^{-(1-e^{-\lambda Q(p)})}] =& p\\ e^{-\lambda Q(p)} =& 1+\log\left(1-\frac{p(e-1)}{e}\right)\end{aligned}$$

The p^{th} quantile function of the KME distribution is obtained as,

$$Q(p) = \frac{-1}{\lambda}\log\left[1+\log\left(1-\frac{p(e-1)}{e}\right)\right]. \tag{3.6}$$

3.2.8 Order statistic

Let $X_1,\ X_2,\cdots,X_n$ be a random sample of size n from the proposed distribution and $X_{(1)},X_{(2)},\cdots,X_{(n)}$ denote the corresponding order statistics. The pdf and cdf of the r^{th} order statistics $f_r(x)$ and $F_r(x)$ are given by

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}F^{r-1}(x)[1-F(x)]^{n-r}f(x)$$

and

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} F^j(x)[1 - F(x)]^{n-j}.$$

The pdf $f_r(x)$ and cdf $F_r(x)$ of r^{th} order statistic of our proposed distributions are obtained by using the pdf and cdf of respective distributions.

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!} \frac{\lambda e^{-(\lambda x - r + 1)} e^{e^{-\lambda x}}}{(e-1)^n} \left(1 - e^{-(1-e^{-\lambda x})}\right)^{r-1} \left(e^{e^{-\lambda x}} - 1\right)^{n-r} \tag{3.7}$$

and

$$F_r(x) = \sum_{j=r}^{n} \binom{n}{j} \frac{e^j}{(e-1)^n} \left(1 - e^{-(1-e^{-\lambda x})}\right)^{j} \left(e^{e^{-\lambda x}} - 1\right)^{n-j}. \tag{3.8}$$

The pdf of the smallest and the largest order statistics $X_{(1)}$ and $X_{(n)}$ are obtained by putting $r = 1$ and $r = n$ respectively in Equation (3.7). The cdf of $X_{(1)}$ and $X_{(n)}$ are obtained by putting $r = 1$ and $r = n$ respectively in Equation (3.8).

3.2.9 Entropy

Entropy is interpreted as the degree of disorder or randomness in the system. The concept of entropy was proposed by Shannon ([131], [132]) in his paper 'A Mathematical Theory of Communication' and is also referred to as Shannon entropy. Entropy has relevance to other areas of mathematics such as combinatorics and machine learning. The definition can be derived from a set of axioms establishing that entropy should be a measure of how 'surprising' the average outcome of a variable is. For a continuous random variable, differential entropy is analogous to entropy. In this section we have obtain three types of entropies.

Reńyi **entropy**

Reńyi entropy (*Reńyi* [133]) is one of the well known entropy measures. If random variable X has the pdf $f(x)$, then the *Reńyi* entropy is defined as,

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} \log \left[\int f^{\gamma}(x) dx \right], \tag{3.9}$$

where $\gamma > 0$ and $\gamma \neq 1$. From Equation (4.5), we get,

$$\begin{aligned} \int_0^{\infty} f^{\gamma}(x) dx =& \frac{\lambda^{\gamma}}{(e-1)^{\gamma}} \int_0^{\infty} (e^{-\lambda x})^{\gamma} (e^{e^{-\lambda x}})^{\gamma} \\ =& \frac{\lambda^{\gamma}}{(e-1)^{\gamma}} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!(\gamma+m)\lambda}. \end{aligned} \tag{3.10}$$

Using above result, Equation (3.9) becomes,

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma} \log \left(\frac{\lambda}{e-1} \right) + \frac{1}{1-\gamma} \log \left(\sum_{m=0}^{\infty} \frac{\gamma^m}{m!(\gamma+m)\lambda} \right). \tag{3.11}$$

Tsallis entropy

The Tsallis entropy measure (Tsallis [134]) is defined by,

$$T_{\gamma}(x) = \frac{1}{\gamma-1} \left[1 - \int_0^{\infty} f^{\gamma}(x) dx \right], \quad \gamma \neq 0,\ \gamma > 0. \tag{3.12}$$

After calculation,

$$T_{\gamma}(x) = \frac{1}{\gamma-1} \left[1 - \frac{\lambda^{\gamma}}{(e-1)^{\gamma}} \sum_{m=0}^{\infty} \frac{\gamma^m}{m!(\gamma+m)\lambda} \right].$$

Havrda and Charvat entropy

The Havrda and Charvat entropy measure (Havrda and Charvat [135]) is defined by:

$$HC_\gamma(x) = \frac{1}{2^{1-\gamma}-1}\left[\left(\int_0^\infty f^\gamma(x)dx\right)^{\frac{1}{\gamma}} - 1\right], \quad \gamma \neq 0,\ \gamma > 0. \tag{3.13}$$

The Havrda and Charvat entropy measure of the proposed model is obtained as,

$$HC_\gamma(x) = \frac{1}{2^{1-\gamma}-1}\left[\frac{\lambda}{e-1}\left(\sum_{m=0}^{\infty}\frac{\gamma^m}{m!(\gamma+m)\lambda}\right)^{\frac{1}{\gamma}} - 1\right].$$

3.2.10 Extropy

An alternative measure of uncertainty, called extropy was introduced by Lad et al. [136]. Suppose an absolutely continuous non-negative random variable X has pdf f and cdf F, then the extropy is defined as

$$I(x) = \frac{-1}{2}\left[\int_0^\infty (f(x))^2 dx\right]. \tag{3.14}$$

Put $\gamma = 2$ in Equation (3.10), we get

$$I(x) = \frac{-1}{2}\left[\frac{\lambda}{e-1}\left(\sum_{m=0}^{\infty}\frac{2^m}{m!(m+2)\lambda}\right)\right].$$

3.2.11 Ordering

By the ageing of a mathematical unit, component or some other physical or biological system, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one. Many criteria of ageing have been developed in the literature. The stochastic comparison of distributions has been an important area of research in many diverse areas of statistics and probability. We are comparing two lifetime variables X and Y in terms of their failure rates $h_F(t)$ and $h_G(t)$, density functions $f(t)$ and $g(t)$, survival functions $\overline{F}(t)$ and $\overline{G}(t)$, mean residual lives $\mu_F(t)$ and $\mu_G(t)$ or other ageing characteristics. Ageing classes can often be characterized by some partial ordering. For example, in Barlow and Proschan [137], IFR and IFRA classes are characterized by convex ordering and star-shaped ordering respectively. Many different types of stochastic orders have been studied in the literature; for example Deshpande et al [11] and a comprehensive discussion of ordering is available in Shaked et al [138]. It is often easy to make value judgements when such ordering exist. Stochastic ordering between two probability distributions, if it holds, is more informative than simply comparing their means or medians only. Similarly, if one wishes to compare the dispersion or spread between two distributions, the simplest way would to be to compare their standard deviations or some such other measures of dispersion.

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. There are different types of stochastic orderings that are useful in ordering random variables in terms of different properties. Here we consider four different stochastic orders, namely, the usual, the hazard rate, the mean residual life, and likelihood ratio order for proposed distributions. If X and Y are two random variables with cumulative distribution functions F_X and F_Y, respectively, then X is said to be smaller than Y in the

- stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

The implication between the ordering is $X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow$

$X \leq_{st} Y$. The KM family distributions proposed here are ordered with respect to the strongest "likelihood ratio" ordering as shown in the following theorem. It shows the flexibility of the proposed distribution.

Theorem 1. *Let $X \sim$ KME(λ_1) and $Y \sim$ KME(λ_2) if $\lambda_1 \geq \lambda_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.*

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1 \; e^{-\lambda_1 x} \; e^{e^{-\lambda_1 x}}}{\lambda_2 \; e^{-\lambda_2 x} \; e^{e^{-\lambda_2 x}}} \tag{3.15}$$

and

$$\log \frac{f_X(x)}{f_Y(x)} = \log \lambda_1 - \lambda_1 x + e^{-\lambda_1 x} - \log \lambda_2 + \lambda_2 x - e^{-\lambda_2 x}$$

thus,

$$\frac{d}{dx} \log \frac{f_X(x)}{f_Y(x)} = -\lambda_1 - \lambda_1 e^{-\lambda_1 x} + \lambda_2 + \lambda_2 e^{-\lambda_2 x} \tag{3.16}$$

If $\lambda_1 \geq \lambda_2$

$\frac{d}{dx} \log \frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. □

3.2.12 Estimation

The maximum likelihood estimation method is one of the most common methods for finding the estimates involved in the distribution. In this method, we maximize the logarithm of the likelihood function for finding the estimates. By this method, we obtain the maximum likelihood estimate of the parameter λ of the KME distribution. The likelihood function is defined as,

$$L(x;\lambda) = \prod_{i=1}^{n} f(x_i, \lambda)$$

In our distribution,

$$L(x;\lambda) = \left(\frac{1}{e-1}\right)^n \lambda^n e^{-\lambda \sum x_i} e^{\sum e^{-\lambda x_i}}.$$

The log-likelihood function of the distribution is given by,

$$\log\ L(x;\lambda) = -n\ \log(e-1) + n\ \log(\lambda) - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} e^{-\lambda x_i}.$$

Partial derivative of the log-likelihood function with respect to the parameter λ is

$$\frac{\partial \log\ L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i e^{-\lambda x_i}.$$

Equating this partial derivative to zero yields a non - linear equation, and the solution provides the maximum likelihood estimate of the parameter λ. The Newton-Raphson method can be used to solve this equation with the help of the available statistical packages. In this work, we use R [139] language for finding the numerical solution of the non-linear system of equations.

3.2.13 Simulation study

In this section, we use the Monte Carlo simulation method to study the estimators' performance of the parameters involved in the newly proposed distribution. In each experiment, for different values of the population parameters and sample sizes, 1000 pseudo-random samples have been generated according to Equation (3.6) as follows: $x_i = \frac{-1}{\lambda} \log \left[1 + \log\left(1 - \frac{u_i(e-1)}{e}\right)\right]$ for $i = 1, 2, ..., n$. Here $u_1, u_2, ..., u_n$ are independent random observations from the standard uniform distribution. We calculate the estimator of the parameter for different values of n and different population parameters for the proposed distribution. Twenty five combinations of the parameter were considered in the KME distribution: $n = 25$, 50, 100, 500, 1000, $\lambda = 0.5$, 1, 1.5, 2 and 2.5. For 1000 repetitions, the standard error (SE) of the estimated parameters is computed as the square root of the average of their corresponding variance. We use R [139] language for simulation studies in this paper. The simulation study result for the distribution is shown in Table 3.1.

From Table 3.1, it is clear that the estimates are quite stable and are close to the true value of the parameter for the given sample sizes. Also, as sample size increases, the standard error of the maximum likelihood estimates decreases as expected.

3.2.14 Real data applications

In this section, we examine how the proposed distribution work in real-life data. For checking their flexibility, we compare the KME distribution with other well-known distributions available in the literature. Here R [139] language is used for all the computation. We consider a set of real data of Wheaton River obtained from Choulakian and Stephens [140] for our proposed KME distribution. Akinsete et al. [141], Lemonte [53], Cordeiro et al. [91], Bourguignon et al. [70], and Nekoukhou and Bidram [61] have also used this data for their comparison purpose. The data represents the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. This data provides 72 exceedances from the years 1958 to 1984, presented in Table 3.2.

Here we use AIC (Akaike Information Criterion), BIC (Bayesian Information

	n	$\widehat{\lambda}$	SE($\widehat{\lambda}$)
$\lambda = 0.5$	25	0.5267	0.1091
	50	0.5195	0.0779
	100	0.5057	0.0555
	500	0.5004	0.0179
	1000	0.5002	0.0176
$\lambda = 1$	25	1.0440	0.2183
	50	1.0289	0.1570
	100	1.0148	0.1113
	500	1.0023	0.0499
	1000	1.0015	0.0353
$\lambda = 1.5$	25	1.5597	0.3245
	50	1.5269	0.2325
	100	1.5116	0.1659
	500	1.5023	0.0748
	1000	1.5017	0.0530
$\lambda = 2$	25	2.0991	0.4361
	50	2.0503	0.3121
	100	2.0281	0.2227
	500	2.0064	0.0999
	1000	2.0002	0.0706
$\lambda = 2.5$	25	2.6035	0.5380
	50	2.5717	0.3920
	100	2.5429	0.2789
	500	2.5076	0.1249
	1000	2.4995	0.0882

Table 3.1: The result of simulation study of KME distribution

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0
12.0	9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0
7.3	22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8
13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5
2.5	27.0								

Table 3.2: Wheaton River Data.

Criterion), K-S (Kolmogorov-Smirnov) test value, and log-likelihood value for the comparison. The distribution which shows minimum AIC, BIC and K-S test values and maximum log-likelihood value is the sign of a better fit for the data set. The AIC and BIC are defined as

$$\text{AIC} = -2\log(\widehat{L}) + 2k,$$

and

$$\text{BIC} = -2\log(\widehat{L}) + k\log(n),$$

where n is the sample size, k is the number of parameters, and $\widehat{L}$ is the maximum value of the likelihood function for the considered distribution. The distributions used for the comparison study are given below:

1. DUS-Exponential(θ) ($DUS_E(\theta)$) distribution Kumar et al. [100] with cdf,

$$F(x) = \frac{1}{(e-1)}\left[e^{(1-e^{-\theta x})} - 1\right], \qquad x > 0, \quad \theta > 0$$

2. SS-Exponential(θ) ($SS_E(\theta)$) distribution Kumar et al. [101] with cdf,

$$F(x) = cos(\frac{\pi}{2}e^{-\theta x}), \qquad x > 0, \quad \theta > 0$$

3. Generalized DUS Exponential (GDUSE) distribution Maurya et al. [60] with cdf,

$$F(x) = \frac{1}{e-1}[e^{(1-e^{-\lambda x})^{\alpha}} - 1], \qquad x > 0, \quad \alpha, \lambda > 0$$

4. Exponentiated Generalized Gumbel (EGGu) distribution Cordeiro et al. [91] with cdf,

$$F(x) = \left[1 - \left(1 - e^{-e^{-(\frac{x-\mu}{\sigma})}}\right)^{\alpha}\right]^{\beta}, \qquad x, \mu \in R, \quad \sigma, \alpha, \beta > 0$$

The comparison of our proposed distribution with these distributions are given in Table 3.3.

Model	ML estimates	LL	AIC	BIC	K-S test value
KME	$\widehat{\lambda} = 0.0632$	-252.0125	506.025	508.3017	0.110
DUS_E	$\widehat{\theta} = 0.09996$	-254.4682	510.9364	513.213	0.186
SS_E	$\widehat{\theta} = 0.04504$	-252.9794	507.9587	510.2354	0.161
GDUSE	$\widehat{\alpha} = 0.6803, \widehat{\lambda} = 0.0812$	-251.6235	507.247	511.8003	0.113
EGGu	$\widehat{\alpha} = 0.0988, \widehat{\beta} = 0.4769, \widehat{\mu} = 2.6317, \widehat{\sigma} = 1.6639$	-256.8889	521.8001	530.8845	0.108

Table 3.3: Maximum likelihood (ML) estimates, log-likelihood (LL), AIC, BIC and K-S test value of the fitted models.

From Table 3.3, we can see that our proposed distribution shows the lowest AIC and BIC values and the largest log-likelihood value among all the distributions considered here. The K-S test value for our distribution is slightly higher than that for the EGGu distribution. Taking everything into account, we conclude that the KME distribution provides a better fit for the data set compared to the other distributions given above. The plot of empirical cdf along with other cdf of the distributions for the first data set is given in Fig. 3.3. for a better understanding of the result. In the comparison plots, we denote DUS-Exponential and SS-Exponential distributions as DUS-Exp and SS-Exp respectively.

Further, we observe that our proposed distribution fits better than the other distributions mentioned in references Akinsete et al. [141], Cordeiro et al. [91], Bourguignon et al. [70], and Nekoukhou and Bidram [61]. The distribution Nadarajah-Haghighi Lindley (NHL) introduced by Pena-Ramirez et al. [142] behaves slightly better than our proposed distribution. It may be noted that the NHL distribution involves more parameters than the KME distribution, making the NHL model's parameter estimation more complicated.

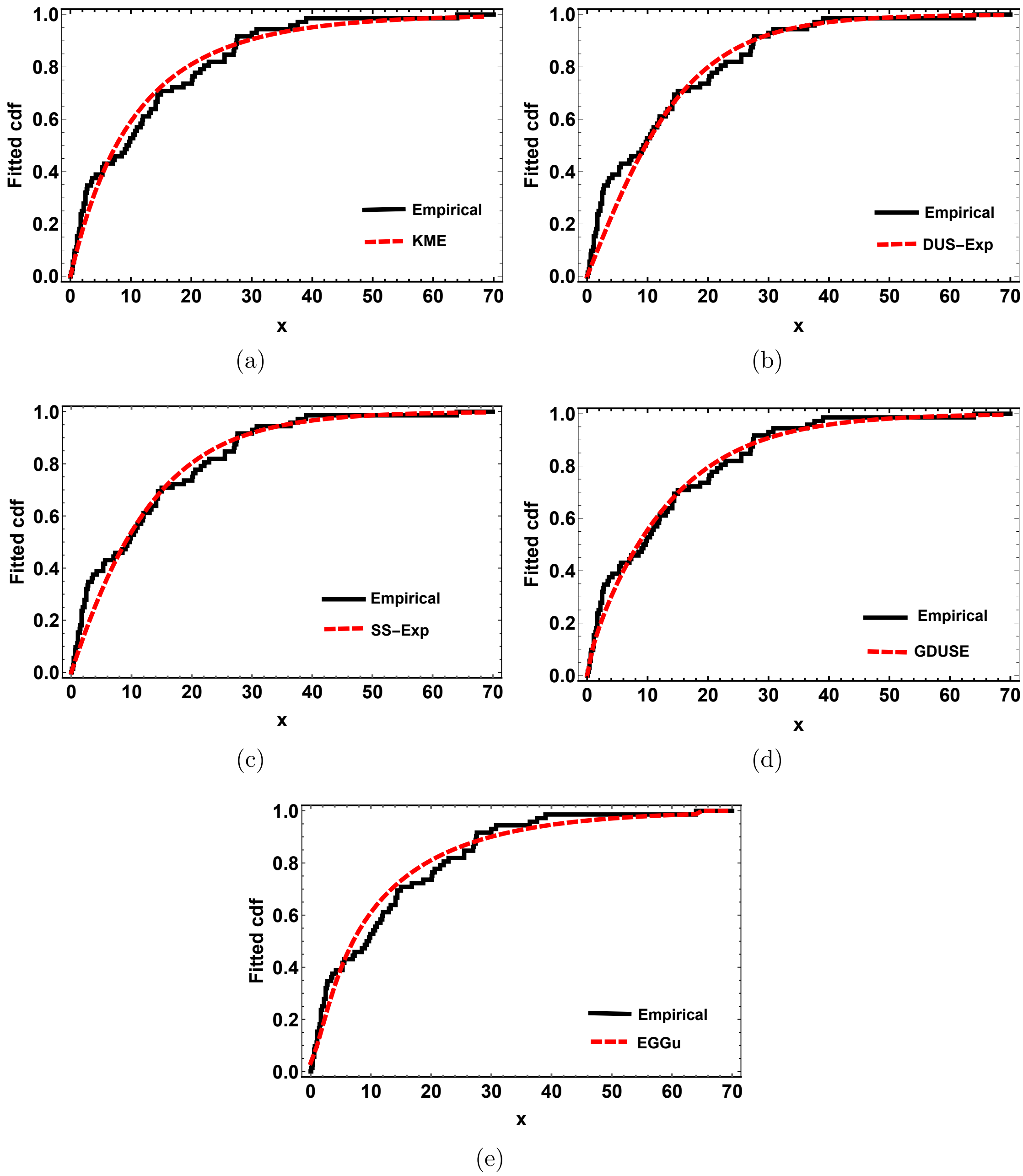


Figure 3.3: The empirical cdf and the cdf plots of the fitted distributions for the Wheaton river data.

3.3 KM-Weibull model

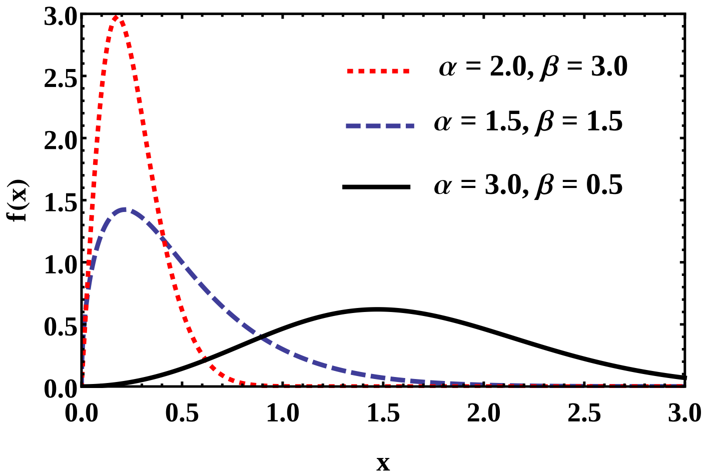
Weibull distribution has wide application in lifetime data analysis. So in this section, we introduce a new distribution called KM-Weibull (KMW) distribution. This is done by using Weibull distribution with cdf $G(x) = 1 - e^{-(\beta x)^{\alpha}}, \quad x >$

0, α, $\beta > 0$ as the baseline distribution in the KM transformation given in Equation (2.1). The cdf, pdf, and hazard rate of the KMW distribution are obtained respectively as

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-(x\beta)^{\alpha}})}], \qquad x > 0, \quad \alpha, \beta > 0 \tag{3.17}$$

$$f(x) = \frac{\alpha\beta^{\alpha}}{e-1} x^{\alpha-1} e^{-(x\beta)^{\alpha}} e^{e^{-(x\beta)^{\alpha}}}, \qquad x > 0, \quad \alpha, \beta > 0 \tag{3.18}$$

The shape of the pdf is given in 3.4.



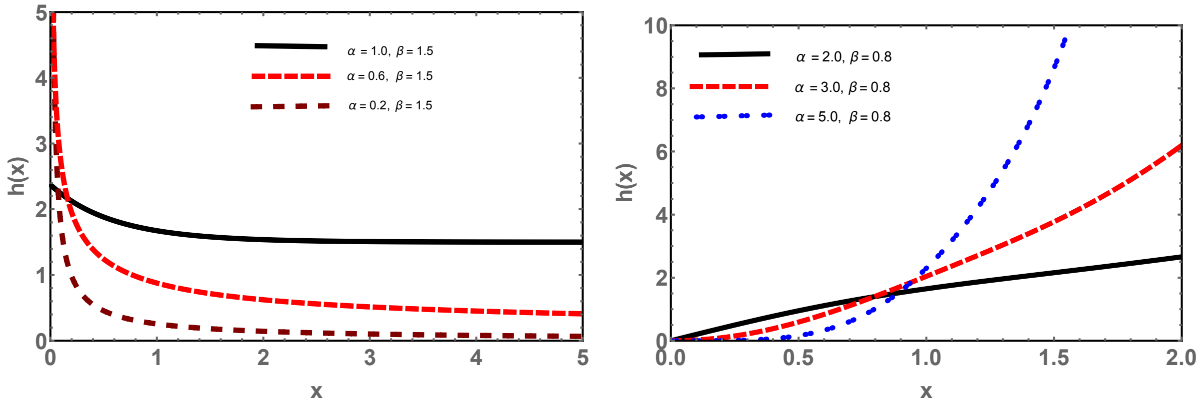
(a) pdf plot of the KMW distribution

Figure 3.4: The pdf plot of KMW distribution for different values of parameters.

The hazard rate function of the model is

$$h(x) = \frac{\alpha\beta^{\alpha}}{e^{e^{-(x\beta)^{\alpha}}} - 1} x^{\alpha-1} e^{-(x\beta)^{\alpha}} e^{e^{-(x\beta)^{\alpha}}} \qquad x > 0, \quad \alpha, \beta > 0. \tag{3.19}$$

The graphical representation of the hazard rate function of KMW distribution for different choices of parameters are given in 3.5.



(a) Decreasing hazard rate plot of the KMW distribution

(b) Increasing hazard rate plot of the KMW distribution

Figure 3.5: The hazard rate plot of KMW distribution for different values of parameters.

3.3.1 Moments of KMW distribution

The r^{th} raw moment of the distribution is obtained as

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r \frac{\alpha\beta^\alpha}{e-1} x^{\alpha-1} e^{-(x\beta)^\alpha} e^{e^{-(x\beta)^\alpha}} \\ &= \frac{1}{\beta(e-1)} \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{\Gamma(\frac{r}{\alpha}-1)}{(m+1)^{\frac{1}{\alpha}}}. \end{aligned}$$

The mean and variance of the proposed model are respectively

$$E(X) = \frac{1}{\beta(e-1)} \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{\Gamma(\frac{1}{\alpha}-1)}{(m+1)^{\frac{1}{\alpha}}}.$$

and

$$V(X) = \frac{1}{\beta(e-1)} \sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{\Gamma(\frac{2}{\alpha}-1)}{(m+1)^{\frac{1}{\alpha}}} - \frac{1}{\beta^2(e-1)^2} \left[\sum_{m=0}^{\infty} \frac{1}{(m+1)!} \frac{\Gamma(\frac{1}{\alpha}-1)}{(m+1)^{\frac{1}{\alpha}}} \right]^2.$$

3.3.2 Moment generating function of the KMW model

The mgf of KMW model is

$$
\begin{aligned}
M_X(t) &= \frac{\alpha\beta^\alpha}{e-1}\int_0^\infty e^{tx}x^{\alpha-1}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} \\
&= \frac{1}{e-1}\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(-1)^n t^n}{(m+1)!n!\beta^n(m+1)^{n\alpha}}\Gamma(\frac{n}{\alpha}+1)
\end{aligned}
$$

3.3.3 Characteristic function of the KMW model

The characteristic function of the KMW model is obtained as,

$$
\begin{aligned}
\phi_X(t) &= \frac{\alpha\beta^\alpha}{e-1}\int_0^\infty e^{itx}x^{\alpha-1}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} \\
&= \frac{1}{e-1}\sum_{n=0}^\infty\sum_{m=0}^\infty \frac{(-1)^n (it)^n}{(m+1)!n!\beta^n(m+1)^{n\alpha}}\Gamma(\frac{n}{\alpha}+1)
\end{aligned}
$$

where $i = \sqrt{-1}$.

3.3.4 Median of KMW model

The median of the KMW distribution is

$$
M = \frac{1}{\beta}\left[-\log\left(1+\log\left(1-\frac{(e-1)}{2e}\right)\right)\right]^{\frac{1}{\alpha}}
$$

3.3.5 Mode of KMW model

We know that the solution of the following nonlinear equation $\frac{d}{dx}f(x) = 0$ provide the mode of the distribution. Here

$$
f'(x) = \frac{\alpha(\alpha-1)\beta^\alpha}{e-1}x^{\alpha-2}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} - \frac{\alpha^2\beta^{2\alpha}}{e-1}x^{2(\alpha-1)}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}}(e^{-(x\beta)^\alpha}+1)
$$

We cannot get the explicit solution in the general case. It has to be obtained numerically.

3.3.6 Mean deviation- KMW model

The mean deviation about mean of the KMW model is

$$\begin{aligned}\delta_1(x) =& 2\mu F(\mu) - 2\mu + \frac{2}{\beta(e-1)} \\ & \sum_{m=0}^{\infty} \frac{1}{(m+1)!(m+1)^{\frac{1}{\alpha}}} \left(\frac{1}{\alpha}\right)! e^{-\frac{u^{1/\alpha}}{\beta(m+1)^{1/\alpha}}} \sum_{h=0}^{1/\alpha} \frac{u^{h/\alpha}}{\beta^h (m+1)^{h/\alpha} h!}.\end{aligned}$$

The mean deviation about median of the KMW model is

$$\begin{aligned}\delta_2(x) =& -\mu + \frac{2}{\beta(e-1)} \sum_{m=0}^{\infty} \frac{1}{(m+1)!(m+1)^{\frac{1}{\alpha}}} \\ & \left(\frac{1}{\alpha}\right)! e^{-\frac{u^{1/\alpha}}{\beta(m+1)^{1/\alpha}}} \sum_{k=0}^{1/\alpha} \frac{u^{k/\alpha}}{\beta^k (m+1)^{k/\alpha} k!}.\end{aligned}$$

3.3.7 Quantile function- KMW model

Let X be a random variable with pdf in Equation (3.18), then the quantile function $Q(p)$ is

$$\begin{aligned}\frac{e}{e-1}[1 - e^{-(1-e^{-(Q(p)\beta)^\alpha})}] =& p \\ (Q(p)\beta)^\alpha =& -\log\left(1 + \log\left(1 - \frac{p(e-1)}{e}\right)\right).\end{aligned}$$

We get

$$Q(p) = \frac{1}{\beta}\left[-\log\left(1 + \log\left(1 - \frac{p(e-1)}{e}\right)\right)\right]^{\frac{1}{\alpha}}. \tag{3.20}$$

3.3.8 Order statistic- KMW model

The pdf of the r^{th} order statistic of KMW distribution is obtained as,

$$\begin{aligned} f_r(x) =& \frac{n!}{(r-1)!(n-r)!}\frac{e^{r-1}\alpha\beta^\alpha}{(e-1)^n}x^{\alpha-1}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} \\ & \left[1-e^{-(1-e^{-(x\beta)^\alpha})}\right]^{r-1}\left[e^{e^{-(1-e^{-(x\beta)^\alpha})}}-1\right]^{n-r} \end{aligned} \tag{3.21}$$

The KMW pdf of first order and n^{th} order statistic is obtained by putting $r=1$ and $r=n$ in Equation (3.21) respectively. Then

$$\begin{aligned} f_1(x) =& \frac{n!}{(n-1)!}\frac{\alpha\beta^\alpha}{(e-1)^n}x^{\alpha-1}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} \\ & \left[e^{e^{-(1-e^{-(x\beta)^\alpha})}}-1\right]^{n-1}, \end{aligned}$$

and

$$\begin{aligned} f_n(x) =& \frac{n!}{(n-1)!}\frac{e^{n-1}\alpha\beta^\alpha}{(e-1)^n}x^{\alpha-1}e^{-(x\beta)^\alpha}e^{e^{-(x\beta)^\alpha}} \\ & \left[1-e^{-(1-e^{-(x\beta)^\alpha})}\right]^{n-1}. \end{aligned}$$

The cdf of the r^{th} order statistic of KMW distribution is obtained as,

$$F_r(x) = \sum_{j=r}^{n}\binom{n}{j}\frac{e^j}{(e-1)^n}\left[1-e^{-(1-e^{-(x\beta)^\alpha})}\right]^j\left[e^{e^{-(1-e^{-(x\beta)^\alpha})}}-1\right]^{n-j}. \tag{3.22}$$

We can easily find the cdf of the first order and n^{th} order statistic of KMW model by putting $r=1$ and $r=n$ in Equation (3.22) respectively.

3.3.9 Ordering- KMW model

In this section, we repeat the procedure given in Section 3.2.11.

Theorem 2. *Let $X \sim KMW(\alpha_1, \beta_1)$ and $Y \sim KMW(\alpha_2, \beta_2)$ if $\alpha_1 = \alpha_2 = \alpha$ and*

$\beta_1 \geq \beta_2$ *and if* $\beta_1 = \beta_2 = \beta$ *and* $\alpha_1 \geq \alpha_2$*, then* $X \leq_{lr} Y$*,* $X \leq_{hr} Y$*,* $X \leq_{mrl} Y$ *and* $X \leq_{st} Y$*.*

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\beta_1 x^{\alpha_1-1}e^{-(x\beta_1)^{\alpha_1}}e^{e^{-(x\beta_1)^{\alpha_1}}}}{\alpha_2\beta_2 x^{\alpha_2-1}e^{-(x\beta_2)^{\alpha_2}}e^{e^{-(x\beta_2)^{\alpha_2}}}} \tag{3.23}$$

and

$$\begin{aligned}\log\frac{f_X(x)}{f_Y(x)} =& \log\alpha_1 - \alpha_1\log\beta_1 + (\alpha_1-1)\log x - (x\beta_1)^{\alpha_1} + e^{-(x\beta_1)^{\alpha_1}} \\ & - \log\alpha_2 - \alpha_2\log\beta_2 - (\alpha_2-1)\log x + (x\beta_2)^{\alpha_2} - e^{-(x\beta_2)^{\alpha_2}}\end{aligned}$$

thus,

$$\begin{aligned}\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} =& \frac{\alpha_1-1}{x} - \alpha_1\beta_1(x\beta_1)^{\alpha_1-1} - \alpha_1\beta_1(x\beta_1)^{\alpha_1-1}e^{-(x\beta_1)^{\alpha_1}} \\ & \frac{\alpha_2-1}{x} + \alpha_2\beta_2(x\beta_2)^{\alpha_2-1} + \alpha_2\beta_2(x\beta_2)^{\alpha_2-1}e^{-(x\beta_2)^{\alpha_2}}\end{aligned} \tag{3.24}$$

1. Case I: $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 \geq \beta_2$

   $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

2. Case II: $\beta_1 = \beta_2 = \beta$, $\alpha_1 \geq \alpha_2$

   $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

□

3.3.10 Maximum likelihood estimation of KMW distribution

We use maximum likelihood method to estimate the parameters of the KMW distribution α and β. The likelihood function of the distribution is given by

$$L(x;\alpha,\beta) = \left(\frac{1}{e-1}\right)^n \alpha^n \beta^{n\alpha} (\prod_{i=1}^{n} x_i^{\alpha-1}) e^{-\beta^\alpha \sum x_i^\alpha} e^{\sum e^{-(\beta x_i)^\alpha}}.$$

The log-likelihood function of the distribution is given by,

$$\log\ L(x;\alpha,\beta) = -\,n\ \log(e-1) + n\ \log(\alpha) + n\alpha\ \log(\beta) + (\alpha-1)\sum_{i=1}^{n}\log(x_i) - \beta^\alpha \sum_{i=1}^{n}(x_i)^\alpha + \sum_{i=1}^{n} e^{-(\beta x_i)^\alpha}.$$

Partial derivatives of the log-likelihood function with respect to the parameter α and β are,

$$\frac{\partial \log\ L}{\partial \alpha} = \frac{n}{\alpha} + n\ \log(\beta) + \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\beta x_i)^\alpha \log(\beta x_i)\left(1 + e^{-(\beta x_i)^\alpha}\right), \tag{3.25}$$

and

$$\frac{\partial \log\ L}{\partial \beta} = \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum_{i=1}^{n}(x_i)^{\alpha-1}\left(x_i + e^{-(x_i\beta)^\alpha}\right). \tag{3.26}$$

The maximum likelihood estimates of α and β are obtained by solving the non-linear system of equations $\frac{\partial \log\ L}{\partial \alpha} = 0$ and $\frac{\partial \log\ L}{\partial \beta} = 0$. The solution can be found numerically by using the R [139] language.

Here we are deriving the existence and uniqueness theorem of maximum likelihood estimates when the other parameter is given.

Theorem 3. *Consider the right hand side of the Equation* (3.25) *and denoted it as* $g_1(\alpha;\beta,x)$*, where* β *is the true value of the parameter. Then there exists at least one root for* $g_1(\alpha;\beta,x)=0$ *for* $\alpha\epsilon(0,\infty)$ *and the solution is unique.*

Proof. We have

$$g_1(\alpha;\beta,x) = \frac{n}{\alpha} + n\ \log(\beta) + \sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}(\beta x_i)^{\alpha}\log(\beta x_i)\left(1+e^{-(\beta x_i)^{\alpha}}\right).$$

We get

$$\lim_{\alpha\to 0} g_1(\alpha;\beta,x) = \infty + n\ \log(\beta) + \sum_{i=1}^{n}\log(x_i) - \log(\beta x_i)\left(1+e^{-1}\right) = \infty,$$

and

$$\lim_{\alpha\to\infty} g_1(\alpha;\beta,x) = 0 + n\ \log(\beta) + \sum_{i=1}^{n}\log(x_i) - \infty = -\infty$$

Hence there exists at least one root say, $\widehat{\alpha}\epsilon(0,\infty)$, such that $g_1(\alpha;\beta,x)=0$. The root is unique when $g_1'(\alpha;\beta,x)<0$ (ie, first derivative of $g_1(\alpha;\beta,x)$), where

$$g_1'(\alpha;\beta,x) = \frac{-n}{\alpha^2} - \sum_{i=1}^{n}(\beta x_i)^{\alpha}(1+e^{-(x_i\beta)^{\alpha}})\log(\beta x_i)[\log(1+e^{-(x_i\beta)}) + \log(\beta x_i)]$$

□

Theorem 4. *Consider the right hand side of the Equation* (3.26) *and denoted it as* $g_2(\beta;\alpha,x)$*, where* α *is the true value of the parameter. Then there exists at least one root for* $g_2(\beta;\alpha,x)=0$ *for* $\beta\epsilon(0,\infty)$ *and the solution is unique when* $\frac{n\alpha}{\beta^2} > \alpha\sum_{i=1}^{n}x_i^{\alpha-1}\left[\alpha x^{\alpha}\beta^{2(\alpha-1)} + (\alpha-1)\beta^{\alpha-2}(x_i+e^{-(x_i\beta)^{\alpha}})\right]$

Proof. We have

$$g_2(\beta;\alpha,x) = \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum_{i=1}^{n}(x_i)^{\alpha-1}\left(x_i+e^{-(x_i\beta)^{\alpha}}\right).$$

We get

$$\lim_{\beta\to 0} g_2(\beta;\alpha,x) = \infty - 0 = \infty,$$

and

$$\lim_{\beta\to\infty} g_2(\beta;\alpha,x) = 0 - \infty = -\infty$$

Hence there exists at least one root say, $\widehat{\beta}\epsilon(0,\infty)$, such that $g_2(\beta;\alpha,x) = 0$. The root is unique when $g_2'(\beta;\alpha,x) < 0$ (ie, first derivative of $g_2(\beta;\alpha,x)$), where

$$g_2'(\beta;\alpha,x) = \frac{-n\alpha}{\beta^2} + \alpha\sum_{i=1}^{n} x_i^{\alpha-1}\left[\alpha x^{\alpha}\beta^{2(\alpha-1)} + (\alpha-1)\beta^{\alpha-2}(x_i + e^{-(x_i\beta)^{\alpha}})\right]$$

□

3.3.11 Simulation study

Here we repeat the procedures performed in Section 2.3. 1000 pseudo–random samples have been generated for different values of parameters and sample sizes from the Equation (3.20) as follows: $x_j = \frac{1}{\beta}\left[-\log\left(1+\log\left(1-\frac{u_j(e-1)}{e}\right)\right)\right]^{\frac{1}{\alpha}}$ for $j = 1, 2, ..., n$. We considered Sixteen combinations of the parameters for the KMW distribution: $n = 50$, 100, 500, 1000, $\alpha = 1.5$, 2, 3, $\beta = 0.5$, 1, 1.5 and 2. The result of the simulation study is given in Tables 3.4, 3.5, and 3.6.

Tables 3.4, 3.5, and 3.6 reveal that all the estimators have the consistency property i.e., standard error decreases as sample size n increases.

3.3.12 Real data application

In this section, we consider a real-life data set of remission times (in months) of a random sample of 128 bladder cancer patients to show the KMW distribution's

$\alpha = 1.5$	n	$\widehat{\alpha}$	$\widehat{\beta}$	SE($\widehat{\alpha}$)	SE($\widehat{\beta}$)
$\beta = 0.5$	50	1.5511	0.5097	0.1635	0.0507
	100	1.5197	0.5034	0.1142	0.0366
	500	1.5032	0.5006	0.0507	0.0166
	1000	1.5024	0.5006	0.03587	0.0118
$\beta = 1$	50	1.5410	1.0133	0.1626	0.1017
	100	1.5244	1.0083	0.1144	0.0731
	500	1.5065	1.0025	0.0508	0.0332
	1000	1.5012	1.0003	0.0358	0.0235
$\beta = 1.5$	50	1.5353	1.5162	0.1619	0.1528
	100	1.5231	1.5081	0.1144	0.1096
	500	1.5045	1.5006	0.0508	0.0498
	1000	1.5013	1.5021	0.0358	0.0354
$\beta = 2$	50	1.5440	2.0345	0.1625	0.2033
	100	1.5224	2.0105	0.1143	0.1460
	500	1.5043	2.0019	0.0508	0.0664
	1000	1.5001	2.0001	0.0358	0.0471

Table 3.4: The result of simulation study of KMW distribution

$\alpha = 2$	n	$\widehat{\alpha}$	$\widehat{\beta}$	SE($\widehat{\alpha}$)	SE($\widehat{\beta}$)
$\beta = 0.5$	50	2.0572	0.5045	0.2165	0.0382
	100	2.0275	0.5034	0.1523	0.0276
	500	2.0089	0.5004	0.0678	0.0124
	1000	1.9999	0.5008	0.0478	0.0089
$\beta = 1$	50	2.0646	1.0109	0.2173	0.0762
	100	2.0260	1.0042	0.1522	0.0550
	500	2.0030	1.0025	0.0676	0.0250
	1000	2.0042	1.0010	0.0478	0.0177
$\beta = 1.5$	50	2.0553	1.5168	0.2164	0.1150
	100	2.0285	1.5096	0.1524	0.0826
	500	2.0016	1.5018	0.0675	0.0375
	1000	2.0011	1.5018	0.0478	0.0265
$\beta = 2$	50	2.0500	2.0217	0.2165	0.1537
	100	2.0382	2.0071	0.1531	0.1093
	500	2.0039	2.0020	0.0676	0.0499
	1000	2.0020	2.0024	0.0478	0.0354

Table 3.5: The result of simulation study of KMW distribution

suitability. This data set is extracted from Lee and Wang [143]. Using this data, Khan et al. [95] studied the applicability of transmuted inverse Weibull

$\alpha = 3$	n	$\widehat{\alpha}$	$\widehat{\beta}$	SE($\widehat{\alpha}$)	SE($\widehat{\beta}$)
$\beta = 0.5$	50	3.0716	0.5030	0.3238	0.0257
	100	3.0349	0.5014	0.2280	0.0184
	500	3.0099	0.5003	0.1016	0.0083
	1000	3.0028	0.5003	0.0717	0.0053
$\beta = 1$	50	3.0984	1.0085	0.3262	0.0509
	100	3.0358	1.0031	0.2279	0.0368
	500	3.0117	1.0011	0.1016	0.0166
	1000	3.0063	1.0001	0.0718	0.0118
$\beta = 1.5$	50	3.0705	1.5105	0.3240	0.0771
	100	3.0306	1.5069	0.2275	0.0554
	500	3.0076	1.5011	0.1015	0.0249
	1000	3.0034	1.5007	0.0717	0.0177
$\beta = 2$	50	3.0949	2.0137	0.3267	0.1019
	100	3.0616	2.0099	0.2295	0.0731
	500	3.0023	2.0019	0.1013	0.0333
	1000	3.0034	2.0009	0.0717	0.023

Table 3.6: The result of simulation study of KMW distribution

distribution (TIWD). Kumar et al ([100], [101]) used this data for the comparison purpose in their two papers. The data set is given in Table 3.7.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97	9.02	13.29
0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06
14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63
17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64
17.36	1.40	3.02	4.34	5.71	7.93	1.46	18.10	11.79	4.40	5.85	8.26	11.98	19.13	1.76
3.25	4.50	6.25	8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	12.07
6.76	21.73	2.07	3.36	6.93	8.65	12.63	22.69							

Table 3.7: Bladder Cancer Patients Data.

We consider four distributions for comparison purpose. The distributions are $DUS_E(\theta)$, $SS_E(\theta)$, GDUSE, and Weibull distribution and their distribution functions are mentioned earlier.

We have used AIC, CAIC (Consistent AIC), K-S test value, and log-likelihood function for the comparison purpose. A smaller value of CAIC for a distribution indicate a better fit for the data. The CAIC is defined as

$$\text{CAIC} = \text{AIC} + \frac{2k(k+1)}{n-k-1}$$

where n is the sample size, and k is the number of parameters.

Model	ML estimates	LL	AIC	CAIC	K-S test value
KMW	$\widehat{\alpha} = 1.1585, \widehat{\beta} = 0.0785$	-413.8836	831.7673	831.8633	0.0604
DUS_E	$\widehat{\theta} = 0.1342$	-416.022	834.044	834.0757	0.0813
SS_E	$\widehat{\theta} = 0.0592$	-415.3	832.6	832.6318	0.0675
GDUSE	$\widehat{\alpha} = 0.9942, \widehat{\lambda} = 0.1338$	-416.0211	836.0422	836.1382	0.1128
Weibull	$\widehat{\alpha} = 1.0528, \widehat{\beta} = 0.1035$	-415.0984	834.1968	834.2928	0.0663

Table 3.8: ML estimates, LL, AIC, CAIC and K-S test values of the fitted models.

Table 3.8 shows that the proposed distribution gives the lowest AIC, CAIC, K-S test values, and the largest log-likelihood value. So we can conclude that our proposed distribution provides a better fit for the data set than the other distributions given in Table 3.8. The plot of Empirical cdf and fitted cdf of the above distributions for the second data set is given in Fig. 3.6.

We compare our result with those in references Khan et al. [95], and Kumar et al. ([100], [101]). We observe that our proposed distribution fits better than the other distributions mentioned in the above papers. The Transmuted DUS-Exponential (θ) ($TD_E(\theta)$) distribution proposed by Yadav et al. [144] shows better result than our KMW distribution. But the transmutation makes the calculation of simulation study and parameter estimation a more difficult task. Hence the analytical tractability in the case of KMW distribution is comparatively better.

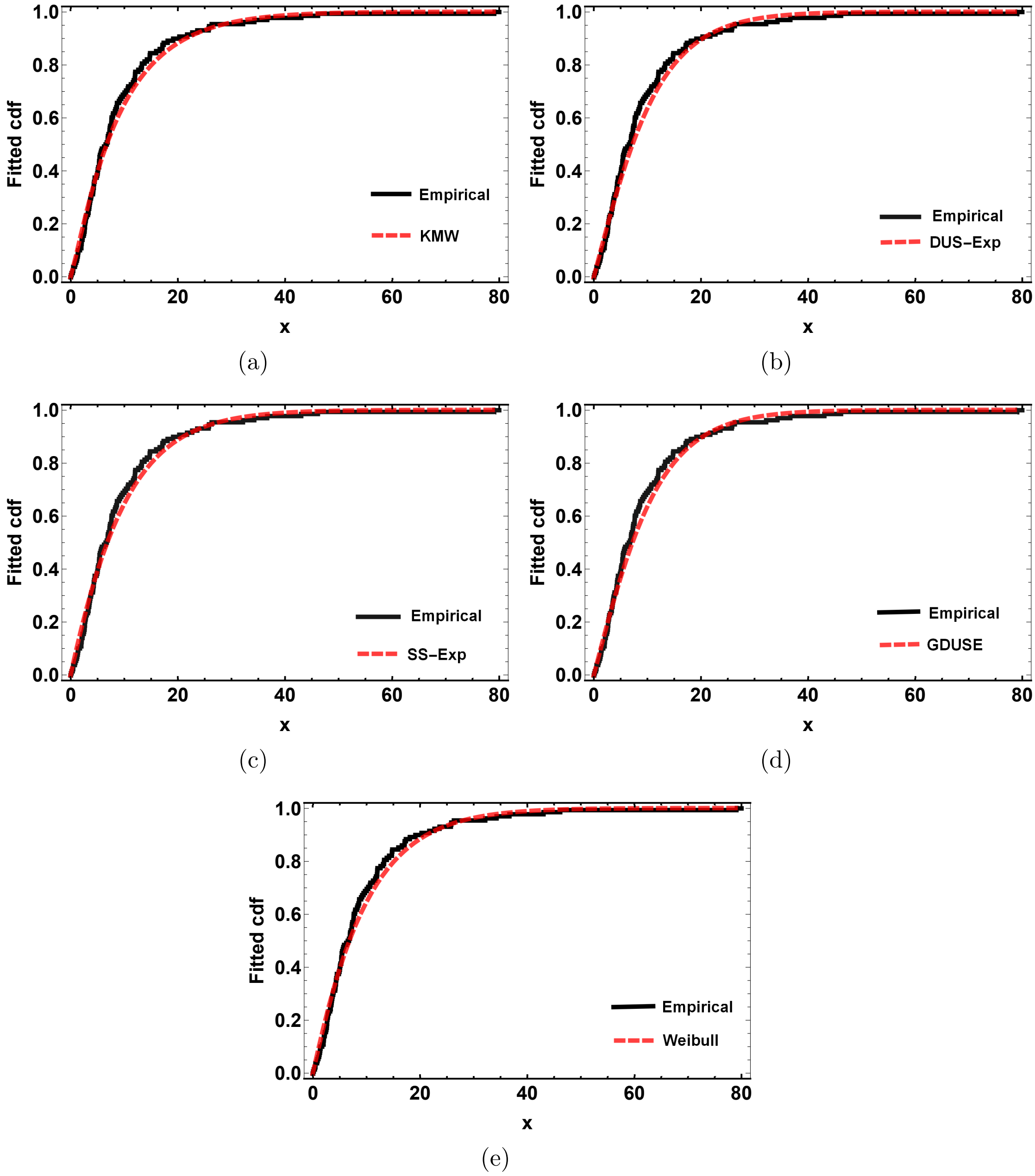


Figure 3.6: The empirical cdf and the cdf plots of the fitted distributions for the bladder cancer patient data.

3.4 KM-Lomax model

Here we have modified the Lomax distribution with cumulative distribution function (cdf)

$$G(x) = 1 - (1 + \beta x)^{-\alpha}, \quad x > 0, \quad \alpha, \ \beta \ > 0, \tag{3.27}$$

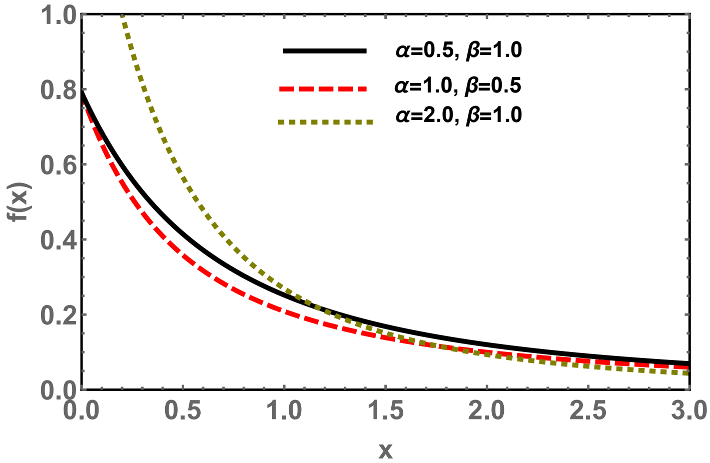


Figure 3.7: Probability density plot of KML distribution

using KM transformation. Here we introduce a new distribution by substituting the cdf of Lomax distribution (3.27) in (2.1). The cdf and pdf of the new distribution are respectively obtained as

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-(1+\beta x)^{-\alpha})}], \quad x > 0, \quad \alpha, \ \beta \ > 0, \tag{3.28}$$

$$f(x) = \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e-1}, \quad x > 0, \quad \alpha, \ \beta \ > 0, \tag{3.29}$$

The graphical representation of pdf is given in Fig. 3.7 for different values of parameters.

Using (2.4), the hazard function of the proposed model is obtained as

$$h(x) = \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e^{(1+\beta x)^{-\alpha}} - 1} \tag{3.30}$$

The shape of the hazard rate function is given in Fig. 3.8.

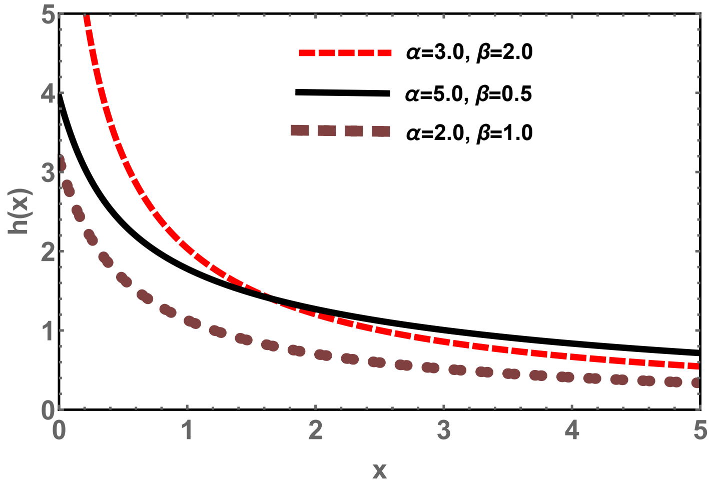


Figure 3.8: Hazard rate plot of KML distribution

3.4.1 Moments of KML distribution

The moments of a random variable, if they exist, are useful for estimating measures of central tendency, dispersion, and shapes. The r^{th} raw moments of the proposed distribution is

$$E(X^r) = \frac{\alpha\beta}{e-1}\int_0^\infty x^r(1+\beta x)^{-(\alpha+1)}e^{(1+\beta x)^{-\alpha}}dx.$$

After transformation, we get,

$$E(X^r) = \frac{1}{\beta^r(e-1)}\int_0^1 (1-u^{\frac{1}{\alpha}})^r u^{-\frac{r}{\alpha}}e^u du,$$

applying binomial expansion, then

$$E(X^r) = \frac{1}{\beta^r(e-1)}\sum_{i=0}^{\infty}(-1)^i\binom{r}{i}\int_0^1 u^{\frac{1}{\alpha}(i-r)}e^u du.$$

Expanding exponential term, and we get the r^{th} raw moment as

$$E(X^r) = \frac{1}{\beta^r(e-1)}\sum_{i=0}^{\infty}\frac{(-1)^i}{j!}\binom{r}{i}\frac{\alpha}{\alpha j+\alpha-r+i}.$$

3.4.2 Moment generating function of the KML model

The moment generating function of the proposed distribution is

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e-1} \\ &= \frac{\alpha\ t^m}{\beta^m (e-1)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!\ n!}\ \frac{(n\alpha+\alpha-2)!}{(n+\alpha)!} \end{aligned}$$

3.4.3 Characteristic function of the KML model

The characteristic function of the proposed distribution is obtained as

$$\begin{aligned} \phi_X(t) &= \int_0^\infty e^{itx} \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)} e^{(1+\beta x)^{-\alpha}}}{e-1} \\ &= \frac{\alpha}{(e-1)} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(it)^m}{m!\ n!\ \beta^m}\ \frac{(n\alpha+\alpha-2)!}{(n+\alpha)!} \end{aligned}$$

where $i = \sqrt{-1}$.

3.4.4 Median of KML model

The median of the KML distribution is obtained as

$$Q(p) = \frac{1}{\beta}\left[\left(1 + \log(1 - \frac{(e-1)}{2e})\right)^{\frac{1}{\alpha}} - 1\right]$$

3.4.5 Mode of KML model

To find the mode of the KML distribution, we have to solve the equation

$$f'(x) = \frac{-\alpha\beta^2}{e-1}\left[\alpha(1+\beta x)^{-2(\alpha+1)} + (\alpha+1)(1+\beta x)^{-(\alpha+2)}\right] = 0.$$

We can only be solve this nonlinear equation numerically.

3.4.6 Quantile function- KML model

The p^{th} quantile function of the proposed distribution is obtained as

$$\frac{e}{e-1}[1-e^{-(1-(1+\beta Q(p))^{-\alpha})}] = p$$
$$(1+\beta Q(p))^{-\alpha} = \left(1+\log(1-\frac{p(e-1)}{e})\right)^{\frac{1}{\alpha}}.$$

Then,

$$Q(p) = \frac{1}{\beta}\left[\left(1+\log(1-\frac{p(e-1)}{e})\right)^{\frac{1}{\alpha}} - 1\right] \tag{3.31}$$

3.4.7 Order statistic- KML model

The pdf of the r^{th} order statistic of the KML model is,

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}\frac{e^{r-1}\alpha\beta}{(e-1)^n}(1+\beta x)^{-(\alpha+1)}e^{(1+\beta x)^{-\alpha}}$$
$$\left[1-e^{-(1-(1+\beta x)^{-\alpha})}\right]^{r-1}\left[e^{e^{-(1-(1+\beta x)^{-\alpha})}} - 1\right]^{n-r}$$

The cdf of the r^{th} order statistic of the KML model is,

$$F_r(x) = \sum_{j=r}^{n}\binom{n}{j}\frac{e^j}{(e-1)^n}\left[1-e^{-(1-(1+\beta x)^{-\alpha})}\right]^j\left[e^{e^{-(1-(1+\beta x)^{-\alpha})}} - 1\right]^{n-j}.$$

Putting $r = 1$ and $r = n$ in the above equations, we have obtain the first order and n^{th} order pdfs and cdfs of the KML distribution.

3.4.8 Ordering- KML model

The KML model satisfies the ordering of random variables as $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Theorem 5. *Let $X \sim KML(\alpha_1, \beta_1)$ and $Y \sim KML(\alpha_2, \beta_2)$ if $\alpha_1 = \alpha_2 = \alpha$ and*

$\beta_1 \geq \beta_2$ and if $\beta_1 = \beta_2 = \beta$ and $\alpha_1 \geq \alpha_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1\beta_1(1+\beta_1 x)^{-(\alpha_1+1)}e^{(1+\beta_1 x)^{-\alpha_1}}}{\alpha_2\beta_2(1+\beta_2 x)^{-(\alpha_2+1)}e^{(1+\beta_2 x)^{-\alpha_2}}} \tag{3.32}$$

and

$$\begin{aligned}\log\frac{f_X(x)}{f_Y(x)} =& \log\alpha_1 + \log\beta_1 - (\alpha_1+1)\log(1+\beta_1 x) + (1+\beta_1 x)^{-\alpha_1} \\ & - \log\alpha_2 - \log\beta_2 + (\alpha_2+1)\log(1+\beta_2 x) - (1+\beta_2 x)^{-\alpha_2}\end{aligned}$$

thus,

$$\begin{aligned}\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} =& -(\alpha_1+1)\frac{\beta_1}{1+\beta_1 x} - \alpha_1\beta_1(1+\beta_1 x)^{-(\alpha_1+1)} \\ & + (\alpha_2+1)\frac{\beta_2}{1+\beta_2 x} + \alpha_2\beta_2(1+\beta_2 x)^{-(\alpha_2+1)}\end{aligned} \tag{3.33}$$

1. Case I: $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 \geq \beta_2$

   $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

2. Case II: $\beta_1 = \beta_2 = \beta$, $\alpha_1 \geq \alpha_2$

   $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)} \leq 0 \Rightarrow X \leq_{lr} Y$ hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

□

3.4.9 Maximum likelihood estimation of KML distribution

In this section we estimate the parameters involved in the distribution using maximum likelihood estimation method. This is one of the most popular methods used for estimation. The likelihood function is defined as,

$$L(x;\lambda) = \prod_{i=1}^{n} f(x_i, \lambda)$$

In our distribution,

$$L(x;\alpha,\beta) = \left(\frac{\alpha\beta}{e-1}\right)^n \prod_{i=0}^{n}(1+\beta x_i)^{-(\alpha+1)} e^{\sum_{i=0}^{n}(1+\beta x_i)^{-\alpha}}.$$

The log-likelihood function of the distribution is given by,

$$\begin{aligned}\log\ L(x;\alpha,\beta) =& -n\ \log(e-1) + n\ \log\alpha + n\log\beta \\ & - (\alpha+1)\sum_{i=1}^{n}\log(1+\beta x_i) + \sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}.\end{aligned}$$

We proceed as follows. First we find partial derivatives of the log-likelihood function with respect to the parameters α and β. The partial derivatives are

$$\frac{\partial \log\ L}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log(1+\beta x_i) + \sum_{i=1}^{n}\log(1+\beta x_i)^{-\alpha}\log(1+\beta x_i),$$

and

$$\frac{\partial \log\ L}{\partial\beta} = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n}\frac{x_i}{1+\beta x_i} - \alpha\sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)}.$$

Two non-linear equations can be obtained by equating these partial derivatives to zero, the solutions for which provide the maximum likelihood estimates of the parameters. The Newton-Raphson method can be used to solve this equation with the help of the available statistical packages. We use R [139] language for finding the numerical solution of the non-linear system of equations.

3.4.10 Application

In this section we are showing the flexibility of the proposed distribution using two real-life data sets. The first data set is the uncensored data set corresponding to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951, published by Maguire et al. [145] and the data set is given in Table 3.9. The second data set is of Wheaton River obtained from Choulakian and Stephens [140] and presented in Table 3.10.

1	4	4	7	11	13	15	15	17	18	19	19
20	20	22	23	28	29	31	32	36	37	47	48
49	50	54	54	55	59	59	61	61	66	72	72
75	78	78	81	93	96	99	108	113	114	120	120
120	123	124	129	131	137	145	151	156	171	176	182
188	189	195	203	208	215	217	217	217	224	228	233
255	271	275	275	275	286	291	312	312	312	315	326
326	329	330	336	338	345	348	354	361	364	369	378
390	457	467	498	517	566	644	745	871	1312	1357	1613
1630											

Table 3.9: Flood Level Data.

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0
12.0	9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1
2.5	14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0
7.3	22.9	1.7	0.1	1.1	0.6	9.0	1.7	7.0	20.1
0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6	5.6	30.8
13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5
2.5	27.0								

Table 3.10: Wheaton River Data.

We use AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), HQC (Hannan-Quinn Information Criterion) and K-S (Kolmogorov-Smirnov) test value for the comparison. The distribution which shows minimum AIC, BIC, HQC and K-S test value is the sign of a better fit for the data set. The HQC is defined as

$$\text{HQC} = -2\ \log(\widehat{L}) + 2m\ \log(\log(n)),$$

where n is the sample size, m is the number of parameters, and $\widehat{L}$ is the maximum value of the likelihood function for the considered distribution. Here R [139] language is used for all the computation. We compare the proposed distribution with the following distributions,

1. DUS-Lomax distribution Deepthi and Chacko [128] with cdf,

$$F(x) = \frac{1}{(e-1)}\left[e^{(1-(1+\theta x)^{-\alpha})} - 1\right], \qquad x > 0, \quad \alpha,\ \theta > 0$$

2. Lomax distribution Lomax [116] with cdf,

$$F(x) = 1 - (1+\theta x)^{-\alpha}, \qquad x > 0, \quad \alpha,\ \theta > 0$$

3. KM-Exponential (KME) distribution with cdf,

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-\lambda x})}], \quad x > 0, \quad \lambda > 0$$

4. KM-Weibul (KMW) distribution with cdf,

$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-(x\beta)^{\alpha}})}], \qquad x > 0, \quad \alpha, \beta > 0$$

5. Weibull distribution Weibull [146] with cdf,

$$F(x) = 1 - e^{-(\beta x)^{\alpha}}, \qquad x > 0, \quad \alpha, \beta > 0$$

The values of AIC, BIC, HQC and K-S test for distributions based on the first data set are given in Table 3.

Model	ML estimates	K-S value	AIC	BIC	HQC
KM-Lomax	$\widehat{\alpha} = 9.4097, \widehat{\beta} = 0.0004$	0.0720	1040.647	1046.03	1042.83
DUS-Lomax	$\widehat{\alpha} = 9.4008, \widehat{\theta} = 0.0004$	0.2703	1187.936	1193.319	1190.119
Lomax	$\widehat{\alpha} = 4.9251, \widehat{\theta} = 0.0011$	0.0642	1405.426	1410.809	1407.609
KME	$\widehat{\lambda} = 0.0033$	0.0761	1404.864	1407.555	1405.955
KMW	$\widehat{\alpha} = 0.9685, \widehat{\beta} = 0.0033$	0.0772	1406.654	1412.037	1408.837
Weibull	$\widehat{\alpha} = 0.8848, \widehat{\beta} = 0.0046$	0.0784	1407.545	1412.927	1409.728

Table 3.11: Maximum likelihood (ML) estimates, K-S test value, AIC, BIC, and HQC of the fitted models.

From Table 3.11, we can see that the new model shows the lowest AIC, BIC and HQC values among all the distributions considered here. The K-S test value of Lomax distribution is smaller than KM-Lomax distribution. In general we can say that our proposed model shows better fit to the data compared to other distributions given in this study. The plot of empirical cdf along with other cdf of the distributions for the first data set is given in Fig. 3.9. for a better understanding of the result. Compared to the distributions mentioned in Mahdavi [147] for this particular data set, the proposed model gives better result.

The comparison table of the considered models for the second data set is given in Table 3.12.

Based on the AIC, BIC and HQC values, we can conclude that the proposed model gives the best fit to the data set compared to other Lomax, KM family and Weibull distributions considered here. The K-S test value of the KMW distribution is slightly lower than the KM-Lomax distribution. The empirical and fitted cdf plot of distributions considered for the comparison for the second data set is given in Fig. 3.10.

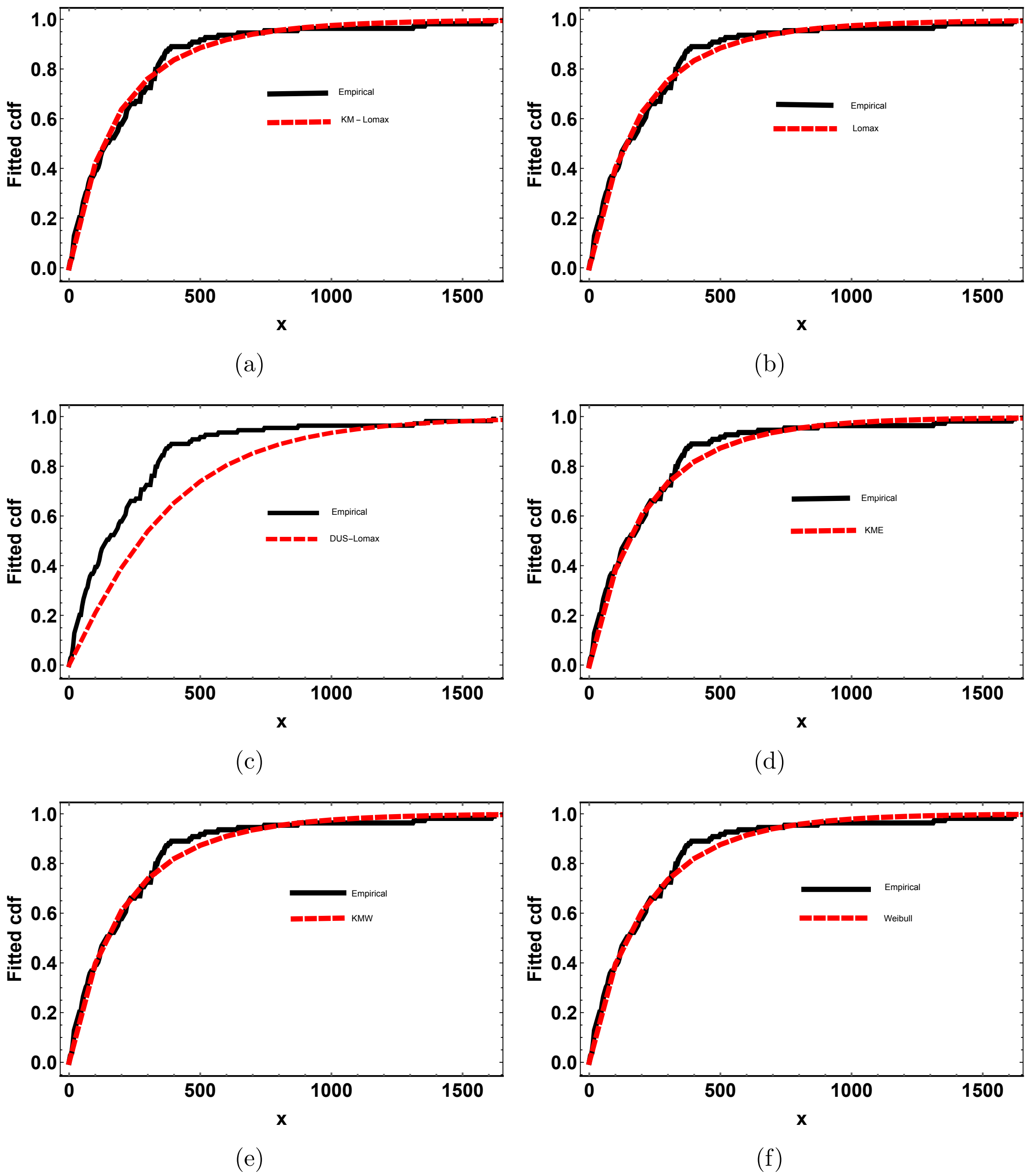


Figure 3.9: Comparison plot for the first data set.

3.5 Summary of the chapter

In this chapter we have developed three new lifetime distributions using the cdfs of exponential, Weibull, and Lomax distributions in KM transformation. The main analytical properties of these newly proposed models like moments, moment

Model	ML estimates	K-S value	AIC	BIC	HQC
KM-Lomax	$\widehat{\alpha} = 243.7254, \widehat{\beta} = 0.000258$	0.11	286.6547	291.2081	288.4674
DUS-Lomax	$\widehat{\alpha} = 251.4388, \widehat{\theta} = 0.00025$	0.24	364.0633	368.6166	365.876
Lomax	$\widehat{\alpha} = 80.7719, \widehat{\theta} = 0.00102$	0.14	508.2621	512.8155	510.0748
KME	$\widehat{\lambda} = 0.0632$	0.11	506.025	508.3017	506.9313
KMW	$\widehat{\alpha} = 0.9722, \widehat{\beta} = 0.0633$	0.10	507.9317	512.4851	509.7444
Weibull	$\widehat{\alpha} = 0.9012, \widehat{\beta} = 0.08597$	0.11	506.9973	511.5506	508.81

Table 3.12: Maximum likelihood (ML) estimates, K-S test value, AIC, BIC, and HQC of the fitted models.

generating function, characteristic function, quantile function, order statistics etc are derived. Stochastic ordering is used for judging the comparative behavior of the random variables proposed here. Parameters involved in newly proposed models are estimated using maximum likelihood estimation method. We checked the performance of the estimated parameters of KME and KMW distributions by simulation technique. Real data applications are included in this chapter to show the flexibility and suitability of the proposed models. The models proposed in this chapter are very promising models in reliability and survival areas.

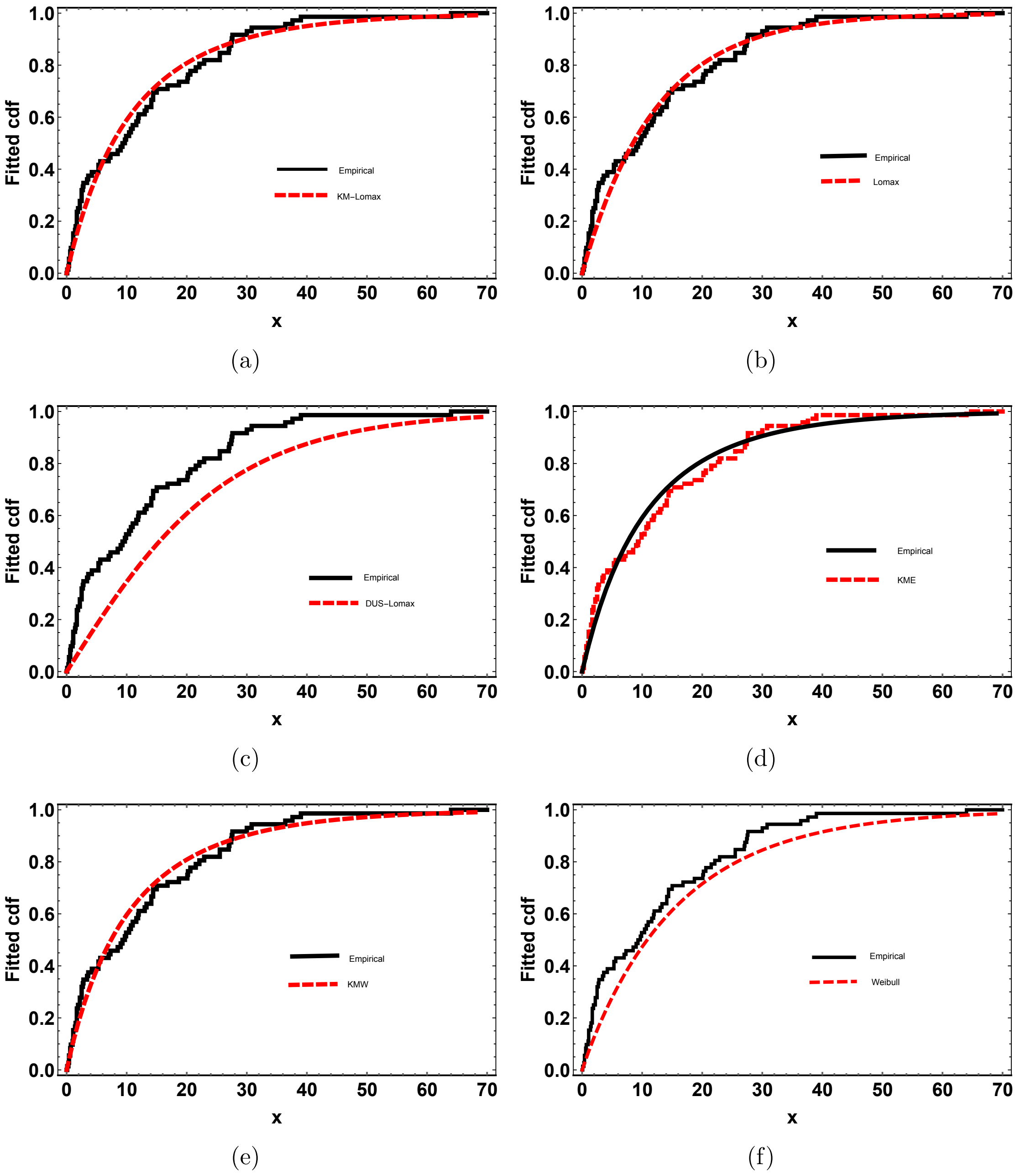


Figure 3.10: Comparison plot for the second data set.

Chapter 4

Generalization of KM Transformation For Lifetime Models

4.1 Introduction

The study of the duration of life of organisms, systems or devices, materials, etc., is of major importance in the biological and engineering sciences[1]. A significant part of such a study is devoted to the mathematical description of the duration of life by a failure distribution. True physical considerations of the failure mechanism often may lead to a specific distribution but sometimes, the choice is made on the basis of how well the actual failure time data appear to be fitted by the distribution. In lifetime studies a search for possible candidates among various distributions is carried out by examining the shape and monotonicity of the failure rate function since they reflect some of the characteristics of the mechanism leading to the conclusion of life. Models with non-monotone failure rate function are useful in reliability analysis, and particularly in reliability-related decision

[1]This chapter is based on Kavya, P., Manoharan, M. (2023)

making and cost analysis. Notable works include those by Mudholkar et al [46], Gupta et al [81] and Gupta and Kundu [109]. In this chapter, a new model, which is useful for modeling this type of failure rate function, is presented.

The inverse bathtub shaped hazard rates are common in biological and reliability studies. For example, the hazard rate curve can be seen in the course of a disease whose mortality reaches a peak after some finite period and then declines gradually. The lifetime models that exhibit inverse bathtub-shaped hazard rates find their application in survival analysis. A study of head and neck cancer data, see Effron [148], and the data of 3878 cases of breast carcinoma seen in Edinburgh from 1954 to 1964, see Langlands et al [149] showed inverse bathtub hazard rates. In reliability theory, the examples of inverse bathtub shaped models can be found in accelerated life testing and repair time situations whenever early failures or occurrences dominate the lifetime distribution. Inverse Weibull, log-logistic, Burr Type III, inverse gamma, and log-normal are some of the statistical distributions that show inverse bathtub hazard rates. For more details, readers may refer to Kotz and Nadarajah [150]. Recently Dimitrakopoulou et al [71], Sharma et al [72], Alkarni [73], Maurya et al [74], and Kavya and Manoharan [75] have developed and studied inverse bathtub shaped hazard rate distributions. The main motivation of our work is to introduce a new lifetime model having non-monotone hazard rate function for different values of parameter and can fit a large variety of failure time data. We also aim for a model with minimum number of parameters so that one can use the model conveniently. For this purpose we generalize the KM (Kavya- Manoharan) transformation which is proposed in Chapter 2. The distributions introduced using the KM transformation are parsimonious in parameter and show monotone hazard rates. In this paper we generalize the above transformation with the aim of developing new lifetime models.

4.2 Development of the new model

In this section we develop a new lifetime model by generalizing KM transformation and use the exponential distribution as the baseline distribution in the transformation.

4.2.1 Generalization of KM transformation

The KM transformation is defined as

$$F(x) = \frac{e}{e-1}[1 - e^{-G(x)}], \tag{4.1}$$

where $G(x)$ is the cumulative distribution function (cdf) of some baseline distribution. The above transformation is generalized using a new parameter α, provided α is always greater than zero. Hereafter we shall refer to the new transformation as the Generalized KM (GKM) transformation. If $G(x)$ is the cdf of some baseline distribution, then the Generalized KM transformation is defined as

$$F(x) = \frac{e}{e-1}[1 - e^{-G^{\alpha}(x)}], \tag{4.2}$$

which is the cdf of the new distribution. The probability density function (pdf) and the hazard rate function are respectively obtained as

$$f(x) = \frac{\alpha e}{e-1} g(x) G^{\alpha-1}(x) e^{-G^{\alpha}(x)}, \tag{4.3}$$

and

$$h(x) = \frac{\alpha e g(x) G^{\alpha-1}(x) e^{-G^{\alpha}(x)}}{e^{1-G^{\alpha}(x)} - 1}. \tag{4.4}$$

The distribution obtained using GKM transformation is shown to possess both monotone and non-monotone hazard rates depending on different values of parameters in the next section.

4.2.2 GKM-Exponential model

Here we introduce a new distribution using exponential distribution as the baseline distribution in the GKM transformation. The reason for choosing exponential distribution is that it has wide application in reliability theory and survival analysis. Substitute the cdf of exponential distribution $G(x) = 1 - e^{-\theta x}, \quad x > 0, \quad \theta > 0$ in Equation (4.2) to get the new distribution called Generalized KM Exponential (GKME) distribution. The pdf and cdf of the new distribution are obtained as,

$$f(x) = \frac{e\alpha\theta}{e-1} e^{-\theta x} \left(1 - e^{-\theta x}\right)^{\alpha - 1} e^{-(1-e^{-\theta x})^{\alpha}}, \quad x > 0, \quad \alpha, \ \theta > 0, \tag{4.5}$$

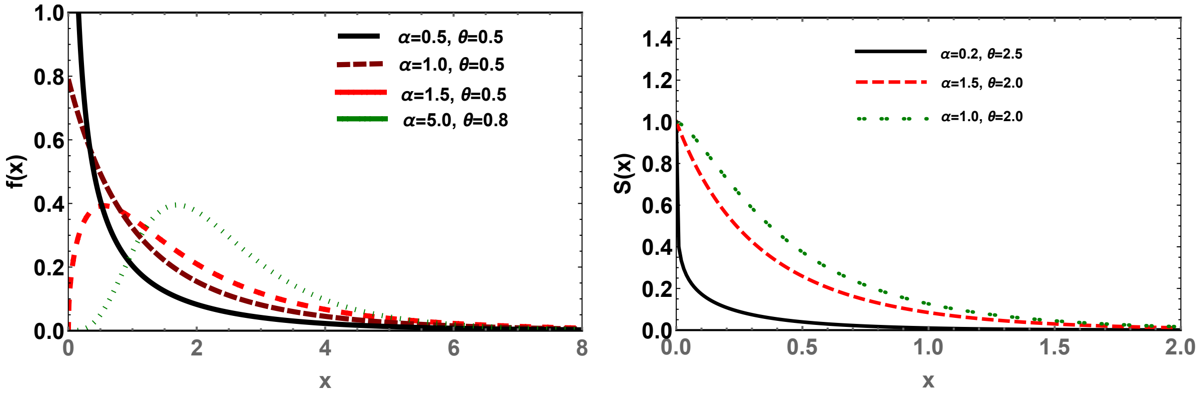
where α is the shape parameter.

$$F(x) = \frac{e}{e-1} [1 - e^{-(1-e^{-\theta x})^{\alpha}}], \quad x > 0, \quad \alpha, \ \theta > 0. \tag{4.6}$$

The survival function of the model is

$$S(x) = \frac{e^{1-(1-e^{-\theta x})^{\alpha}} - 1}{e-1}, \quad x > 0, \quad \alpha, \ \theta > 0. \tag{4.7}$$

The plot of the pdf and survival function for different values of parameters is given in Fig. 4.1.



(a) Probability density plot (b) Survival function plot

Figure 4.1: The probability density and survival function plot.

4.3 Hazard rate function

The hazard rate function of a distribution is defined as

$$h(x) = \frac{f(x)}{1 - F(x)}. \tag{4.8}$$

Here we study the hazard rate function of GKME distribution. Substituting Equations (4.5) and (4.7) in Equation (4.8) we get the hazard rate function of the GKME distribution as,

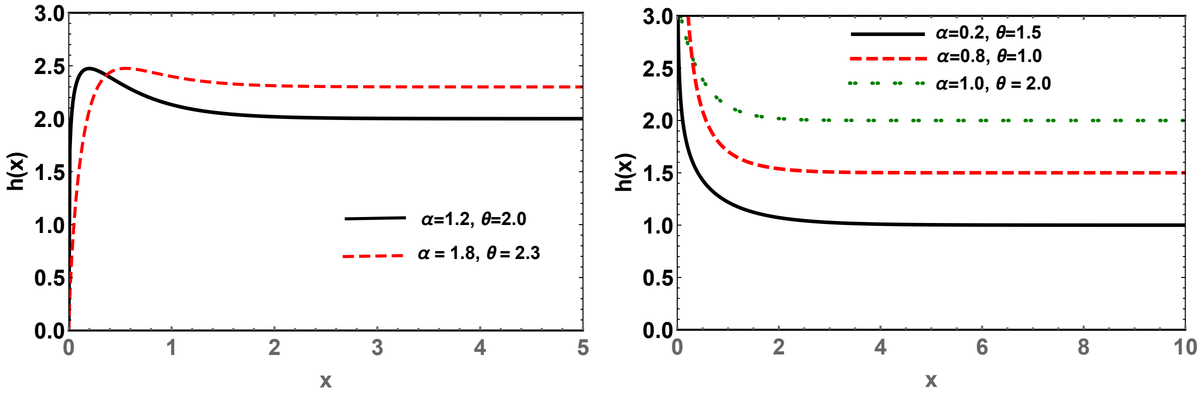
$$h(x) = \frac{e\alpha\theta}{\left(e^{1-(1-e^{-\theta x})^{\alpha}} - 1\right)} e^{-\theta x} \left(1 - e^{-\theta x}\right)^{\alpha - 1} e^{-(1-e^{-\theta x})^{\alpha}}, \quad x > 0, \ \ \alpha, \ \theta > 0. \tag{4.9}$$

Various shapes for hazard rate function of the proposed distribution for different values of parameters are shown in Fig. 4.2.

From Fig. 4.2 we can see that the hazard rate function has two shapes,

- Hazard rate function is decreasing when $\alpha \leq 1$.
- Hazard rate function has inverse bathtub shape when $\alpha > 1$.

We use the method proposed by Glaser [36] for the theoretical study of the shapes of the hazard rate. We first calculate $\vartheta(x)$. If $f(x)$ is the density function



(a) Inverse bathtub shaped hazard rate

(b) Decreasing hazard rate

Figure 4.2: The hazard rate plot of GKME distribution for different values of parameters.

and $f'(x)$ is the first derivative of the density function then $\vartheta(x)$ is defined as

$$\vartheta(x) = \frac{-f'(x)}{f(x)}.$$

For the GKME distribution,

$$\vartheta(x) = \theta - \theta(\alpha - 1)e^{-\theta x}(1 - e^{-\theta x})^{-1} + \alpha\theta e^{-\theta x}(1 - e^{-\theta x})^{\alpha-1}$$

and

$$\vartheta'(x) = \theta^2 e^{-\theta x}(1-e^{-\theta x})^{-1}\left[(1-e^{-\theta x})^{\alpha-1}e^{-\theta x}(\alpha^2-1)+(\alpha-1)-\alpha(1-e^{-\theta x})^{\alpha}\right]. \tag{4.10}$$

Following Glaser [36] we obtain two results from Equation (4.10):

1. $\vartheta'(x) < 0$ for all $x > 0$ when $\alpha \leq 1$. Then the distribution has decreasing failure rate (DFR).

2. When $\alpha > 0$, there exists a x^* such that $\vartheta'(x) > 0$ for $0 < x < x^*$ and $\vartheta'(x) < 0$ for $x > x^*$ where x^* depends on the values of α and θ, but the

exact functional form of x^* in terms of α and θ could not be obtained.

We can easily verify that $\lim_{x\to 0} f(x) = 0$, therefore the distribution has inverse bathtub hazard rate.

4.4 Moments and generating function

We now study the r^{th} raw moment, conditional moment and moment generating function of the proposed model.

4.4.1 r^{th} moment

The importance and necessity of the moments in any statistical analysis, mainly in applied work, cannot be over emphasized. We can study some of the most important features and characteristics of a distribution through moments like tendency, dispersion, skewness, and kurtosis. So here we derive the r^{th} raw moment of the GKME distribution.

$$E(X^r) = \frac{e\alpha\theta}{e-1}\int_0^\infty x^r e^{-\theta x}(1-e^{-\theta x})^{\alpha-1}e^{-(1-e^{-\theta x})^\alpha}dx, \quad r \geq 1$$

Expanding exponential term, we get,

$$\begin{aligned}
E(X^r) &= \frac{e\alpha\theta}{e-1}\int_0^\infty x^r e^{-\theta x}(1-e^{-\theta x})^{\alpha-1}\sum_{m=0}^{\infty}\frac{(-1)^m(1-e^{-\theta x})^{\alpha m}}{m!}dx \\
&= \frac{e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\int_0^\infty x^r e^{-\theta x}(1-e^{-\theta x})^{\alpha m+\alpha-1}dx,
\end{aligned}$$

applying binomial expansion in the above expression and after some calculations, we get,

$$
\begin{aligned}
E(X^r) =& \frac{e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\int_0^{\infty} x^r e^{-\theta x}e^{-\theta jx}dx \\
=& \frac{e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\int_0^{\infty} x^r e^{-\theta x(j+1)}dx \\
=& \frac{e\alpha}{\theta^r(e-1)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{\Gamma r+1}{(j+1)^{r+1}}.
\end{aligned}
$$

4.4.2 Conditional moment

The conditional moment of a lifetime model is an interesting property. We denote it as $\delta_k(x)$ and

$$
\delta_k(x) = E(X^k|X>x) = \frac{1}{S(x)}\int_x^{\infty} x^k\ f(x)\ dx,
$$

The conditional moment of the GKME distribution is,

$$
\begin{aligned}
\delta_k(x) &= \frac{e\alpha\theta}{e^{1-(1-e^{-\theta x})^{\alpha}}}\int_0^{\infty} x^k e^{-\theta x}(1-e^{-\theta x})^{\alpha-1}e^{-(1-e^{-\theta x})^{\alpha}}dx \\
&= \frac{e\alpha\theta}{e^{1-(1-e^{-\theta x})^{\alpha}}}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\int_0^{\infty} x^r e^{-\theta x(j+1)}dx.
\end{aligned}
$$

Applying complementary incomplete gamma function,

$$
\delta_k(x) = \frac{\alpha e\theta}{e^{1-(1-e^{-\theta x})^{\alpha}}}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{k-1}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}(k-1)!e^{-\theta x(j+1)}\frac{x^n}{n!}.
$$

Using this expression, we can easily obtain the mean residual life function.

4.4.3 Moment generating function

The moment generating function (mgf) is one of the key properties of a distribution. The mgf of the new distribution is

$$M_X(t) = \frac{e\alpha}{\theta^r(e-1)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{1}{\theta(j+1) - t}, \quad \text{for } t < \theta.$$

4.5 Mean residual and mean past lifetime function

The mean residual life and mean past lifetime functions are discussed in this section.

4.5.1 Mean residual life function

The expected remaining life $(T-t)$, given that the item has survived up to time t is the mean residual life (MRL). For the proposed model,

$$\begin{aligned}
\nu(t) =& E(T-t|T>t) \\
=& \frac{\int_t^\infty x\ f(x)\ dx}{1-F(t)} - t \\
=& \alpha e\theta \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} e^{-\theta(j+1)t}(1+\theta(j+1)t) \\
& - t(j+1)\theta[e^{1-(1-e^{-\theta\ t})^\alpha} - 1] \Big[(j+1)\theta[e^{1-(1-e^{-\theta\ t})^\alpha}) - 1]\Big]^{-1}.
\end{aligned}$$

4.5.2 Mean past lifetime

Systems are often not monitored continuously in real life situations. When we are interested in the history of the system, (for instance, in situations where the individual components fail) the mean past lifetime (MPL) of a component is a quantity of importance. Let us suppose a component with lifetime T has failed at a time t ($t \geq 0$), or before that. The random variable $t - T|T \leq t$ is the time elapsed from the failure of the component, given that its lifetime is less than or equal to t. The MPL of the component can now be defined as,

$$\begin{aligned}\omega(t) =& E(t - T|T \leq t)\\ =& t - \frac{\int_0^t x\ f(x)\ dx}{F(t)}.\end{aligned}$$

The MPL of the proposed model is obtained as

$$\begin{aligned}\omega(t) =& t(1 - e^{-(1-e^{-\theta t})^{\alpha}}) - \alpha\theta \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j}\\ & \frac{1}{\theta(j+1)} \left[\frac{1}{\theta(j+1)} - e^{-\theta t(j+1)} \left(t + \frac{1}{\theta(j+1)} \right) \right] \left(1 - e^{-(1-e^{-\theta t})^{\alpha}} \right)^{-1}\end{aligned}$$

4.6 Mean deviation

Here we derive the expressions of mean deviation about mean and median of the proposed model.

4.6.1 Mean deviation about mean

The mean deviation about mean is defined as

$$\begin{aligned}\zeta_1(x) =& \int_0^{\infty} |x - \mu| f(x) dx\\ =& \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x - \mu) f(x) dx,\end{aligned}$$

where μ is the mean. After simplification, we get

$$\zeta_1(x) = 2\mu F(\mu) - 2\mu + 2\int_{\mu}^{\infty} xf(x)dx,$$

where $F(.)$ is the proposed cdf.

$$\begin{aligned}\int_{\mu}^{\infty} xf(x)dx =& \frac{e\alpha\theta}{e-1}\int_{\mu}^{\infty} xe^{-\theta x}\left(1-e^{-\theta x}\right)^{\alpha-1} e^{-(1-e^{-\theta x})^{\alpha}}dx, \\ =& \frac{e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{e^{-\theta(j+1)\mu}(1+\theta(j+1)\mu)}{(j+1)\theta}.\end{aligned}$$

Therefore,

$$\zeta_1(x) = 2\mu F(\mu) - 2\mu + \frac{2e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{e^{-\theta(j+1)\mu}(1+\theta(j+1)\mu)}{(j+1)\theta}.$$

4.6.2 Mean deviation about median

Mean deviation about median is defined as

$$\begin{aligned}\zeta_2(x) =& \int_0^{\infty} |x-M|f(x)dx, \\ =& \int_0^{M}(M-x)f(x)dx + \int_M^{\infty}(x-M)f(x)dx.\end{aligned}$$

where M denotes median. After simplification, we get

$$\zeta_2(x) = -\mu + 2\int_M^{\infty} xf(x)dx.$$

The mean deviation about median is obtained as

$$\zeta_2(x) = -\mu + \frac{2e\alpha\theta}{e-1}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{e^{-\theta(j+1)M}(1+\theta(j+1)M)}{(j+1)\theta}.$$

4.7 Quantile function of the model

The p^{th} quantile function $Q(p)$ is obtained by the equation $F(Q(p)) = p$. The p^{th} quantile function of the GKME distribution is,

$$Q(p) = \frac{-1}{\theta} \log \left[1 - \left(-\log \left(1 - \frac{p(e-1)}{e} \right) \right)^{\frac{1}{\alpha}} \right] \tag{4.11}$$

After substituting $p = \frac{1}{4}, \frac{1}{2},$ and $\frac{3}{4}$ in Equation (4.11), we get the first, second and third quartile functions respectively.

A random sample X with GKME(α, θ) distribution can be simulated using

$$X = \frac{-1}{\theta} \log \left[1 - \left(-\log \left(1 - \frac{u_i(e-1)}{e} \right) \right)^{\frac{1}{\alpha}} \right], \tag{4.12}$$

where $u_1, u_2, ..., u_n$ are independent random observations from the standard uniform distribution.

4.8 *Reńyi* entropy

Entropy quantifies the amount of information contained in a random observation pertaining to its parent distribution (population). A large entropy corresponds to greater uncertainty in the data. The concept of entropy finds its application in areas such as probability and statistics, communication theory, physics, and economics. Different measures of entropy have been developed and studied extensively. Here we derive the *Reńyi* entropy (*Reńyi* [133]) of the GKME distribution. If a random variable X has the pdf f(x), then the *Reńyi* entropy is defined as,

$$\mathcal{J}_R(\gamma) = \frac{1}{1-\gamma} log \left[\int f^{\gamma}(x) dx \right], \tag{4.13}$$

where $\gamma > 0$ and $\gamma \neq 1$. From Equation (4.5), we get,

$$\begin{aligned}\int_0^\infty f^\gamma(x)dx &= \left(\frac{e\alpha\theta}{e-1}\right)^\gamma \int_0^\infty (e^{-\theta x})^\gamma (1-e^{-\theta x})^{\gamma(\alpha-1)} e^{-\gamma(1-e^{\theta x})} \\ &= \left(\frac{e\alpha\theta}{e-1}\right)^\gamma \sum_{m=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{m+j}\gamma^m}{m!} \binom{\gamma\alpha+\alpha m-\gamma}{j} \frac{1}{\theta(j+\gamma)}.\end{aligned}$$

Using above result, Equation (4.13) becomes,

$$\mathcal{J}_R(\gamma) = \frac{\gamma}{1-\gamma} log\left(\frac{e\alpha\theta}{e-1}\right) + \frac{1}{1-\gamma} log\left(\sum_{m=0}^\infty \sum_{j=0}^\infty \frac{(-1)^{m+j}\gamma^m}{m!} \binom{\gamma\alpha+\alpha m-\gamma}{j} \frac{1}{\theta(j+\gamma)}\right). \tag{4.14}$$

4.9 Estimation of parameters

In this section we discuss some estimation methods for estimating the parameters in the model. We provide the theory of maximum likelihood, moment estimation, and least square methods.

4.9.1 Maximum likelihood estimation

Maximum likelihood estimation method is the most popular method to find the estimates of the parameters in a distribution. Here we maximize the logarithm of likelihood function to find the estimates. In this section we use this method to obtain the maximum likelihood estimates of the GKME distribution's parameters. The likelihood function is defined as,

$$L(x;\alpha,\theta) = \prod_{i=1}^n f(x_i,\alpha,\theta).$$

In our model,

$$L(x;\alpha,\theta)=\left(\frac{e\theta\alpha}{e-1}\right)^n e^{-\theta\sum_{i=1}^n x_i}\prod_{i=1}^n(1-e^{-\theta x_i})^{\alpha-1}e^{-\sum_{i=1}^n(1-e^{-\theta x_i})^{\alpha}}.$$

The log-likelihood function of the distribution is given by,

$$\begin{aligned}\log\ L(x;\alpha,\theta)=&n\left(\log(\frac{e}{e-1})\right)+n\ \log\alpha+n\ \log\theta-\theta\sum_{i=1}^n x_i+\\ &(\alpha-1)\sum_{i=1}^n\log(1-e^{-\theta x_i})-\sum_{i=1}^n(1-e^{-\theta x_i})^{\alpha}.\end{aligned}\tag{4.15}$$

Partial derivative of the log-likelihood function with respect to the parameters α and θ are

$$\frac{\partial\log\ L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^n\log(1-e^{-\theta x_i})-\sum_{i=1}^n\log(1-e^{-\theta x_i})(1-e^{-\theta x_i})^{\alpha},\tag{4.16}$$

and

$$\frac{\partial\log\ L}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^n x_i-\sum_{i=1}^n x_ie^{-\theta x_i}\frac{\alpha-1}{(1-e^{-\theta x_i})}+\alpha\sum_{i=1}^n x_ie^{-\theta x_i}(1-e^{-\theta x_i})^{\alpha-1}.\tag{4.17}$$

Equating Equations (4.16) and (4.17) to zero yields two non - linear equations. The maximum likelihood estimate of the parameter α and θ are obtained as the solution of these equations. Newton-Raphson method can be used to solve these equations with the help of the R [139] language.

Here we study the existence and uniqueness of the maximum likelihood estimates when other parameters are known.

Theorem 6. *Consider the right hand side of the Equation* (4.16) *and is denote as* $k_1(\alpha;\theta,x)$, *where* θ *is the true value of the parameter. Then there exists at* *least one root for* $k_1(\alpha;\theta,x)=0$ *for* $\alpha\in(0,\infty)$ *and the solution is unique.*

Proof. We have

$$k_1(\alpha;\theta,x)=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1-e^{-\theta x_i})-\sum_{i=1}^{n}\log(1-e^{-\theta x_i})(1-e^{-\theta x_i})^{\alpha}.$$

We get

$$\lim_{\alpha\to 0}k_1(\alpha;\theta,x)=\infty$$

and

$$\lim_{\alpha\to\infty}k_1(\alpha;\theta,x)=-\infty<0$$

Hence, there exists at least one root say, $\widehat{\alpha}\in(0,\infty)$, such that $k_1(\widehat{\alpha};\theta,x)=0$. The root is unique when the first derivative of $k_1(\widehat{\alpha};\theta,x)$, ie, $k_1^{'}(\widehat{\alpha};\theta,x)<0$, where

$$k_1^{'}(\widehat{\alpha};\theta,x)=\frac{-n}{\alpha^2}-\left[\sum\log(1-e^{-\theta x_i})\right]^2(1-e^{-\theta x_i})^{\alpha}$$

□

Theorem 7. *Consider the right hand side of the Equation* (4.17) *and is denote as* $k_2(\theta;\alpha,x)$, *where* α *is the true value of the parameter. Then there exists at least one root for* $k_2(\theta;\alpha,x)=0$ *for* $\theta\in(0,\infty)$ *and the solution is unique.*

Proof. We have

$$k_2(\theta;\alpha,x)=\frac{n}{\theta}-\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}x_ie^{-\theta x_i}\frac{\alpha-1}{(1-e^{-\theta x_i})}+\alpha\sum_{i=1}^{n}x_ie^{-\theta x_i}(1-e^{-\theta x_i})^{\alpha-1}.$$

We get

$$\lim_{\theta \to 0} k_2(\theta; \alpha, x) = \infty$$

and

$$\lim_{\theta \to \infty} k_1(\alpha; \theta, x) = -\sum x_i < 0$$

Hence, there exists at leasts one root say, $\widehat{\theta} \in (0, \infty)$, such that $k_2(\widehat{\theta}; \alpha, x) = 0$. The root is unique when the first derivative of $k_2(\widehat{\theta}; \alpha, x)$, ie, $k_2'(\widehat{\theta}; \alpha, x) < 0$, where

$$\begin{aligned} k_2'(\widehat{\theta}; \alpha, x) =& \frac{-n}{\theta^2} + \frac{\sum x_i^2 (\alpha - 1) e^{-\theta x_i}}{(1 - e^{-\theta x_i})^2} - \\ & \alpha \sum (x_i^2) e^{-\theta x_i} \left[(1 - e^{-\theta x_i})^{\alpha - 1} - (\alpha - 1) e^{-\theta x_i} (1 - e^{-\theta x_i})^{\alpha - 2} \right] \end{aligned}$$

□

The joint likelihood for both location and scale parameters has exactly one point of maximum and atmost one stationary point then the maximum likelihood function is unimodal. Here we show the likelihood surface of the proposed model is unimodal using a 3D-plot.

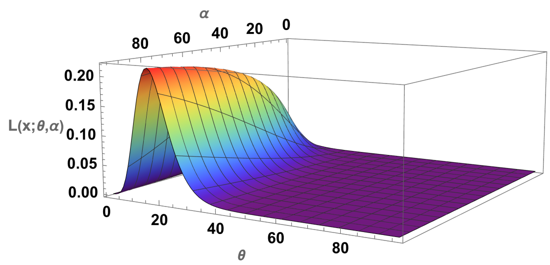


Figure 4.3: The likelihood function plot

From the Fig. 4.3, the likelihood function has exactly one point of maximum.

Asymptotic Confidence bounds

In this subsection, we compute the observed Fisher information for the MLE. Here we derive the asymptotic confidence intervals of the parameters involved in the proposed distribution when $\alpha > 0$ and $\theta > 0$, by using variance covariance matrix. We now derive the observed Fisher information for the likelihood using (4.16) and (4.17). We have

$$I = \begin{pmatrix} \frac{-\partial^2 \log\ L}{\partial \alpha^2} & \frac{-\partial^2 \log\ L}{\partial \alpha \theta} \\ \frac{-\partial^2 \log\ L}{\partial \theta \alpha} & \frac{-\partial^2 \log\ L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log\ L}{\partial \alpha^2} = \frac{-n}{\alpha^2} - \sum \log(1 - e^{-\theta x_i})^2 (1 - e^{-\theta x_i})^{\alpha},$$

$$\begin{aligned} \frac{\partial^2 \log\ L}{\partial \alpha \theta} =& \frac{x_i e^{-\theta x_i}}{\sum (1 - e^{-\theta x_i})} \left[1 - (1 - e^{-\theta x_i}) \right] - \\ & \alpha x_i e^{-\theta x_i} (1 - e^{-\theta x_i})^{\alpha - 1} \sum \log(1 - e^{-\theta x_i}), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 \log\ L}{\partial \theta^2} =& \frac{-n}{\theta^2} + (\alpha - 1) \sum x_i \left[\frac{x_i e^{-\theta x_i} (1 - e^{-\theta x_i}) + x_i e^{-2\theta x_i}}{(1 - e^{-\theta x_i})^2} \right] + \\ & \alpha \sum X_i \left[(\alpha - 1) x_i e^{-2\theta x_i} (1 - e^{-\theta x_i})^{\alpha - 2} - x_i e^{-\theta x_i} (1 - e^{-\theta x_i})^{\alpha - 1} \right]. \end{aligned}$$

We can derive the $(1 - \gamma)100\%$ confidence intervals of the parameters α and θ by using variance matrix as in the form

$$\widehat{\alpha} \pm Z_{\frac{\gamma}{2}} \sqrt{Var(\widehat{\alpha})}, \tag{4.18}$$

and

$$\widehat{\theta} \pm Z_{\frac{\gamma}{2}} \sqrt{Var(\widehat{\theta})} \tag{4.19}$$

where $Z_{\frac{\gamma}{2}}$ is the upper $(\frac{\gamma}{2})^{th}$ percentile of the standard normal distribution.

4.9.2 Method of moment estimation

The r^{th} moment of the proposed model is

$$E(X^r) = \mu_r' = \frac{e\alpha}{\theta^r(e-1)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{\Gamma r + 1}{(j+1)^{r+1}}.$$

Taking $r = 1,\ 2$ in the above equation, we get the first and second raw moments as

$$\mu_1' = \frac{\alpha e}{\theta(e-1)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{1}{(j+1)^2}, \tag{4.20}$$

and

$$\mu_2' = \frac{\alpha e}{\theta^2(e-1)} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{2}{(j+1)^3}. \tag{4.21}$$

The variance and coefficient of variation (CV) of the proposed model are

$$\begin{aligned} \text{V(X)} =& \frac{\alpha e}{\theta^2(e-1)} \left[\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{2}{(j+1)^3} - \right. \\ & \left. \frac{\alpha e}{e-1} \left(\sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+j}}{m!} \binom{\alpha m + \alpha - 1}{j} \frac{1}{(j+1)^2} \right)^2 \right], \end{aligned}$$

and

$$
\begin{aligned}
\text{CV} =& \frac{1}{\theta}\left[\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{2}{(j+1)^3}-\right.\\
&\left.\frac{\alpha e}{e-1}\left(\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{1}{(j+1)^2}\right)^2\right]\\
&\left[\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\alpha m+\alpha-1}{j}\frac{1}{(j+1)^2}\right]^{-1}
\end{aligned}
$$

The MMEs of the proposed model are obtained by equating Equations (4.20) and (4.21) with the sample moments respectively.

$$
\frac{1}{n}\sum_{i=1}^{n}x_i = \frac{\widehat{\alpha}e}{\widehat{\theta}(e-1)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\widehat{\alpha}m+\widehat{\alpha}-1}{j}\frac{1}{(j+1)^2},
$$

and

$$
\frac{1}{n}\sum_{i=1}^{n}x_i^2 = \frac{\widehat{\alpha}e}{\widehat{\theta}^2(e-1)}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{m+j}}{m!}\binom{\widehat{\alpha}m+\widehat{\alpha}-1}{j}\frac{2}{(j+1)^3}.
$$

4.9.3 Method of least square estimation

Swain et al [151] proposed the method of least square estimation for estimating the parameters of Beta distribution. Here we use the same technique for the proposed distribution to estimate the unknown parameters involved in the model. To find the least square estimate of the unknown parameters, we need to minimize the following function

$$
\sum_{i=1}^{n}\left[F(X_{(i)})-\frac{i}{n+1}\right]^2.
$$

Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the ordered random variables of a set of random variables $\{X_1,\ X_2,\ ..., X_n\}$ of size n. $F(X_{(i)})$ is the distribution function of the i^{th} order statistic. The least square estimates of the parameters α and θ are obtained by minimizing

$$\sum_{i=1}^{n} \left[\frac{e}{e-1} \left(1 - e^{-\left(1-e^{-\theta x(i)}\right)^{\alpha}} \right) - \frac{i}{n+1} \right]^2 \tag{4.22}$$

with respect to α and θ.

4.10 Simulation study

In this section we conduct a simulation study using Monte Carlo simulation method. Here we evaluate the performance of the maximum likelihood estimators of the parameters involved in the proposed distribution. In each experiment, using Equation (4.12), thousand pseudo-random samples have been generated for different values of population parameters ($\alpha = 0.6,\ 1.5,\ 2,\ 2.5,$ and $\theta = 0.5,\ 1,\ 1.5,\ 2,\ 2.5$) and sample sizes ($n$= 50, 100, 500, 1000). For 1000 repetitions, the standard error (SE) of the estimated parameters are computed as the square root of the average of their corresponding variance. The results are obtained using R [139] language and presented in Tables 4.1, 4.2, 4.3 and 4.4.

From Tables 4.1, 4.2, 4.3 and 4.4 we can conclude that all the estimators show the property of consistency as a matter of fact that, the standard error decreases as sample size increases.

Next we compute the 95% confidence interval for the parameters using (4.18) and (4.19), and also calculate the coverage percentage. The confidence interval and coverage percentage for the parameters in the proposed model are shown in Tables 4.5, 4.6 and 4.7.

When observing Tables 4.5, 4.6 and 4.7, we can see that the sample size

	n	$\widehat{\alpha}$	$\widehat{\theta}$	$SE(\widehat{\alpha})$	$SE(\widehat{\theta})$
$\theta = 0.5$	50	0.6252	0.5473	0.0902	0.1238
	100	0.6114	0.5144	0.0639	0.0871
	500	0.6031	0.5052	0.0288	0.0398
	1000	0.6015	0.5021	0.0203	0.0281
$\theta = 1$	50	0.6235	1.0837	0.0897	0.2438
	100	0.6121	1.0402	0.0640	0.1751
	500	0.6031	1.0132	0.0288	0.0797
	1000	0.6010	1.0024	0.0203	0.0561
$\theta = 1.5$	50	0.6221	1.6094	0.0898	0.3656
	100	0.6091	1.5408	0.0638	0.2619
	500	0.6013	1.5109	0.0287	0.1191
	1000	0.6006	1.5008	0.0203	0.0841
$\theta = 2$	50	0.6278	2.1581	0.0904	0.4892
	100	0.6111	2.0764	0.0639	0.3511
	500	0.6017	2.0089	0.0287	0.1583
	1000	0.6007	2.0064	0.0203	0.1124
$\theta = 2.5$	50	0.6239	2.7107	0.0901	0.6138
	100	0.6122	2.6252	0.0640	0.4433
	500	0.6015	2.5220	0.0287	0.1987
	1000	0.6011	2.5163	0.02032	0.1410

Table 4.1: The result of simulation study of GKME distribution when $\alpha = 0.6$

increases the length of the confidence interval decreases and the coverage percentage increases.

4.11 Real data applications

In this section, we compare our proposed distribution with some of the well-known distributions in the literature to prove the suitability of the GKME distribution. The distributions which used for comparison are Generalized DUS Exponential (GDUSE) proposed by Maurya et al. [60], KM-Exponential (KME) and KM-Weibull (KMW) introduced in Chapter 3, inverse Weibull (IW), exponential, and Weibull. Additionally we compare the proposed model with Alpha-power transformed Lomax (APTL) by Dey et al [64] and a new lifetime model introduced by Dimitrakopoulou et al [71] for the first data set, Transmuted burr

	n	$\widehat{\alpha}$	$\widehat{\theta}$	$SE(\widehat{\alpha})$	$SE(\widehat{\theta})$
$\theta = 0.5$	50	1.5793	0.5219	0.2658	0.0913
	100	1.5232	0.5063	0.1870	0.0650
	500	1.5064	0.5023	0.0854	0.0296
	1000	1.5049	0.5017038	0.0605	0.0210
$\theta = 1$	50	1.5720	1.0395	0.2629	0.1819
	100	1.5390	1.0284	0.1895	0.1314
	500	1.5050	1.0021	0.0853	0.0591
	1000	1.5041	1.0026	0.0605	0.0419
$\theta = 1.5$	50	1.6062	1.5832	0.2706	0.2757
	100	1.5488	1.5449	0.1905	0.1971
	500	1.5102	1.5085	0.0857	0.0890
	1000	1.5056	1.5052	0.0606	0.0630
$\theta = 2$	50	1.5944	2.1148	0.2666	0.3675
	100	1.5333	2.0410	0.1887	0.2613
	500	1.5059	2.0066	0.0854	0.1183
	1000	1.5022	2.0000	0.0604	0.0837
$\theta = 2.5$	50	1.5884	2.6256	0.2690	0.4585
	100	1.5520	2.5647	0.1920	0.3277
	500	1.5063	2.5130	0.0854	0.1482
	1000	1.5047	2.5063	0.0605	0.1048

Table 4.2: The result of simulation study of GKME distribution when $\alpha = 1.5$

type III (for convenience we use TB Type III in the comparison plot) proposed by Abul-Moniem [155] and a new model by Dimitrakopoulou et al [71] for the second data set and APTL for the third data set. For this purpose we use AIC, BIC, HQC, LL value, p-value and K-S test value for the comparison of the data sets. The distribution which gets minimum AIC, BIC, HQC and K-S test values and maximum LL and p-value is more suitable to the data set. In this section we analyzed the introduced model using R [139] language.

4.11.1 Data set I

Here we consider a real data set which represents the maximum flood levels (in millions cubic of feet per second) of the Susquehanna River at Harrisburg, Pennsylvania, observed over a 20-year period (Dumonceaux and Antle [152]),

	n	$\widehat{\alpha}$	$\widehat{\theta}$	$SE(\widehat{\alpha})$	$SE(\widehat{\theta})$
$\theta = 0.5$	50	2.1326	0.5213	0.3775	0.0845
	100	2.0562	0.5091	0.2682	0.0605
	500	2.0178	0.5027	0.1222	0.0275
	1000	2.0061	0.5012	0.0861	0.0195
$\theta = 1$	50	2.1321	1.0473	0.3775	0.1698
	100	2.0648	1.0208	0.2696	0.1212
	500	2.0198	1.0055	0.1224	0.0551
	1000	2.0084	1.0023	0.0863	0.0390
$\theta = 1.5$	50	2.1186	1.5633	0.3762	0.2539
	100	2.0549	1.5362	0.2685	0.1830
	500	2.0137	1.5074	0.1218	0.0826
	1000	2.0034	1.5034	0.0860	0.0585
$\theta = 2$	50	2.1285	2.0902	0.3749	0.3386
	100	2.0658	2.0548	0.2706	0.2444
	500	2.0040	2.0024	0.1211	0.1099
	1000	2.0037	2.0007	0.0861	0.0779
$\theta = 2.5$	50	2.1281	2.5990	0.3763	0.4218
	100	2.0716	2.5662	0.2711	0.3049
	500	2.0018	2.5074	0.1211	0.1377
	1000	2.0034	2.5023	0.0860	0.0974

Table 4.3: The result of simulation study of GKME distribution when $\alpha = 2$

presented in Table 4.8. Using this data set, Maurya et al. [60] and Kavya and Manoharan [75] studied the suitability of their proposed models.

The comparison table of the proposed model with other six models are given in Table 4.9.

From Table 4.9, we can see that our proposed distribution has lowest AIC, BIC, HQC and K-S test values and largest LL and p-value compared to other given distributions except IW. When compared to IW, the AIC, BIC, and HQC values are slightly lower and LL value is larger than the proposed model. Our model has the lowest K-S test value and the largest p-value compared to IW distribution. The comparison plot is given in Fig. 4.4.

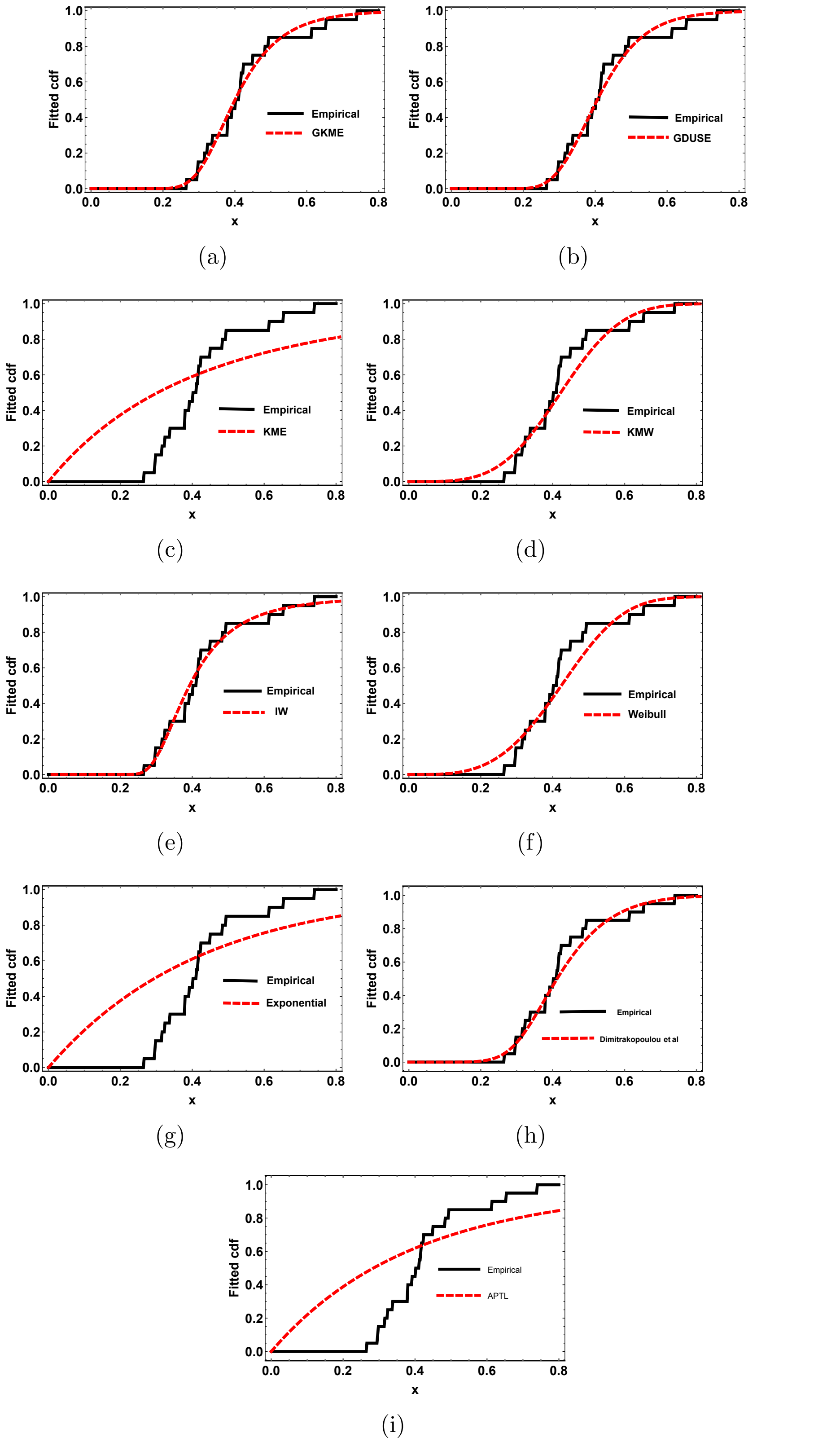


Figure 4.4: The empirical cdf and the cdf plots of the fitted distributions for the first data set.

	n	$\widehat{\alpha}$	$\widehat{\theta}$	$SE(\widehat{\alpha})$	$SE(\widehat{\theta})$
$\theta = 0.5$	50	2.6552	0.5221	0.4906	0.0805
	100	2.5572	0.5079	0.3494	0.0575
	500	2.5248	0.5033	0.1611	0.0262
	1000	2.5101	0.5014	0.1137	0.0186
$\theta = 1$	50	2.6808	1.0417	0.4946	0.1601
	100	2.6106	1.0278	0.3569	0.1158
	500	2.5211	1.0043	0.1607	0.0523
	1000	2.5020	1.0011	0.1132	0.0371
$\theta = 1.5$	50	2.6697	1.5496	0.4979	0.2397
	100	2.5910	1.5342	0.3548	0.1730
	500	2.5177	1.5081	0.1606	0.0787
	1000	2.5120	1.5058	0.1138	0.0557
$\theta = 2$	50	2.6653	2.0782	0.4869	0.3185
	100	2.6024	2.0610	0.3572	0.2324
	500	2.5207	2.0113	0.1607	0.1048
	1000	2.5059	2.0017	0.1134	0.0741
$\theta = 2.5$	50	2.6652	2.6003	0.4924	0.4011
	100	2.5990	2.5579	0.3557	0.2883
	500	2.5173	2.5106	0.1606	0.1310
	1000	2.5111	2.5086	0.1137	0.0928

Table 4.4: The result of simulation study of GKME distribution when $\alpha = 2.5$

4.11.2 Data set II

The second data set gives the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang [143]), given in Table 6. Khan et al. [95], and Kumar et al ([100] studied the applicability of their proposed models using this data. Table 7 shows how well the proposed model fits the data compared to other six models.

Our proposed model shows minimum AIC, BIC, HQC and K-S test value and maximum LL and p-value. So we can say that GKME distribution provides better fit for the data set compared to other distributions mentioned in this study. The plot of empirical cdf and fitted cdf of the distributions for the second data set is given in Fig. 4.5

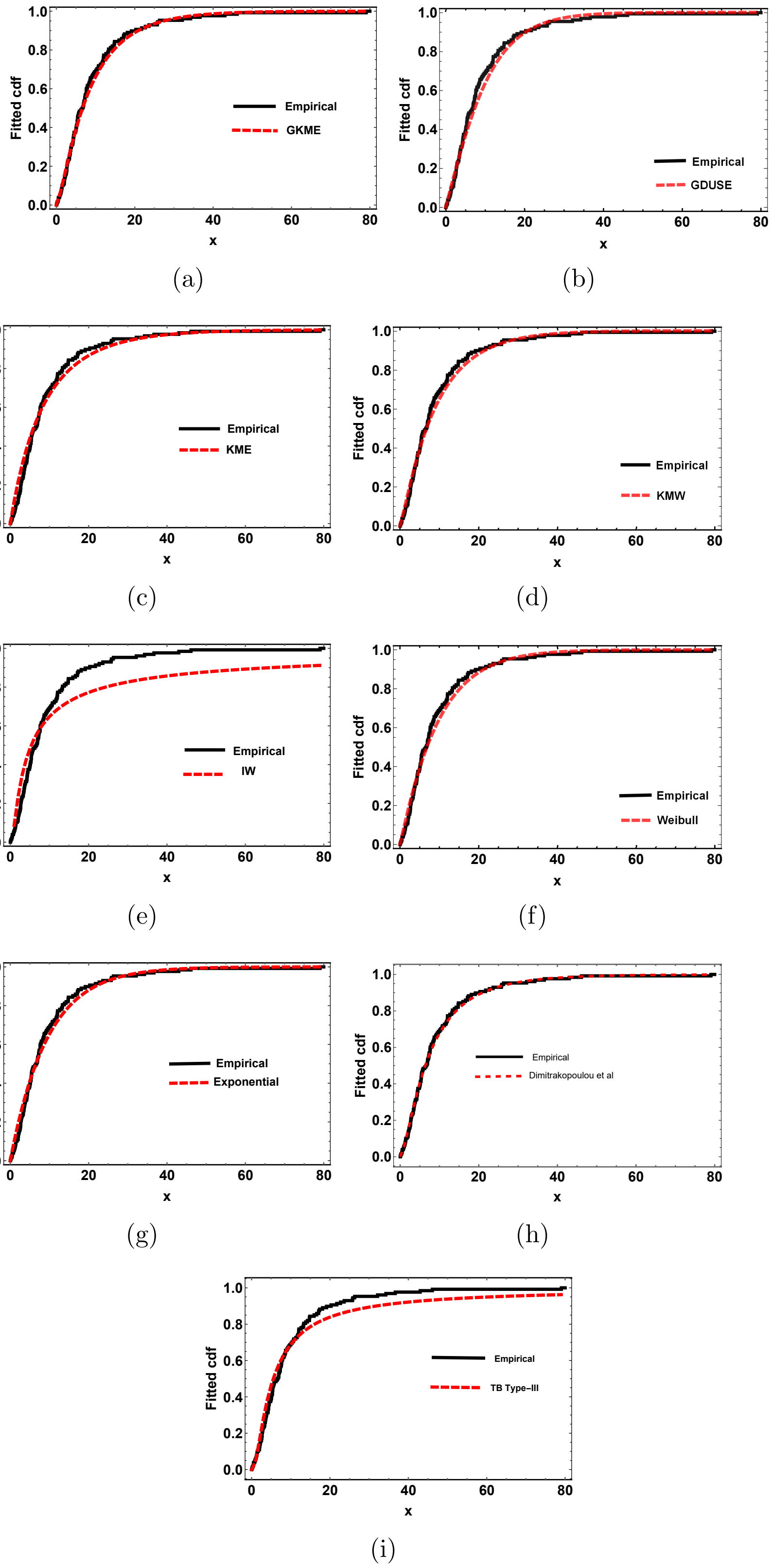


Figure 4.5: The empirical cdf and the cdf plots of the fitted distributions for the second data set.

	n	CI ($\widehat{\alpha}$)	CI ($\widehat{\theta}$)	CP ($\widehat{\alpha}$)	CP ($\widehat{\theta}$)
$\theta = 0.5$	50	(0.2937,0.6478)	(0.2683, 0.7459)	92.9	92.8
	100	(0.6349, 0.8858)	(0.5308, 0.8753)	94.4	94.3
	500	(0.4944, 0.6072)	(0.3993, 0.5550)	94.6	94.9
	1000	(0.5819, 0.6614)	(0.4392, 0.5495)	94.8	96.2
$\theta = 1$	50	(0.4759, 0.8258)	(0.6512, 1.6121)	93.0	92.6
	100	(0.5422, 0.7933)	(0.5109, 1.2003)	93.7	95.0
	500	(0.5222, 0.6348)	(0.8755, 1.1870)	94.3	95.1
	1000	(0.5710, 0.6506)	(0.9593, 1.1797)	95.7	95.2
$\theta = 1.5$	50	(0.4450, 0.7973)	(1.2457, 2.6887)	92.8	93.9
	100	(0.6347, 0.8856)	(1.5926, 2.6262)	94.4	94.2
	500	(0.4946, 0.6074)	(1.1981, 1.6651)	94.6	94.9
	1000	(0.5818, 0.6613)	(1.3170, 1.6479)	94.8	96.2
$\theta = 2$	50	(0.4759, 0.8258)	(1.3014, 3.2232)	93.0	92.6
	100	(0.5424, 0.7935)	(1.0221, 2.4009)	93.7	94.9
	500	(0.5222, 0.6348)	(1.7509, 2.3738)	94.3	95.1
	1000	(0.5690, 0.6485)	(1.8330, 2.2738)	95.7	95.1

Table 4.5: The result of 95% confidence interval (CI) and coverage percentage (CP) for the parameters when $\alpha = 0.6$

	n	CI ($\widehat{\alpha}$)	CI ($\widehat{\theta}$)	CP ($\widehat{\alpha}$)	CP ($\widehat{\theta}$)
$\theta = 0.5$	50	(0.8086, 1.8512)	(0.2865, 0.6461)	91.2	92.3
	100	(0.8963, 1.6387)	(0.2761, 0.5327)	94.0	93.8
	500	(1.3081, 1.6439)	(0.4491, 0.5653)	94.4	94.8
	1000	(1.3340, 1.5713)	(0.4466, 0.5288)	94.9	95.3
$\theta = 1$	50	(1.1862, 2.2409)	(0.8667, 1.5862)	90.6	92.7
	100	(1.0251, 1.7743)	(0.6364, 1.1512)	93.5	93.4
	500	(1.3061, 1.6399)	(0.8438, 1.0755)	95.5	95.2
	1000	(1.4588, 1.6962)	(0.9679, 1.1325)	96.3	95.9
$\theta = 1.5$	50	(0.7027, 1.7547)	(0.5160, 1.6021)	92.4	93.7
	100	(1.0600, 1.8017)	(1.0236, 1.7932)	94.2	94.8
	500	(1.2418, 1.5770)	(1.2331, 1.5822)	95.3	95.2
	1000	(1.4119, 1.6486)	(1.3990, 1.6454)	95.3	95.4
$\theta = 2$	50	(0.7467, 1.7955)	(0.9234, 2.3649)	93.3	94.0
	100	(1.1979, 1.9450)	(1.6764, 2.7054)	94.2	94.2
	500	(1.4864, 1.8213)	(1.7814, 2.2467)	94.5	94.5
	1000	(1.3990, 1.6364)	(1.7546, 2.0836)	94.6	95.5

Table 4.6: The result of 95% confidence interval (CI) and coverage percentage (CP) for the parameters when $\alpha = 1.5$

	n	CI ($\widehat{\alpha}$)	CI ($\widehat{\theta}$)	CP ($\widehat{\alpha}$)	CP ($\widehat{\theta}$)
$\theta = 0.5$	50	(2.0682, 3.5502)	(0.6053, 0.9364)	93.4	93.1
	100	(1.5519, 2.6080)	(0.4847, 0.7223)	93.8	94.3
	500	(1.8436, 2.3190)	(0.4542, 0.5621)	93.9	94.4
	1000	(1.8515, 2.1900)	(0.4631, 0.5395)	94.0	94.7
$\theta = 1$	50	(1.3893, 2.8389)	(0.6921, 1.3472)	91.2	93.2
	100	(1.4746, 2.5313)	(0.7224, 1.1991)	93.6	93.8
	500	(1.9302, 2.4074)	(0.8095, 1.2254)	94.0	93.9
	1000	(1.6945, 2.0314)	(0.8594, 1.0122)	95.2	95.2
$\theta = 1.5$	50	(2.3483, 3.8177)	(1.3207, 2.3184)	92.0	94.2
	100	(1.6874, 2.7464)	(0.9651, 1.6810)	95.0	94.8
	500	(1.7350, 2.21186)	(1.3893, 1.7129)	95.3	95.3
	1000	(1.8887, 2.2268)	(1.4064, 1.6358)	95.4	95.4
$\theta = 2$	50	(1.4252, 2.8890)	(1.9073, 3.2213)	92.5	92.4
	100	(1.4454, 2.4958)	(1.4704, 2.4252)	93.8	94.3
	500	(1.8178, 2.2960)	(1.8457, 2.2776)	93.9	94.9
	1000	(1.7974, 2.1356)	(1.8533, 2.1591)	94.2	95.8

Table 4.7: The result of 95% confidence interval (CI) and coverage percentage (CP) for the parameters when $\alpha = 2$

0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423	0.379	0.324
0.296	0.740	0.418	0.412	0.494	0.416	0.338	0.392	0.484	0.265

Table 4.8: Flood Level Data.

Model	ML estimates	K-S test value	p-value	LL	AIC	BIC	HQC
GKME	$\widehat{\alpha} = 55.65603, \widehat{\theta} = 10.13937$	0.1132	0.9599	16.7992	-29.598	-27.607	-28.848
GDUSED	$\widehat{\alpha} = 72.8797, \widehat{\beta} = 12.3894$	0.1342	0.8635	16.4813	-28.963	-26.971	-28.574
KME	$\widehat{\lambda} = 1.5747$	0.4573	0.0005	-4.6911	11.3822	12.378	11.5766
KMW	$\widehat{\alpha} = 3.9833, \widehat{\lambda} = 1.9669$	0.1982	0.4121	14.0157	-24.0314	-22.04	-23.6427
IW	$\widehat{\alpha} = 4.529, \widehat{\beta} = 0.3617$	0.1452	0.7928	16.8281	-29.6563	-27.6648	-29.2675
Exponential	$\widehat{\lambda} = 2.3557$	0.4643	0.0004	-2.8631	7.7263	8.722	7.9206
Weibull	$\widehat{a} = 3.5634, \widehat{b} = 2.1278$	0.203	0.382	13.4538	-22.9076	-20.9161	-22.5188
Dimitrakopoulou et al [71]	$\widehat{\alpha} = 0.2429, \widehat{\beta} = 7.9232, \widehat{\lambda} = 8642.9526$	0.1635	0.6591	16.2675	-26.5351	-23.5479	-25.9520
APTL	$\widehat{\alpha} = 1.44034e^{-06}, \widehat{\beta} = 2.455945e^{+03}, \widehat{\lambda} = 1.317631e^{+04}$	0.4770	0.0002	-3.6030	13.2059	16.1931	13.7891

Table 4.9: ML estimates, K-S test Value, p-value, LL, AIC, BIC, and HQC of the fitted models for first data set.

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97	9.02	13.29
0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81	2.62	3.82	5.32	7.32	10.06
14.77	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63
17.12	46.12	1.26	2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64
17.36	1.40	3.02	4.34	5.71	7.93	1.46	18.10	11.79	4.40	5.85	8.26	11.98	19.13	1.76
3.25	4.50	6.25	8.37	12.02	2.02	13.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	12.07
6.76	21.73	2.07	3.36	6.93	8.65	12.63	22.69							

Table 4.10: Bladder Cancer Patients Data.

Model	ML estimates	K-S test value	p-value	LL	AIC	BIC	HQC
GKME	$\widehat{\alpha}=1.3738, \widehat{\theta}=0.1022$	0.0519	0.8806	-412.4914	828.9827	834.6868	831.3003
GDUSED	$\widehat{\alpha}=0.9942, \widehat{\beta}=0.1338$	0.1128	0.3725	-416.0211	836.0422	841.7462	838.3597
KME	$\widehat{\lambda}=0.079$	0.1075	0.1039	-416.3697	834.7393	837.5914	835.8981
KMW	$\widehat{\alpha}=1.1585, \widehat{\lambda}=0.0785$	0.0604	0.7382	-413.8836	831.7673	837.4713	834.0848
IW	$\widehat{\alpha}=0.7496, \widehat{\beta}=3.2884$	0.1429	0.0108	-445.7943	895.5886	901.2927	897.9062
Exponential	$\widehat{\lambda}=0.1059$	0.0832	0.3384	-415.4052	832.8104	835.6624	833.9682
Weibull	$\widehat{a}=1.0528, \widehat{b}=0.1035$	0.0663	0.6272	-415.0984	834.1968	839.9009	836.5144
Transmuted Burr-Type-III	$\widehat{k}=2.9358, \widehat{c}=1.1224, \widehat{\lambda}=-0.76498$	0.0861	0.2993	-423.5252	1217.472	1226.028	1220.948
Dimitrakopoulou et al [71]	$\widehat{\alpha}=0.4359, \widehat{\beta}=1.5335, \widehat{\lambda}=0.1366$	0.03723	0.9943	-411.6133	829.2266	837.7827	832.703

Table 4.11: ML estimates, K-S test Value, p-value, LL, AIC, BIC, and HQC of the fitted models for second data set.

4.11.3 Data set III

The data set consists of survival times (in days) of guinea pigs injected with different doses of tubercle bacilli (Bjerkedal [153]), presented in Table 4.12. Kundu and Howlader [154] and Abd-Elrahman [69] used this data set in their papers for comparison purpose.

12	15	22	24	24	32	32	33	34	38
38	43	44	48	52	53	54	54	55	56
57	58	58	59	60	60	60	60	61	62
63	65	65	67	68	70	70	72	73	75
76	76	81	83	84	85	87	91	95	96
98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341
341	376								

Table 4.12: The survival times (in days) of guinea pigs injected with different doses of tubercle bacilli.

The ML estimates, AIC, BIC, HQC, LL, K-S test values and p-values of the distributions for the third data set are given in Table 4.13.

Model	ML estimates	K-S test value	p-value	LL	AIC	BIC	HQC
GKME	$\widehat{\alpha}=2.7193, \widehat{\theta}=0.015$	0.1163	0.2842	-391.4704	786.9408	791.4941	788.7535
GDUSED	$\widehat{\alpha}=2.0251, \widehat{\beta}=0.0184$	0.1429	0.1058	-394.9731	793.9462	798.4995	795.7589
KME	$\widehat{\lambda}=0.0072$	0.23	0.001	-406.3072	814.6144	816.891	815.5207
KMW	$\widehat{\alpha}=1.5531, \widehat{\lambda}=0.0074$	0.139	0.1237	-395.4254	794.8509	799.4042	796.6636
IW	$\widehat{\alpha}=1.4148, \widehat{\beta}=54.1888$	0.152	0.0718	-395.6491	795.2982	799.8515	797.1109
Exponential	$\widehat{\lambda}=0.01$	0.2116	0.0032	-403.4421	808.8843	811.1609	809.7906
Weibull	$\widehat{a}=1.3932, \widehat{b}=0.009$	0.1465	0.0909	-397.1477	798.2953	802.84879	800.108
APTL	$\widehat{\alpha}=9099.1457, \widehat{\beta}=3.0499, \widehat{\lambda}=56.6277$	0.0919	0.5768	-389.852	785.704	792.534	788.423

Table 4.13: ML estimates, K-S test Value, p-value, LL, AIC, BIC, and HQC of the fitted models for third data set.

Based on AIC, BIC, HQC, K-S test value, p-value and LL, we can conclude that the proposed model is more suitable for the data set compared to other models except APTL distribution. The AIC, BIC, and HQC of APTL model is slightly lower and P-value is higher thant the GKME model. Note that the number of parameters in the APTL distribution is greater than the GKME model. The plot of Empirical cdf and fitted cdf of the above distributions for the third data set is given in Fig. 4.6.

4.11.4 Other applications

Apart from the failure rate data sets studied above, we compare the proposed distribution with other distributions given in Abdul-Moniem [155]. So we consider the data set of the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed, so we have complete data with the exact times of failure. The ML estimates of the GKME distribution $(\alpha,\ \theta)$ for the given data set are $(1.863406,\ 0.589529)$ and the corresponding -LL, K-S value, AIC, AICC (Corrected AIC), and BIC values are 122.532, 0.0888, 249.0641, 248.0641, and 253.7255 respectively. Here our proposed model gives better fit to the data set compared to the distributions given in Abdul-Moniem [155]. Next we consider the data set of the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada (Choulakian and Stephens [140]). ML estimates, -LL, K-S value, p-value and AIC of the GKME distribution for this data set are respectively $\alpha = 0.92555$, $\theta = 0.05918$, 251.8458, 0.09241, 0.5701, and 507.6916. Therefore we can say that the GKME distribution works better than the distributions given in references Akinsete et al. [141], Cordeiro et al. [91], Bourguignon et al. [70], Lemonte [53], and Nekoukhou and Bidram [61].

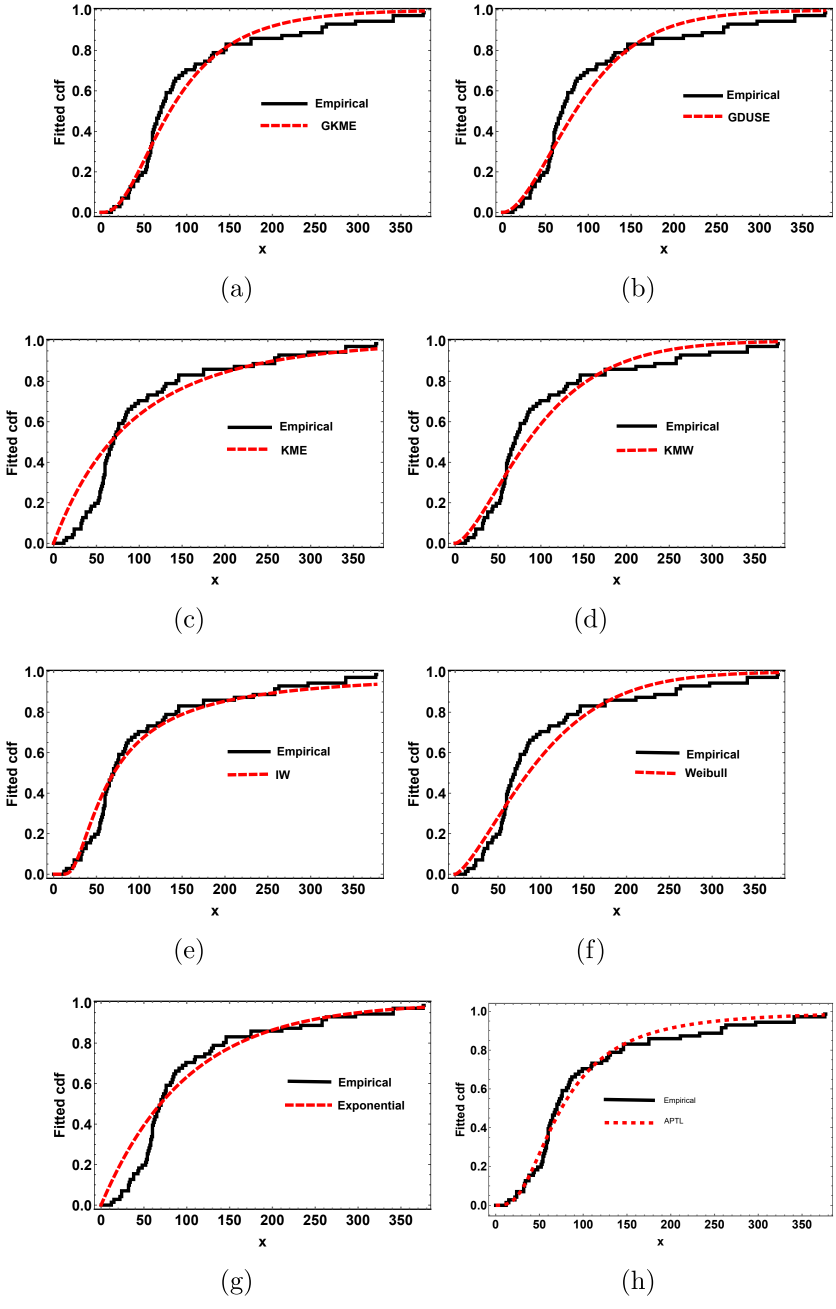


Figure 4.6: The empirical cdf and the cdf plots of the fitted distributions for the third data set.

4.12 Summary of the chapter

In this chapter, we develop a new lifetime model with decreasing and inverse bathtub shaped hazard rate function. The distribution is obtained by generalizing the KM transformation. The basic properties of the new distribution like moments, moment generating function, conditional moments, quantile function, mean deviation and entropy are derived. The consistency of the maximum likelihood estimators of the parameters involved in the distribution is demonstrated through simulation studies. With the help of three real data sets, we study and illustrate the flexibility of the new distribution. Our studies show that the GKME distribution is a promising model for a large variety of lifetime data in medical and reliability areas.

Chapter 5

Estimation of Stress-Strength Reliability Based on KME Model

5.1 Introduction

In reliability theory the estimation of stress-strength reliability is an important problem. It has many applications in engineering and physics areas. In many practical situations, the assumption of identical strength distributions may not be quite realistic because components of a system are of different structure.

The term stress is defined as a failure inducing variable. That means the stress (load) which tends to produce a failure of a component or of a device of a material. For example, environment, pressure, load, velocity, resistance, temperature, humidity, vibrations, and voltage etc. The term strength is defined as it is failure resisting variable. The ability of component, device or a material to accomplish its required function (mission) satisfactorily without failure when subjected to the external loading and environment.

The stress-strength reliability model depicts the life of a component or item with a random strength X and is subjected to a random stress Y. If the stress on the component surpasses the strength, it fails instantaneously. Whenever $Y < X$

the item functions satisfactorily. The component reliability is defined as

$$R = P(Y < X) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x,\ y)\ dy\ dx,$$

where $f(x,\ y)$ is the joint pdf of X and Y. Suppose the random variable X and Y are independent, then R can be written as

$$R = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x)\ g(y)\ dy\ dx,$$

where $f(x)$ and $g(y)$ are the marginal pdfs of X and Y. This is also can be written as

$$R = \int_{-\infty}^{\infty} f(x)\ G_y(x)\ dx.$$

where $G_y(x)$ is the cdf of $g(y)$.

The germ of this idea was proposed by Birnbaum [156] and was developed by Birnbaum and McCarty [157]. The formal term “stress–strength” firstly appears in the title of Church and Harris [158]. Based on certain parametric assumptions regarding X and Y, the first attempt to study R was undertaken by Owen et al. [159]. They also calculated the confidence interval for R when X and Y are independent or dependent normally distributed random variables. The estimation of R for major distributions like normal (Church and Harris [158], Downton [160];, Woodward and Kelley [161]), exponential (Kelly et al [162], Tong [163]), Pareto (Beg and Singh [164]), and exponential families (Tong [165]) was derived by the end of seventies. Enis and Geisser [166] contribute the Bayes estimation of R for exponentially or normally distributed X and Y. The other major works of the seventies include the introduction of a non-parametric empirical Bayes estimation of R by Ferguson [167] and Hollander and Korwar [168], and the study of system reliability (Bhattacharya and Johnson [169]).

Both stress and strength depend on some known covariates, Guttman et al. [170] and Weerahandi and Johnson [171] discussed the estimation and associated confidence interval of R. Using Bayesian approach Sun et al. [172] estimated the stress-strength reliability. Raqab and Kundu [173] carried out the estimation of stress-strength reliability, when Y and X two independent scaled Burr type X distribution. A comprehensive treatment of the different stress-strength models till 2001 can be found in the excellent monograph by Kotz et al. [174]. Some of the work on the estimation of stress-strength reliability can be obtained in Kundu and Gupta ([175], [176]), Kundu and Raqab [177], Krishnamoorthy et al. [178], Raqab et al. [179], Rezaei et al. [180], and Baklizi [181]. Baklizi and Eidous [182] introduced an estimator of stress-strength reliability based on kernel estimators. Estimation of stress-strength reliability using empirical likelihood method was studied by Jing et al. [183].

Basirat et al. [184] studied the estimation of stress-strength parameter using record values from proportional hazard model. Estimation of stress-strength reliability based on the generalized exponential distribution was developed by Asgharzadeh et al.[185]. Bai et al. [186] considered reliability inference of stress-strength model under progressively Type-II censored samples when stress and strength have truncated proportional hazard rate distributions. Bi and Gui [187] derived Bayesian estimation of R using inverse Weibull distribution. Ghitany et al. [188] discussed inference on stress-strength reliability based on power Lindley distribution. Sharma [189] proposed an upside-down bathtub shape distribution and estimate of stress-strength reliability of inverse Lindley distribution.

In recent times the study of the stress-strength reliability estimation of single and multi-component systems using various generalizations of half logistic distribution was done by Jose et al. [190] and Xavier and Jose [191]. Domma et al. [192] developed the stress-strength reliability based on the m-generalized order statistics and the corresponding concomitant. Estimation of R using inverse Weibull distribution based on progressive first failure censoring was proposed by Krishna et al. [193]. Inference on R in bivariate Lomax model introduced by Musleh et al. [194]. Kohansal and Nadarajah [195] considered estimation of R using Kumaraswamy distribution based on Type-II hybrid progressive censored samples. Estimation of stress-strength reliability of single and multi-component systems based on discrete phase type distribution was studied by Joby et al. [196].

In reliability analysis the multi-component stress-strength (MSS) reliability modeling gets greater attention because in many real life situations the system consist of two or more components. Bhattacharyya and Johnson [197] noticed the performance of a system depends on more than one component and these components have their own strength. Mokhlis and Khames [198] derived the reliability of some parallel and series MSS model using multivariate Marshall-Olkin Exponential distribution. The MSS reliability based on generalized exponential distribution was introduced by Rao [199]. Recently many authors studied MSS reliability, the main works include Rao et al. [200], Dey et al. [201], Khalil [202], Kohansal [203], Abouelmagd et al. [204], Hassan and Alohal [205], Fatma [206], Jamal et al. [207], Jha et al. [208], and Hassan et al. [209].

In this chapter we study the estimation of stress-strength reliability for KME distribution. Here we consider the case that the strength variables X and stress variable Y are independent. The maximum likelihood estimation and asymptotic distribution of R are derived. The simulation study is carried out and using simulated data set we perform the analysis.

5.2 Preliminaries of KME model

We obtain the KME model using the cdf of exponential distribution in KM transformation given in Equation (2.1). Recall the pdf and cdf of the KME distribution from Chapter 3. The pdf and cdf are

$$f(x) = \frac{\lambda e^{-\lambda x} e^{e^{-\lambda x}}}{e-1}, \quad x > 0, \quad \lambda > 0,$$
$$F(x) = \frac{e}{e-1}[1 - e^{-(1-e^{-\lambda x})}], \quad x > 0, \quad \lambda > 0,$$

5.3 Stress-strength reliability based on the model

The stress-strength reliability model depicts the life of a component or item with a random strength X and is subjected to a random stress Y. If the stress on the component surpasses the strength, it fails instantaneously. Whenever $Y < X$ the item functions satisfactorily. The component reliability is defined as $R = P(Y < X)$. It has applications in engineering fields such as failure of aircraft structures, deterioration of rocket motors, and the aging of concrete pressure vessels.

Suppose X and Y are two independent random variables. If $X \sim \text{KME}(\lambda_1)$ and $Y \sim \text{KME}(\lambda_2)$, then the stress-strength reliability is obtained as

$$R = P(Y < X) = \int_0^\infty \frac{\lambda_1 e^{-\lambda_1 x} e^{e^{-\lambda_1 x}}}{e-1} \left[\frac{e}{e-1}\left(1 - e^{-(1-e^{-\lambda_2 x})}\right)\right] dx$$
$$= \frac{\lambda_1 e}{(e-1)^2} \sum_{m=0}^{\infty} \frac{1}{m!} \left[I_1 + I_2\right] \tag{5.1}$$

where $I_1 = \int_0^\infty e^{-\lambda_1 x(m+1)} dx$ and $I_2 = \int_0^\infty e^{-\lambda_1 x(m+1)} e^{-(1-e^{-\lambda_2 x})} dx$. After integration, we get the values of $I_1 = \frac{1}{\lambda_1(m+1)}$ and $I_2 = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{n+i}}{n!}\binom{n}{i}\frac{1}{\lambda_1(m+1)+\lambda_2 i}$. Substituting these values in (5.1), the stress-strength reliability based on the KME model is obtained as

$$\begin{aligned} R &= P(Y < X) \\ &= \frac{\lambda_1 e}{(e-1)^2}\sum_{m=0}^{\infty}\frac{1}{m!}\left[\frac{1}{\lambda_1(m+1)} - \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{n+i}}{n!}\binom{n}{i}\frac{1}{\lambda_1(m+1)+\lambda_2 i}\right]. \end{aligned} \tag{5.2}$$

5.4 Estimation of R

Suppose we drawn a random sample $x_1, x_2, ..., x_p$ of size p from KME(λ_1) and $y_1, y_2, ..., y_q$ of size q from KME(λ_2). The likelihood function is obtained as

$$L = \left(\frac{1}{e-1}\right)^p \lambda_1^p e^{-\lambda_1\sum_{i=1}^p x_i} e^{\sum_{i=1}^p e^{-\lambda_1 x_i}} \left(\frac{1}{e-1}\right)^q \lambda_2^q e^{-\lambda_2\sum_{j=1}^q y_j} e^{\sum_{j=1}^q e^{-\lambda_2 y_j}} \tag{5.3}$$

The log likelihood function is

$$\begin{aligned} \log\ L =& p\log(\frac{1}{e-1}) + p\log(\lambda_1) - \lambda_1\sum_{i=1}^p x_i + \sum_{i=1}^p e^{-\lambda_1 x_i} \\ & q\log(\frac{1}{e-1}) + q\log(\lambda_2) - \lambda_2\sum_{j=1}^q y_j + \sum_{j=1}^q e^{-\lambda_2 y_j} \end{aligned} \tag{5.4}$$

The partial derivatives of the log likelihood function with respect to λ_1 and λ_2 are

$$\frac{\partial \log\ L}{\partial \lambda_1} = \frac{p}{\lambda_1} - \sum_{i=1}^p x_i\left(1 + e^{-\lambda_1 x_i}\right)$$

and

$$\frac{\partial \log\ L}{\partial \lambda_2} = \frac{q}{\lambda_2} - \sum_{j=1}^{q} y_j \left(1 + e^{-\lambda_2 y_j}\right)$$

The maximum likelihood estimates of the parameters are obtained as the solution of the above non-linear equations.

The second partial derivatives of the log likelihood function with respect to λ_1 and λ_2 are

$$\frac{\partial^2 \log\ L}{\partial \lambda_1^2} = \frac{-p}{\lambda_1^2} + \sum_{i=1}^{p} x_i^2 e^{-\lambda_1 x_i}$$

and

$$\frac{\partial^2 \log\ L}{\partial \lambda_2^2} = \frac{-q}{\lambda_2^2} + \sum_{j=1}^{q} y_j^2 e^{-\lambda_2 y_j}$$

The maximum likelihood estimate of Stress-Strength reliability R is

$$\widehat{R}_{ML} = \frac{\widehat{\lambda_1} e}{(e-1)^2} \sum_{m=0}^{\infty} \frac{1}{m!} \left[\frac{1}{\widehat{\lambda_1}(m+1)} - \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{n+i}}{n!} \binom{n}{i} \frac{1}{\widehat{\lambda_1}(m+1) + \widehat{\lambda_2} i} \right] \tag{5.5}$$

We obtain the expression of the maximum likelihood estimate of Stress-Strength reliability R by substituting the estimated parameters in the Equation (5.2).

5.5 Asymptotic distribution and confidence interval

In this section we focused on the asymptotic distribution and confidence interval of the maximum likelihood estimate of R. To obtain the asymptotic variance of the maximum likelihood estimate of R, we consider the Fisher information matrix of λ and is denoted as I.

$$I = -\begin{pmatrix} E(\frac{\partial^2 \log\ L}{\partial\lambda_1^2}) & E(\frac{\partial^2 \log\ L}{\partial\lambda_1\partial\lambda_2}) \\ E(\frac{\partial^2 \log\ L}{\partial\lambda_2\partial\lambda_1}) & E(\frac{\partial^2 \log\ L}{\partial\lambda_2^2}) \end{pmatrix}$$

Using the standard method of asymptotic properties of maximum likelihood estimate, we derive the asymptotic normality of R as

$$d(\lambda) = \left(\frac{\partial R}{\partial\lambda_1}, \frac{\partial R}{\partial\lambda_2}\right)' = (d_1, d_2)'$$

Here

$$\frac{\partial R}{\partial\lambda_1} = -\frac{e}{e-1}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{n+i}}{n!}\binom{n}{i}\frac{\widehat{\lambda_2}i}{(\widehat{\lambda_1}(m+1)+\widehat{\lambda_2}i)^2}$$

and

$$\frac{\partial R}{\partial\lambda_2} = -\frac{\widehat{\lambda_1}e}{(e-1)^2}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^{n+i}}{n!}\binom{n}{i}\frac{i}{(\widehat{\lambda_1}(m+1)+\widehat{\lambda_2}i)^2}$$

Now we obtain the asymptotic distribution of $\widehat{R}_{ML}$ as

$$\sqrt{p+q}(\widehat{R}_{ML} - R) \rightarrow^d N(0, d'(\lambda)I^{-1}d(\lambda)).$$

The asymptotic variance of the $\widehat{R}_{ML}$ is

$$\begin{aligned} AV(\widehat{R}_{ML}) =& \frac{1}{p+q}0, d'(\lambda)I^{-1}d(\lambda) \\ =& V(\widehat{\lambda_1})d_1^2 + V(\widehat{\lambda_2})d_2^2 + 2d_1d_2Cov(\widehat{\lambda_1}, \widehat{\lambda_2}). \end{aligned}$$

Hence an asymptotic $100(1-\xi)\%$ confidence interval for R can be obtained as

$$\widehat{R}_{ML} \pm Z_{\frac{\xi}{2}} \sqrt{AV(\widehat{R}_{ML})},$$

where $Z_{\frac{\xi}{2}}$ is the upper $\frac{\xi}{2}$ quantile function of the standard normal distribution.

5.6 Simulation study

In this section we check the performance of estimators in R using simulation technique. For this purpose we generate 1000 pseudo random samples using Newton-Raphson method. The random samples are generated for different population parameters of (λ_1, λ_2) as (0.5,1), (0.9,0.5), and (1.0,0.9) and sample sizes (p, q) as (10,10),(15,25),(20,20),(30,30),(40,40), and (50,50). The maximum likelihood estimates, their mean square error (MSE) and 95% confidence interval (CI) are calculated and the results are given in the following tables.

(p, q)	ML estimates	MSEs	CI
(10,10)	λ_1 =0.57077	0.24634	(0.08795, 1.05359)
	λ_2 =1.14347	0.06295	(0.80201, 1.26686)
(15,25)	λ_1 =0.53939	0.20729	(0.13312, 0.94566)
	λ_2 =1.03657	0.12229	(0.80124, 1.06082)
(20,20)	λ_1 =0.53241	0.32406	(-0.10276, 1.16757)
	λ_2 =1.05715	0.04907	(0.96097, 1.15333)
(30,30)	λ_1 =0.52109	0.13712	(0.25233, 0.78985)
	λ_2 =1.04110	0.01237	(0.80140, 1.28079)
(40,40)	λ_1 =0.51019	0.22229	(0.07450, 0.94588)
	λ_2 =1.02810	0.02570	(0.97773, 1.07847)
(50,50)	λ_1 =0.51125	0.19753	(0.12409, 0.89841)
	λ_2 =1.02751	0.02640	(0.97577, 1.07925)

Table 5.1: The ML estimates, MSEs and confidence interval of different estimators of R when $\lambda_1 = 0.5$ and $\lambda_2 = 1.0$

Results from the simulation study reveals that sample sizes p and q increase, the estimated parameter values tends to population parameter values. Also the

(p,q)	ML estimates	MSEs	CI
(10,10)	λ_1 =1.00947	0.29601	(0.42928, 1.58966)
	λ_2 =0.55836	0.10123	(0.35994, 0.75677)
(15,25)	λ_1 =0.96646	0.12012	(0.73102, 1.2019)
	λ_2 =0.52620	0.01846	(0.35258, 0.52656)
(20,20)	λ_1 =0.95902	0.08818	(0.78620, 1.13185)
	λ_2 =0.53293	0.01209	(0.43045, 0.53541)
(30,30)	λ_1 =0.94372	0.06563	(0.81508, 1.07237)
	λ_2 =0.51937	0.00126	(0.49567, 0.54307)
(40,40)	λ_1 =0.92773	0.00185	(0.82411, 0.93135)
	λ_2 =0.51179	0.00033	(0.41115, 0.51243)
(50,50)	λ_1 =0.91653	0.00167	(0.89133, 0.91980)
	λ_2 =0.51249	0.00017	(0.45121, 0.51283)

Table 5.2: The ML estimates, MSEs and confidence interval of different estimators of R when $\lambda_1 = 0.9$ and $\lambda_2 = 0.5$

(p,q)	ML estimates	MSEs	CI
(10,10)	λ_1 =1.70398	0.11763	(1.06944, 1.73853)
	λ_2 =1.01417	0.04031	(0.13567, 1.01478)
(15,25)	λ_1 =1.62178	0.09751	(1.10266, 1.64089)
	λ_2 =0.93769	0.03302	(0.27297, 1.00240)
(20,20)	λ_1 =1.59399	0.08241	(1.23654, 1.65144)
	λ_2 =0.95547	0.00594	(0.43819, 0.96712)
(30,30)	λ_1 =1.56481	0.08071	(1.39415, 1.54773)
	λ_2 =0.93981	0.00556	(0.63479, 0.98262)
(40,40)	λ_1 =1.55355	0.05150	(1.39382, 1.71329)
	λ_2 =0.92423	0.00625	(0.71198, 0.93647)
(50,50)	λ_1 =1.53238	0.00228	(1.32790, 1.53686)
	λ_2 =0.92832	0.00279	(0.81279, 0.94385)

Table 5.3: The ML estimates, MSEs and confidence interval of different estimators of R when $\lambda_1 = 1.5$ and $\lambda_2 = 0.9$

MSEs are decreasing with increase in sample sizes (p,q).

5.7 Application

In this section we have generated two data sets ($p = q = 20$) using KME model with parameter values $\lambda_1 = 1$ and $\lambda_2 = 0.5$. Therefore the value of R is obtained as 0.17633. The data points are adjusted in two decimal points and the data sets are presented in the following tables.

4.02	0.44	1.43	0.09	0.49	0.27	0.54	0.02	0.48	1.77
3.04	4.30	0.94	3.08	1.42	0.09	3.05	2.17	0.21	0.64

Table 5.4: Data set I

0.09	0.26	0.20	0.11	2.08	1.48	0.85	2.57	1.04	0.26
0.01	0.40	1.37	0.71	0.29	1.10	0.81	0.13	1.73	2.25

Table 5.5: Data set II

In this case the maximum likelihood estimates of λ_1 and λ_2 are obtained respectively as 0.854 and 0.542. Here the estimated value of R, $\widehat{R}_{ML}$ is obtained as 0.21336. The corresponding 95% confidence interval based on asymptotic distribution is (0.19153, 0.23519).

5.8 Summary of the chapter

In this chapter we consider the estimation of the stress-strength reliability for the KME model for independent stress and strength random variables when the parameters are unknown. The maximum likelihood estimators of the unknown parameters are calculated. Then provide the asymptotic distributions of the maximum likelihood estimators, which have been used to construct the asymptotic confidence intervals. Simulation study is carried out to examine the performance of the estimators. The study reveals that MSEs are decreasing with increase in sample sizes. Using a simulated data set, we find the estimates of the parameters, $\widehat{R}_{ML}$ value and 95% confidence interval.

Chapter 6

Conclusions and Future Works

6.1 Conclusions and future works

The theory of reliability is a well established scientific discipline with its own principles and methods of problem solving. Probability theory and mathematical statistics play an important role in most problems in reliability theory. The study of life length of human beings, structures, organisms, materials, etc., is of great importance in the biological, actuarial, engineering and medical sciences. Therefore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions based on some aspects of ageing. So it is clear that research on ageing properties (univariate, bivariate, and multivariate) is currently being vigorously pursued. Many of the univariate definitions do have physical interpretations such as arising from shock models. The simple ageing classes, that is, IFR, IFRA, NBU, NBUE, DMRL etc have been shown to be very useful in reliability related decision making, such as replacement and maintenance studies.

While positive ageing concepts are well understood, negative ageing concepts (life improved by age) are less intuitive. Nevertheless, negative ageing phenomenon does occur quite frequently. There have been cases reported by several authors where the failure rate functions decrease with time. A population is expected to exhibit decreasing failure rate (DFR) when its behaviors over time is characterized by 'work hardening' (in engineering terms), or 'immunity' (in biological terms). Modern phenomenon of DFR includes reliability growth (in software reliability). Non-monotonic ageing concepts have been found useful in many reliability and survival analysis such as burn-in time decision.

In Chapter 1, the basic concepts of the reliability theory and review of the lifetime distributions are given. The relevance and scope of the study is also discussed in this chapter.

In chapter 2, a new transformation for lifetime models has been proposed and the transformation is called Kavya- Manoharan (KM) transformation. Shapes of the pdf and the hazard rate function are discussed in this chapter. New lifetime distributions can be developed using this transformation. The main advantage of the newly proposed models is that they do not introduce any additional parameters. In other words, the distributions are parsimonious in parameters. Consequently, the newly introduced models using KM transformation allow for simple mathematical computations and parametric estimation.

In Chapter 3, Three new distributions are proposed using the newly developed transformation. Exponential, Weibull, and Lomax distributions are used as the baseline distribution in the transformation and the new distributions are called respectively KM-Exponential (KME), KM-Weibull (KMW) and KM-Lomax (KML) distributions. KME and KML distributions show decreasing hazard rate function and the KMW distribution shows both increasing and decreasing hazard rate function. The analytical characteristics of these newly proposed models are explained. The suitability of these distributions are illustrated using real data sets which are available in the literature. The proposed lifetime models give better fit to the data sets compared to some of the other well known distributions given in this study.

In chapter 4, we have generalized the KM transformation using a new parameter. A new lifetime model is developed by substituting the exponential distribution in the generalized KM transformation and is called Generalized KM Exponential (GKME) distribution. The new lifetime model shows decreasing and inverse bathtub failure rate function. Many statistical characteristics of the new lifetime model have been derived. Three real data sets are used to show the flexibility of the proposed model. The new lifetime model shows better result than other distributions given in this study.

In chapter 5, we have derived the stress-strength reliability of the KME model. The stress-strength reliability R plays an important role in many practical fields including medicine, quality control and engineering. It measures the probability that the random strength X exceeds the stress Y of a component. So we carried out the estimation of stress-strength parameter using maximum likelihood method and also establish the asymptotic distribution and confidence interval for R. To check the performance of the maximum likelihood estimators, simulation study is carried out. With the help of simulated data sets, the maximum likelihood estimates and the estimate of R are calculated.

In this thesis we focused to enhance the literature of ageing of life distributions using newly proposed models which are more suitable to the real lifetime data sets. Many authors incorporated more parameters to enhance the existing lifetime models so the mathematical calculations, estimation of parameters, analysis of the real life data sets become more complicated. In this scenario, our work gets more attention because of its simplicity. Without any hectic exercise we can simply calculate the analytical characteristics of the models, estimation of parameters and analysis.

Many works are open for the interested scholars in this area. Our main interests is to study the general properties of the KM family of distributions, which is the work we are now focused on. Along with this, we are going to perform different estimation methods to estimate the parameters of the introduced models and show the consistency of parameters using simulation study. Researchers can introduce new lifetime models using existing distributions as baseline distribution in the KM transformation. To study the change point of failure rate and mean residual functions of the proposed model (GKME) is also a possible future work. The GKME distribution has a compact distribution function so we can use in a convenient manner for censored data. Therefore, we plan to explore it as one of our future work. Burn-in process and maintenance policy are often used in real life situations. So we currently work on the computation of optimal burn-in time and optimal age under age replacement policy for KM-Kumaraswamy model, which shows bathtub shaped failure rate model. In addition, we aim to find the appropriate testing parameters to minimize the total of testing, manufacturing, quality and reliability costs and also to study about the application of the new model in decision making.

List of Published Works

1. Kavya, P., Manoharan, M. (2020). On a generalized lifetime model using DUS transformation. In: Joshua V., Varadhan S., Vishnevsky V. (eds) Applied Probability and Stochastic Processes, Infosys Science Foundation Series. Springer Nature, Singapore, pp. 281-291. DOI: 10.1007/978-981-15-5951-8_17.

2. Kavya, P., Manoharan, M. (2021). Some parsimonious models for lifetimes and applications. *Journal of Statistical Computation and Simulation*, vol. 91, no. 18, pp. 3693-3708. https://doi.org/10.1080/00949655.2021.1946064.

3. Manoharan, M., Kavya, P. (2022). A new reliability model and applications. *Reliability:Theory and Applications*, Vol. 17, pp. 65-75. https://doi.org/10.24412/1932-2321-2022-167-65-75.

4. Kavya, P., Manoharan, M. (2023). A new lifetime model for non-monotone failure rate data. *Journal of the Indian Society for Probability and Statistics.* https://doi.org/10.1007/s41096-023-00152-x.

5. Gauthami P., Mariyamma K. D., Kavya P. (2023). A modified inverse Weibull distribution using KM transformation. *Reliability:Theory and Applications*, Vol. 18, No. 1(72), pp. 35-42. https://doi.org/10.24412/1932-2321-2023-172-35-42.

6. Kavya, P., Manoharan, M. (2023). Estimation of stress-strength reliability based on KME model. Accepted in Reliability:Theory and Applications.

Presentations in Conference/Seminars

1. 'On a Generalized Lifetime Model Using DUS Transformation', International Conference on Advances in Applied Probability and Stochastic Processes organized by Centre for Research in Mathematics, Department of Mathematics, CMS College Kottayam during 7-10 January 2019.

2. 'On A New Parsimonious Lifetime Distribution', National Seminar on Recent Trends in Statistical Theory and Applications-2020 (NSSTA-2020) Webinar organized by Indian Society for Probability and Statistics (ISPS), Kerala Statistical Association (KSA) and the Department of Statistics, University of Kerala, Trivandrum during June 29-July 01, 2020.

3. 'A new lifetime model for non-monotone failure rate', International Conference (virtual mode) on Emerging Trends in Statistics and Data Science in Conjunction with 40th Annual Convention of Indian Society for Probability and Statistics (ISPS) jointly organized by the Departments of Statistics of CUSAT, Cochin, M.D. University, Rohtak, University of Kerala, Trivandrum, Bharathiar University, Coimbatore, The Madura College (Autonomous), Madurai during 7-10 September, 2021.

Bibliography

[1] Barlow, P. E., Proschan, F. (1965). Mathematical Theory of Reliability. John Wiley and Sons, Hoboken.

[2] Lawless, J. (1982). Statistical Models and Methods for Lifetime Data. John Wiley and Sons, New York.

[3] Marshall, A. W., Olkin, I. (2007). Life Distributions. Structure of Nonparametric Semiparametric and Parametric Families. Springer, New York.

[4] Deevey, E. S. (1947). Life Tables for Natural Populations of Animals. *Quarterly Review of Biology*, Vol.22, pp. 283-314.

[5] Chiang, C. L. (1968). Introduction to Stochastic Processes in Biostatistics. Wiley, New York.

[6] Cox, D. R. (1962). Renewal Theory. Methuen, London.

[7] Kotz, S., Shanbhag, D. N. (1980). Some New Approaches to Probability Distributions. *Journal of Applied Probability*, Vol. 12, pp. 903-921.

[8] Hall, W. J., Wellner, J. A. (1981). Mean Residual Life. Statistics and Related Topics. North-Holland, Amsterdam, pp. 169-184.

[9] Morrison, D. G. (1978). On Linearly Increasing Mean Residual Lifetimes. *Journal of Applied Probability*, Vol. 15 (3).

[10] Bhattacharjee, M. C. (1982). The Class of Mean Residual Lives and Some Consequences. *SIAM Journal on Matrix Analysis and Applications*, Vol. 3 (1), DOI:10.1137/0603006.

[11] Deshpande, J. V., Kochar, S. C., Singh, H. (1986). Aspects of Positive Ageing. *Journal of Applied Probability*, 23, pp. 748-758.

[12] Lai, C. D., Xie, M. (2006). Stochastic ageing and dependence for reliability. Springer.

[13] Rolski, T. (1975). Mean Residual Life. *Bulletin International Statistical Institute*, Vol. 46, pp. 266-270.

[14] Klefsjo, B. (1981). HNBUE Survival Under Some Shock Models. *Scandinavian Journal of Statistics*, Vol. 8, No. 1, pp. 39-47.

[15] Klefsjo, B. (1983). A Useful Ageing Property Based on The Laplace Transform. *Journal of Applied Probability*, Vol. 20, No. 3, pp. 615-626.

[16] Launer, R. L. (1984). Inequalities for NBUE and NWUE Life Distributions. *Operations Research*, Vol. 32(3), pp. 660-667.

[17] Gupta, R. C., Kirmani, S. N. U. A., Launer, R. L. (1987). On Life Distributions Having Monotone Residual Variance. *Probability in the Engineering and Informational Sciences*, Vol. 1 (3), pp. 299-307.

[18] Deshpande, J. V., Singh, H., Bagai, I., Jain, K. (1990). Some Partial Orders Describing Positive Ageing. *Communication in Statistics- Stochastic Model*, Vol. 6, pp. 471-481.

[19] Singh, H., Deshpande, J. V. (1985). On Some New Ageing Properties. *Scandinavian Journal of Statistics*, Vol. 12 (3), pp. 213-220.

[20] Barlow, R. E., Proschan, F. (1975). Statistical theory of Reliability and Life testing: Probability models. Holt, Rinehart and Winston, INC.

[21] Shaked, M.,, Shanthikumar, J. G. (2007). Stochastic orders. Springer, New York.

[22] Koutras, V. P. (2011). Two-level software rejuvenation model with increasing failure rate degradation. In: *Zamojski W., Kacprzyk J., Mazurkiewicz J., Sugier J., Walkowiak T. (eds) Dependable Computer Systems. Advances in Intelligent and Soft Computing*, Springer, Berlin, Heidelberg, vol 97, pp. n101-115. 39:35–42.

[23] Ross, S. M., Shanthikumar, G. J., Zhu, Z. (2005). On incresing-failure-rate random variables. *Jornal of Applied Probability*, Vol 42, pp. 797-809.

[24] Barlow, R. E., Marshall, A. W., Proschan, F. (1963). Properties of probability distributions with monotone hazard rate. *Annals of Mathematical Statistics*, Vol 34, pp. 375-389.

[25] Barlow, R. E., Marshall, A. W. (1964). Bounds for distributions with monotone hazard rate I and II. *Annals of Mathematical Statistics*, Vol 35, pp. 1234-1274.

[26] Barlow, R. E., Marshall, A. W. (1965). Tables of bounds for distributions with monotone hazard rate. *Journal of Amererican Statistical Association*, Vol 60, pp. 872-890.

[27] Gupta, R. D. (2001). Exponentiated exponential family: an alternative to gamma and Weibull distributions. *Biometrica Journal*, Vol 43, pp. 117-130.

[28] Gupta, R. D., Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, Vol 43, pp. 621-634.

[29] Cancho, V. G., Louzada-Neto, F., Barriga, G. D. C. (2011). The Poisson-exponential lifetime distribution. *Computational Statistics and Data Analysis*, Vol 55, pp. 677-686.

[30] Cozzolino, J.M. (1968). Probabilistic models of decreasing failure rate processes. *Naval Research Logistics Quarterly*, Vol 15, pp. 361-74.

[31] Proschan, F.(1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, Vol 5, pp. 375-383.

[32] Adamidis, K., Loukas,S. (1998). A lifetime distribution with decreasing failure rate.*Statistics and Probability Letters*, Vol 39 (1), pp. 35-42.

[33] Tahmasbi, R., Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. *Computational Statistics and Data Analysis*, Vol 52 (8), pp. 3889-901.

[34] Chahkandi, M., Ganjali, M. (2009). On some lifetime distributions with decreasing failure rate. *Computational Statistics and Data Analysis*, Vol 53 (12), pp. 4433-40.

[35] Kus, C. (2007). A new lifetime distribution. *Computational Statistics and Data Analysis*, Vol 51 (9), pp. 4497-509.

[36] Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of American Stattistical Association*, Vol 75, 667-672.

[37] Lai, C. D., Xie, M. and Murthy, D. N. P. (2001). Bathtub shaped failure rate life distributions. *Balakrishnan, N. and Rao, C.R. (Editors), Elsevier Science, Amsterdam. Handbook of Statistics: Advances in Reliability*, Vol 20, pp. 69-104.

[38] Hjorth, U. (1980). A reliability distribution with increasing, decreasing, constant and Bathtub shaped failure rate. *Technometrics*, Vol 22, pp. 99-107.

[39] Xie, M., Tang, Y. and Goh, T.N. (2002). A modified Weibull extension with Bathtub shaped failure rate function. *Reliability Engineering and System Safety*, Vol 76(3), pp. 279-285.

[40] Xie, M., Lai, C.D. and Murthy, D.N.P. (2003). Weibull-related distributions for modelling of Bathtub shaped failure rate functions. *Mathematical and Statistical Methods in Reliability, Lindqvist, B.H. and Doksum, K.A. (Eds), World Scientific*, Singapore, pp. 283-297.

[41] Navarro, J. and Hernandez, P.J. (2004). How to obtain Bathtub shaped failure rate models from normal mixtures. *Probability in the Engineering and Informational Sciences*, Vol 18, pp. 511-531.

[42] Kundu, D. (2004). Exponentiated Exponential distribution. Encyclopedia of Statistical Sciences, 2nd edition, Wiley, New York.

[43] Wondmagegnehu, E.T., Navarro, J., and Hernandez, P.J. (2005). Bathtub shaped failure rates from mixtures: a practical point of view. *IEEE Transactions on Reliability*, Vol. 54 (2), pp. 270-275.

[44] Block, H. W., Li, Y., Savits, T. H. and Wang, J. (2008). Continuous mixtures with Bathtub shaped failure rates. *Journal of Applied Probability*, Vol. 45(1), pp. 260-270.

[45] Sarhan, A. and Kundu, D. (2009). Generalized linear failure rate distribution. *Communications in Statistics - Theory and Methods*, Vol. 38(5), pp. 642-660.

[46] Mudholkar, G. S. and Srivastava, D. K. (1993). ExponentiatedWeibull family for analyzing Bathtub failure rate data. *IEEE Transactions on Reliability*, Vol. 42(2), pp. 299-302.

[47] Xie, M. and Lai, C. D. (1995). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, Vol. 52, pp. 87-93.

[48] Xie, M., Tang, Y. and Goh, T. N. (2002). A modified Weibull extension with bathtub shaped failure rate function. *Reliability Engineering and System Safety*, Vol. 76(3), pp. 279-285.

[49] Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics and Probability Letters*, Vol. 49(2), pp. 155-161.

[50] Wang, F.K. (2000). A new model with Bathtub shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety*, Vol. 70, pp. 305-312.

[51] Wang, R., Sha, N., Gu, B. and Xu, X. (2014). Statistical analysis of a Weibull extension with bathtub shaped failure rate function. *Advances in Statistics*, Vol. 2014, Article ID 304724.

[52] Lemonte, A. J., Cordeiro, G. M. (2013). An extended Lomax distribution *A Journal of Theoretical and Applied Statistics*, vol. 47(4), pp. 800-816.

[53] Lemonte, A.J. (2013). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics and Data Analysis*, 62:149-170.

[54] Zhang, T., Dwight, R. and El-Akruti, K. (2013). On aWeibull related distribution model with decreasing, increasing and upside-down Bathtub shaped

failure rate. *Proceedings Annual Reliability and Maintainability Symposium (RAMS)*, Orlando, pp. 1-6.

[55] Parsa, M., Motashami Borzadaran, G.R. and Rezaei Roknabadi, A.H. (2014). On the change points of mean residual life and failure rate functions for some generalized Gamma type distributions. *International Journal of Metrology and Quality Engineering*, 5(1), DOI: 10.1051/ijmqe/2014002.

[56] Wang, X., Yu, C. and Li, Y. (2015). A new finite interval lifetime distribution model for fitting Bathtub shaped failure rate curve. *Mathematical Problems in Engineering*, pp. 1-6. DOI:10.1155/2015/954327.

[57] Shehla, R. and Ali-Khan, A. (2016). Reliability analysis using an exponential power model with Bathtub shaped failure rate function: A Bayes study. *Springer plus*, 5(1), DOI: 10.1186/s40064-016-2722-3.

[58] Zeng, H., Lan, T. and Chen, Q. (2016). Five and four-parameter lifetime distributions for Bathtub shaped failure rate using Perks mortality equation. *Reliability Engineering and System Safety*, Vol. 152, pp. 307-315.

[59] Cordeiro, G. M., Silva, G. O.,and Ortega, E. M. M. (2010).The Beta extended Weibull family. *Life time data analysis*, 16, 409-430.

[60] Maurya, S.K., Kaushik, A., Singh, R.K., Singh, S.K., and Singh, U. (2017). A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. *Communications in Statistics - Theory and Methods*, Vol. 46(20), pp. 10359-10372.

[61] Nekoukhou, V., and Bidram, H. (2017). A new generalization of the weibull-geometric distribution with bathtub failure rate. *Communications in Statistics - Theory and Methods*, Vol. 46(9), pp. 4296-4310.

[62] Shafiq, M. and Viertl, R. (2017). Bathtub hazard rate distributions and fuzzy life times. *Iranian Journal of Fuzzy Systems*, Vol. 14(5), pp. 31-41.

[63] Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., and Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, Vol. 90(3). https://doi.org/10.1590/0001-3765201820170635.

[64] Dey, S., Nassar, M., Kumar, D., Alzaatreh, A. and Tahir, M.H. (2019). A new lifetime distribution with decreasing and upside-down Bathtub shaped hazard rate function. *Statistica*, Vol. 79(4), pp. 399-426.

[65] Al abbasi, J. N., Khaleel, M. A., Abdal-hammed, M. Kh., Loh, Y.F. and Ozel, G. (2019). A new uniform distribution with bathtub shaped failure rate with simulation and application. *Mathematical Sciences*, Vol. 13(2), pp. 105-114.

[66] Shoaee, S. (2019). Statistical analysis of bivariate failure time data based on Bathtub shaped failure rate model. *Journal of Iranian Statistical Society*, Vol. 18(1), pp. 53-87.

[67] Ahsan, S., Lemma, T. A. and Gebremariam, M. A. (2019). Reliability analysis of gas turbine engine by means of Bathtub shaped failure rate distribution. *Process safety progress*, Vol. 39, DOI: 10.1002/prs.12115.

[68] Chen, S. and Gui, W. (2020). Statistical analysis of a lifetime distribution with a bathtub shaped failure rate function under adaptive progressive type-II censoring. *Mathematics*, Vol. 8(5), pp. 670-691.

[69] Abd-Elrahman, A. M. (2017). A new two-parameter lifetime distribution with decreasing, increasing or upside-down bathtub-shaped failure rate. *Com-*

munications in Statistics - Theory and Methods, Vol. 46(18), pp. 8865-8880.

[70] Bourguignon, M., Silva, R.B., Zea, L.M., and Cordeiro, G.M. (2013). The Kumaraswamy Pareto distribution. *Journal of Statistical Theory and Applications*, Vol. 12(2), pp. 129-144.

[71] Dimitrakopoulou, T., Adamidis, K., and Loukas, S. (2007). A lifetime distribution with an upside-Down bathtub-shaped hazard function. *IEEE Transactions On Reliability*, Vol.56(2), pp. 308-311.

[72] Sharma, V. K., Singh, S. K, and Singh, U. (2014). A new upside-down bathtub shaped hazard rate model for survival data analysis. *Applied Mathematics and Computation*, Vol. 239, pp. 242-253.

[73] Alkarni, S. H. (2015). Extended inverse lindley distribution: properties and application. *SpringerPlus*, Vol. 4.

[74] Maurya, S.K., Singh, S.K., and Singh, U. (2020). A new right-skewed upside down bathtub shaped heavytailed distribution and its applications. *Journal of Modern Applied Statistical Methods*, Vol. 19(1), eP2888.

[75] Kavya, P., and Manoharan, M. (2020). On a generalized lifetime model using DUS transformation. In: *Joshua V., Varadhan S., Vishnevsky V. (eds) Applied Probability and Stochastic Processes*, Infosys Science Foundation Series. Springer Nature, Singapore.

[76] Pearson, K. (1895). Notes on regression and inheritance in the case of two parents. *Proceedings of the Royal Society of London*, Vol. 58, pp. 240-242. https://doi.org/10.1098/rspl.1895.0041.

[77] Burr, I. W. (1942). Cumulative frequency functions. *The Annals of Mathematical Statistics*, Vol. 13, pp. 215-232. http://dx.doi.org/10.1214/aoms/1177731607.

[78] Johnson, N. L. (1949). Systems of frequency curves generated by methods of translation. *Biometrika*, Vol. 36(1-2), pp. 149-176. https://doi.org/10.1093/biomet/36.1-2.149.

[79] Hastings, J. C., Mosteller,F., Tukey, J. W., and Windsor, C. (1947). Low Moments for Small Samples: A Comparative Study of Order Statistics. *The Annals of Mathematical Statistics*, Vol. 18, pp. 413-426. doi:10.1214/aoms/1177730388.

[80] Tukey, J. W. (1960). A survey of sampling from contaminated distributions. In: *Oklin, I., Ed., Contributions to Probability and Statistics*, Stanford University Press, Redwood City, CA.

[81] Gupta, R.C., Gupta,R.D., and Gupta, P.L. (1998). Modeling failure time data by lehman alternatives. *Communication in Statistics- Theory and Methods*, Vol. 27, pp. 887-904.

[82] Nadarajah, S., and Kotz, S. (2006). The exponentiated type distributions. *Acta Applicandae Mathematicae*, Vol. 92(2), pp. 97-111

[83] Nadarajah, S. (2006). The exponentiated gumbel distribution with climate application. *Environmetrics*, Vol. 17(1), pp. 13-23.

[84] Lee, C., Famoye, F., and Alzaatreh, A. (2013). Methods for generating families of continuous distribution in the recent decades. *Wiley Interdisciplinary Reviews: Computational Statistics* , Vol. 5(3), pp. 219-238.

[85] Azzalini, A. (1985). A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, Vol. 12(2), pp. 171-178.

[86] Azzalini, A. (1986). Further results on the class of distributions which includes the normal ones. *Statistica*, Vol. 46, pp. 199-208.

[87] Kus, C. (2007). A new lifetime distribution. *Computational Statistics and Data Analysis*, Vol. 51(9), pp. 4497-4509.

[88] Mahmoudi, E., and Jafari, A.A. (2012). Generalized exponential-power series distributions. *Computational Statistics and Data Analysis*, Vol. 56(12), pp. 4047-4066.

[89] Silva, R.B., Bourguignon, M., Dias, C.R.B., and Cordeiro, G.M. (2013). The compound class of extended weibull power series distributions. *Computational Statistics and Data Analysis*, Vol. 58, pp. 352-67.

[90] Marshall, A. W., and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and weibull families. *Biometrika*, Vol. 84(3), pp. 641-652.

[91] Cordeiro, G.M., Ortega, E.M.M., and da Cunha, D.C.C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, Vol. 11, pp. 1-27.

[92] Shaw, W.T., and Buckley, I.R. (2007). The alchemy of probability distribution: Beyond Gram-Charlier Cornish–Fisher expansions, and skew-normaland kurtotic-normal distribution. Research Report, 1-8.

[93] Aryal, G.R., and Tsokos, C.P. (2009). On the transmuted extreme value distribution with application. *Nonlinear Analalysis: Theory Method and Applications*, Vol. 71, pp. 1401-1407.

[94] Aryal, G.R. (2013). Transmuted log-logistic distribution. *Journal of Statistics Applications and Probability*, Vol. 2(1), pp. 11-20.

[95] Khan, M.S., King, R., and Hudson, I.L. (2014). Characterisations of the transmuted inverse Weibull distribution. *ANZIAM Journal 55 (EMAC-2013)*, pp. C197-C217.

[96] Eugene, N., Lee, C. and Famoye, F. (2002). Beta-Normal Distribution and Its Applications. *Communications in Statistics—Theory and Methods*, Vol. 31, pp. 497-512. https://doi.org/10.1081/STA-120003130.

[97] Alzaatreh, A., Lee, C. and Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, Vol. 71(1). DOI: 10.1007/s40300-013-0007-y.

[98] Cooray, K. and Ananda, M.M.A. (2005). Modeling actuarial data with composite lognormal-pareto model. *Scandinavian Actuarial Journal*, pp. 321-334. https://doi.org/10.1080/03461230510009763.

[99] Ferreira, J. T. A. S., and Steel, M. (2006). A constructive representation of univariate skewed distributions. *Journal of the American Statistical Association*, vol. 101, pp. 823-829.

[100] Kumar, D., Singh, U., and Singh, S.K. (2015a). A method of proposing new distribution and its application to bladder cancer patient data. *Journal of Statistics and Probability letters*, Vol. 2, pp. 235-245.

[101] Kumar, D., Singh, U., and Singh, S.K. (2015b). A new distribution using Sine function- Its application to bladder cancer patients data. *Journal of Statistics and Probability letters*, Vol. 4(3), pp. 417-427.

[102] Kotz, S., and Vicari, D., (2005). Survey of developments in the theory of continuous skewed distributions. *Metron - International Journal of Statistics, Dipartimento di Statistica, Probabilità e Statistiche Applicate - University of Rome*, vol. 0(2), pp. 225-261.

[103] Alexander, C.,Cordeiro, G. M., E. M. M. Ortega, E. M. M., and Sarabia, J. M. (2012). Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, Vol. 56, pp. 1880-97. doi:10.1016/j.csda.2011.11.015.

[104] Hansen, B. E. (1994). Autoregressive conditional density estimation. *International Economic Review*, Vol. 35, pp. 705-730. doi:10.2307/2527081.

[105] Azzalini, A., and Capitanio, A. (2003). Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. *Journal of the Royal Statistical Society: Series B*, Vol. 65, pp. 367-389. doi:10.1111/ 1467-9868.00391.

[106] Barreto-Souza, W., De Morais, A. L., and Cordeiro, G. M. (2011). The Weibull-Geometric distribution. *Journal of Statistical Computation and Simulation*, Vol. 81, pp. 645-657. doi:10.1080/00949650903436554.

[107] Khan, M. S. H.,and Jain, G. C. (1978). A class of distributions in the first emptiness of a semi-infinite reservoir. *Biometrical Journal*, Vol. 20(3), pp. 243-252. doi.org/10.1002/bimj.4710200305.

[108] Gupta, R. D, and Kundu, D. (1999). Generalized exponential distributions. *Australian and New Zealand Journal of Statistics*, Vol. 41(2), pp. 173-188.

[109] Gupta, R. D, and Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull distributions. *BiometricalJournal*, Vol. 43(1), pp. 117-130.

[110] Adamidis, K., Dimitrakopoulou, T., and Loukas, S. (2005). On a generalization of the exponential-geometric distribution. *Statistics and Probability Letters*, Vol. 73(3), pp. 259-269. doi: 10.1016/j.spl.2005.03.013.

[111] Oguntunde, P. E., Balogun, O. S., Okagbue, H. I., and Bishop, S. A. (2015). The Weibull-exponential distribution: its properties and applications. *Journal of Applied Sciences*, Vol. 15, pp. 1305-1311.

[112] Ahmad, Z., Elgarhy, M., and Hamedani, G. G. (2018). A new Weibull-X family of distributions: properties, characterizations and applications. *Journal of Statistical Distributions and Applications*, Vol. 5(5).

[113] Kharazmi, O.(2016). A new lifetime distribution and its application in reliability. *2016 12th International Conference on Industrial Engineering (ICIE)*, pp. 46-57. doi: 10.1109/INDUSENG.2016.7519348.

[114] Elbatala, I., Zayed, M., Rasekhic, M., Afifyb, A. Z., and Iqbald, Z. (2019). A new extended weibull model for lifetime data. *Journal of Applied Probability and Statistics*, Vol. 14(1), pp. 57-73.

[115] Ishaq, A. I., and Abiodun, A. (2020). The Maxwell–Weibull distribution in modeling lifetime datasets. *Annals of Data Science*, Vol. 7(4), pp. 639-662.

[116] Lomax, K. S. (1954). Business failures: another example of the analysis of failure data. *Journal of the American Statistical Association*, vol. 49(268), pp. 847-852.

[117] Hassan, A., Al-Ghamdi, A. (2009). Optimum step stress accelerated life testing for Lomax distribution. *Journal of Applied Sciences Research*, vol. 5, pp. 2153-2164.

[118] Kilany, N.M. (2016). Weighted Lomax distribution. *SpringerPlus*, vol. 5(1), article no. 1862.

[119] El-Bassiouny, A. H., Abdo, N. F., Shahen, H. S. (2015). Exponential lomax distribution. *International Journal of Computer Applications*, vol. 121(13), pp. 24-29.

[120] Salem, H. M. (2014). The exponentiated lomax distribution: different estimation methods. *American Journal of Applied Mathematics and Statistics*, vol. 2(6), pp. 364-368.

[121] Cordeiro, G.M., Ortega, E.M.M., Popovi'c, B. V. (2015). The gamma-Lomax distribution. *Journal of Statistical Computation and Simulation*, vol. 85(2), pp. 305-319.

[122] Ashour, S. K., Eltehiwy, M. A. (2013). Transmuted Lomax distribution. *American Journal of Applied Mathematics and Statistics*, vol. 1(6), pp. 121-127.

[123] Al-Zahrani, B., Sagor, H. (2014). The Poisson-Lomax distribution. *Revista Colombiana de Estadystica*, vol. 37(1), pp. 225-245.

[124] Tahir, M. H., Cordeiro, G. M., Mansoor, M., Zubair, M. (2015). The weibull-lomax distribution: properties and applications. *Hacettepe Journal of Mathematics and Statistics*, vol. 44(2), pp. 461-480.

[125] Rady, E.-H. A., Hassanein, W. A., Elhaddad, T. A. (2016). The power Lomax distribution with an application to bladder cancer data. *SpringerPlus*, vol. 5, 1838 pages.

[126] Shams, T. M. (2013). The kumaraswamy-generalized lomax distribution. *Middle-East Journal of Scientific Research*, 17 (5), 641-646.

[127] Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., Adejumo, A. O., Odetunmibi, O. A. (2017). A new generalization of the lomax distribution with increasing, decreasing, and constant failure rate. *Modelling and Simulation in Engineering*, Volume 2017, Article ID 6043169.

[128] Deepthi, K. S., Chacko, V. M. (2020). An upside-down bathtub-shaped failure rate model using a dus transformation of lomax distribution. Stochastic Models in Reliability Engineering, 1st Edition, CRC Press.

[129] Al-Zahrani, B., Al-Sobhi, M. (2013). On parameters estimation of Lomax distribution under general progressive censoring. *Journal of Quality and Reliability Engineering*, vol. 2013, Article ID 431541.

[130] Wolfram Research, Inc. (2016). Mathematica, Version 11.0. Champaign, Illinois.

[131] Shannon, C. E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, Vol. 27(3), pp. 379-423. doi:10.1002/j.1538-7305.1948.tb01338.

[132] Shannon, Claude E. (October 1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, Vol. 27(4), pp.623-656. doi:10.1002/j.1538-7305.1948.tb00917.

[133] *Reńyi*, A. (1961). On measures of entropy and information. In: *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability* (vol. 1, pp. 547-561). Berkeley: University of California Press.

[134] Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, Vol. 52, pp. 479-487.

[135] Havrda, J., Charvat, F. (1967). Quantification method of classification processes, concept of structural a-entropy. *Kybernetika*, Vol. 3, pp. 30-35.

[136] Lad, F., Sanfilippo, G., Agro, G. (2015). Extropy: Complementary dual of entropy. *Statistical Science*, Vol. 30, pp. 40-58.

[137] Barlow, R. E., Proschan, F. (1981). Statistical theory of Reliability and Life testing. To begin with, Silver Springs, Maryland.

[138] Shaked, M.,Shanthikumar, J. G. (1994). Stochastic orders and their applications. Academic press, New York.

[139] R Core Team (2019). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria.

[140] Choulakian, V., Stephens, M.A. (2001). Goodness-of-fit for the generalized Pareto distribution. *Technometrics*, Vol. 43, pp. 478-484.

[141] Akinsete, A., Famoye, F., Lee, C. (2008). The beta-Pareto distribution. *Statatistical Journal of Theory and Applications*, Vol. 42(6), pp. 547-563.

[142] Pena-Ramirez, F.A., Guerra, R.R., Cordeiro, G.M. (2019). The Nadarajah-Haghighi Lindley distribution. *Anais da Academia Brasileira de Ciências*, Vol. 91(1), e20170856.

[143] Lee, E.T., Wang, J.W. (2003). Statistical Methods for Survival Data Analysis. Wiley, New York

[144] Yadav, A.S., Kumar, D., Kumar, P., Singh, S.K., Singh, U. (2020). A new transmuted lifetime distribution: Statistical properties and application to survival data. *Journal of Scientific Research*, Vol. 64(1), pp. 127-137.

[145] Maguire, B. A., Pearson, E., andWynn, A. (1952). The time intervals between industrial accidents. *Biometrika*, 168-180.

[146] Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 293-297.

[147] Mahdavi, A., Kundu, D. (2016): A new method for generating distributions with an application to exponential distribution. *Communications in Statistics - Theory and Methods*.

[148] Effron, B. (1988). Logistic regression, survival analysis, and the Kaplan-Meier curve. *Journal of American Statistical Association*, Vol. 83, pp. 414-425.

[149] Langlands, A., Pocock, S., Kerr, G., Gore, S. (1997) Long-term survival of patients with breast cancer: a study of the curability of the disease. *British Medical Journal*, Vol. 2, pp. 1247-1251

[150] Kotz, S., Nadarajah, S. (2000). Extreme Value Distributions: Theory and Applications. London, UK: World Scientific.

[151] Swain, J. J., Venkatraman, S., Wilson, J. R. (1988). Least-squares estimation of distribution functions in johnson's translation system. *Journal of Statistical Computation and Simulation*, 29(4), pp. 271-297. DOI : https : //doi.org/10.1080/00949658808811068.

[152] Dumonceaux, R., Antle, C. (1973). Discrimination between the lognormal and the Weibull distributions. *Technometrics*, Vol. 15, pp. 923-926.

[153] Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *American Journal of Epidemiology*, Vol. 72, pp. 130-148.

[154] Kundu, D., Howlader, H. (2010). Bayesian inference and prediction of the inverseWeibull distribution for Type-II censored data. *Computational Statistics and Data Analysis*, Vol. 54, pp. 1547-1558.

[155] Abdul-Moniem, I. (2015): Transmuted Burr Type III distribution. *Journal of Statistics: Advances in Theory and Applications*, Vol. 14(1), pp. 37-47.

[156] Birnbaum, Z.W. (1956). On a use of the Mann-Whitney statistic. Contributions to the theory of statistics. *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, University of California, Vol. 1, pp. 13-17.

[157] Birnbaum, Z. W., McCarty, B.C. (1958). A distribution-free upper confidence bounds for $Pr(Y < X)$ based on independent samples of X and Y. *The Annals of Mathematical Statistics*, Vol. 29(2), pp. 558-562.

[158] Church, J. D., Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics*, vol. 12 (1), pp. 49-54.

[159] Owen, D. B., Craswell, K. J., Hanson, D. L. (1964). Nonparametric upper confidence bounds for $PrY < X$ and confidence limits for $PrY < X$ when X and Y are normal. *Journal of the American Statistical Association*, Vol. 59(307), pp. 906-924. https://doi.org/10.1080/01621459.1964.10480739.

[160] Downton, F. (1973). The estimation of $P(X > Y)$ in the normal case. *Technometrics*, Vol. 15, pp. 551-558.

[161] Woodward, W. A., Kelley, G. D (1977). Minimum variance unbiased estimation of $P[Y < X]$ in the normal case. *Technometrics*, Vol. 19(1), pp. 95-98. DOI: 10.1080/00401706.1977.10489505.

[162] Kelley, G. D, Kelley, J. A, Schucany, W. R. (1976). Efficient estimation of $P(Y < X)$ in the exponential case *Technimetrics*, pp. 359-360

[163] Tong, H. (1974). A note on the estimation of $PrY < X$ in the exponential case. *Technimetrics*, Vol. 16(4).

[164] Beg, M. A., Singh, N. (1979). Estimation of $Pr(Y < X)$ for the pareto distribution. *IEEE Transaction on Reliability*, Vol. 28(5), pp. 411-414. DOI: 10.1109/TR.1979.5220665

[165] Tong, H. (1977). On the estimation of $P(Y < X)$ exponential families. *IEEE Transaction on Reliability*, Vol. 26(1), pp. 54-56.

[166] Enis, P., Geisser, s. (1971). Estimation of the probability that $Y < X$. *Journal of the American Statistical Association*, Vol. 66, pp. 162-168. https://doi.org/10.1080/01621459.1971.10482238.

[167] Ferguson, T. S. (1973). A Bayesian analysis of some nonparametric problems. *Annals of Statistics*, Vol. 1(2), pp. 209-230. https://doi.org/10.1214/aos/1176342360.

[168] Hollander, M., Korwar, R. M. (1976). Nonparametric empirical bayes estimation of the probability that $x \geq y$. *Communications in Statistics-Theory and Methods*, Vol. 5(14), pp. 1369-1383. https://doi.org/10.1080/03610927608827448.

[169] Bhattacharya, G. K., Johnson, R. A. (1974). Estimation of reliability in a multicomponent stress-strength model. *Journal of the American Statistical Association*, Vol. 69, pp. 966-970.

[170] Guttman, I., Johnson, R. A., Bhattacharyya, G. K., Reiser, B. (1988). Confidence limits for stress-strength models with explanatory variables. *Technometrics*, Vol. 30(2), pp. 161-168.

[171] Weerahandi, S., Johnson, R. A. (1992). Testing reliability in a stress-strength model when X and Y are normally distributed. *Technometrics*, Vol. 34(1), pp. 83-91.

[172] Sun, D., Ghosh, M., Basu, A. P. (1998). Bayesian analysis for a stress-strength system under non-informative priors. *The Canadian Journal of Statistics*, Vol. 26(2), pp. 323-332.

[173] Raqab, M.Z., Kundu, D. (2005). Comparison of different estimators of $P[Y < X]$ for a scaled Burr type X distribution. *Communications in Statistics-Simulation and Computation*, Vol. 34, pp. 465-483.

[174] Kotz, S., Lumelskii, Y., Pensky, M. (2003). The Stress-Strength Model and its Generaliza- tions: Theory and Applications. Singapore: World Scientific Press.

[175] Kundu, D., Gupta, R.D. (2005). Estimation of $P[Y < X]$ for generalized exponential distribution. *Metrika*, Vol. 61, pp. 291-308.

[176] Kundu, D., Gupta, R.D. (2006). Estimation of $R = P[Y < X]$ for Weibull distributions. *IEEE Transactions on Reliability*, Vol. 55, pp. 270-280.

[177] Kundu, D., Raqab, M.Z. (2009). Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution. *Statistics and Probability Letters*, Vol. 79, pp. 1839-1846.

[178] Krishnamoorthy, K., Mukherjee, S., Guo, H. (2007). Inference on reliability in two-parameter exponential stress-strength model. *Metrika*, Vol. 65, pp. 261-273.

[179] Raqab, M. Z, Madi, T., Kundu, D. (2008). Estimation of $P(Y < X)$ for the three-parameter generalized exponential distribution. *Communications in Statistics- Theory and Methods*, Vol. 37, pp. 2854-2865.

[180] Rezaei, S., Tahmasbi, R., Mahmoodi, M. (2010). Estimation of $P(Y < X)$ for generalized Pareto distribution. *Journal of Statistical Planning and Inference*, Vol. 140(2), pp. 480-494.

[181] Baklizi, A. (2012). Inference on $P(X < Y)$ in the two-parameter Weibull model based on records. *ISRN Probability and Statistics*, pp. 1-11.

[182] Baklizi, A., Eidous, O. (2006). Nonparametric estimation of $P(X < Y)$ using kernel methods. *Metron-International Journal of Statistics*, Vol. 64(1), pp. 47-60.

[183] Jing, B.Y., Yuan, J., Zhou, W. (2009). Jackknife empirical likelihood. *Journal of American Statistical Association*, Vol. 104, pp. 1224-1232.

[184] Basirat, M., Baratpour, S., Ahmadi, J. (2016). On estimation of stress-strength parameter using record values from proportional hazard rate models. *Communications in Statistics- Theory and Methods*, Vol. 45(19), pp. 5787-5801.

[185] Asgharzadeh, A., Valiollahi, R., Raqab, M. Z. (2017). Estimation of $P(Y < X)$ for the two-parameter of generalized exponential records. *Communications in Statistics- Simulation and Computation*, Vol. 46(1), pp. 379-394.

[186] Bai, X., Shi, Y., Liu, Y., Liu, B. (2018). Reliability inference of stress-strength model for the truncated proportional hazard rate distribution under progressively Type-II censored samples. *Applied Mathematical Modelling*, Vol.65, pp. 377-389.

[187] Bi, Q., Gui, W. (2017). Bayesian and classical estimation of stress- strength reliability for Inverse Weibull lifetime models. *Algorithms*, Vol. 10(2), pp. 1-16.

[188] Ghitany, M. E., Al-Mutairi, D. K., Aboukhamseen, S. M. (2014). Estimation of the reliability of a stress-strength system from power Lindley distributions. *Communications in Statistics- Theory and Methods*, Vol. 44(1), pp. 118-136.

[189] Sharma, V. K. (2014). Bayesian analysis of head and neck cancer data using generalized inverse Lindley stress-strength reliability model. *Communications in Statistics- Theory and Methods*, Vol. 47(5), pp. 1155-1180.

[190] Jose, J. K., Xavier, T., Drisya, M. (2019). Estimation of stress-strength reliability using Kumaraswamy half logistic distribution. *Journal of Probability and Statistical Science*, Vol. 17(2), pp. 141-154.

[191] Xavier, T., Jose, J. K. (2020). A study of stress-strength reliability using a generalization of power transformed half-logistic distribution. *Communications in Statistics- Theory and Methods.* DOI: 10.1080/03610926.2020.1716250.

[192] Domma, F., Eftekharian, A., Razmkhah, M. (2019). Stress-strength based on m-generalized order statistics and concomitant for dependent families. *Applications of Mathematics*, Vol. 64, pp. 437-467.

[193] Krishna, H., Dube, M., Garg, R. (2019). Estimation of stress-strength reliability of Inverse Weibull distribution under progressive first failure censoring. *Austrian Journal of Statistics*, Vol. 48, pp. 14-37.

[194] Musleh, R. M., Helua, A., Samawib, H. (2019). Inference on $P(X < Y)$ in bivariate Lomax model. *Electronic Journal of Applied Statistical Analysis*, Vol. 13(3), pp. 619-636.

[195] Kohansal, A., Nadarajah, S. (2019). Stress-strength parameter estimation based on type-II hybrid progressive censored samples for a Kumaraswamy distribution. *IEEE Transactions on Reliability*, Vol. 68(4), pp. 1296-1310.

[196] Joby, K. J, Drisya, M., Manoharan, M. (2020). Estimation of stress-strength reliability using discrete phase type distribution.

Communications in Statistics- Theory and Methods, pp. 1-19. DOI: 10.1080/03610926.2020.1749663.

[197] Bhattacharyya, G. K., Johnson, R. A. (1974). Estimation of reliability in multicomponent stress-strength model. *Journal of the American Statistical Association*, Vol. 69, pp. 966-970.

[198] Mokhlis, N. A., Khames, S. K. (2011). Reliability of multi-component stress-strength models. *Journal of the Egyptian Mathematical Society*, Vol. 19, pp. 106-111.

[199] Rao, G. S. (2012). Estimation of reliability in multicomponent stress-strength model based on generalized exponential distribution. *Colombian Journal of Statistics*, Vol. 35(1), pp. 67-76.

[200] Rao, G. S., Aslam, M., Kundu, D. (2015). Burr-XII distribution parametric estimation and estimation of reliability of multicomponent stress-strength. *Communications in Statistics- Simulation and Computation*, Vol. 44(23), pp. 4953-4961.

[201] Dey, S., Mazucheli, J., Anis, M. Z. (2016). Estimation of reliability of multicomponent stress-strength for a Kumaraswamy distribution. *Communications in Statistics- Theory and Methods*, Vol. 46(4), pp. 1560-1572.

[202] Khalil, M. (2017). Estimation a stress-strength model for using the Lindley distribution. *Colombian Journal of Statistics*, Vol. 40(1), pp. 105-121.

[203] Kohansal, A. (2017). On estimation of reliability in a multicomponent stress-strength model for a Kumaraswamy distribution based on progressively censored sample. *Statistical Papers*, vol. 60, pp. 2185-2224.

[204] Abouelmagd, T. H. M., Al-mualim, S., Afify, A. Z., Ahmad, M., Al-Mofleh, H. (2018). The odd Lindley Burr XII distribution with applications. *Pakistan Journal of Statistics*, Vol. 34(1), pp. 15-32.

[205] Hassan, M. K. H., Alohali, M. I. (2018). Estimation of reliability in a multicomponent stress-strength model based on generalized linear failure rate distribution. *Hacettepe Journal of Mathematics and Statistics*, Vol. 47(6), pp. 1634-1651.

[206] Fatma, G. A. (2019). Reliability estimation in multicomponent stress-strength model for Topp-Leone distribution. *Journal of Statistical Computation and Simulation*, Vol. 89(15), pp. 2914-2929.

[207] Jamal, Q. A., Arshad, M., Nancy, K. (2019). Multicomponent stress-strength reliability estimation for Pareto distribution based on upper record values. arXiv:1909.13286.

[208] Jha, M. K., Dey, S., Refah, M. A., Yogesh, M. T. (2020). Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive type II censoring. *Quality and Reliability Engineering International*, Vol. 36(3), pp. 1-23.

[209] Hassan, A. S., Nagy, H. F., Muhammed, H. Z., Saad, M. S. (2020). Estimation of multicomponent stress-strength reliability following Weibull distribution based on upper record values. *Journal of Taibah University for Science*, Vol. 14(1), pp. 244-253.