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A Study On The Spectrum Of Zero Divisor Graph On The Ring Of Integers Modulo n

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by

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I hereby certify that the thesis entitled "**A Study On The Spectrum Of Zero Divisor Graph On The Ring Of Integers Modulo n**" submitted by **Magi P M** to St. Mary's College, Thrissur-680020 for the award of the Degree of **Doctor of Philosophy** is a bona-fide record of the research work carried out by her under my supervision and guidance. The contents of this thesis, in full or parts, have not been submitted and will not be submitted to any other Institute or University for the award of any degree or diploma.

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DECLARATION

I hereby declare that the thesis entitled "**A Study On The Spectrum Of Zero Divisor Graph On The Ring Of Integers Modulo n**" is based on the original work done by me under the supervision of **Dr. Sr. Magie Jose**, Research Supervisor, Post Graduate and Research Department of Mathematics, St. Mary's College, Thrissur-680020 and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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This is to certify that, the adjudicators for the Ph. D thesis entitled **"A Study On The Spectrum Of Zero Divisor Graph On The Ring Of Integers Modulo n"** submitted by Magi P M under my guidance, have not suggested any corrections or modifications for the thesis and have unanimously recommended the award of the Ph. D degree in Mathematics for the above thesis. I also certify that the contents in the thesis and the soft copy are one and the same.

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ABSTRACT OF THE Ph.D THESIS

A Study On The Spectrum Of Zero Divisor Graph On The Ring Of Integers Modulo n

Graph Theory is an important branch of Discrete Mathematics, which is a key tool to model network systems involved in major domains of real life. Graph Theory extends its countless applications to various walks of science like Network Theory, Operational Research, Chemistry, Quantum Physics, Biology, Economics, Artificial Intelligence, Sociology and so on. Exploring algebraic structures through graph theory has become a captivating research field over the past three decades. Researchers have extensively studied graphs associated with algebraic structures such as groups and rings, viz Cayley graphs, power graphs, zero-divisor graphs and co-maximal graphs, etc. Such study provides interconnections between Algebra and Graph Theory. The zero divisor graph $\Gamma(R)$ of a commutative ring R is the simple undirected graph with vertices non-zero zero-divisors of R and two distinct vertices x, y are adjacent if $xy = 0$. This thesis focuses on the study of different matrices associated with the zero divisor graph on the ring of integers modulo n and explores its spectra.

Usually, the eigenvalues of a graph can be computed by finding the roots of its characteristic polynomial. But there is no algebraic method to solve a polynomial equation of degree greater than or equal to five. This makes the computation of spectrum of graphs tedious. However, for a graph with large size and complicated combinatorial structure, the determination of spectra is really challenging. Sometimes, it becomes a convenient practice that the spectrum of a fairly large graph can be described in terms of the spectra of smaller graphs using some simple graph operations, like union, join, corona, edge corona etc.

The analysis of the adjacency matrix of the zero divisor graph on $\mathbb{Z}_n$, for $n = p^2q^2, p^2q, p^k, k > 1$, where p, q are distinct primes, leads to some intriguing results about the graph parameters of these graphs as well as their characteristic polynomials.

Analogous to the Laplacian and signless Laplacian matrix of a graph, the definition of distance Laplacian and distance signless Laplacian matrix was introduced and studied by M. Aouchiche and P. Hansen. In this thesis, the study on the distance, distance Laplacian and distance signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ has been initiated. The eigenvalues of the distance and distance Laplacian matrix of $\Gamma(\mathbb{Z}_n)$ for some values of n, are found along with multiplicities, by direct computation using matrix tools. The distance Laplacian eigenvalues of

$\Gamma(\mathbb{Z}_{p^k})$, where p is any prime and $k > 1$ is any positive integer, are completely explored with multiplicities. Also, a general method is proposed for finding the characteristic polynomial of the distance and distance Laplacian matrix of $\Gamma(\mathbb{Z}_n)$ for any n.

H.S. Ramane et al defined Seidel Laplacian and Seidel signless Laplacian matrix of graphs. In this thesis, Seidel, Seidel Laplacian and Seidel signless Laplacian spectrum of the generalised union of regular graphs is investigated and extended these results to the zero divisor graph on the ring of integers modulo n.

പഠനസംഗ്രഹം

ഗണിതശാസ്ത്രത്തിലെ സജീവവും രസകരവുമായ ഒരു ഗവേഷണമേഖലയാണ് ഗ്രാഫ് തിയറി. ഗ്രാഫ് എന്നത് ഒരു നെറ്റ്‌വർക്കിന്റെ ഗണിതശാസ്ത്ര പ്രതിനിധാനമാണ്. ഇന്റർനെറ്റ് പോലെയുള്ള സാങ്കേതിക ഉപാധികളുടെ ശൃംഖലകളെയും സാമുഹിക ശൃംഖലകളെയും മറ്റും പ്രതിനിധീകരിക്കുവാൻ ഗ്രാഫുകൾ വളരെ വ്യാപകമായി ഉപയോഗിച്ചുവരുന്നു. സങ്കീർണ്ണമായ നെറ്റ്‌വർക്കുകളുടെ രൂപകൽപനക്കും വിശകലനത്തിനും വളരെ സഹായകരമായ സാധ്യതകൾ ഗ്രാഫ് തിയറിയിലുണ്ട്. നെറ്റ്‌വർക്ക് തിയറി, ഓപ്പറേഷണൽ റിസർച്ച്, കെമിസ്ട്രി, ക്വാണ്ടം ഫിസിക്സ്, ബയോളജി, ഇക്കണോമിക്സ്, ആർട്ടിഫിഷ്യൽ ഇന്റലിജൻസ്, സോഷ്യോളജി തുടങ്ങി വിവിധ മേഖലകളിൽ വളരെ ഫലപ്രദമായി ഗ്രാഫ് തിയറി ഉപയോഗിച്ചു വരുന്നു.

ഗ്രൂപ്പ്, റിങ്ങ് തുടങ്ങിയ ബീജഗണിതഘടനകളെ, ഗ്രാഫ് തിയറിയുടെ സഹായത്താൽ വിശകലനം ചെയ്യുന്ന രീതി, കഴിഞ്ഞ മൂന്ന് പതിറ്റാണ്ടുകളായി ഉയർന്നുവരുന്ന ഗവേഷണസാധ്യയുള്ള ഒന്നാണ്. അമൂർത്ത ബീജഗണിതത്തിൽ ഇടത് അല്ലെങ്കിൽ വലത് പൂജ്യം വിഭജനമായ ഒരു മൂലകത്തെ "പൂജ്യം വിഭജനം" അഥവാ സീറോ ഡിവൈസർ എന്ന് വിളിക്കുന്നു. ഒരു കമ്യൂട്ടേറ്റീവ് റിങ്ങ് R ലെ പൂജ്യമല്ലാത്തതും പൂജ്യം വിഭജനമായതുമായ മൂലകങ്ങൾ ശീർഷകങ്ങളായി Z_n വരികയും ഇത്തരം രണ്ട് മൂലകങ്ങളുടെ ഗുണനഫലം പൂജ്യമായി വന്നാൽ അവ തമ്മിൽ ബന്ധിപ്പിച്ചിരിക്കുന്നതുമായ ഗ്രാഫിനെ R ന്റെ സീറോ ഡിവൈസർ ഗ്രാഫ് എന്ന് വിളിക്കുന്നു . n എന്നത് ഒരു നിസർഗസംഖ്യയായിരിക്കെ Z_n എന്ന റിങ്ങിന്റെ സീറോ ഡിവൈസർ ഗ്രാഫുമായി ബന്ധപ്പെട്ട ചിലതരം മാട്രിക്സുകളുടെയും അവയുടെ സ്പെക്ട്രത്തെയും കുറിച്ചുള്ള വിശകലനമാണ് ഈ ഗവേഷണ പ്രബന്ധത്തിൽ ഉൾപ്പെടുത്തിയിരിക്കുന്നത് .

"A study on the Spectrum of Zero Divisor Graph on the Ring of Integers Modulo n " എന്ന ഈ പ്രബന്ധത്തിൽ ഏഴ് അദ്ധ്യായങ്ങളാണ് ഉൾപ്പെടുത്തിയിട്ടുള്ളത് . ഈ പ്രബന്ധത്തിന്റെ അവതാരികയും ഇതിലേക്ക് ആവശ്യമുള്ള അടിസ്ഥാന നിർവചനങ്ങളും ആശയങ്ങളുമാണ് Z_n ആദ്യത്തെ രണ്ട് അദ്ധ്യായങ്ങളിൽ ഉള്ളത് Γ . (Z_n) എന്ന സീറോ ഡിവൈസർ ഗ്രാഫിന്റെ അഡ്ജസൻസി മാട്രിക്സ് , ഡിസ്റ്റൻസ് മാട്രിക്സ് , ഡിസ്റ്റൻസ് ലാപ്ലേഷ്യൻ മാട്രിക്സ്, ഡിസ്റ്റൻസ് സൈൻലെസ് ലാപ്ലേഷ്യൻ മാട്രിക്സ്, സീഡൽ മാട്രിക്സ്, സീഡൽ ലാപ്ലേഷ്യൻ മാട്രിക്സ്, സീഡൽ സൈൻലെസ് ലാപ്ലേഷ്യൻ മാട്രിക്സ് എന്നിവയെക്കുറിച്ചും അവയുടെ സ്പെക്ട്രത്തെക്കുറിച്ചും ഈ ഗ്രാഫുകളുടെ പലതരം സ്ഥിരരാശികളെക്കുറിച്ചുമാണ് മൂന്ന് മുതൽ ഏഴ് വരെയുള്ള

അദ്ധ്യായങ്ങളിൽ പ്രതിപാദിച്ചിരിക്കുന്നത് . ഈ പഠനത്തിൽ നിന്നുള്ള നിഗമനങ്ങളും ഭാവിയിലേക്കുള്ള സാധ്യതകളുമാണ് അവസാനത്തെ അദ്ധ്യായത്തിൽ ഉൾപ്പെടുത്തിയിരിക്കുന്നത് .

ഈ പ്രബന്ധത്തിൽ പ്രതിപാദിച്ചിരിക്കുന്ന വിവിധതരം മാട്രിക്സുകൾ, അവയുടെ സ്പെക്ട്രം, അവയുടെ അവലോകനങ്ങൾ എന്നിവ പലതരം സാങ്കേതിക പ്രശ്നങ്ങളുടേയും ശാസ്ത്രപ്രശ്നങ്ങളുടേയും പരിഹാരസാധ്യതകൾക്ക് ഉതകുമെന്ന് പ്രതീക്ഷിക്കുന്നു.

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Contents

Chapter 1

Introduction

Graph Theory is an important branch of Discrete Mathematics, which is a key tool to model network systems involved in major domains of real life. Graph Theory extends its countless applications to various walks of science like Network Theory, Operational Research, Chemistry, Quantum Physics, Biology, Economics, Artificial Intelligence, Sociology and so on. It marks its origin in 1735 with Leonard Euler's solution [51] to the famous Konigsberg bridge problem and now it has been evolved as a tool to analyse the structure of networks arising from real world system and to study the impact of the structure on the dynamic processes taking place in it.

Spectral Graph Theory, is an emerging and flourishing area in Graph Theory, which studies the relation between graph properties and the spectrum of graph theoretic matrices, like adjacency matrix and Laplacian matrix. The largest (Perron Frobenius) eigenvalue of the network adjacency matrix has emerged as a significant quantity for the analysis of various dynamical processes. The second largest eigenvalue of a graph gives information regarding expansion and randomness properties. Moreover, many other spectral properties reflect the structural properties of a graph.

It is a recent trend that graphs are crafted out of algebraic structures like groups and rings. Cayley graphs, Annihilator graphs, Co-maximal graphs, Annihilating ideal graphs, Essential ideal graphs, Total graphs and Zero divisor graphs, are good examples for graphs framed out from the algebraic properties of a ring. In-depth research have been carried out in classifying the rings on the structural properties of these algebraic graphs derived from these.

In this thesis, the focal objective is to investigate various spectra including adjacency, distance, distance Laplacian, distance signless Laplacian, Seidel, Seidel Laplacian and Seidel signless Laplacian of the zero divisor graph on the ring on integers modulo n.

The study of zero divisor graph on commutative rings has attracted the attention of many researchers since 1988. In [17], Ivan Beck introduced the concept of zero divisor graph associated to a commutative ring R, in connection with some colouring problems and was further studied by D.D. Anderson and M. Naseer in [8]. In [7], D.F. Anderson and Livingston redefined this concept and the authors studied the interplay between the ring-theoretic properties of a commutative ring and the graph theoretic properties of its zero divisor graph, $\Gamma(R)$. This really embarked on a new phase and attracted many ring theorists to graph theory with a purpose of exploring the algebraic structure among the zero divisors of a ring through the graph theoretic structure of its zero divisor graph. In [5], these authors studied how the algebraic properties of a ring reflect on the size and shape of its zero divisor graph and a complete characterization is established for planar and toroidal zero divisor graphs.

The analysis of graph parameters associated with zero divisor graphs of commutative rings can be found in [14, 54]. In [46], A. Haouaoui et al focus on the properties of zero divisor graph of power series rings and in [13], M. Axtell et al find the preservation of diameter and girth of the zero divisor graph under extension to polynomial and power series rings. In [68], the authors study the zero divisor graph for the ring of Gaussian integers modulo n. A wide and exclusive survey on the study of distances in zero divisor graphs and total graphs on commutative rings can be found in [81]. In [82], R.G. Tirop et. al analyse the adjacency matrices of the zero-divisor graphs of Galois rings. We refer to [69, 71, 1, 64] for a survey of results regarding the adjacency matrix of zero-divisor graphs on various finite commutative rings. The main source of motivation for this study, is the analysis of the adjacency matrix of zero divisor graph of the ring of integers modulo n, initiated by M Young [85]. Pranjali et al [70] describe results regarding the adjacency matrix of the zero-divisor graph over finite ring of Gaussian integers. For more literature for the study of zero divisor graphs on commutative rings, refer [78, 4, 3]. Later, zero divisor graphs were also defined and investigated for non-commutative rings, near rings, modules, semi groups, lattices, semi rings and posets. S. Chattopadhyay and P. Panigrahi [24] have initiated the study of Laplacian spectrum of power graphs of finite cyclic and dihedral groups and later Z. Mehranian et al [60] have continued the study of spectra of power graphs of finite groups. Motivated by these works, we have initiated the investigation of spectra of the zero divisor graph on the ring of integers modulo n.

Usually, the eigenvalues of a graph can be computed by finding the roots of its characteristic polynomial. But there is no algebraic method to solve a polynomial equation of degree greater than or equal to five. This makes the computation of spectrum of graphs tedious. However, some typical graphs like complete graphs, complete bi-partite graphs, cycles, paths etc. have some kind of symmetry that allows the eigenvalues to be evaluated in smarter and less computational ways. Kindly refer [31, 66, 38, 9, 76] for the extensive study on the spectra of graphs. The eigenvalues of a simple connected graph are either rational integers or algebraic irrationals. Hence, in order to find the eigenvalues of matrices associated to a graph, it is not much appropriate to resort to computer algorithms by which the irrational eigenvalues are roughly approximated to rationals.

However, for a graph with large size and complicated combinatorial structure, the determination of spectra is really challenging. Sometimes, it becomes a convenient practice that the spectrum of a fairly large graph can be described in terms of the spectra of smaller graphs using some simple graph operations, like union, join, corona, edge corona etc. See [59, 27, 53] and the references therein for the study on different types of spectra (Laplacian, signless Laplacian) of graphs obtained by means of operations like disjoint union, corona, edge corona etc. In this thesis, the tools of matrix have been used to explore different kinds of spectrum of the zero divisor graph on the ring of integers modulo n, in terms of its induced subgraphs. Besides the adjacency spectrum, the distance, distance Laplacian, distance signless Laplacian, Seidel, Seidel Laplacian and Seidel signless Laplacian spectrum of this graph are also investigated in this thesis.

The notion of distance and transmission of vertices finds applications in different domains including Design of Communication Networks and Graph Embedding. The distance spectral radius has gained a keen focus by researchers in Spectral Graph Theory. Though the distance matrix marks its origin in 1841 in the very first paper of Cayley [23], extensive study was initiated in 1971, as Graham and Pollack [40] established a relationship between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. They also established that the determinant of the distance matrix of a tree is a function of the number of vertices only. Thereafter, many researchers were interested in studying the spectral properties of the distance matrix of a connected graph.

Some of the important domains of application of the distance matrix are the Design of Communication Networks [34, 40], Network Flow Algorithms [30, 36], Graph Embedding Theory [29, 33, 39] as well as Molecular Stability [47, 88]. Balaban, Ciubotariu and Medeleanu [15] proposed the use of the distance spectral radius as a molecular descriptor. Gutman and Medeleanu [41] applied the distance spectral radius to infer the extent of branching and model boiling points of an alkane.

Analogous to the Laplacian and signless Laplacian matrix of a graph, the definition of distance Laplacian and distance signless Laplacian matrix was introduced and studied by M. Aouchiche and P. Hansen [11]. For more literature, refer [12, 28] and the references therein. In this thesis, the study on the distance, distance Laplacian and distance signless Laplacian spectrum of the zero divisor graph on the ring $\mathbb{Z}_n$ has been initiated.

Seidel matrices originally appeared in [84] in connection with equiangular lines in Euclidean spaces and was further studied in [83, 50]. The study on the

Seidel eigenvalues of graphs can be seen in [87, 52, 77]. The Seidel energy of graphs was defined in [43] analogous to normal graph energy. H.S. Ramane et al [72] defined Seidel Laplacian and Seidel signless Laplacian matrix of graphs. The computation of the seidel Laplacian and Seidel signless Laplacian spectrum of the disjoint union of regular graphs can be seen in [73]. In this thesis, these spectrum of the generalised union of regular graphs and in particular, the zero divisor graph on the ring of integers modulo n, has been studied.

In addition to the introductory chapter, the thesis is divided into seven other chapters and chapters into sections.

Chapter 1 is the introduction.

Chapter 2 provides a brief description of the basic definitions of Graph Theory and elementary tools of Matrix Theory and Linear Algebra to give a better understanding of the upcoming chapters.

In **Chapter 3**, the straight forward computation of the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ for some values of n is given. **Section 1** contains some basic definitions and preliminaries. In **Section 2**, the analysis of the adjacency matrix of the zero divisor graph $\mathit{\Gamma}(\mathbb{Z}_n)$ for $n = p^2q^2$, where $p < q$ are distinct primes is done. Also, the girth, diameter, stability number and clique number of this class of graphs are traced from the adjacency matrix. In **Section 3**, the characteristic polynomial of this graph and the multiplicity of the two eigenvalues, 0 and 1, are found out by direct computation, using matrix tools. In **Section 4**, the adjacency matrix and eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$ for distinct primes $p < q$ are investigated.

In **Chapter 4**, the attention is restricted to the prime-power values of n.

In **Section 2** of this chapter, the characteristic polynomial and the spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^3, p^4$ where p is any prime are found. In **Section 3**, the combinatorial structure of $\Gamma(\mathbb{Z}_n)$ as a generalised join of its induced subgraphs, is described for any n. **Section 4** focuses on the structure of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ for any $k \geqslant 3$. The fascinating structure of the adjacency matrix of this class of graphs; for both even and odd values of k, leads to the determination of its graph parameters like, stability number, clique number and girth. This analysis also helps to evaluate the multiplicity of the eigenvalues 0 and 1. In **Section 5**, a general method to compute the eigenvalues of $\Gamma(\mathbb{Z}_n)$ for any n, is proposed, which involves the quotient matrix of equitable partition of its vertex set and is illustrated with some examples. Also, as a special case, the characteristic polynomial of $\Gamma(\mathbb{Z}_{p^k})$ for any $k \geqslant 3$ is found out and the multiplicities of its two eigenvalues 0 and 1 are also computed.

Chapter 5 contains the direct computation of distance, distance Laplacian and distance signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for some values of n. **Section 1** of this chapter contains the definition of distance Laplacian and distance signless Laplacian matrix of a connected matrix which is analogous to the Laplacian and signless Laplacian matrix. In **Section 2** , some tools of matrix theory are applied to find the distance spectra of $\Gamma(\mathbb{Z}_n)$ for $n = pq, p^3$, where $p < q$ are distinct primes. Also, the distance spectral radius of the above mentioned graphs are determined. In **Section 3** , the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^3})$ are computed. In **Section 4**, the distance signless Laplacian spectra of these graphs are found, using the same techniques. It is worth mentioning that the matrix tools applied in this chapter seems inconvenient if n is the product of powers of more primes.

Chapter 6 is an attempt to generalise the results of **Chapter 5** to any value of n. Some tools of Linear Algebra are applied for the same. The Fiedler's Lemma and its generalization is highlighted in **Section 2**. **Section 3** details the role of Fiedler's Lemma in the computation of the distance spectrum of the generalized join of regular graphs. Also, the distance between any two vertices in the proper divisor graph of n as well as the partitioned structure of the distance matrix of $\Gamma(\mathbb{Z}_n)$ are found. The investigation of the distance spectrum of $\Gamma(\mathbb{Z}_n)$ for any n and in particular for $n = p^k$ for any prime p and $k \geqslant 3$, is described in this section and illustrated with examples. Also, it is established that -1 and -2 are the distance eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$, for n and their multiplicities are counted. In the fourth section, the distance Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$, for any n, are computed and described with examples. Also, the distance Laplacian spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^k}), k \geqslant 3$ is completely determined.

Chapter 7 is about the computation of the Seidel related spectrum of the zero divisor graph $\mathit{\Gamma}(\mathbb{Z}_n)$. **Section 1** outlines the definition of Seidel, Seidel Laplacian and Seidel signless Laplacian matrix of a graph. The **Section 2** is begun with the computation of the Seidel spectrum of the join of two regular graphs. The regularity allows the use of Coronal to make the method of finding the Seidel characteristic polynomial of this graph, less computational. The result is extended to the joined union of regular graphs, using Fiedler's Lemma, as described in the previous chapter and is illustrated with example. These results are applied to compute the Seidel spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$ for any n. Also, the Seidel spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$ is evaluated for $n = pq, p^3, p^2q$, where $p < q$ are distinct primes. The computation of the Seidel spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ is described for $k \geqslant 3$. In **Section 3**, the Seidel Laplacian spectrum of the join of two regular graphs is investigated and then extended to the generalised join of regular graphs. This section focuses on the computation of Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for any n. It is also established that, 0 is a simple Seidel Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ for any n. The Seidel Laplacian spectrum for particular values of n, say $n = pq, p^3, p^4$ are found. **Section 4** focuses on the computation of the Seidel signless Laplacian spectrum of the generalised join of regular graphs and explores these spectrum for $\Gamma(\mathbb{Z}_n)$, for any n.

Chapter 8 provides conclusion and further scope of research.

Chapter 2

Preliminaries

This chapter provides a brief account of the preliminary definitions from Graph Theory which are very useful for the upcoming chapters.

2.1 Basic Definitions

Definition 2.1.1. [19] A graph G is an ordered triple $G = (V(G), E(G), \psi(G))$ consisting of a non-empty set $V(G)$ of vertices, a set $E(G)$ of edges and an incidence function $\psi(G)$ which associates with each element of $E(G)$, an unordered pair of vertices (not necessarily distinct) of G.

Definition 2.1.2. [19] Two vertices are said to be adjacent, if there is an edge between them. Otherwise, they are non adjacent. If a vertex u is adjacent to a vertex v in a graph G, we denote it as $u \sim v$. If two edges are incident with a common vertex, then they are adjacent. Otherwise, non adjacent.

Definition 2.1.3. [19] An edge with identical end points is called a loop. Edges joining the same pair of vertices are called multiple edges.

Definition 2.1.4. [19] A graph G is finite if both the vertex set $V(G)$ and the edge set $E(G)$ are finite. Otherwise, the graph G is said to be infinite.

Definition 2.1.5. [19] The number of vertices in a graph G is known as the order of the graph and it is denoted by $O(G)$.

Definition 2.1.6. [19] A graph which has no loops and multiple edges is called a simple graph. A graph is trivial if it has only one point.

Definition 2.1.7. [19] Two graphs G and H are said to be isomorphic if there exists bijections $\theta : V(G) \to V(H)$ and $\phi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\phi(e)) = \theta(u)\theta(v)$. If G and H are isomorphic, we write $G \approxeq H$.

Definition 2.1.8. [19] A graph G is complete if every pair of distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition 2.1.9. [19] An empty graph (null graph) is a graph which has no edges.

Definition 2.1.10. [19] A graph G is called bipartite if the vertex set $V(G)$ can be partitioned into two subsets X and Y such that each edge of G has one end in X and the other end in Y. (X, Y) is called a partition of G.

Definition 2.1.11. [19] A graph G is said to be complete bipartite if G is simple, bipartite with bipartition (X, Y) and each vertex of X is joined to every vertex of Y. If $\mid X \mid= m, \mid Y \mid= n$, then G is denoted by $K_{m,n}$.

Definition 2.1.12. [19] The complement of a simple graph G, denoted by $\overline{G}$, is a simple graph with vertex set $V(G)$ and such that two vertices are adjacent in $\overline{G}$ if and only if they are non adjacent in G.

Definition 2.1.13. [19] A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ψ_H is the restriction of ψ_G to $E(H)$.
If H is a subgraph of G, then it is denoted by $H \subseteq G$. G is the super graph of H. If H is a subgraph of G and $H \neq G$, then H is a proper subgraph of G $(H \subset G)$. H is a spanning subgraph of G if H is a subgraph and $V(H) = V(G)$.
Let G be a graph. By deleting all loops and for every pair of adjacent vertices all except one link joining them, we obtain a simple spanning subgraph of G called the underlying simple graph of G.

Definition 2.1.14. [19] Let V^1 be a non-empty subset of the vertex set V of G. The subgraph of G whose vertex set is V^1 and whose edge set is the set of those edges of G that have both ends in V^1 is called the subgraph of G induced by V^1 and is denoted by $G[V^1]$. We say $G[V^1]$ is the induced subgraph of G.

Definition 2.1.15. [19] If $u \in V(G)$, the open neighborhood of u; denoted by $N_G(u)$ is the set of vertices adjacent to u in G.

Definition 2.1.16. [19] Let G be a graph and v be it's vertex. Then, the degree of v is the number of edges of G incident with v, counting each loop as two edges. The degree of v is denoted as $deg(v)$. For a simple graph, $deg(v)$ is the cardinality of $N_G(v)$. The vertex with zero degree is called an isolated vertex. A vertex with degree one is called an end vertex or pendant vertex.
We denote by $\delta(G)$ and $\Delta(G)$, the minimum degree and the maximum degree of vertices in G respectively.

Definition 2.1.17. [19] A graph G is k-regular if $deg(v) = k$ for all $v \in V$. A regular graph is a graph which is k-regular for some $k \geqslant 0$.

Definition 2.1.18. [19] A walk in a graph is a finite, non-null or non-empty sequence $w = v_0e_1v_1e_2v_2, \ldots, e_kv_k$ whose terms are alternatively vertices and edges so that for $1 \leqslant i \leqslant k$, the ends of e_i are v_{i-1} and v_i. We say that w is a walk from v_0 to v_k or w is a (v_0, v_k) walk. v_0 is called the origin and v_k is called the terminus of w. $v_1, v_2, \ldots, v_{k-1}$ are called the internal vertices. The integer k is called the length of w.

Definition 2.1.19. [19] A walk in which every edge is distinct is called a trail.

Definition 2.1.20. [19] A walk in which every vertex is distinct (hence edges are distinct) is called a path.

Definition 2.1.21. [19] A cycle is a closed trail whose origin and internal vertices are different. A cycle of length k is called a k-cycle and it is denoted by C_k.

Definition 2.1.22. [19] Two vertices u and v of G are said to be connected if there is a (u, v) path between them.

Definition 2.1.23. [19] The distance between two vertices u and v, denoted by $d_G(u, v)$, is the length of a shortest path connecting them. Clearly $d_G(u, u) = 0$ and $d_G(u, v) = \infty$, if there is no path connecting u and v.

Definition 2.1.24. [19] Diameter of a graph G is the maximum distance between the vertices of G. That is, $diam(G) = Max\{d(x, y)$: x and y are vertices of $G\}$.

Definition 2.1.25. [19] In a connected graph G, the transmission degree of a vertex v is defined as $Tr(v) = \sum_{u \in V(G)} d_G(u, v)$.

Definition 2.1.26. [19] Clique of a graph is a set of mutually adjacent vertices. The maximum size of a clique of a graph G, called the clique number of G, is denoted by $\omega(G)$.

Definition 2.1.27. [19] For a graph G, a stable set is a set of vertices, no two of which are adjacent. The maximum cardinality of a stable set in a graph G is called the stability number, denoted by $\alpha(G)$.

Definition 2.1.28. [19] The girth of G, denoted by $gr(G)$, is the length of a shortest cycle in G. ($gr(G) = \infty$ if G contains no cycles).

Definition 2.1.29. [75] A partition $\{V_1, V_2, ..., V_k\}$ of the vertex set of $V(G)$ is said to be an equitable partition, if for any two vertices in V_i have the same number of neighbours in V_j for $1 \leqslant i \leqslant j \leqslant k$.

2.2 Matrix Theory

The study on graph spectra is impossible without a basic understanding of some necessary facts in Matrix Theory. Refer [67, 37, 61, 16, 9] for basic definitions and fundamental results in Matrix Theory.

Let A be an $n \times n$ matrix. The determinant $det(A - \lambda I)$ is a polynomial in the (complex) variable λ of degree n and is called the characteristic polynomial of A. The equation $det(A - \lambda I) = 0$, is called the characteristic equation of A. By the Fundamental Theorem of Algebra, the equation has n complex roots and these roots are called the eigenvalues of A. The eigenvalues might not all be distinct.

Definition 2.2.1. [16] The number of times an eigenvalue occurs as a root of the characteristic equation is called the algebraic multiplicity of the eigenvalue.

Definition 2.2.2. [16] The spectrum of a square matrix A, denoted by $\sigma(A)$, is the multi set of all the eigenvalues of A. If $\lambda_1, \lambda_2, ..., \lambda_r$, are the distinct eigenvalues of A with respective algebraic multiplicities $m_1, m_2, ..., m_r$, then we shall denote the spectrum of A, by $\sigma(A)$.

$$\sigma(A) = \left\{ \begin{array}{cccc} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ m_1 & m_2 & \cdots & m_r \end{array} \right\}.$$

Definition 2.2.3. [9] The spectral radius of a matrix is the maximum of the absolute values of its eigenvalues.

Definition 2.2.4. [66] The geometric multiplicity of the eigenvalue λ of A is defined to be the dimension of the null space of $A - \lambda I$. The geometric multiplicity of an eigenvalue does not exceed its algebraic multiplicity.

Definition 2.2.5. [66] An eigenvalue of a matrix is said to be simple, if its algebraic multiplicity is 1.

If $\lambda_1, ..., \lambda_n$ are the eigenvalues of A, then $detA = \lambda_1 \cdots \lambda_n$, while $traceA = \lambda_1 + ... + \lambda_n$.

Definition 2.2.6. [67] Let A be a matrix with real entries. A is said to be positive semidefinite if, for any vector x with real components, the dot product of Ax and x is nonnegative. That is if, $\langle Ax, x \rangle \geqslant 0$. Note that the eigenvalues of a positive semi definite matrix are nonnegative.

Definition 2.2.7. [67] A principal sub matrix of a square matrix is a sub matrix formed by a set of rows and the corresponding set of columns. A principal minor of A is the determinant of a principal sub matrix. A leading principal minor is a principal minor involving rows and columns $1, ..., k$ for some k.

Definition 2.2.8. [67] A matrix $A \in M_n$ is said to be reducible or decomposable if there is a permutation matrix $P \in M_n$ such that $P^T AP = \begin{bmatrix} B & C \\ O_{n-r,r} & D \end{bmatrix}$, and $1 \leqslant r \leqslant n-1$.

A matrix is said to be irreducible if it is not reducible.

Definition 2.2.9. [9] (Equitable partitions)

Consider a real symmetric matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{23} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & \dots & & A_{mm} \end{bmatrix}$$

whose rows and columns are indexed by $X = \{1, ..., n\}$ and partitioned according to a partition $P = \{X_1, ..., X_m\}$ of X. The characteristic matrix S is the $n \times m$ matrix whose $j-$th column is the characteristic vector of $X_j, (j = 1, ..., m)$. Let Q be the $m \times m$ matrix whose entries are the average row sums of the blocks A_{ij} of A. The partition P is called equitable if each block A_{ij} of A has constant row (and column) sum and in such case the matrix Q is called the equitable quotient matrix. Generally, the eigenvalues of Q interlace the eigenvalues of A. The following result is well-known and useful.

Lemma 2.2.10. [9] If, for an equitable partition, v is an eigenvector of Q for an eigenvalue λ , then Sv is an eigenvector of A for the same eigenvalue λ. That is, if the rows and columns of a real symmetric matrix A is partitioned according to an equitable partition, then each eigenvalue of its quotient matrix, is an eigenvalue of A.

Definition 2.2.11. [67] A matrix $[a_{ij}]$ is strictly diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry is strictly larger than the sum of the magnitudes of all other non-diagonal entries in that row, that is if, $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$, for all i.

Theorem 2.2.12. [67] (Levy- Desplanques Theorem) A strictly diagonally dominant matrix is non-singular.

Theorem 2.2.13. [9] (Perron Frobenius Theorem) If A is a non-negative matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n$, then $|\lambda_1| \geqslant |\lambda_k|$, for $k = 1, 2, ..., n$ and the eigenvalue λ_1 has an eigenvector with all entries non-negative. If A is irreducible (indecomposable), then the eigenvalue λ_1 is simple and the eigenvector has all entries positive.

2.3 Some matrices associated with simple graphs

Definition 2.3.1. [16] Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The adjacency matrix, $A(G)$ of G is a $0-1$ matrix $[a_{i,j}]$ of order $n \times n$ with entries $a_{i,j}$ such that $a_{i,j} = 1$ if $v_i \sim v_j$ and $a_{i,j} = 0$ if v_i and v_j are nonadjacent. The (i, i)-th entry of $A(G)$ is 0 for $i = 1, ..., n$.

Definition 2.3.2. [16] For a graph G, the Laplacian matrix is defined as $L(G) = Deg(G) - A(G)$, and signless Laplacian matrix of G is defined as $Q(G) = Deg(G) + A(G)$ where $Deg(G)$ is the diagonal matrix of degree of vertices. Note that $L(G)$ and $Q(G)$ are positive semi definite matrices (Recall Definition 2.2.6).

Definition 2.3.3. [16] The distance matrix of a simple connected graph G of order n is a symmetric matrix $D = [d_{i,j}]_{n\times n}$, where $d_{i,j}$ denotes the distance between two distinct vertices v_i and v_j.

Definition 2.3.4. [11] Let $Tr(G)$ be the diagonal matrix of transmission degree of vertices of a connected graph G (Recall Definition 2.1.25). Then, the distance

Laplacian matrix of G is defined as

$$D^L(G) = Tr(G) - D(G),$$

where $D(G)$ is the distance matrix. The distance signless Laplacian matrix of any connected graph G is defined as

$$D^Q(G) = Tr(G) + D(G).$$

Definition 2.3.5. [84] The Seidel matrix of G is a (-1, 0, 1) adjacency matrix given by, $S(G) = [s_{i,j}]$ where $s_{i,j} = -1$ if the vertices v_i and v_j are adjacent and $s_{i,j} = 1$ if the vertices v_i and v_j are non adjacent and $s_{i,j} = 0$ if $i = j$. That is, $S(G) = J - I - 2A(G)$. Clearly, if $\overline{G}$ denotes the complement of a graph G, then $S(\overline{G}) = -S(G)$.

Definition 2.3.6. [72] Let $D_S(G) = diag(n-2d_1-1, n-2d_2-1, ..., n-2d_n-1)$ be the diagonal matrix with diagonal entries $n-2d_i-1$, where d_i denotes the degree of the i^{th} vertex. The Seidel Laplacian matrix of a graph G is defined as

$$S^L(G) = D_S(G) - S(G)$$

and the Seidel signless Laplacian matrix of a graph G is defined as

$$S^Q(G) = D_S(G) + S(G).$$

2.4 Some graph operations

Definition 2.4.1. [45] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. Then, the graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$, is called the union of graphs G_1 and G_2 and is denoted by $G_1 \cup G_2$. If $V_1 \cap V_2 = \phi$, then $G_1 \cup G_2$ is usually denoted by $G_1 + G_2$, called the sum of the graphs G_1 and G_2.

Definition 2.4.2. [45] Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs with $V_1 \cap V_2 = \phi$. Then the join, $G_1 \bigtriangledown G_2$, of G_1 and G_2 is the super graph of $G_1 + G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2.

Definition 2.4.3. [74] Let G be a finite graph with vertices labeled as $1, 2, 3, ..., n$ and let $H_1, H_2, ..., H_n$ be a family of vertex disjoint graphs. The generalized join of $H_1, H_2, ..., H_n$ denoted by $G\left[H_1, H_2, ..., H_n\right]$ is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j depending on whether or not i and j are adjacent in G. ie, $G[H_1, H_2, ..., H_n]$ is obtained by taking the union of $H_1, H_2, ..., H_n$ and joining each vertex of H_i to all vertices of H_j if and only if $ij \in E(G)$.

Chapter 3

Adjacency matrix and graph parameters of the zero divisor graph

In this Chapter, the characteristic polynomial of $\mathit{\Gamma}(\mathbb{Z}_n)$ is found for some values of n. The first Section contains some basic definitions and preliminaries. In Section 2, the adjacency matrix of the zero divisor graph $\mathit{\Gamma}(\mathbb{Z}_n)$ is explored for $n = p^2q^2$, where $p < q$ are distinct primes. Also, the girth, diameter, stability number and clique number of this graph are traced. In Section 3, the characteristic polynomial of this graph along with the multiplicity of two eigenvalues is found by direct computation using matrix tools. In Section 4, the adjacency matrix and eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^2q})$ for distinct primes $p < q$ are investigated.

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3.1 Introduction

Associating various graphs to an algebraic structure to understand its properties, is an exciting recent trend in the research of Algebraic Graph Theory. Also the interplay of various graph parameters and the algebraic properties of these structures can be investigated and it leads to a better understanding of its theory. One among these graphs which can be associated to a commutative ring, is zero divisor graph.

Definition 3.1.1. [7] *Let R be a commutative ring with unity and $Z^*(R)$ be the set of non-zero zero divisors of R. The zero divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero divisors as vertices and two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Thus $\Gamma(R)$ is the null graph if and only if R is an integral domain.*

The characteristic polynomial of the adjacency matrix of a graph G, denoted by $\Phi(G; \lambda)$, is often referred to as the characteristic polynomial of G and the eigenvalues of G are the roots of $\Phi(G; \lambda)$. The spectrum of a finite graph G is by definition the spectrum of the adjacency matrix $A(G)$, that is, its set of eigenvalues together with their multiplicities. Clearly, $A(G)$ is a real symmetric matrix and hence its eigenvalues are all real numbers and the algebraic multiplicity of each eigenvalue is same as its geometric multiplicity [66].

3.2 Adjacency Matrix of $\mathit{\Gamma}(\mathbb{Z}_{p^2q^2})$

The zero-divisor graph of $\mathbb{Z}_n$, the ring of integers modulo n, is a simple and undirected graph. ie. even though, for an idempotent element x, $x \cdot x = 0$, x is not adjacent with itself in its zero divisor graph. Thus the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_n)$ is a symmetric matrix with entries 0 and 1, where all diagonal entries are zeroes. In this section, the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_n)$ for $n = p^2q^2$, where p and q, are distinct prime numbers with $p < q$, is investigated with an objective of computing its characteristic polynomial. While indexing the rows and columns of the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_n)$, a special interest is taken on dividing its non-zero zero divisors into an equitable partition and labeling accordingly, so that the vertices of the least degree corresponds to the first row of blocks in its adjacency matrix.

Recall that in any finite commutative ring, any non-zero element is either a unit or a zero divisor. Using elementary number theory, it is easy to calculate the order of $\Gamma(\mathbb{Z}_n)$ for any n.

Proposition 3.2.1. *The number of nonzero zero divisors of* $\mathbb{Z}_n$ *is* $n - \phi(n) - 1$.

Proof. Let m be a positive integer. Then $m \in Z^*(\mathbb{Z}_n)$ if and only if m and n have at least one common prime factor. The co-totient function, $n - \phi(n)$ counts the number of positive integers less than or equal to n which have at least one prime factor in common with n. Hence the number of nonzero zero divisors of n is $n - \phi(n) - 1$. $\square$

Theorem 3.2.2. *The adjacency matrix of* $\mathit{\Gamma}(\mathbb{Z}_n)$ *for* $n = p^2q^2$, *where* p *and* q *are distinct primes,* $p < q$, *is given by*

$A(\Gamma(\mathbb{Z}_n)) =$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_4 \\ A_5 \\ A_3 \\ A_6 \\ A_7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} A_1 & A_2 & A_4 & A_5 & A_3 & A_6 & A_7 \end{array} \\
\begin{bmatrix}
O & O & O & O & O & O & J \\
O & O & O & O & O & J & O \\
O & O & O & J & O & O & J \\
O & O & J & O & O & J & O \\
O & O & O & O & J-I & J & J \\
O & J & O & J & J & J-I & J \\
J & O & J & O & J & J & J-I
\end{bmatrix}
\end{array}$$

where J is a matrix of all ones and I is an identity matrix. The order of $A(\Gamma(\mathbb{Z}_n))$ is $pq(p+q-1)-1$.

Proof. Let $n = p^2q^2$, $p < q$. The divisors of n are $p, q, pq, p^2, q^2, p^2q, pq^2$. By proposition 3.2.1, the number of non-zero zero divisors of $\Gamma(\mathbb{Z}_{p^2q^2})$ is $pq(p+q-1)-1$. We partition the non-zero zero divisors of $\Gamma(\mathbb{Z}_{p^2q^2})$ into seven classes as multiples of the divisors of p^2q^2 as follows.

$A_1 = \{k_1p : k_1 = 1, 2, ...pq^2 - 1,\text{where } p \nmid k_1 \text{ and } q \nmid k_1\}$.

$A_2 = \{k_2q : k_2 = 1, 2, ...p^2q - 1,\text{where } p \nmid k_2 \text{ and } q \nmid k_2\}$.

$A_3 = \{k_3pq : k_3 = 1, 2, ...pq - 1,\text{where } p \nmid k_3 \text{ and } q \nmid k_3\}$.

$A_4 = \{k_4p^2 : k_4 = 1, 2, ...q^2 - 1,\text{where } q \nmid k_4\}$.

$A_5 = \{k_5q^2 : k_5 = 1, 2, ...p^2 - 1,\text{where } p \nmid k_5\}$.

$A_6 = \{k_6p^2q : k_6 = 1, 2, ...q - 1\}$.

$A_7 = \{k_7pq^2 : k_7 = 1, 2, ...p - 1\}$.

Using elementary number theory, it can be easily seen that the cardinality of A_1 is

$|A_1| = q(p-1)(q-1)$. Similarly,
$|A_2| = p(p-1)(q-1)$, $|A_3| = (p-1)(q-1)$, $|A_4| = q(q-1)$, $|A_5| = p(p-1)$, $|A_6| = (q-1)$, $|A_7| = (p-1)$. We also observe that,

1. $xy \neq 0, \forall x \in A_1$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$ and $xy = 0, \forall x \in A_1$ and $\forall y \in A_7$.

2. $xy \neq 0, \forall x \in A_2$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_7$ and $xy = 0, \forall x \in A_2$ and $\forall y \in A_6$.

3. $xy \neq 0, \forall x \in A_3$ and $\forall y \in A_1 \cup A_2 \cup A_4 \cup A_5$ and $xy = 0, \forall x \in A_3$ and $\forall y \in A_3 \cup A_6 \cup A_7$.

4. $xy \neq 0, \forall x \in A_4$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_6$ and $xy = 0, \forall x \in A_4$ and $\forall y \in A_5 \cup A_7$.

5. $xy \neq 0, \forall x \in A_5$ and $\forall y \in A_1 \cup A_2 \cup A_3 \cup A_5 \cup A_7$ and $xy = 0, \forall x \in A_5$ and $\forall y \in A_4 \cup A_6$.

6. $xy \neq 0, \forall x \in A_6$ and $\forall y \in A_1 \cup A_4$ and
   $xy = 0, \forall x \in A_6$ and $\forall y \in A_2 \cup A_3 \cup A_5 \cup A_6 \cup A_7$.

7. $xy \neq 0, \forall x \in A_7$ and $\forall y \in A_2 \cup A_5$ and
   $xy = 0, \forall x \in A_7$ and $\forall y \in A_1 \cup A_3 \cup A_4 \cup A_6 \cup A_7$.

Also for any $x,\, y \in A_i, i = 1, 2, 4, 5, \quad xy \neq 0$.

These observations describe the adjacency of vertices in $\Gamma(\mathbb{Z}_{p^2q^2})$. For example, no vertex in A_1 is adjacent to vertices in $A_1, A_2, A_3, A_4, A_5,$ or A_6 and correspondingly, it gives blocks of zeros in the adjacency matrix of $\Gamma(\mathbb{Z}_{p^2q^2})$. Also, all vertices of A_1 are adjacent to every vertex of A_7 and correspondingly, it gives a block of all ones and so on. Recall that the zero divisor graph of a commutative ring is a simple, undirected graph. Since $xy = 0, \forall x, y \in A_i$, for $i = 3, 6, 7$; $A_3 \cup A_6 \cup A_7$ induces a complete subgraph , but self adjacency of vertices are omitted. The non-zero zero divisors of Z_n are rearranged, such that the elements of A_1 appear first and then A_2, A_4, A_5, A_3, A_6, and A_7. Thus the adjacency matrix of $\Gamma(\mathbb{Z}_{p^2q^2})$ is a 7×7 block matrix consisting of 49 blocks of zeros and ones in the following form,

$$A(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} O & O & O & O & O & O & J \\ O & O & O & O & O & J & O \\ O & O & O & J & O & O & J \\ O & O & J & O & O & J & O \\ O & O & O & O & J-I & J & J \\ O & J & O & J & J & J-I & J \\ J & O & J & O & J & J & J-I \end{bmatrix} \tag{3.1}$$

Thus the order of this matrix is $\Sigma_{i=1}^{7}|A_i| = pq(p+q-1) - 1$.

□

Theorem 3.2.3. *Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The clique number $\omega(G) = pq - 1$.*

Proof. The principal sub matrix $\begin{bmatrix} J-I & J & J \\ J & J-I & J \\ J & J & J-I \end{bmatrix}$ of $A(\Gamma(\mathbb{Z}_n))$ corresponds to a complete subgraph of maximum order, induced by the vertices in

$A_3 \cup A_6 \cup A_7$. Hence the clique number of G is given by

$$\begin{aligned} \omega(G) &= |A_3| + |A_6| + |A_7| \\ &= pq - 1. \end{aligned}$$

□

Theorem 3.2.4. *Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The stability number $\alpha(G) = p(q-1)(p+q-1)$.*

Proof. Since, for any $x, y \in A_1 \cup A_2 \cup A_4$, $xy \neq 0$; no two vertices of $A_1 \cup A_2 \cup A_4$, are adjacent. Thus $A_1 \cup A_2 \cup A_4$ is a maximum independent set. Hence the stability number of G, $\alpha(G) = |A_1| + |A_2| + |A_4| = p(q-1)(p+q-1)$. □

Theorem 3.2.5. *Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The girth, $gr(G) = 3$.*

Proof. Since G has a clique of cardinality $pq - 1 > 3$, it contains a cycle of length 3. Hence gr(G)=3. . □

It was shown in [81] that, for a commutative ring R, $diam\Gamma(R) \leqslant 3$. In the next theorem, it is seen that $\Gamma(\mathbb{Z}_{p^2q^2})$ attains this upper bound.

Theorem 3.2.6. *Let $G = \Gamma(\mathbb{Z}_n)$, for $n = p^2q^2$, where p and q are distinct primes, $p < q$. The diameter, $diam(G) = 3$.*

Proof. For $x \in A_1, y \in A_2$, any shortest (x,y)-path contains an intermediate vertex from A_7 and a vertex from A_6. Thus $diamG = 3$. □

3.3 Eigenvalues of $\Gamma(\mathbb{Z}_{p^2q^2})$

Among the zero divisor graphs belonging to this class, $\Gamma(\mathbb{Z}_{36})$ is of minimum order, ie for $p = 2, q = 3$ and the order of $\Gamma(\mathbb{Z}_{36})$ is 23. As the order increases, the difficulty level of extracting eigenvalues of the graph increases. In this section, two eigenvalues of $\Gamma(\mathbb{Z}_{p^2q^2})$ for any $p < q$, are found with multiplicities. Also, the polynomial of degree seven which gives the remaining eigenvalues is explored through a long computation. The diction is made as precise as possible.

The circulant matrix of the form $C_{(a,b,n)}$ plays a vital role in the following computations. The following Lemmas are used to find the determinant and the inverse of $C_{(a,b,n)}$. Using the properties of determinant of a square matrix, the following Propositions are proved.

Definition 3.3.1. *Let $M_n(F)$ denote the vector space of all square matrices of size $n \times n$ with entries from a field F. A circulant matrix of size $n \times n$, with entries a and b, $a, b \in \mathbb{R}$, denoted by $C_{(a,b,n)}$ is of the form*

$$C_{(a,b,n)} = \begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n\times n}$$

The complexity of computing the characteristic polynomial of a $n \times n$ block matrix is often reduced to some extent by the application of the following Lemmas.

Proposition 3.3.2. *Let* $C_{(a,b,n)}=\begin{bmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ b & b & a & \dots & b \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \dots & a \end{bmatrix}_{n\times n}$ *be a circulant matrix of size $n \times n$; with entries a and b.*

Then, $detC_{(a,b,n)}$*; denoted by δ, is given by* $\delta = (a + (n-1)b)\,(a-b)^{n-1}$.

Proposition 3.3.3. *If $C_{(a,b,n)}$ is nonsingular, then its inverse is given by*

$$C^{-1}_{(a,b,n)} = \frac{1}{\delta}\begin{bmatrix} \delta_{n-1} & \Delta_{n-1} & \dots & \Delta_{n-1} \\ \Delta_{n-1} & \delta_{n-1} & \dots & \Delta_{n-1} \\ \vdots & & \ddots & \vdots \\ \Delta_{n-1} & \dots & & \delta_{n-1} \end{bmatrix} = \frac{1}{\delta}C(\delta_{n-1}, \Delta_{n-1}, n),$$

where $\delta_{n-1} = (a + (n-2)b)\,(a-b)^{n-2}$ *and* $\Delta_{n-1} = -b \cdot (a-b)^{n-2}$.

A matrix $A \in M_n(F)$, where F is a field of numbers(real or complex); of the form $A=\begin{bmatrix} A_{11} & O & \dots & O \\ O & A_{22} & \dots & O \\ \vdots & & \ddots & \vdots \\ O & \dots & & A_{nn} \end{bmatrix}$ in which $A_{ii} \in M_{n_i}(F), i = 1, 2, ...k$, $\Sigma_{i=1}^{k} n_i = n$, and all blocks above and below the block diagonal are the zero blocks, is called a block diagonal matrix.

Thus $A=A_{11} \oplus A_{22} ... \oplus A_{kk} = \bigoplus_{i=1}^{k} A_{ii}$, is the direct sum of matrices $A_{11}, A_{22}, ...A_{kk}$.

Lemma 3.3.4. [67] $det(\bigoplus_{i=1}^{k} A_{ii}) = \prod_{i=1}^{k} det(A_{ii})$.

In particular, if $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$, then,

$$det \begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = det(A_{11}) \cdot det(A_{22}).$$

Lemma 3.3.5. [67] *If $A_{11} \in M_n(F)$ and $A_{22} \in M_m(F)$ are non singular, then,*
$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}$$

Lemma 3.3.6. [31] *Let M, N, P, Q be matrices and let M be invertible. Let* $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$,
then $detS = detM.\ det(Q - PM^{-1}N)$.

$(Q - PM^{-1}N)$ is called the Schur complement of M in S.

Theorem 3.3.7. *Let $G = \Gamma(\mathbb{Z}_{p^2q^2})$ and let λ be an eigenvalue of G. Then 0 and -1 are eigenvalues of G with multiplicities $pq(p+q-2)-4$ and $pq-4$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies $\lambda^7 - b_6\lambda^6 + b_5\lambda^5 + b_4\lambda^4 + b_3\lambda^3 - b_2\lambda^2 + b_1\lambda + b_0 = 0$, where,*

$b_6 = pq - 4$

$b_5 = p^2(q-1) - q(2p+1) + 6 - (p-1)(q-1)(3pq + p + 1),$

$b_4 = pq(p-1)(q-1)(3pq - p - q - 8) - pq + 2,$

$b_3 = pq(p-1)(q-1)\left[pq(3pq - 4p - 4q + 7) + p^2 + q^2 - 9\right],$

$$b_2 = [pq(p-1)(q-1)]^2\,[3pq - 3p - 3q - 2] + \\ pq(p-1)(q-1)\left[pq(2p+2q+1) - 2(p+q)^2 + 4\right],$$

$$b_1 = pq(p-1)^3(q-1)^4(pq + p + q - p^2) + \\ [pq(p-1)(q-1)]^2\left[1 - p(p-1)(q-1)^2 - p(q-1)\right] + \\ pq(p-1)^2(q-1)^2\,[pq - (p+q)]\,[p(p-2)(q-1) - q(q-2)],$$

$b_0 = p^2q^2(p-1)^4(q-1)^4\,[pq - (p+q)].$

Proof. Let the adjacency matrix of $G = \Gamma(\mathbb{Z}_{p^2q^2})$; given by equation 3.1 be denoted by M. Then the eigenvalues of G are given by $|M - \lambda I| = 0$. Now,

$$M - \lambda I = \left[\begin{array}{ccc|cccc} C_{(-\lambda,0)} & O & O & O & O & O & J \\ O & C_{(-\lambda,0)} & O & O & O & J & O \\ O & O & C_{(-\lambda,0)} & J & O & O & J \\ \hline O & O & J & C_{(-\lambda,0)} & O & J & O \\ O & O & O & O & C_{(-\lambda,1)} & J & J \\ O & J & O & J & J & C_{(-\lambda,1)} & J \\ J & O & J & O & J & J & C_{(-\lambda,1)} \end{array}\right]$$

$$= \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

By Lemma 3.3.6, if $\lambda \neq 0$,

$$det(M - \lambda I) = detA \cdot det(D - CA^{-1}B) \tag{3.2}$$

Since $A = \begin{bmatrix} C_{(-\lambda,0)} & O & O \\ O & C_{(-\lambda,0)} & O \\ O & O & C_{(-\lambda,0)} \end{bmatrix}$ is a scalar matrix of size $p(q-1)(p+q-1)$, using Lemma 3.3.4,

$$detA = (-\lambda)^{p(q-1)(p+q-1)}$$

(Note that size of A is sum of cardinalities of A_1, A_2 and A_4 which is equal to $p(q-1)(p+q-1)$). Thus, equation (3.2)becomes

$$det(M - \lambda I) = (-\lambda)^{p(q-1)(p+q-1)} \cdot det(D - CA^{-1}B) \tag{3.3}$$

Also $A^{-1} = \dfrac{-1}{\lambda} I$, where I is the identity matrix of size $p(q-1)(p+q-1)$.

Let,

$$a = |A_4| = q(q-1)$$

$$b = |A_2| = p(p-1)(q-1)$$

$$c = |A_1| = q(p-1)(q-1).$$

Then,

$$CA^{-1}B = \frac{-1}{\lambda}CB$$

$$= \frac{-1}{\lambda}\begin{bmatrix} a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a & \cdots & a \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a & \cdots & a \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b & \cdots & b & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b & \cdots & b & 0 & \cdots & 0 \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a+c & \cdots & a+c \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ a & \cdots & a & 0 & \cdots & 0 & 0 & \cdots & 0 & a+c & \cdots & a+c \end{bmatrix}$$

Hence $D - CA^{-1}B=$

$$\begin{bmatrix}
\frac{a}{\lambda}-\lambda & \frac{a}{\lambda} & \cdots & \frac{a}{\lambda} & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} \\
\frac{a}{\lambda} & \frac{a}{\lambda}-\lambda & \cdots & \frac{a}{\lambda} & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
\frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda}-\lambda & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & \frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} \\
0 & \cdots & \cdots & 0 & -\lambda & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & 1 & -\lambda & \cdots & 1 & \vdots & & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & -\lambda & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{b}{\lambda}-\lambda & \frac{b}{\lambda}+1 & \cdots & \frac{b}{\lambda}+1 & 1 & \cdots & \cdots & 1 \\
\vdots & & & \vdots & \vdots & & & \vdots & \frac{b}{\lambda}+1 & \frac{b}{\lambda}-\lambda & & \vdots & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \ddots & \vdots & \vdots & & & \vdots \\
1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{b}{\lambda}+1 & \cdots & \cdots & \frac{b}{\lambda}-\lambda & 1 & \cdots & \cdots & 1 \\
\frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{a+c}{\lambda}-\lambda & \frac{a+c}{\lambda}+1 & \cdots & \frac{a+c}{\lambda}+1 \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \frac{a+c}{\lambda}+1 & \frac{a+c}{\lambda}-\lambda & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots & \vdots & & & \vdots & \vdots & & \ddots & \vdots \\
\frac{a}{\lambda} & \cdots & \cdots & \frac{a}{\lambda} & 1 & \cdots & \cdots & 1 & 1 & \cdots & \cdots & 1 & \frac{a+c}{\lambda}+1 & \cdots & \cdots & \frac{a+c}{\lambda}-\lambda
\end{bmatrix}$$

Thus,

$$D{-}CA^{-1}B = \left[\begin{array}{cc|cc}
X_{1_{p(p-1)}} & O_{p(p-1)\times(p-1)(q-1)} & Q_{p(p-1)\times(q-1)} & Y_{p(p-1)\times(p-1)} \\
O_{(p-1)(q-1)\times p(p-1)} & X_{2_{(p-1)(q-1)}} & R_{(p-1)(q-1)\times(q-1)} & S_{(p-1)(q-1)\times(p-1)} \\
\hline
Q^T_{(q-1)\times p(p-1)} & R^T_{(q-1)\times(p-1)(q-1)} & X_{3(q-1)} & U_{(q-1)\times(p-1)} \\
Y^T_{(p-1)\times p(p-1)} & S^T_{(p-1)\times(p-1)(q-1)} & U^T_{(p-1)\times(q-1)} & X_{4_{(p-1)}}
\end{array}\right]$$

where $X_1 = C_{\left(\frac{a}{\lambda}-\lambda,\ \frac{a}{\lambda},\ p(p-1)\right)}$, $\quad X_2 = C_{(-\lambda,\ 1,\ (p-1)(q-1))}$, $\quad X_3 = C_{\left(\frac{b}{\lambda}-\lambda,\ \frac{b}{\lambda}+1,\ q-1\right)}$,

$X_4 = C_{\left(\frac{a+c}{\lambda}-\lambda,\ \frac{a+c}{\lambda}+1,\ p-1\right)}$, $\quad Y = \frac{a}{\lambda}J_{p(p-1)\times(p-1)}$, where J is a matrix of all ones and Q, R, S, U are matrices of all ones.

Let n, m be the size of X_1 and X_2 respectively. Hence $n = p(p-1)$ and $m = (p-1)(q-1)$. While doing tedious computations hereafter, the following substitutions are made.

$f(\lambda) = na - \lambda^2$,

$g(\lambda) = m - 1 - \lambda$, and

$h(\lambda) = f(\lambda)g(\lambda)\{p(p-1)(q-1)^2 + \lambda(q-2) - \lambda^2\} - (q-1)\{n\lambda^2 g(\lambda) + m\lambda f(\lambda)\}$.

Using Proposition 3.3.2 and Proposition 3.3.3, it can be seen that,

$$detX_1 = (-1)^{n-1}(\lambda)^{n-2} f(\lambda) \tag{3.4}$$

$$detX_2 = (-1)^{m-1}(\lambda+1)^{m-1} g(\lambda) \tag{3.5}$$

$X_1^{-1} = \frac{-1}{f(\lambda)} \cdot C_{\left(\frac{(n-1)a}{\lambda} - \lambda,\ \frac{-a}{\lambda},\ n\right)}$, $X_2^{-1} = \frac{-1}{(\lambda+1)g(\lambda)} \cdot C_{(m-\lambda-2,\ -1,\ m)}$.

Using Lemma 3.3.4, Lemma 3.3.5 and Lemma 3.3.6, we see that

$$\begin{aligned} det\left(D - CA^{-1}B\right) &= detX_1 \cdot\ detX_2 \cdot \\ det &\left(\begin{bmatrix} X_3 & U \\ U^T & X_4 \end{bmatrix} - \begin{bmatrix} Q^T & R^T \\ Y^T & S^T \end{bmatrix} \begin{bmatrix} X_1^{-1} & O \\ O & X_2^{-1} \end{bmatrix} \begin{bmatrix} Q & Y \\ R & S \end{bmatrix}\right) \end{aligned} \tag{3.6}$$

Applying Lemma 3.3.6 again,

$det\left(\begin{bmatrix} X_3 & U \\ U^T & X_4 \end{bmatrix} - \begin{bmatrix} Q^T & R^T \\ Y^T & S^T \end{bmatrix} \begin{bmatrix} X_1^{-1} & O \\ O & X_2^{-1} \end{bmatrix} \begin{bmatrix} Q & Y \\ R & S \end{bmatrix}\right)$ gets surprisingly simplified to be the determinant of the circulant matrix $C_{(v-z,\ w-z,\ p-1)}$, where

$z = \{f(\lambda)g(\lambda) - (nag(\lambda) + mf(\lambda))\}^2 \dfrac{\lambda(q-1)}{f(\lambda)g(\lambda)h(\lambda)}$ and

$v = \dfrac{f(\lambda)g(\lambda)(a + c - \lambda^2) - m\lambda f(\lambda) - na^2 g(\lambda)}{\lambda f(\lambda)g(\lambda)}$

Also, applying Proposition 3.3.2,

$$detC_{(v-z,\ w-z,\ p-1)} = (-1)^{p-2}(\lambda+1)^{p-2}\{(v-z)(p-1) + (\lambda+1)(p-2)\}$$

Simplifying equation (3.6), using equations (3.4), and (3.5), and substituting in (3.3), the characteristic equation of $\Gamma(\mathbb{Z}_{p^2q^2})$ is

$\Phi(G;\lambda) = (\lambda)^{pq(p+q-2)-4} \cdot (\lambda+1)^{pq-4} \cdot \phi(\lambda) = 0,$ where

$$\phi(\lambda) = \lambda^7 - b_6\lambda^6 + b_5\lambda^5 + b_4\lambda^4 + b_3\lambda^3 - b_2\lambda^2 + b_1\lambda + b_0, \tag{3.7}$$

where,

$$\begin{aligned}
b_6 &= pq - 4\\
b_5 &= p^2(q-1) - q(2p+1) + 6 - (p-1)(q-1)(3pq+p+1),\\
b_4 &= pq(p-1)(q-1)(3pq-p-q-8) - pq + 2,\\
b_3 &= pq(p-1)(q-1)\left[pq(3pq-4p-4q+7) + p^2 + q^2 - 9\right],\\
b_2 &= \left[pq(p-1)(q-1)\right]^2\left[3pq-3p-3q-2\right] + \\
&\qquad pq(p-1)(q-1)\left[pq(2p+2q+1) - 2(p+q)^2 + 4\right],\\
b_1 &= pq(p-1)^3(q-1)^4(pq+p+q-p^2) + \\
&\qquad \left[pq(p-1)(q-1)\right]^2\left[1 - p(p-1)(q-1)^2 - p(q-1)\right] + \\
&\qquad pq(p-1)^2(q-1)^2\left[pq-(p+q)\right]\left[p(p-2)(q-1) - q(q-2)\right],\\
b_0 &= p^2q^2(p-1)^4(q-1)^4\left[pq-(p+q)\right].
\end{aligned}$$

Clearly, if $\lambda \neq 0, -1$, then $\phi(\lambda) = 0$. □

Remark 3.3.8. Performing elementary row transformations, it can be seen that the number of zero rows in any row echelon form of M is $|A_1|+|A_2|+|A_4|+|A_5|-4$. Hence the nullity of M is $pq(p+q-2)-4$ and thus $\lambda = 0$ is an eigenvalue of G with the geometric multiplicity of $\lambda = 0$ is $pq(p+q-2)-4$ which is same as its algebraic multiplicity. In a similar way, it can be seen that the algebraic multiplicity of $\lambda = -1$ is the nullity of $M+I$, which is equal to $|A_3|+|A_6|+|A_7|-3 = pq-4$. Since $\phi(\lambda)$ is a polynomial of degree 7, the total number of eigenvalues is $[pq(p+q-2)-4] + (pq-4) + 7$, which is found to be the same as $\sum_{i=1}^{7} |A_i|$.

Remark 3.3.9. Since M is a symmetric matrix with all diagonal entries zero, the sum of the eigenvalues is equal to the trace of M, which is zero . The non-zero eigenvalues of M are precisely -1 and the zeros of $\phi(\lambda)$. Also, from equation (3.7), the sum of the roots of $\phi(\lambda) = 0$; is $pq - 4$. Hence it is convinced that sum of the eigenvalues of M is $(pq - 4)(-1) + (pq - 4) = 0$.

The above Theorem gives the characteristic polynomial of a class of graphs, namely $\Gamma(\mathbb{Z}_{p^2q^2})$ where $p < q$ are distinct primes. For example, the characteristic polynomial of $\mathbb{Z}_{36}$, $\mathbb{Z}_{100}$, $\mathbb{Z}_{225}$ can be obtained using the above Theorem.

3.4 Adjacency Matrix and Eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$

Let $n = p^2q$ where p and q are prime integers with $p < q$ and let $G = \Gamma(\mathbb{Z}_{p^2q})$. By proposition 3.2.1, n has $p(p+q-1)-1$ non-zero zero divisors. The computation of the spectrum of this graph is not as much difficult as in the previous section, since the equitable partition of the set of non-zero zero divisors of $\mathbb{Z}_{p^2q}$ contains only four disjoint sets as given below.

$E_1 = \{k_1p : k_1 = 1, 2, ...pq - 1,\text{where } p \nmid k_1 \text{ and } q \nmid k_1 \}$.

$E_2 = \{k_2q : k_2 = 1, 2, ...p^2 - 1,\text{where } p \nmid k_2 \}$.

$E_3 = \{k_3pq : k_3 = 1, 2, ...p - 1\}$.

$E_4 = \{k_4p^2 : k_4 = 1, 2, ...q - 1\}$.

Clearly, $|E_1| = (p - 1)(q - 1)$, $|E_2| = p(p - 1)$, $|E_3| = (p - 1)$, $|E_4| = (q - 1)$.

The adjacency matrix of G is given by ,

$$A(\Gamma(\mathbb{Z}_{p^2q})) = \left[\begin{array}{cc|cc} O_{(p-1)(q-1)} & O_{(p-1)(q-1)\times p(p-1)} & J_{(p-1)(q-1)\times(p-1)} & O_{(p-1)(q-1)\times(q-1)} \\ O_{p(p-1)\times(p-1)(q-1)} & O_{p(p-1)} & O_{p(p-1)\times(p-1)} & J_{p(p-1)\times(q-1)} \\ \hline J_{(p-1)\times(p-1)(q-1)} & O_{(p-1)\times p(p-1)} & J-I_{(p-1)} & J_{(p-1)\times(q-1)} \\ O_{(q-1)\times(p-1)(q-1)} & J_{(q-1)\times p(p-1)} & J_{(q-1)\times(p-1)} & O_{(q-1)} \end{array}\right]$$

Remark 3.4.1. The stability number $\alpha(G) = |E_1| + |E_2| = (p-1)(p+q-1)$.

Remark 3.4.2. The clique number $\omega(G) = |E_3| = p-1$.

Remark 3.4.3. Since E_3 induces a complete subgraph of order $p-1$, the girth $gr(G) = 4$ if $p = 2$ and $gr(G) = 3$, if $p \geqslant 3$.

Theorem 3.4.4. *Let $G = \Gamma(\mathbb{Z}_{p^2q})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $(p-1)(p+q-1)+(q-4)$ and $p-2$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies,*
$\lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2 = 0.$

Proof. Let $M = A(\Gamma(\mathbb{Z}_{p^2q}))$. The eigenvalues of G are given by

$$det(M-\lambda I) = 0$$

$$M - \lambda I = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} -\lambda & 0 & \cdots & 0 \\ 0 & -\lambda & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & -\lambda \end{bmatrix}_{(p-1)(p+q-1)}$$

$$B = \begin{bmatrix} 1 & \cdots\cdots & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 1 & \cdots\cdots & 1 & 0 & \cdots & 0 \\ 0 & \cdots\cdots & 0 & 1 & \cdots & 1 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots\cdots & 0 & 1 & \cdots & 1 \end{bmatrix}_{(p-1)(p+q-1)\times(p+q-2)} \quad \text{and}$$

$$C = \begin{bmatrix} -\lambda & 1 & \cdots & 1 & 1 & \cdots & \cdots & 1 \\ 1 & -\lambda & & 1 & \vdots & & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 1 & \cdots & \cdots & -\lambda & 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 & -\lambda & 0 & \cdots & 0 \\ \vdots & & & \vdots & 0 & -\lambda & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & -\lambda \end{bmatrix}_{p+q-2}$$

If $\lambda \neq 0$, then A is invertible and by Proposition 3.3.2 and Proposition 3.3.3

$$detA = (-\lambda)^{(p-1)(p+q-1)} \tag{3.8}$$

and $A^{-1} = \dfrac{-1}{\lambda} I$.

Also $B^T A^{-1} B = \dfrac{-1}{\lambda} \left[\begin{array}{c|c} (p-1)(q-1)J_{p-1} & O_{(p-1)\times(q-1)} \\ \hline O_{(q-1)\times(p-1)} & p(p-1)J_{q-1} \end{array}\right]$, where J denotes a matrix of all ones.

Now, the Schur complement of A in C is given by

$$C - B^T A^{-1} B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} = C_{\left(-\lambda+\frac{(p-1)(q-1)}{\lambda},\ 1+\frac{(p-1)(q-1)}{\lambda},\ p-1\right)}$,

$A_{12} = J_{(p-1)\times(q-1)}$,

$A_{21} = J_{(q-1)\times(p-1)}$ and

$A_{22} = C_{\left(-\lambda+\frac{p(p-1)}{\lambda},\ \frac{p(p-1)}{\lambda},\ q-1\right)}$. Thus by Lemma 3.3.6 and equation (3.8),

$$\begin{aligned} det(M-\lambda I) &= detA.\ det(C-B^TA^{-1}B) \\ &= (-\lambda)^{(p-1)(p+q-1)}.det(C-B^TA^{-1}B) \end{aligned} \tag{3.9}$$

Applying Lemma 3.3.6 once again ,

$$det(C-B^TA^{-1}B) = detA_{11}.\ det(A_{22}-A_{21}A_{11}^{-1}A_{12})$$

By Proposition 3.3.2 it can be seen that,

$$detA_{11} = \frac{h(\lambda)}{\lambda}(-\lambda-1)^{p-2}$$

where $h(\lambda) = -\lambda^2 + (p-2)\lambda + (p-1)^2(q-1)$.

Also by Proposition 3.3.3,

$$A_{11}^{-1} = \frac{1}{h(\lambda)(\lambda+1)} C_{(\lambda^2-(p-3)\lambda-(p-1)(q-1)(p-2)),\ \lambda+(p-1)(q-1),\ p-1)}$$

Also,

$$A_{22}-A_{21}A_{11}^{-1}A_{12} = C_{\left(-\lambda+\frac{p(p-1)}{\lambda}-\frac{\lambda(p-1)}{h(\lambda)},\ \frac{p(p-1)}{\lambda}-\frac{\lambda(p-1)}{h(\lambda)}),\ q-1\right)}$$

and

$$det(A_{22}-A_{21}A_{11}^{-1}A_{12}) = (-1)^{q-1}\lambda^{q-2}\frac{\lambda^2h(\lambda)-h(\lambda)p(p-1)(q-1)+\lambda^2(p-1)(q-1)}{\lambda h(\lambda)}$$

Applying these in equations (3.8), and (3.9), the characteristic equation of $\mathit{\Gamma}(\mathbb{Z}_{p^2q})$ is obtained as

$$\Phi(G;\lambda) = \lambda^{(p-1)(p+q-1)+q-4}(\lambda+1)^{p-2}\phi(\lambda) = 0$$

where $\phi(\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2$

Hence $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $(p-1)(p+q-1)+(q-4)$ and $p-2$ respectively. Also if $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies,

$\phi(\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2 = 0$ $\square$

Chapter 4

The adjacency matrix and eigenvalues of $\Gamma(\mathbb{Z}_{p^k})$

The spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^3, p^4$ where p is any prime, is found in Section 2 of this Chapter. The adjacency matrix and the characteristic polynomial of $\Gamma(\mathbb{Z}_{p^k})$ for any $k \geqslant 3$, along with two eigenvalues with their multiplicities, are explored in section 3. A general method is proposed to compute the eigenvalues of $\Gamma(\mathbb{Z}_n)$ for any n, using the quotient matrix of equitable partition of its vertex set in Section 4.

4.1 Introduction

As it is seen in the previous Chapter, the difficulty level of finding the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ increases as the number of classes in the equitable partition of the zero divisors of $\mathbb{Z}_n$ increases or in other words, if the number of proper divisors of n increases. In this chapter, the study is focused to the class

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of zero divisor graphs of $\mathbb{Z}_n$ where n is any finite power of an arbitrary prime p.

4.2 Adjacency matrix and the computation of the spectrum of $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^4})$

Recall that $\mathbb{Z}_p$ is an integral domain and it does not contain any non-zero zero divisors. $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph and hence it is trivial to find its spectrum. Hence, the attempt is initiated with $\Gamma(\mathbb{Z}_{p^3})$ and $\Gamma(\mathbb{Z}_{p^4})$.

4.2.1 The adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$

Theorem 4.2.1. *The adjacency matrix of* $\Gamma(\mathbb{Z}_n)$ *for* $n = p^3$*, where* p *is a prime integer, is*

$$A(\Gamma(\mathbb{Z}_{p^3})) = \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J - I_{(p-1)} \end{array}\right]$$

where J *is a matrix of all ones and* I *is an identity matrix. The order of this matrix is* $p^2 - 1$.

Proof. Let $n = p^3$. By Proposition 3.2.1, the number of non-zero zero divisors of $\mathbb{Z}_{p^3}$ is $p^2 - 1$. These $p^2 - 1$ non-zero zero divisors are partitioned as follows.

$P_1 = \{k_1 p : k_1 = 1, 2, ...p^2 - 1, \text{where } p \nmid k_1\}$.

$P_2 = \{k_2 p^2 : k_2 = 1, 2, ...p - 1, \text{where } p \nmid k_2\}$.

Using elementary number theory, it can be easily seen that the cardinality of P_1 is $|P_1| = p^2 - p$. Similarly,

$|P_2| = p - 1$. It is also observed that,

1. $xy \neq 0, \forall x, y \in P_1$.

2. $xy = 0, \forall x \in P_1$ and $\forall y \in P_2$.

3. $xy = 0, \forall x, y \in P_2$.

These simple observations give rise to the partitioned structure of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$. The non-zero zero divisors of n are rearranged such that the elements of P_1 appear first and then P_2. Since no two vertices in P_1 are adjacent, it is an independent set in $\Gamma(\mathbb{Z}_{p^3})$ and hence it corresponds to a block of zeroes in the adjacency matrix. Also since all vertices of P_1 are adjacent to every vertex of P_2, it corresponds to a block of all ones and so on. Thus the adjacency of vertices among P_2 corresponds to the block $J - I$. Thus the adjacency matrix of $\Gamma(\mathbb{Z}_{p^3})$ is a 2×2 block matrix consisting of blocks of zeros and ones in the following form,

$$A(\mathit{\Gamma}(\mathbb{Z}_{p^3})) = \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J - I_{(p-1)} \end{array}\right] \tag{4.1}$$

The order of this matrix is $|P_1| + |P_2| = p^2 - 1$. □

4.2.2 Spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$

The order of the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ is $p^2 - 1$. To reduce the complexity in the direct computation of the characteristic polynomial, some tools of Matrix Theory are adopted, among which Schur complement and coronal play vital role.

Definition 4.2.2. [59] *Let $\mathbf{1}_n$ denote an all-one vector. The coronal of a matrix A, denoted by $\Gamma_A(x)$, is defined as the sum of the entries of the matrix $(xI - A)^{-1}$. That is, $\Gamma_A(x) = (\mathbf{1}_n)^T.(xI - A)^{-1}.\mathbf{1}_n$*

Lemma 4.2.3. [27] *Let G be a r-regular graph on n vertices, with adjacency matrix A. Then, $\Gamma_A(x) = \dfrac{n}{x-r}$.*

Lemma 4.2.4. [53] *Let A be an $n \times n$ matrix and $J_{n\times n}$ denote an all one matrix. Then, $det(xI_n - A - \alpha J_{n\times n}) = (1-\alpha\Gamma_A(x)).det(xI_n - A)$, where α is a real number.*

Let $G = \mathit{\Gamma}(\mathbb{Z}_{p^3})$ and $M = A(\mathit{\Gamma}(\mathbb{Z}_{p^3}))$. The vertex set of G is partitioned into P_1 and P_2, where P_1 induces the null subgraph $\overline{K}_{p^2-p}$ and P_2 induces a complete subgraph K_{p-1} both of which are regular of degree 0 and $p-2$ respectively.

Theorem 4.2.5. *Let $G = \Gamma(\mathbb{Z}_{p^3})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities p^2-p-1 and $p-2$ respectively. If $\lambda \neq 0$, $\lambda \neq -1$, then λ satisfies $\lambda^2 - (p-2)\lambda - p(p-1)^2 = 0$.*

Proof. Let the adjacency matrix of $G = \mathit{\Gamma}(\mathbb{Z}_{p^3})$ be denoted by M. From equation (4.1),

$$M = \left[\begin{array}{c|c} O_{(p^2-p)} & J_{(p^2-p)\times(p-1)} \\ \hline J_{(p-1)\times(p^2-p)} & J - I_{(p-1)} \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

Clearly A_1 and A_2 are the adjacency matrices of the induced subgraphs $\overline{K}_{p^2-p}$ and K_{p-1} of G respectively. By Lemma 4.2.3, $\Gamma_{A_1}(\lambda) = \dfrac{p^2-p}{\lambda}$ and
$\Gamma_{A_4}(\lambda) = \dfrac{p-1}{\lambda - p + 2}$.

The eigenvalues of G are given by $det(\lambda I - M) = 0$.

By Lemma 3.3.6 ,

$$det(\lambda I - M) = det(\lambda I - A_1).det\left[(\lambda I - A_4) - {A_3}^T.(\lambda I - A_1)^{-1}.A_2\right], \qquad (4.2)$$

where $det(\lambda I - A_1) = \lambda^{p^2-p}$ and $det(\lambda I - A_4) = (\lambda - p + 2).(\lambda + 1)^{p-2}$.

Also, using Lemma 4.2.3 and Lemma 4.2.4 and equation (4.2),

$$\begin{aligned} det\left[(\lambda I - A_4) - {A_3}^T.(\lambda I - A_1)^{-1}.A_2\right] &= det\left[(\lambda I - A_4) - \Gamma_{A_1}(\lambda).J_{p-1\times p-1}\right] \\ &= det\left[(\lambda I - A_4) - (\frac{p^2-p}{\lambda}).J_{p-1\times p-1}\right] \\ &= \left(1 - (\frac{p^2-p}{\lambda})\Gamma_{A_4}(\lambda)\right).det(\lambda I - A_4) \\ &= \left(1 - (\frac{p^2-p}{\lambda})(\frac{p-1}{\lambda-p+2})\right) \\ &\qquad .(\lambda - p + 2).(\lambda+1)^{p-2}. \end{aligned}$$

Thus, the characteristic polynomial of G is given by,

$$\Phi(G;\lambda) = \lambda^{p^2-p-1}.(\lambda+1)^{p-2}.\left(\lambda^2 - (p-2)\lambda - p(p-1)^2\right)$$

Thus the spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ is

$$\left\{\begin{array}{cccc} 0 & -1 & \dfrac{(p-2)+\sqrt{4p^3-7p^2+4}}{2} & \dfrac{(p-2)-\sqrt{4p^3-7p^2+4}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array}\right\} \quad \square$$

4.2.3 Adjacency matrix and eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^4})$

Partitioning the non-zero zero divisors of $\mathbb{Z}_{p^4}$ into multiples of p, p^2, p^3 and labeling the vertices of the zero-divisor graph $\mathit{\Gamma}(\mathbb{Z}_{p^4})$ properly, the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_{p^4})$ is obtained as,

$$A(\mathit{\Gamma}(\mathbb{Z}_{p^4})) = \left[\begin{array}{c|cc} O_{p^2(p-1)} & O_{p^2(p-1)\times p(p-1)} & J_{p^2(p-1)\times(p-1)} \\ O_{p(p-1)\times p^2(p-1)} & J - I_{p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p^2(p-1)} & J_{(p-1)\times p(p-1)} & J - I_{(p-1)} \end{array}\right]$$

The characteristic polynomial can be computed as in Section 4.2.2.

Theorem 4.2.6. *Let $G = \Gamma(\mathbb{Z}_{p^4})$ and let λ be an eigenvalue of G. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $p^3 - p^2 - 1$ and $p^2 - 3$*

respectively. If $\lambda \neq 0$, $\lambda \neq -1$, *then* λ *satisfies* $\phi(\lambda) = \lambda^3 - (p^2-3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p-1)^2(p^2-p-1) = 0$.

4.3 The generalised join structure of $\Gamma(\mathbb{Z}_n)$

It is observed that, for higher powers of p, the number of blocks in the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ is not feasible for applying Lemma 4.2.3 and Lemma 4.2.4 to extract the spectrum in full and this problem becomes severe when n has more prime power factors. At this juncture, the structural properties of the graphs $\Gamma(\mathbb{Z}_n)$ may be analyzed in terms of the induced subgraphs. S. Chattopadhyay et al have thrown light into the generalised join structure of $\Gamma(\mathbb{Z}_n)$ by the subgraphs induced by the vertices which forms an equitable partition of its vertex set.

By a proper divisor of n, we mean a positive divisor d such that d/n , $1 < d < n$. Let $\xi(n)$ denote the number of proper divisors of n. Then, $\xi(n) = \sigma_0(n) - 2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1. It is convenient to denote the proper divisors of n by $d_1, d_2, ..., d_{\xi(n)}$. Consider the canonical decomposition $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, and $r, n_1, n_2, ..., n_r$ are positive integers. Then,

$$\xi(n) = \prod_{i=1}^{r}(n_i+1) \quad - 2$$

. Let $\mathcal{A}(d) = \{k \in \mathbb{Z}_n : gcd(k,n) = d\}$. Then $\{\mathcal{A}(d_1), \mathcal{A}(d_2), ..., \mathcal{A}(d_{\xi(n)})\}$ is an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$ such that $\mathcal{A}(d_i) \cap \mathcal{A}(d_j) = \phi, i \neq j$, and any two vertices in $\mathcal{A}(d_i)$ have the same number of neighbours in $\mathcal{A}(d_j)$ for all divisors d_i, d_j of n. Refer [25] for the following Lemma.

Lemma 4.3.1. [25] $|\mathcal{A}(d_i)| = \phi(\frac{n}{d_i})$, *for every* $i = 1, 2, ...\xi(n)$.

Lemma 4.3.2. [25]

$$\Gamma(\mathcal{A}(d_j)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ K_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases}.$$

For example, in $\Gamma(\mathbb{Z}_{p^3})$, $\mathcal{A}(p)$ induces $\overline{K}_{p(p-1)}$ and $\mathcal{A}(p^2)$ induces K_{p-1}. In $\Gamma(\mathbb{Z}_{p^2q})$, $\mathcal{A}(p), \mathcal{A}(q), \mathcal{A}(p^2)$ induce $\overline{K}_{(p-1)(q-1)}$, $\overline{K}_{p(p-1)}, \overline{K}_{q-1}$ respectively while $\mathcal{A}(pq)$ induces K_{p-1}; which is visible from the diagonal blocks $0, J-I$ in the adjacency matrices of respective graphs.

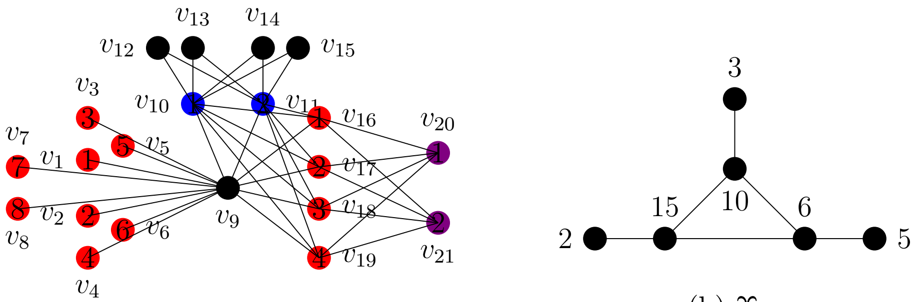
It is relevant to define the proper divisor graph of n which is closely associated with $\Gamma(\mathbb{Z}_n)$ in describing its joined union structure.

Definition 4.3.3. [49] *The proper divisor graph of n, denoted by Υ_n is a simple connected graph with vertices labeled as $d_1, d_2, ..., d_{\xi(n)}$; in which two distinct vertices d_i and d_j are adjacent if and only if $n/d_i d_j$.*

The following Lemma is very crucial in finding the spectrum of $\Gamma(\mathbb{Z}_n)$ for any n; which states that the zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is a generalised join of its subgraphs $\Gamma(\mathcal{A}(d_i))$, for $i = 1, 2, ..., \xi(n)$.

Lemma 4.3.4. [25] $\Gamma(\mathbb{Z}_n) = \Upsilon_n \left[\Gamma(\mathcal{A}(d_1)), \Gamma(\mathcal{A}(d_2)), ..., \Gamma(\mathcal{A}(d_{\xi(n)}))\right]$

For example, consider $\Gamma(\mathbb{Z}_{30})$. The proper divisors of 30 are $2, 3, 5, 6, 10, 15$. The vertices of $\Gamma(\mathbb{Z}_{30})$ are partitioned into disjoint sets as follows.
$\mathcal{A}(2) = \{2, 4, 8, 14, 16, 22, 26, 28\}$, $\mathcal{A}(3) = \{3, 9, 21, 27\}$, $\mathcal{A}(5) = \{5, 25\}$,
$\mathcal{A}(6) = \{6, 12, 18, 24\}$, $\mathcal{A}(10) = \{10, 20\}$ and $\mathcal{A}(15) = \{15\}$. The zero-divisor graph $\Gamma(\mathbb{Z}_{30})$ and its proper divisor graph are given in Figure 4.1.



(b) Υ_{30}

(a) $\Gamma(\mathbb{Z}_{30})$

Figure 4.1: The zero divisor graph on $\mathbb{Z}_{30}$ and the proper divisor graph Υ_{30}

Example 4.3.5. For example, consider $\Gamma(\mathbb{Z}_{36})$. The number of proper divisors of 36 is 7. They are precisely $2, 3, 4, 6, 9, 12, 18$. The non-zero divisors of $\Gamma(\mathbb{Z}_{36})$ is partitioned into 7 classes as follows. $\mathcal{A}(2) = \{2, 10, 14, 22, 26, 34\}$,
$\mathcal{A}(3) = \{3, 15, 21, 33\}$, $\quad \mathcal{A}(4) = \{4, 8, 16, 20, 28, 32\}$, $\quad \mathcal{A}(6) = \{6, 30\}$,
$\mathcal{A}(9) = \{9, 27\}$, $\quad \mathcal{A}(12) = \{12, 24\}$, $\quad \mathcal{A}(18) = \{18\}$.
The graphs $\Gamma(\mathbb{Z}_{36})$ and Υ_{36} are given in Figure 4.2 and Figure 4.3. Note that $\Gamma(\mathcal{A}(6))$, $\Gamma(\mathcal{A}(12))$ and $\Gamma(\mathcal{A}(18)))$ are complete subgraphs, while the others are null graphs.

4.4 Adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$, $k \geqslant 3$

In this section, the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$ is analysed. Also, we note that the proper divisors of p^k are $p, p^2, ..., p^{k-1}$ and the number of non-zero zero-divisors of $\mathbb{Z}_{p^k}$ is $p^{k-1} - 1$, by Proposition 3.2.1.

The adjacency matrix of $\Gamma(\mathbb{Z}_{p^4})$, $\Gamma(\mathbb{Z}_{p^5})$, $\Gamma(\mathbb{Z}_{p^6})$, $\Gamma(\mathbb{Z}_{p^7})$, analysis of which led to some interesting results, are given in Figure 4.4, Figure 4.5, Figure 4.6, and

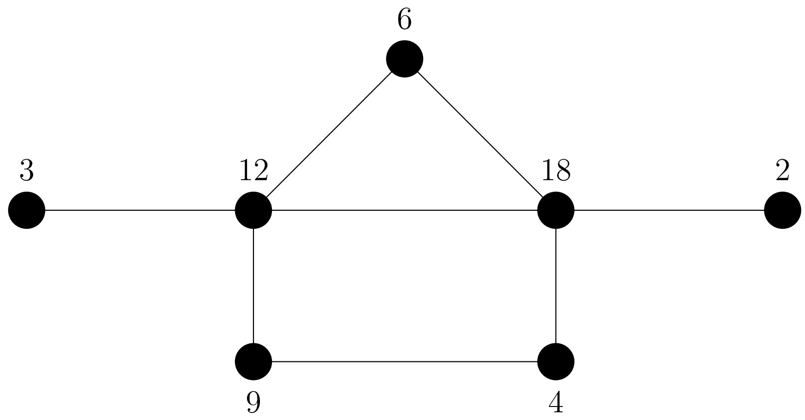


Figure 4.2: Υ_{36}

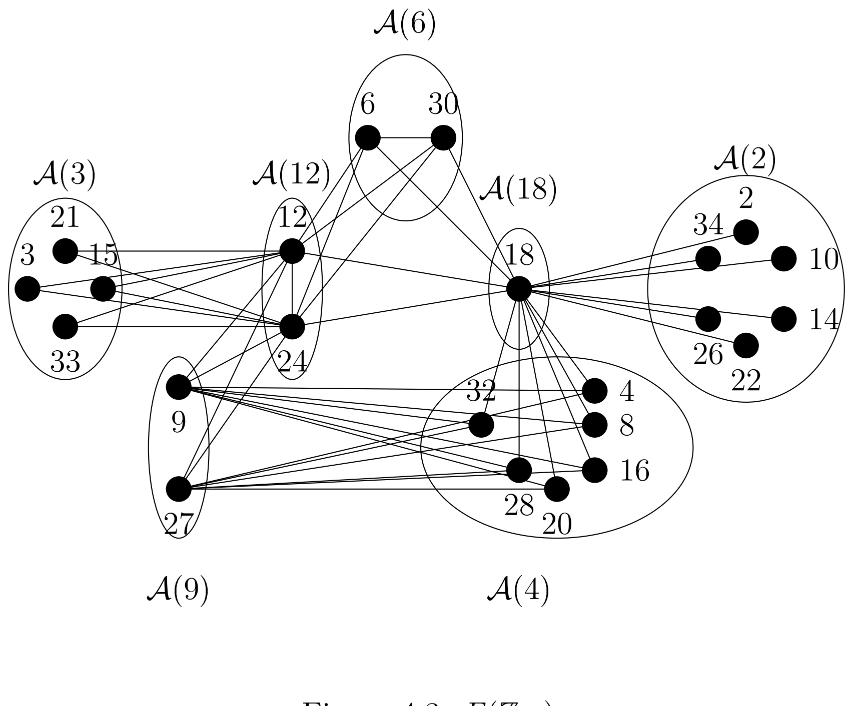


Figure 4.3: $\Gamma(\mathbb{Z}_{36})$

Figure 4.7

As illustrated in section 3.2, the adjacency matrix of $\varGamma(\mathbb{Z}_{p^k})$ contains blocks

$$\begin{array}{cc} & \begin{array}{ccc}\mathcal{A}(p) & \mathcal{A}(p^2) & \mathcal{A}(p^3)\end{array} \\ \begin{array}{c}\mathcal{A}(p)\\ \mathcal{A}(p^2)\\ \mathcal{A}(p^3)\end{array} & \left[\begin{array}{c|cc} O & O & J \\ \hline O & J-I & J \\ J & J & J-I \end{array}\right]\end{array}$$

Figure 4.4: $A(\varGamma(\mathbb{Z}_{p^4}))$

$$\begin{array}{cc} & \begin{array}{cccc}\mathcal{A}(p) & \mathcal{A}(p^2) & \mathcal{A}(p^3) & \mathcal{A}(p^4)\end{array} \\ \begin{array}{c}\mathcal{A}(p)\\ \mathcal{A}(p^2)\\ \mathcal{A}(p^3)\\ \mathcal{A}(p^4)\end{array} & \left[\begin{array}{cc|cc} O & O & O & J \\ O & O & J & J \\ \hline O & J & J-I & J \\ J & J & J & J-I \end{array}\right]\end{array}$$

Figure 4.5: $A(\varGamma(\mathbb{Z}_{p^5}))$

$$\begin{array}{cc} & \begin{array}{ccccc}\mathcal{A}(p) & \mathcal{A}(p^2) & \mathcal{A}(p^3) & \mathcal{A}(p^4) & \mathcal{A}(p^5)\end{array} \\ \begin{array}{c}\mathcal{A}(p)\\ \mathcal{A}(p^2)\\ \mathcal{A}(p^3)\\ \mathcal{A}(p^4)\\ \mathcal{A}(p^5)\end{array} & \left[\begin{array}{cc|ccc} O & O & O & O & J \\ O & O & O & J & J \\ \hline O & O & J-I & J & J \\ O & J & J & J-I & J \\ J & J & J & J & J-I \end{array}\right]\end{array}$$

Figure 4.6: $A(\varGamma(\mathbb{Z}_{p^6}))$

$$\begin{array}{cc} & \begin{array}{cccccc}\mathcal{A}(p) & \mathcal{A}(p^2) & \mathcal{A}(p^3) & \mathcal{A}(p^4) & \mathcal{A}(p^5) & \mathcal{A}(p^6)\end{array} \\ \begin{array}{c}\mathcal{A}(p)\\ \mathcal{A}(p^2)\\ \mathcal{A}(p^3)\\ \mathcal{A}(p^4)\\ \mathcal{A}(p^5)\\ \mathcal{A}(p^6)\end{array} & \left[\begin{array}{ccc|ccc} O & O & O & O & O & J \\ O & O & O & O & J & J \\ O & O & O & J & J & J \\ \hline O & O & J & J-I & J & J \\ O & J & J & J & J-I & J \\ J & J & J & J & J & J-I \end{array}\right]\end{array}$$

Figure 4.7: $A(\varGamma(\mathbb{Z}_{p^7}))$

of all zero matrices, all one matrices and identity matrices. If all vertices of $\mathcal{A}(p^i)$ are adjacent to every vertex of $\mathcal{A}(p^j)$, we write $\mathcal{A}(p^i) \sim \mathcal{A}(p^j)$. Clearly, $\mathcal{A}(p^i) \sim \mathcal{A}(p^j)$ iff $i + j \geqslant k$. Also, $\mathcal{A}(p^i) \sim \mathcal{A}(p^i)$, indicates that every vertex of $\mathcal{A}(p^i)$ is adjacent to every other vertex of $\mathcal{A}(p^i)$ and clearly the equivalent condition of adjacency of vertices among $\mathcal{A}(p^i)$ is that; $\mathcal{A}(p^i) \sim \mathcal{A}(p^i)$ iff $i \geqslant \lceil \frac{k}{2} \rceil$. Thus, the adjacency matrix of $\varGamma(\mathbb{Z}_{p^k})$ is obtained as in Figure 4.8 and Figure 4.9.

$$\begin{array}{rl}
 & \begin{array}{ccccccccc} \mathcal{A}(p) & \mathcal{A}(p^2) & \cdots & \mathcal{A}(p^{\lceil \frac{k}{2} \rceil - 1}) & \mathcal{A}(p^{\lceil \frac{k}{2} \rceil}) & \cdots & \cdots & & \mathcal{A}(p^{k-1}) \end{array} \\
\begin{array}{c} \mathcal{A}(p) \\ \mathcal{A}(p^2) \\ \vdots \\ \mathcal{A}(p^{\lceil \frac{k}{2} \rceil - 1}) \\ \mathcal{A}(p^{\lceil \frac{k}{2} \rceil}) \\ \vdots \\ \\ \vdots \\ \mathcal{A}(p^{k-1}) \end{array} &
\left[\begin{array}{cccc|ccccc}
O & \cdots & \cdots & O & O & \cdots & & O & J \\
\vdots & \ddots & & \vdots & \vdots & & O & J & J \\
\vdots & & \ddots & \vdots & \vdots & & & \vdots & \vdots \\
O & \cdots & \cdots & O & O & J & \cdots & \cdots & J \\
\hline
 & \cdots & \cdots & O & J-I & J & \cdots & \cdots & J \\
\vdots & \cdots & O & J & J & J-I & J & \cdots & J \\
\vdots & & & \vdots & \vdots & & \ddots & & \vdots \\
O & J & \cdots & J & \vdots & & & \ddots & J \\
J & J & \cdots & J & J & J & \cdots & \cdots & J-I
\end{array}\right]
\end{array}$$

Figure 4.8: $A(\Gamma(\mathbb{Z}_{p^k}))$; when k is even

4.4.1 Some graph parameters of $\Gamma(\mathbb{Z}_{p^k})$

The above analysis of the structure of $\Gamma(\mathbb{Z}_{p^k})$ leads to some results regarding the stability number, clique number and girth of $\Gamma(\mathbb{Z}_{p^k})$. As the matrix narrates the adjacency between $\mathcal{A}(p^i)$ and $\mathcal{A}(p^j)$, for $i, j = 1, 2, ..., k-1$ and the adjacency among the vertices of each $\mathcal{A}(p^i)$, $i = 1, 2, ..., k-1$, it is clear that, any principal sub matrix of zero blocks corresponds to an independent set in $\Gamma(\mathbb{Z}_{p^k})$.

Theorem 4.4.1. *Let $G = \Gamma(\mathbb{Z}_{p^k})$, $k \geqslant 3$. Then, $\alpha(G) = p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor}$*

Proof. From the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$, it is clear that, the maximum size of a principal sub matrix of zero blocks is $|\mathcal{A}(p)| + |\mathcal{A}(p^2)| + ... + |\mathcal{A}(p^{\lceil \frac{k}{2} \rceil - 1})|$.

$$
\begin{array}{c|cccc|ccccc|}
 & \mathcal{A}(p) & \mathcal{A}(p^2) & \cdots & \mathcal{A}(p^{\lceil\frac{k}{2}\rceil-1}) & \mathcal{A}(p^{\lceil\frac{k}{2}\rceil}) & \cdots & & & \mathcal{A}(p^{k-1}) \\
\hline
\mathcal{A}(p) & O & \cdots & \cdots & O & O & O & \cdots & O & J \\
\mathcal{A}(p^2) & \vdots & \ddots & & \vdots & O & \vdots & \cdots & J & J \\
\vdots & \vdots & & \ddots & \vdots & \vdots & J & \cdots & \vdots & \vdots \\
\mathcal{A}(p^{\lceil\frac{k}{2}\rceil-1}) & O & \cdots & \cdots & O & J & J & \cdots & \cdots & J \\
\hline
\mathcal{A}(p^{\lceil\frac{k}{2}\rceil}) & O & O & \cdots & J & J-I & J & \cdots & \cdots & J \\
\vdots & O & \cdots & J & J & J & J-I & J & \cdots & J \\
 & \vdots & & \vdots & \vdots & \vdots & & \ddots & & \vdots \\
 & O & J & \cdots & J & \vdots & & & \ddots & \vdots \\
S(p^{k-1}) & J & J & \cdots & J & J & J & \cdots & \cdots & J-I \\
\hline
\end{array}
$$

Figure 4.9: $A(\varGamma(\mathbb{Z}_{p^k}))$; when k is odd

Thus, by Proposition 4.3.1,

$$
\begin{aligned}
\alpha(G) &= \phi(\frac{p^k}{p}) + \phi(\frac{p^k}{p^2}) + ... + \phi(\frac{p^k}{p^{\lceil\frac{k}{2}\rceil-1}}) \\
&= \phi(p^{k-1}) + \phi(p^{k-2}) + ... + \phi(p^{k-\lceil\frac{k}{2}\rceil+1}) \\
&= p^{k-1} - p^{\lfloor\frac{k}{2}\rfloor}.
\end{aligned}
$$

$\square$

Theorem 4.4.2. *Let $G = \varGamma(\mathbb{Z}_{p^k})$, $k \geqslant 3$. Then,*

$$
\omega(G) = \begin{cases} p^{\frac{k}{2}} - 1; & \textit{if } k \textit{ is even}, \\ p^{\lfloor\frac{k}{2}\rfloor}; & k \textit{ is odd.} \end{cases}
$$

Proof. A clique of a graph G is a subset of $V(G)$ which induces a complete subgraph in G. Thus, the maximum size of a clique in $\varGamma(\mathbb{Z}_{p^k})$ is $|\mathcal{A}(p^{\lceil\frac{k}{2}\rceil})| + |\mathcal{A}(p^{\lceil\frac{k}{2}\rceil+1})| + ... + |\mathcal{A}(p^{k-1})|$; if k is even and one more than this number if k is odd.

$$\begin{aligned}|\mathcal{A}(p^{\lceil \frac{k}{2} \rceil})| + |\mathcal{A}(p^{\lceil \frac{k}{2} \rceil + 1})| + ... + |\mathcal{A}(p^{k-1})| &= \phi(\frac{p^k}{p^{\lceil \frac{k}{2} \rceil}}) + ... + \phi(\frac{p^k}{p^{k-1}}) \\ &= \phi(p) + ... + \phi(p^{\lfloor \frac{k}{2} \rfloor}) \\ &= p^{\lfloor \frac{k}{2} \rfloor} - 1.\end{aligned}$$

Thus, $\omega(G) = \begin{cases} p^{\frac{k}{2}} - 1; & \text{if k is even,} \\ p^{\lfloor \frac{k}{2} \rfloor}; & \text{if k is odd.} \end{cases}$ □

Theorem 4.4.3. *Let $G = \Gamma(\mathbb{Z}_{p^k})$, $k \geqslant 3$ for any prime p. Then, $gr(G) = 3$ except that $gr(\Gamma(\mathbb{Z}_8)) = \infty$*

Proof. Consider $k \geqslant 3$.
If k is even, from the above Theorem, we see that $\omega(G) \geqslant 3$, for any prime p.
If k is odd, $\omega(G) \geqslant 3$, for any prime $p \geqslant 3$. Thus, the length of the shortest cycle is 3 in these cases. Also for $p = 2$ and $k = 3$, we see that the zero divisor graph $\Gamma(\mathbb{Z}_8)$ is $K_{1,2}$, which contains no cycle. □

4.4.2 The eigenvalues $\lambda = 0$ and $\lambda = -1$ of $\Gamma(\mathbb{Z}_{p^k})$

Matrix Theory is a mode of conveying very important information regarding both structural and algebraic parameters of a graph. Here, from the point of view of Linear Algebra, the multiplicities of the eigenvalues 0 and -1 are calculated.

Theorem 4.4.4. *Let $G = \Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$. Then $\lambda = 0$ and $\lambda = -1$ are eigenvalues of G with multiplicities $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$ and $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ respectively.*

Proof. The adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ contains repeated rows. Hence the determinant is zero. This indicates that $\lambda = 0$ is an eigenvalue. Since the adjacency matrix of any simple graph is real and symmetric, it follows that the algebraic multiplicity of $\lambda = 0$ is the nullity of the adjacency matrix, which is exactly the number of dependent rows in the adjacency matrix of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$. If $M = A(\mathit{\Gamma}(\mathbb{Z}_{p^k}))$, then in each of the first $\lceil \frac{k}{2} \rceil - 1$ blocks of M , all but one, are dependent rows. Thus,

$$\begin{aligned} nullity(M) &= |\mathcal{A}(p)| + |\mathcal{A}(p^2)| + ... + |\mathcal{A}(p^{\lceil \frac{k}{2} \rceil - 1})| - (\lceil \frac{k}{2} \rceil - 1) \\ &= \phi(\frac{p^k}{p}) + \phi(\frac{p^k}{p^2}) + ... + \phi(\frac{p^k}{p^{\lceil \frac{k}{2} \rceil - 1}}) - (\lceil \frac{k}{2} \rceil - 1) \\ &= p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1. \end{aligned}$$

Thus, multiplicity of $\lambda = 0$ is $\quad p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$.

Also, we can see that $det(M + I) = 0$. Hence $\lambda = -1$ is an eigenvalue of M and multiplicity of $\lambda = -1$ is the nullity of $M + I$. In each of the last $\lfloor \frac{k}{2} \rfloor$ blocks of $M + I$, all but one, are dependent rows. Thus nullity of $M + I$ is given by,

$$\begin{aligned} nullity(M + I) &= |\mathcal{A}(p^{\lceil \frac{k}{2} \rceil})| + |\mathcal{A}(p^{\lceil \frac{k}{2} \rceil + 1})| + ... + |\mathcal{A}(p^{k-1})| - \lfloor \frac{k}{2} \rfloor \\ &= p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1. \end{aligned}$$

Thus multiplicity of the eigenvalue $\lambda = -1$ is $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$. □

The other eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ are computed in Section: 4.5.

4.5 Computation of eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$

It is very exciting to observe that the induced subgraphs $\Gamma(\mathcal{A}(d_i))$; being complete or null; are regular for $i = 1, 2, 3, ..., \xi(n)$. The regularity of these induced subgraphs of $\mathit{\Gamma}(\mathbb{Z}_n)$ and its joined union structure makes it easy to apply the technique described by A.J. Schwenk [75] in finding the spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$.

Definition 4.5.1. [75] *Let $V_1, V_2, ...V_m$ be an equitable partition of a graph G, with $|N(v) \cap V_j| = t_{ij}, 1 \leqslant i, j \leqslant m$, for all $v \in V_i$. Then, $T = [t_{ij}]$ is called the matrix associated with the partition.*

Theorem 4.5.2. [75] *Let G be a graph on p vertices. If $H_i, 1 \leqslant i \leqslant p$ are all r_i-regular graphs, then $V_1 \cup V_2 \cup ... \cup V_p$ is an equitable partition of $G[H_1, H_2, ...H_p]$. Let T denote the matrix associated with this partition. Then, the characteristic polynomial of the generalised composition is*

$$\Phi(G[H_1, H_2, ...H_p]; \lambda) = \Phi(T; \lambda). \prod_{i=1}^{p} \frac{\Phi(H_i; \lambda)}{(\lambda - r_i)}.$$

The above theorem leads to a very exciting way of computing the eigenvalues of $\Gamma(\mathbb{Z}_n)$.

First, we determine T, the matrix associated with the partition $\mathcal{A}(d_1) \cup \mathcal{A}(d_2) \cup ... \cup \mathcal{A}(d_{\xi(n)})$. of the graph $\Gamma(\mathbb{Z}_n)$. Let $|N(v) \cap \mathcal{A}(d_j)| = t_{ij}, 1 \leqslant i, j \leqslant \xi(n)$, for all $v \in \mathcal{A}(d_i)$. Note that $\mathcal{A}(d_i) \sim \mathcal{A}(d_j)$ if and only if $n/d_i d_j$ and $\mathcal{A}(d_i)$ induces a complete subgraph in G, if and only if n/d_i^2 and a null graph if and only if $n \nmid d_i^2$. Thus, $T = [t_{ij}]_{\xi(n) \times \xi(n)}$ is defined as follows.

$$t_{ij} = \begin{cases} \phi(\frac{n}{d_j}); & \text{if } n \ / \ d_i d_j; \quad i \neq j \\ \phi(\frac{n}{d_i}) - 1; & \text{if } n \ / \ d_i^2; \quad i = j \\ 0; & \text{otherwise} \end{cases} \tag{4.3}$$

This description completely determines T and subsequently $\Phi(T; \lambda)$ and makes it possible to explore the characteristic polynomial of $\Gamma(\mathbb{Z}_n)$ in a very convenient manner.

Theorem 4.5.3. *Let $G = \Gamma(\mathbb{Z}_n), n \neq p, p^2$, for any prime p. Then the characteristic polynomial of G is $\Phi(G; \lambda) = \Phi(T; \lambda). \prod_{n/d_i^2} (\lambda + 1)^{\phi(\frac{n}{d_i}) - 1}. \prod_{n \nmid d_i^2} \lambda^{\phi(\frac{n}{d_i}) - 1}$,* *where* $T = [t_{ij}]$,

$$t_{ij} = \begin{cases} \phi(\frac{n}{d_j}); & \text{if } n \,/\, d_i d_j; \;\; i \neq j \\ \phi(\frac{n}{d_i}) - 1; & \text{if } n \,/\, d_i^2; \;\; i = j \\ 0; & \text{otherwise} \end{cases}$$

Proof. Let $n = p_1^{n_1} p_2^{n_2} ... p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, $r, n_1, n_2, ..., n_r$ are positive integers and assume that n is neither a prime nor the square of a prime (to avoid triviality). Let $d_1, d_2, ... d_{\xi(n)}$ be the proper divisors of n. Let m_i denote the cardinality of $\mathcal{A}(d_i), i = 1, 2, ..., \xi(n)$. Thus, $m_i = \phi(\frac{n}{d_i}), i = 1, 2, ..., \xi(n)$. It is already seen that $\Gamma(\mathcal{A}(d_i))$; the subgraph induced by $\mathcal{A}(d_i)$ is either K_{m_i} or $\overline{K}_{m_i}$ which are regular of order $m_i - 1$ or 0 respectively; accordingly as n divides d_i^2 or not.

Thus, $\Phi(\Gamma(\mathcal{A}(d_i)); \lambda) = \Phi(K_{m_i}; \lambda) = (\lambda + 1)^{m_i - 1}.(\lambda - m_i + 1)$; if $n \,/\, d_i^2$ and $\Phi(\Gamma(\mathcal{A}(d_i)); \lambda) = \Phi(\overline{K}_{m_i}; \lambda) = \lambda^{m_i}$; if $n \nmid d_i^2$.

Thus, the conclusion follows from Lemma 4.3.4 and Theorem 4.5.2 .

□

Example 4.5.4. Consider $n = p^2 q$, where p and q are distinct primes $p < q$. The proper divisors of $p^2 q$ are $d_1 = p$, $d_2 = q$, $d_3 = p^2$ and $d_4 = pq$.

$\mathcal{A}(d_1) = \{k_1 p : k_1 = 1, 2, ... pq - 1; p \nmid k_1, q \nmid k_1\}$.

$\mathcal{A}(d_2) = \{k_2 q : k_2 = 1, 2, ... p^2 - 1; p \nmid k_2\}$.

$\mathcal{A}(d_3) = \{k_3 p^2 : k_3 = 1, 2, ... q - 1\}$.

$\mathcal{A}(d_4) = \{k_4 pq : k_4 = 1, 2, ... p - 1\}$.

Clearly, $|\mathcal{A}(d_1)| = (p-1)(q-1)$, $|\mathcal{A}(d_2)| = p(q-1)$, $|\mathcal{A}(d_3)| = q - 1$, and $|\mathcal{A}(d_4)| = p - 1$.

These sets form an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$, as seen in

Section 3.2.

Also, the matrix of partition,

$$T = \begin{bmatrix} 0 & 0 & 0 & p-1 \\ 0 & 0 & q-1 & 0 \\ 0 & p(p-1) & 0 & p-1 \\ (p-1)(q-1) & 0 & q-1 & p-2 \end{bmatrix}.$$

The characteristic polynomial of this matrix is given by, $\Phi(T;\lambda) = det(T - \lambda I)$. Thus,

$\Phi(T;\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2$.

Let $G_i = \Gamma(\mathcal{A}(d_i)), i = 1,2,3,4.$. Note that G_1, G_2, G_3 are null graphs of order $(p-1)(q-1)$, $p(p-1)$ and $q-1$ respectively and G_4 is a complete graph of order $p-1$ which is regular of degree $p-2$. Thus,

$\Phi(G_1;\lambda) = \lambda^{(p-1)(q-1)}$

$\Phi(G_2;\lambda) = \lambda^{p(p-1)}$

$\Phi(G_3;\lambda) = \lambda^{q-1}$

$\Phi(G_4;\lambda) = (\lambda+1)^{p-2}(\lambda - p + 2)$.

Hence the characteristic polynomial of $\mathit{\Gamma}(\mathbb{Z}_{p^2q})$ is,

$$\Phi(\mathit{\Gamma}(\mathbb{Z}_{p^2q};\lambda)) = (\lambda+1)^{p-2}.\lambda^{(p-1)(p+q-1)+(q-4)}.\Phi(T;\lambda);$$

where $\Phi(T;\lambda) = \lambda^4 - (p-2)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p(p-1)(p-2)(q-1)\lambda + p(p-1)^3(q-1)^2$.

Remark 4.5.5. In the above example, the order of the zero divisor graph $\mathit{\Gamma}(\mathbb{Z}_{p^2q})$ is $p^2 + pq - p - 1$, by Proposition 3.2.1. Theorem 4.5.3 reduces the inconvenience of handling a huge matrix of order $p^2 + pq - p - 1$ in finding the eigenvalues, by means of a 4×4 matrix of partition and thereby serves the purpose of bypassing the tedious traffic of direct computation using matrix operations.

Corollary 4.5.6. *The characteristic polynomial of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 2$ is given by*

$$\Phi(\Gamma(\mathbb{Z}_{p^k});\lambda) = \Phi(T;\lambda).\prod_{i<\lceil\frac{k}{2}\rceil}\lambda^{(p-1)p^{(k-i-1)}}.\prod_{i\geqslant\lceil\frac{k}{2}\rceil}(\lambda+1)^{(p-1)p^{(k-i-1)}},$$

where $T = [t_{ij}]_{(k-1)\times(k-1)}$,

$$t_{ij} = \begin{cases} (p-1)p^{k-j-1}; & \text{if } i+j \geqslant k;\ \ i \neq j \\ (p-1)p^{k-j-1}-1; & \text{if } i+j \geqslant k;\ \ i = j \\ 0; & \text{otherwise} \end{cases}$$

Proof. $\mathcal{A}(d_i)$ induces a complete subgraph in $\Gamma(\mathbb{Z}_n)$ if and only if n/d_i^2; and a null graph otherwise. hence if $n = p^k; k \geqslant 2$, $\mathcal{A}(p^i)$ induces a complete subgraph of order $p^{k-i-1}(p-1)$ if $i \geqslant \lceil\frac{k}{2}\rceil$ or a null graph otherwise. □

Example 4.5.7. For $G = \Gamma(\mathbb{Z}_{p^4})$, it can be seen from section 3.1 that the matrix of partition of the vertex set of G is given by,

$$T = \begin{bmatrix} 0 & 0 & p-1 \\ 0 & p(p-1)-1 & p-1 \\ p^2(p-1) & p(p-1) & p-2 \end{bmatrix}.$$

Thus $\Phi(T;\lambda)$ is obtained as,

$\Phi(T;\lambda) = \lambda^3-(p^2-3)\lambda^2-(p^4-2p^3+2p^2-2)\lambda+p^2(p-1)^2(p^2-p-1)$. Applying Corollary 4.5.6, we see that the characteristic polynomial of $\Gamma(\mathbb{Z}_{p^4})$ is,

$$\Phi(\Gamma(\mathbb{Z}_{p^4});\lambda) = (\lambda+1)^{p^2-3}.\lambda^{p^3-p^2-1}.\Phi(T;\lambda),$$

where $\Phi(T;\lambda) = \lambda^3-(p^2-3)\lambda^2-(p^4-2p^3+2p^2-2)\lambda+p^2(p-1)^2(p^2-p-1)$

Thus Theorem 4.5.3 and Corollary 4.5.6 are the generalisation of results in Section 4.2.2 and Section 4.2.3.

Chapter 5

Direct computation of distance related spectrum of $\Gamma(\mathbb{Z}_n)$ for some values of n

In this Chapter, the distance matrix, distance Laplacian matrix and distance signless Laplacian matrix and the spectra of these matrices for the graphs $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^3})$, for $p < q$, are found.

5.1 Introduction

The notion of distance and transmission of vertices in graph theory finds wide applications in different realms of the physical world including the design of communication networks, graph embedding and molecular stability. This Chapter is devoted to the determination of distance, distance Laplacian and distance signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ for some values of n. Recall Definition 2.3.3 and Definition 2.3.4 of distance, distance Laplacian and distance signless Laplacian matrix of a connected graph G.

[1]This chapter has been published in *Journal of Mathematical and Computational Science,* Volume 11, Issue 1, 2021, Pages 365-379.

Since the distance matrix of a connected graph is symmetric, its eigenvalues are real and can be ordered as $\partial_1 \geqslant \partial_2 \geqslant ... \geqslant \partial_n$. Let $\Phi_D(G;\lambda)$ denote the characteristic polynomial of the distance matrix of G and $Spec_D(G)$ denote its distance spectrum. The largest distance eigenvalue of G is referred to as the distance spectral radius. The distance energy of G is given by, $\mathcal{E}_D(G) = \Sigma_{i=1}^{n}|\partial_i|$. [48] $Spec_{D^L}(G)$ and $Spec_{D^Q}(G)$ denote the spectrum of G related to the distance Laplacian and the distance signless Laplacian matrix of G respectively. Let us denote $det(\lambda I - D^L(G))$ and $det(\lambda I - D^Q(G))$ by $\Phi_{D^L}(G;\lambda)$ and $\Phi_{D^Q}(G;\lambda)$ respectively. For a connected graph G on n vertices, $\partial_1^L \geqslant \partial_2^L \geqslant ... \geqslant \partial_n^L$ denote the distance Laplacian eigenvalues and $\partial_1^Q \geqslant \partial_2^Q \geqslant ... \geqslant \partial_n^Q$ denote the distance signless Laplacian eigenvalues.

5.2 Distance spectrum of $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^3})$ where p and q are distinct primes, $p < q$

This Section, illustrates the direct computation of the distance spectrum of $\Gamma(\mathbb{Z}_n)$, where $n = p^3, pq$, where p and q are distinct primes, $p < q$. Recall that $\Gamma(\mathbb{Z}_n)$ is a complete graph if and only if n is the square of a prime and a complete bipartite graph if and only if n is the product of two distinct primes or $n = 8$.

Theorem 5.2.1. *For any two distinct primes p and q, the distance spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by,*

$$Spec_D(\Gamma(\mathbb{Z}_{pq})) = \begin{Bmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{Bmatrix}.$$

Proof. Let $G = \Gamma(\mathbb{Z}_{pq})$. The only proper divisors of pq are p and q and the number of non-zero zero-divisors of $\mathbb{Z}_{pq}$ is $p+q-2$. Labeling these $p+q-2$ vertices of G properly, it can be seen that,

$$\mathcal{A}(p) = \{k_1 p : k_1 = 1, 2, ..., q-1\},$$

$$\mathcal{A}(q) = \{k_2 q : k-2 = 1, 2, ..., p-1\}.$$

And these two subsets of vertices of G, from an equitable partition of $V(G)$. Then, clearly $|\mathcal{A}(p)| = q-1$ and $|\mathcal{A}(q)| = p-1$. While labeling the vertices, let those from $\mathcal{A}(p)$ be arranged first and then $\mathcal{A}(q)$. Also, $\mathcal{A}(p)$ and $\mathcal{A}(q)$ induce null graphs of order $q-1$ and $p-1$ respectively. Thus, $G = \Gamma(\mathbb{Z}_{pq})$ is the K_2-join of $\overline{K}_{q-1}$ and $\overline{K}_{p-1}$.

That is,

$$\Gamma(\mathbb{Z}_{pq}) = \overline{K}_{q-1} \vee \overline{K}_{p-1}.$$

Clearly, the distance between any two vertices of $\mathcal{A}(q)$ as well as $\mathcal{A}(p)$ in G, is 2. Also, the distance between a vertex in $\mathcal{A}(q)$ and a vertex in $\mathcal{A}(p)$ is 1 in G. Thus, if J denotes an all-one matrix, the distance matrix of G is given by,

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$
$$= \left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array}\right]$$

where $A = 2(J-I)_{(q-1)\times(q-1)}$, $B = 2(J-I)_{(p-1)\times(p-1)}$.

Using Lemma 3.3.6, it can be easily seen that,

$$\Phi_D(G;\lambda) = det(\lambda I - A).det\left[(\lambda I - B) - J^T(\lambda I - A)^{-1}J\right]$$

Using Lemma: 4.2.4,

$$\Phi_D(G;\lambda) = det(\lambda I - A) \cdot det(\lambda I - B) \cdot (1 - \Gamma_A(\lambda).\Gamma_B(\lambda)) \tag{5.1}$$

Since A and B are symmetric matrices with constant row sums $2(q-2)$ and $2(p-2)$ respectively, it follows from Lemma 4.2.3 that,

$$\Gamma_A(\lambda) = \frac{q-1}{\lambda - 2(p-2)}$$

and

$$\Gamma_B(\lambda) = \frac{p-1}{\lambda - 2(q-2)}.$$

Since $A = 2(J - I)$, where J and I are all-one matrix of size $q-1$ and identity matrix of size $q-1$ respectively, again by Lemma 4.2.4,

$$det(\lambda I - A) = det(\lambda I - 2J + 2I) = (\lambda + 2)^{q-2}.\left(\lambda - 2(q-2)\right).$$

Similarly, since $B = 2(J - I)$ is of size $p - 1$,

$$det(\lambda I - B) = det(\lambda I - 2J + 2I) = (\lambda + 2)^{p-2}.\left(\lambda - 2(p-2)\right).$$

Thus, from equation (5.1), it follows that, $\Phi_D(G;\lambda) = (\lambda+2)^{q-2} \cdot (\lambda+2)^{p-2}.Q(\lambda)$, where $Q(\lambda) = \lambda^2 - 2\lambda(p+q-4) + 3pq - 7(p+q) + 15$. Thus,

$Spec_D(\Gamma(\mathbb{Z}_{pq})) =$

$$\left\{ \begin{array}{ccc} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{array} \right\}$$

$\square$

Since the distance matrix of a connected graph is irreducile, non-negative and symmetric, it follows from Perron Frobenius Theorem that, its largest eigenvalue is simple . It is obvious that the number of distinct distance eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ is 3 and thus immediately the next Corollary follows.

Corollary 5.2.2. *The distance spectral radius of $\Gamma(\mathbb{Z}_{pq})$, when p and q are distinct primes , $p < q$, is $p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1}$.*

The following Corollary gives a lower bound for the distance energy of $\Gamma(\mathbb{Z}_{pq})$.

Corollary 5.2.3. *For any two distinct prime p, q, $\mathcal{E}_D(\Gamma(\mathbb{Z}_{pq})) \geqslant 2(p+q-4)$.*

Theorem 5.2.4. *For any prime $p \neq 2$, the distance spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by,*

$$Spec_D(\Gamma(\mathbb{Z}_{p^3})) = \left\{ \begin{array}{cccc} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array} \right\}.$$

Proof. Let $G = \Gamma(\mathbb{Z}_{p^3})$. As described in Theorem 4.2.1, it can be seen that the number of non-zero zero divisors of $\mathbb{Z}_{p^3}$ is p^2-1. These p^2-1 vertices of G are partitioned (equitable partition) as follows.

$\mathcal{A}(p) = \{k_1 p : k_1 = 1, 2, ...p^2-1, \text{where } p \nmid k_1 \}$.

$\mathcal{A}(p^2) = \{k_2 p^2 : k_2 = 1, 2, ...p-1, \text{where } p \nmid k_2\}$.

Clearly, $\mathcal{A}(p)$ induces a null graph of order $p(p-1)$ and $\mathcal{A}(p^2)$ induces a complete graph on $p-1$ vertices, as described in Section:4.2. Thus, $\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \vee K_{p-1}$. Recall that the distance matrix of a complete graph is the same as its adjacency matrix. Thus, the distance matrix of G is given as follows.

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array} \right]$$
$$= \left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array} \right]$$

where $A = 2(J-I)_{p(p-1)\times p(p-1)}$, $B = (J-I)_{(p-1)\times(p-1)}$,

Proceeding as in the previous theorem, we get,

$$Spec_D(\Gamma(\mathbb{Z}_{p^3})) = \left\{ \begin{array}{cccc} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array} \right\}$$

$\square$

Corollary 5.2.5. *The distance spectral radius of $\Gamma(\mathbb{Z}_{p^3})$, is,* $\frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2}$.

Corollary 5.2.6. *For any prime p, $\mathcal{E}_D(\Gamma(\mathbb{Z}_{p^3})) \geqslant 3(p^2-p-6)$.*

5.3 Distance Laplacian spectrum for $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$, where p and q are distinct primes, $p < q$.

In this Section, the Laplacian spectra of $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$ are computed, where p and q are distinct primes, $p < q$. Each row sum of $D^L(G)$ is zero and for a connected graph G of order n, and 0 is a simple eigenvalue of $D^L(G)$ with $\mathbf{1}_n$ as the corresponding eigen vector [11].

Theorem 5.3.1. *For any two distinct primes p and q, the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by,*

$$Spec_{D^L}(\Gamma(\mathbb{Z}_{pq})) = \left\{ \begin{array}{cccc} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{array} \right\}$$

Proof. Let $G = \Gamma(\mathbb{Z}_{pq})$. Then $G = \overline{K}_{q-1} \vee \overline{K}_{p-1}$. Also, $\mathcal{A}(p) = \{k_1 p : k_1 = 1, 2, ..., q-1\}$ and $\mathcal{A}(q) = \{k_2 q : k-2 = 1, 2, ..., p-1\}$ form an equitable partition of $V(G)$. Thus, for any vertex $v \in \mathcal{A}(p)$,

$$\begin{aligned}
Tr(v) &= \sum_{u \in V(G)} d_G(u, v) \\
&= \sum_{u \in \mathcal{A}(p)} d_G(u, v) + \sum_{u \in \mathcal{A}(q)} d_G(u, v) \\
&= 2(q-2) + (p-1) = p + 2q - 5.
\end{aligned}$$

Similarly, for any vertex $w \in \mathcal{A}(q)$,

$$\begin{aligned}
Tr(w) &= \sum_{u \in V(G)} d_G(u, w) \\
&= \sum_{u \in \mathcal{A}(p)} d_G(u, w) + \sum_{u \in \mathcal{A}(q)} d_G(u, w) \\
&= (q-1) + 2(p-2) = 2p + q - 5.
\end{aligned}$$

Thus, the transmission matrix $Tr(G)$ is given by

$$Tr(G) = \left[\begin{array}{c|c} (p+2q-5)I_{(q-1)\times(q-1)} & O_{(q-1)\times(p-1)} \\ \hline O_{(p-1)\times(q-1)} & (2p+q-5)I_{(p-1)\times(p-1)} \end{array}\right]$$

And the distance matrix of G is given by,

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$

Thus, the distance Laplacian matrix of G is given by

$$\begin{aligned}
D^L(G) &= \left[\begin{array}{c|c} (p+2q-5)I - 2(J-I)_{(q-1)\times(q-1)} & -J_{(q-1)\times(p-1)} \\ \hline -J_{(p-1)\times(q-1)} & (2p+q-5)I - 2(J-I)_{(p-1)\times(p-1)} \end{array}\right] \\
&= \left[\begin{array}{c|c} A & -J \\ \hline -J^T & B \end{array}\right]
\end{aligned}$$

where $A = (p+2q-3)I - 2J_{(q-1)\times(q-1)}$, $\quad B = (2p+q-3)I - 2J_{(p-1)\times(p-1)}$,

Proceeding as in the previous Section,

$$Spec_{D^L}(\Gamma(\mathbb{Z}_{pq})) = \begin{Bmatrix} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{Bmatrix}$$

□

Theorem 5.3.2. *For any prime $p \neq 2$, the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by*

$$Spec_{D^L}(\Gamma(\mathbb{Z}_{p^3})) = \begin{Bmatrix} 0 & p^2-1 & 2p^2-p-1 \\ 1 & p-1 & p^2-p-1 \end{Bmatrix}.$$

Proof. Let $G = \Gamma(\mathbb{Z}_{p^3})$. $\mathcal{A}(p) = \{k_1 p : k_1 = 1, 2, ...p^2 - 1, \text{where } p \nmid k_1\}$, $\mathcal{A}(p^2) = \{k_2 p^2 : k_2 = 1, 2, ...p - 1, \text{where } p \nmid k_2\}$ form an equitable partition for the vertex set $V(G)$. Also, $\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \vee K_{p-1}$. The transmission degree of any vertex in $\mathcal{A}(p)$ in G, is $2p^2-p-3$ and of any vertex in $\mathcal{A}(p^2)$ is p^2-2. Thus, the transmission matrix and the distance matrix and of G are given as follows.

$$Tr(G) = \left[\begin{array}{c|c} (2p^2-p-3)I_{p(p-1)\times p(p-1)} & O_{p(p-1)\times(p-1)} \\ \hline O_{(p-1)\times p(p-1)} & (p^2-2)I_{(p-1)\times(p-1)} \end{array}\right].$$

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

Thus, it can be easily seen, that the distance Laplacian matrix of G is

$$D^L(G) = \left[\begin{array}{c|c} (2p^2-p-1)I - 2J_{p(p-1)\times p(p-1)} & -J_{p(p-1)\times(p-1)} \\ \hline -J_{(p-1)\times p(p-1)} & (p^2-1)I - J_{(p-1)\times(p-1)} \end{array}\right].$$

A similar computation shows that,

$$\Phi_{D^L}(G;\lambda) = \left[\lambda - (2p^2-p-1)\right]^{p^2-p-1}.\left[\lambda - (p^2-1)\right]^{p-2}.Q(\lambda),$$

where $Q(\lambda) = \lambda\,(\lambda - (p^2 - 1))$. Thus,

$$Spec_{D^L}(\Gamma(\mathbb{Z}_{p^3})) = \left\{ \begin{array}{ccc} 0 & p^2-1 & 2p^2-p-1 \\ 1 & p-1 & p^2-p-1 \end{array} \right\}.$$

□

5.4 Distance signless Laplacian spectrum for $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$, where p and q are distinct primes, $p < q$.

In this Section, the distance signless Laplacian spectra of $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^3})$, where p and q are distinct primes, $p < q$ are computed. Note that, if G is a connected graph, $D^Q(G)$ is a real, symmetric, nonnegative, irreducible and positive semi definite matrix. Thus, all eigenvalues of $D^Q(G)$ are real and nonnegative and also by the Perron Frobenius Theorem, the largest eigenvalue of $D^Q(G)$, called the distance signless Laplacian spectral radius of G, denoted by $\partial^Q(G)$, is positive and simple.

Theorem 5.4.1. *For any two distinct primes p and q, the distance signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ are* $\dfrac{5(p+q)-18+\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2}$, $\dfrac{5(p+q)-18-\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2}$, $p+2q-7$ *and* $2p+q-7$ *with multiplicities* $1, 1, q-2$ *and* $p-2$ *respectively.*

Proof. let $G = \Gamma(\mathbb{Z}_{pq})$. Then, as in the previous Section, counting the transmission degree of each vertex $v \in V(G)$ it can be seen that,

$$Tr(G) = \left[\begin{array}{c|c} (p+2q-5)I_{(q-1)\times(q-1)} & O_{(q-1)\times(p-1)} \\ \hline O_{(p-1)\times(q-1)} & (2p+q-5)I_{(p-1)\times(p-1)} \end{array}\right].$$

And the distance matrix of G is given by,

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

Thus, the distance signless Laplacian matrix of G is given by

$$D^Q(G) = \left[\begin{array}{c|c} (p+2q-5)I + 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & (2p+q-5)I + 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$
$$= \left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array}\right]$$

where $A = (p+2q-7)I + 2J_{(q-1)\times(q-1)}$, $\quad B = (2p+q-7)I + 2J_{(p-1)\times(p-1)}$.

Thus, computing the characteristic polynomial of $D^Q(G)$,

$$\Phi_{D^Q}(G;\lambda) = (\lambda - (p+2q-7))^{q-2} . (\lambda - (2p+q-7))^{p-2} .Q(\lambda),$$

where,

$$Q(\lambda) = \lambda^2 - (5p+5q-18)\lambda + 4\left[(p-1)(p-2) + (q-1)(q-2) + 4(p-2)(q-2)\right],$$

which is a quadratic polynomial with zeroes $\dfrac{5(p+q) - 18 + \sqrt{9(p-q)^2 + 4(p-1)(q-1)}}{2}$ and $\dfrac{5(p+q) - 18 - \sqrt{9(p-q)^2 + 4(p-1)(q-1)}}{2}$. Thus, the distance signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ are $\dfrac{5(p+q) - 18 + \sqrt{9(p-q)^2 + 4(p-1)(q-1)}}{2}$, $\dfrac{5(p+q) - 18 - \sqrt{9(p-q)^2 + 4(p-1)(q-1)}}{2}$, $p+2q-7$ and $2p+q-7$ with multiplicities $1, 1, q-2$ and $p-2$ respectively. $\square$

Corollary 5.4.2. *Let $G = \Gamma(\mathbb{Z}_{pq})$. Then, the distance signless Laplacian spectral radius,* $\partial^Q(G) = \dfrac{5(p+q) - 18 + \sqrt{9(p-q)^2 + 4(p-1)(q-1)}}{2}$.

Theorem 5.4.3. *For any prime $p \neq 2$, the distance signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by*
$Spec_{D^Q}(\Gamma(\mathbb{Z}_{p^3})) =$

$$\left\{\begin{array}{cccc} 2p^2-p-5 & p^2-3 & \frac{5p^2-2p-9+\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} & \frac{5p^2-2p-9-\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array}\right\}$$

Proof. Let $G = \Gamma(\mathbb{Z}_{p^3})$. $\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \vee K_{p-1}$. The transmission degree of any vertex in $S(p)$ in G, is $2p^2 - p - 3$ and of any vertex in $S(p^2)$ is $p^2 - 2$. The transmission matrix and the distance matrix and of G are described in the previous Section. Thus, it can be easily seen that the distance signless Laplacian matrix of G is

$$D^Q(G) = \left[\begin{array}{c|c} (2p^2-p-5)I + 2J_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (p^2-3)I + J_{(p-1)\times(p-1)} \end{array}\right]$$

A similar computation shows that,

$$\Phi_{D^Q}(G;\lambda) = \left[\lambda - (2p^2-p-5)\right]^{p^2-p-1} . \left[\lambda - (p^2-3)\right]^{p-2} .Q(\lambda),$$

where $Q(\lambda) = \lambda^2 - (5p^2 - 2p - 9)\lambda + (4p^4 - 22p^2 + 6p + 20)$. Thus,
$Spec_{D^Q}(\Gamma(\mathbb{Z}_{p^3})) =$

$$\left\{\begin{array}{cccc} 2p^2-p-5 & p^2-3 & \frac{5p^2-2p-9+\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} & \frac{5p^2-2p-9-\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array}\right\}$$

□

Corollary 5.4.4. *Let $G = \Gamma(\mathbb{Z}_{p^3})$. Then, the distance signless Laplacian spectral radius,* $\partial^Q(G) = \dfrac{5p^2 - 2p - 9 + \sqrt{9p^4 - 20p^3 + 2p^2 + 12p + 1}}{2}$.

Chapter 6

Computation of distance, distance Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for any n

The role of Fiedler's Lemma and its generalization, to the computation of the distance spectrum of the generalized join of regular graphs, is described in the Section 2. The investigation of the distance spectrum of $\Gamma(\mathbb{Z}_n)$ for any n and in particular for $n = p^k$ for any prime p and $k \geqslant 3$, using Fiedler's Lemma, is described in the Section 3. Also, it is shown that, -1 and -2 are the distance eigenvalues of $\Gamma(\mathbb{Z}_n)$, for n and their multiplicities are counted. In the Section 4, the computation of the distance Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$, for any n is described and the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$ is completely determined.

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6.1 Introduction

The method proposed by A.J. Schwenk which is described in Section: 4.5, is confined to the adjacency spectrum. The distance matrix of $\Gamma(\mathbb{Z}_n)$, contains blocks of entries other than 0 and 1. Also, the effort for the computation of distance related spectrum described in the previous Chapter is seamless, especially when n has more prime power factors. So some tools of Linear Algebra are used to explore the spectrum of such complicated structures. The combinatorial structure of $\Gamma(\mathbb{Z}_n)$ as the Υ_n-join of the induced subgraphs, where Υ_n is the proper divisor graph of n, is well utilized while applying Fiedler's Lemma and its generalisation towards this end.

6.2 Fiedler's Lemma and its its generalisation

Lemma 6.2.1. [35] *Let A and B be symmetric matrices of orders m and n, respectively, with corresponding eigen pairs $(\alpha_i, \boldsymbol{u}_i)$, $i = 1, 2, ..., m$ and $(\beta_i, \boldsymbol{v}_i)$, $i = 1, 2, ..., n$, respectively. Suppose that $\|\boldsymbol{u}_1\| = 1 = \|\boldsymbol{v}_1\|$. Then, for each arbitrary constant ρ, the matrix*
$C = \begin{bmatrix} A & \rho \boldsymbol{u}_1 \boldsymbol{v}_1^T \\ \rho \boldsymbol{v}_1 \boldsymbol{u}_1^T & B \end{bmatrix}$ *has eigenvalues $\alpha_2, ..., \alpha_m, \beta_2, ..., \beta_n, \gamma_1, \gamma_2$, where γ_1, γ_2, are the eigenvalues of the matrix* $\widehat{C} = \begin{bmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{bmatrix}$.

The above Lemma, popularly known as Fiedler's Lemma, has been extended by D.M. Cardoso et.al in [20, 21], to larger block diagonal symmetric matrices and it was applied to the exploration of spectra of the generalised join of regular graphs.

6.2.1 Generalization of Fiedler's Lemma

For $j \in \{1, 2, ..., k\}$, let M_j be a $m_j \times m_j$ symmetric matrix, with corresponding eigenpairs $(\alpha_{rj}, \mathbf{u}_{rj})$, $1 \leqslant r \leqslant m_j$. Moreover, for $q \in \{1, 2, ..., k-1\}$ and $l \in \{q+1, ..., k\}$, let $\rho_{q,l}$ be arbitrary constants. Let $\widehat{\alpha}$ be the $k-$ tuple

$$\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k}) \tag{6.1}$$

where each $\alpha_{i_j,j}$ is chosen from the elements of $\{\alpha_{1,j}, ..., \alpha_{m_j,j}\}$ with $j \in \{1, 2, ..., k\}$. Then, considering an arbitrary $\frac{k(k-1)}{2}$- tuple of reals

$$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k}), \tag{6.2}$$

consider the symmetric matrices

$C_{\widehat{\alpha}}(\widehat{\rho}) =$

$$\begin{bmatrix} M_1 & \rho_{1,2}\mathbf{u}_{i_1,1}\mathbf{u}_{i_2,2}^T & \rho_{1,3}\mathbf{u}_{i_1,1}\mathbf{u}_{i_3,3}^T & \cdots & \rho_{1,k}\mathbf{u}_{i_1,1}\mathbf{u}_{i_k,k}^T \\ \rho_{1,2}\mathbf{u}_{i_2,2}\mathbf{u}_{i_1,1}^T & M_2 & \rho_{2,3}\mathbf{u}_{i_2,2}\mathbf{u}_{i_3,3}^T & \cdots & \rho_{2,k}\mathbf{u}_{i_2,2}\mathbf{u}_{i_k,k}^T \\ \rho_{1,3}\mathbf{u}_{i_3,3}\mathbf{u}_{i_1,1}^T & \rho_{2,3}\mathbf{u}_{i_3,3}\mathbf{u}_{i_2,2}^T & M_3 & \cdots & \rho_{3,k}\mathbf{u}_{i_3,3}\mathbf{u}_{i_k,k}^T \\ \vdots & \vdots & & \ddots & \vdots \\ \rho_{1,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_1,1}^T & \rho_{2,k-1}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_2,2}^T & \cdots & M_{k-1} & \rho_{k-1,k}\mathbf{u}_{i_{k-1},k-1}\mathbf{u}_{i_k,k}^T \\ \rho_{1,k}\mathbf{u}_{i_k,k}\mathbf{u}_{i_1,1}^T & \rho_{2,k}\mathbf{u}_{i_k,k}\mathbf{u}_{i_2,2}^T & \cdots & \cdots & M_k \end{bmatrix} \tag{6.3}$$

$$\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho}) = \begin{bmatrix} \alpha_{i_1,1} & \rho_{1,2} & \cdots & \rho_{1,k-1} & \rho_{1,k} \\ \rho_{1,2} & \alpha_{i_2,2} & \cdots & \rho_{2,k-1} & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{1,k-1} & \rho_{2,k-1} & \cdots & \alpha_{i_{k-1},k-1} & \rho_{k-1,k} \\ \rho_{1,k} & \rho_{2,k} & \cdots & \rho_{k-1,k} & \alpha_{i_k,k} \end{bmatrix}_{k\times k}. \tag{6.4}$$

Theorem 6.2.2. [12] *For $j \in \{1, 2, ..., k\}$, let M_j be an $m_j \times m_j$ symmetric matrix, with eigen pairs $(\alpha_{rj}, \boldsymbol{u}_{rj}), \forall r \in I_j = \{1, 2, ..., m_j\}$ and suppose that for each j, the system of eigenvectors $\{\boldsymbol{u}_{rj}, r \in I_j\}$ is orthonormal. Consider a $\frac{k(k-1)}{2}$-tuple of scalars,*
$\widehat{\rho} = (\rho_{1,2}, \rho_{1,3}, ..., \rho_{1,k}, \rho_{2,3}, ..., \rho_{2,k}, ..., \rho_{k-1,k})$ and the k- tuple $\widehat{\alpha} = (\alpha_{i_1,1}, ..., \alpha_{i_k,k})$ as defined in equation (6.1) and equation (6.2). Then, the matrix $C_{\widehat{\alpha}}(\widehat{\rho})$ in equation (6.3) has the multi set of eigenvalues
$\left(\bigcup_{j=1}^{k}\{\alpha_{1,j}, ..., \alpha_{m_j,j}\}\backslash\{\alpha_{i_j,j}\}\right) \cup \{\gamma_1, ..., \gamma_k\}$, where $\gamma_1, \gamma_2, ..., \gamma_k$ are eigenvalues of the matrix $\widetilde{C}_{\widehat{\alpha}}(\widehat{\rho})$ in equation (6.4).

6.3 Computation of the distance spectrum of $\Gamma(\mathbb{Z}_n)$ for any n

Consider $G[H_1, H_2, ..., H_k]$ where G is a connected graph with vertices labeled as $1, 2, ..., k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $A(H_j)$ denote the adjacency matrix of H_j.
Take $M_j = 2(J - I)_{n_j} - A(H_j)$. Then, clearly $\alpha_{i_j,j} = 2(n_j - 1) - r_j$ is the Perron eigenvalue for M_j for every $j = 1, 2, ..., k$ with corresponding Perron eigenvector, $\mathbf{1}_{n_j}$. (Note that since H_j is r_j- regular, r_j is the Perron eigenvalue of H_j with $\mathbf{1}_{n_j}$

as the corresponding eigenvector, for $j = 1, 2, ..., k$). Thus, since G is connected and H_j is regular, M_j, $j = 1, 2, ..., k$ correspond to the diagonal blocks in the distance matrix of $G\left[H_1, H_2, ..., H_k\right]$.

As in equation (6.3), taking

$$M_j = 2(J-I)_{n_j} - A(H_j), \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = \left(2(n_j-1) - r_j \ , \ \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j}\right)$$

and the real numbers $\quad \rho_{l,q} = d_{l,q} \cdot \sqrt{n_l n_q}, \quad$ for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, where $d_{l,q} = d_{q,l} = d_G(l, q)$, is the distance between the vertices l and q in the connected graph G, it can be seen that the distance matrix of $G\left[H_1, H_2, ..., H_k\right]$ is obtained as in the following Theorem,

Theorem 6.3.1. [22] *Consider $G\left[H_1, H_2, ..., H_k\right]$, where G is a connected graph with vertices labeled as $1, 2, ..., k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$, and let d_{lq} denote the distance between the distinct vertices l snd q in G for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$. Let $A(H_j)$ denote the adjacency matrix of H_j and $M_j = 2(J-I)_{n_j} - A(H_j)$. Then, the distance matrix of the generalized G-join of the graphs $H_1, H_2, ..., H_k$ is given by,*
$D(G\left[H_1, H_2, ..., H_k\right])=$

$$\begin{bmatrix} M_1 & d_{1,2}J_{n_1\times n_2} & d_{1,3}J_{n_1\times n_3} & & \cdots & d_{1,k}J_{n_1\times n_k} \\ d_{1,2}J^T_{n_1\times n_2} & M_2 & d_{2,3}J_{n_2\times n_3} & & \cdots & d_{2,k}J_{n_2\times n_k} \\ d_{1,3}J^T_{n_1\times n_3} & d_{2,3}J^T_{n_2\times n_3} & M_3 & & \cdots & d_{3,k}J_{n_3\times n_k} \\ \vdots & \vdots & & \ddots & & \vdots \\ d_{1,k-1}J^T_{n_1\times n_{k-1}} & d_{2,k-1}J^T_{n_2\times n_{k-1}} & \cdots & & M_{k-1} & d_{k-1,k}J_{n_{k-1}\times n_k} \\ d_{1,k}J^T_{n_1\times n_k} & d_{2,k}J^T_{n_2\times n_k} & \cdots & & \cdots & M_k \end{bmatrix}. \tag{6.5}$$

Also, applying Theorem 6.2.2, the distance spectrum of $D(G\left[H_1, H_2, ..., H_k\right])$

is given by the following theorem,

Theorem 6.3.2. [22] *Consider $G[H_1, H_2, ..., H_k]$, where G is a connected graph with vertices labeled as $1, 2, ..., k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $A(H_j)$ denote the adjacency matrix of H_j and $M_j = 2(J-I)_{n_j} - A(H_j)$. Then, the distance spectrum of $G[H_1, H_2, ..., H_k]$ is given by,*

$$Spec_D(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} Spec(M_j) \backslash \{\ 2(n_j - 1) - r_j\} \right) \cup Spec(\widetilde{C}). \tag{6.6}$$

where

$$\widetilde{C} = \begin{bmatrix} 2(n_1-1)-r_1 & d_{1,2}\sqrt{n_1 n_2} & \dots & d_{1,k}\sqrt{n_1 n_k} \\ d_{1,2}\sqrt{n_1 n_2} & 2(n_2-1)-r_2 & \dots & d_{2,k}\sqrt{n_2 n_k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k}\sqrt{n_1 n_k} & d_{2,k}\sqrt{n_2 n_k} & \dots & 2(n_k-1)-r_k \end{bmatrix}. \tag{6.7}$$

When M_js belong to a class of graphs with known spectrum, the only remaining task is to find the spectrum of the matrix $\widetilde{C}$ as evident from equation (6.6). But the non-diagonal entries of $\widetilde{C}$ in equation (6.7) are not so appealing to the computation of spectrum. Hence, a matrix similar to it is defined in the following way.

Consider the graph G as a vertex weighted graph by assigning the weight $n_j = |V(H_j)|$ to the vertex j of G for $j = 1, 2, ..., k$ and consider the diagonal matrix of vertex weights,

$$W = \begin{bmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & n_k \end{bmatrix}.$$

A combinatorial Laplacian matrix with vertex weights was defined by F.R.K Chung et al in [26] as follows, to generalize the Matrix-Tree Theorem for counting the number of rooted directed spanning trees. Let the vertex v has a weight α_v. Consider the matrix with rows and columns labeled by vertices,

$$\mathbf{L}(u,v) = \begin{cases} \Sigma_{z\sim u}\alpha_z & \text{if } u = v \\ -\alpha_v & \text{if } u \sim v \\ 0 & \text{otherwise.} \end{cases}$$

Analogous to this, a combinatorial vertex weighted distance matrix of G, denoted by $T_D(G)$, can be defined as,

$$T_D(G) = \begin{bmatrix} 2(n_1-1)-r_1 & d_{1,2}n_2 & \dots & d_{1,k}n_k \\ d_{1,2}n_1 & 2(n_2-1)-r_2 & \dots & d_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k}n_1 & d_{2,k}n_2 & \dots & 2(n_k-1)-r_k \end{bmatrix}$$

Note that the (i,j)-th entry of $T_D(G)$ is $d_{i,j}$ times the vertex weight n_j and the (i,i)th entry is the sum of all other entries of the i-th row.

Remark 6.3.3. *It is easy to show that, $T_D(G) = W^{-\frac{1}{2}}\widetilde{C}W^{\frac{1}{2}}$. Hence, it follows that $\widetilde{C}$ and $T_D(G)$ are similar. Thus $\sigma(\widetilde{C}) = \sigma(T_D(G))$. Thus, the distance spectrum of $G[H_1, H_2, ..., H_k]$ is completely determined by the matrices M_j for $j = 1, 2, ..., k$ and the combinatorial distance matrix $T_D(G)$ associated to G.*

Thus, the next Corollary follows.

Corollary 6.3.4. *For the graphs G, $T_D(G)$ and $H_j s$ as defined above,*

$$Spec_D(G\,[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} Spec(M_j)\backslash\{\ 2(n_j-1)-r_j\}\right) \cup Spec(T_D(G)). \tag{6.8}$$

6.3.1 Distance matrix of $\mathit{\Gamma}(\mathbb{Z}_n)$

As seen in Section 4.3, the zero divisor graph on the ring of integers modulo n can be constructed as

$$\mathit{\Gamma}(\mathbb{Z}_n) = \Upsilon_n[\Gamma(\mathcal{A}(d_1)), \mathcal{A}\Gamma((d_2)), ..., \Gamma(\mathcal{A}(d_{\xi(n)}))] \tag{6.9}$$

where $d_1, d_2, ..., d_{\xi(n)}$ are the proper divisors of n. To facilitate the study of spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$, the proper divisor graph of n is to be understood and analysed in a better way. Throughout this Chapter, $d_{i,j}$ denotes the distance between the vertices d_i and d_j in Υ_n. That is, $d_{i,j} = d_{\Upsilon_n}(d_i, d_j)$. The following Lemmas are used in the main Theorem of this Section.

Lemma 6.3.5. *For any two distinct vertices d_i and d_j in the proper divisor graph Υ_n ,*

$$d_{i,j} = \begin{cases} 1 & \textit{if } n/d_i d_j \\ 2 & \textit{if } n \nmid d_i d_j, \qquad gcd(d_i, d_j) \neq 1. \\ 3 & \textit{otherwise.} \end{cases}$$

Proof. $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, and $r, n_1, n_2, ..., n_r$ are positive integers. Let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors of n.

Case(i) $n/d_i d_j$ is trivial.

Case(ii) Let $n \nmid d_i d_j$, and let $gcd(d_i, d_j) = g > 1$. Clearly, $n/(\frac{n}{g})d_i$ and $n/(\frac{n}{g})d_j$. Thus, $\frac{n}{g} \sim d_i$ and $\frac{n}{g} \sim d_j$. Hence, $\frac{n}{g}$ is a common neighbour of d_i and d_j in Υ_n.

Conversely, let $n \nmid d_i d_j$ and d_i and d_j have a common neighbour in Υ_n. Let d_k be the common neighbour of d_i and d_j. Then, $n/d_i d_k$ and $n/d_j d_k$. Thus it is obvious that $(\frac{n}{d_k})/d_i$ and $(\frac{n}{d_k})/d_j$. Hence, $\frac{n}{d_k}$ is a common divisor of d_i and d_j. Thus, the $gcd(d_i, d_j) \geqslant \frac{n}{d_k} > 1$.

Thus, $n \nmid d_i d_j$, d_i and d_j have a common neighbour in Υ_n, iff $gcd(d_i, d_j) > 1$.

case(iii) Let $n \nmid d_i d_j$, and $gcd(d_i, d_j) = 1$. Then, as proved in the above case, d_i and d_j do not have a common neighbour. Now

$$d_i \sim \frac{n}{d_i}, \quad d_j \sim \frac{n}{d_j}$$

Since n divides $\frac{n}{d_i} . \frac{n}{d_j}$, it follows that $\frac{n}{d_i} \sim \frac{n}{d_j}$. Thus, $d_i \sim \frac{n}{d_i} \sim \frac{n}{d_j} \sim d_j$ is a shortest path between d_i and d_j. Thus $d_{\Upsilon_n}(d_i, d_j) = 3$, in this case. Thus, the result immediately follows since for any finite commutative ring R, $diam(\Gamma(R)) \leqslant 3$ [81]. $\square$

Lemma 6.3.6. *Let $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_r^{n_r}$, where $p_1, p_2, ..., p_r$ are distinct primes, and $r, n_1, n_2, ..., n_r$ are positive integers. Then, the number of proper divisors d of n such that $n \nmid d^2$ is* $\prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) - 2$

Proof. The number of proper divisor of n is given by

$$\xi(n) = \prod_{i=1}^{r} (n_i + 1) \quad - 2.$$

The number of proper divisors d of n such that $n \nmid d^2$ is exactly the number of proper divisors of $p_1^{\lceil \frac{n_1}{2} \rceil} \cdot p_2^{\lceil \frac{n_2}{2} \rceil} \cdots p_r^{\lceil \frac{n_r}{2} \rceil}$ which is $\prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) - 2$ $\square$

The following Lemma is the immediate consequence of Lemma 6.3.6

Lemma 6.3.7. *The number of proper divisors d of n such that n divides d^2, is* $\prod_{i=1}^{r}(n_i+1) - \prod_{i=1}^{r}\left(\lceil\frac{n_i}{2}\rceil+1\right)$

The next Theorem describes the distance matrix of $\Gamma(\mathbb{Z}_n)$ for any n.

Theorem 6.3.8. *Let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors of n. Then, the distance matrix of $\Gamma(\mathbb{Z}_n)$ is given by*

$$D(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} M_1 & d_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \cdots & d_{1,\xi(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} \\ d_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & M_2 & \cdots & d_{2,\xi(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\xi(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} & d_{2,\xi(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} & \cdots & M_{\xi(n)} \end{bmatrix}$$

where,

$$M_j = \begin{cases} 2(J-I)_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2 \end{cases}$$

and for $l \in \{1, 2, ..., \xi(n)-1\}$ and $q \in \{l+1, ..., \xi(n)\}$, $l \neq q$,

$$d_{l,q} = \begin{cases} 1 & \text{if } n/d_l d_q \\ 2 & \text{if } n \nmid d_l d_q, \quad gcd(d_l, d_q) \neq 1 \\ 3 & \text{otherwise.} \end{cases}$$

Proof. Recall from equation (6.9) that, $\Gamma(\mathbb{Z}_n)$ is the Υ_n - join of $\Gamma(\mathcal{A}(d_1)), \Gamma(\mathcal{A}(d_2)), ...,$ $\Gamma(\mathcal{A}(d_{\xi(n)}))$. Also, the adjacency matrix of $\Gamma(\mathcal{A}(d_j))$ is given by,

$$A(\Gamma(\mathcal{A}(d_j))) = \begin{cases} O_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2. \end{cases}$$

Thus, taking $G = \Upsilon_n$, and $H_j = \Gamma(\mathcal{A}(d_j))$ and $M_j = 2(J-I)_{\phi(\frac{n}{d_j})} - A(\Gamma(\mathcal{A}(d_j)))$, the conclusion is an immediate consequence of Theorem 6.3.1 and Lemma 6.3.5.

□

6.3.2 Distance eigenvalues of $\Gamma(\mathbb{Z}_n)$

Consider the proper divisors $d_1, ..., d_{\xi(n)}$, and the matrices,

$$M_j = \begin{cases} 2(J-I)_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2. \end{cases}$$

The distance spectrum of $\Gamma(\mathbb{Z}_n)$ is completely determined by the matrices M_j, for $j = 1, 2, ..., \xi(n)$ and the matrix $T_D(\Upsilon_n)$, as in equation (6.8). The spectrum of M_j as described above are, if $n \nmid d_j^2$,

$$Spec(M_j) = \left\{ \begin{array}{cc} -2 & 2(\phi(\frac{n}{d_j}) - 1) \\ \phi(\frac{n}{d_j}) - 1 & 1 \end{array} \right\} \tag{6.10}$$

and if n/d_j^2,

$$Spec(M_j) = \left\{ \begin{array}{cc} -1 & \phi(\frac{n}{d_j}) - 1 \\ \phi(\frac{n}{d_j}) - 1 & 1 \end{array} \right\}. \tag{6.11}$$

Also, note that the subgraphs $\Gamma(\mathcal{A}(d_j))$, for $j = 1, 2, ...\xi(n)$ are r_j- regular, where

$$r_j = \begin{cases} \phi(\frac{n}{d_j}) - 1 & \text{if } n/d_j^2 \\ 0 & \text{if } n \nmid d_j^2. \end{cases}$$

While taking the union of all eigenvalues of M_j as described above in (6.10) and (6.11), the multiplicity of -2 as the distance eigenvalues of $\Gamma(\mathbb{Z}_n)$ is $\sum_{d/n, n \nmid d^2}(\phi(\frac{n}{d}) - 1)$, where the Σ runs over all divisors d of n such that d/n and

$n \nmid d^2$. By Lemma 6.3.6, this count amounts to $\sum_{d/n, n \nmid d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) + 2$. Similarly, while taking the union of all eigen values of M_j, the multiplicity of -1 as the distance eigenvalues of $\Gamma(\mathbb{Z}_n)$ is $\sum_{d/n, n/d^2} (\phi(\frac{n}{d}) - 1)$, which counts to $\sum_{d/n, n/d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} (n_i + 1) + \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right)$, by Lemma 6.3.7. Thus, applying Theorem 6.3.4, the next Theorem follows.

Theorem 6.3.9. *For any n, the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ has distance eigenvalues -2 and -1 with multiplicities $\sum_{d/n, n \nmid d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right) + 2$ and $\sum_{d/n, n/d^2} \phi(\frac{n}{d}) - \prod_{i=1}^{r} (n_i + 1) + \prod_{i=1}^{r} \left(\lceil \frac{n_i}{2} \rceil + 1\right)$ respectively and the remaining distance eigenvalues are the eigenvalues of the vertex weighted distance matrix of Υ_n, as follows*

$$T_D(\Upsilon_n) = \begin{bmatrix} t_1 & d_{1,2}\phi(\frac{n}{d_2}) & \dots & d_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ d_{1,2}\phi(\frac{n}{d_1}) & t_2 & \dots & d_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,\xi(n)}\phi(\frac{n}{d_1}) & d_{2,\xi(n)}\phi(\frac{n}{d_2}) & \dots & t_{\xi(n)} \end{bmatrix}$$

where,

$$t_j = \begin{cases} 2(\phi(\frac{n}{d_j}) - 1) & \text{if } n \nmid d_j^2 \\ \phi(\frac{n}{d_j}) - 1 & \text{if } n/d_j^2, \end{cases}$$

and for $i \in \{1, 2, ..., \xi(n) - 1\}, j \in \{i + 1, ..., \xi(n)\}$, $i \neq j$,

$$d_{i,j} = \begin{cases} 1 & \text{if } n/d_i d_j \\ 2 & \text{if } n \nmid d_i d_j, \quad gcd(d_i, d_j) \neq 1. \\ 3 & \text{otherwise.} \end{cases}$$

Hence, the distance eigenvalues of $\Gamma(\mathbb{Z}_n)$ are completely determined by the vertex weighted distance matrix $T_D(\Upsilon_n)$. Thus, the next Corollary follows.

Corollary 6.3.10. *$\mathit{\Gamma}(\mathbb{Z}_n)$ is distance integral if and only if $T_D(\Upsilon_n)$ is integral.*

Example 6.3.11. Consider $\mathit{\Gamma}(\mathbb{Z}_{pq})$, where $p < q$ are distinct primes. Counting the number of non-zero zero divisors of $\mathbb{Z}_{pq}$, it can be easily seen that the zero-divisor graph $\mathit{\Gamma}(\mathbb{Z}_{pq})$, has $p+q-2$ vertices. The proper divisors of pq are p and q and the proper divisor graph $\Upsilon_{pq} \equiv K_2$, with vertices labeled as p and q. Clearly $\mathit{\Gamma}(\mathbb{Z}_{pq}) = K_2[\Gamma(\mathcal{A}(p)), \Gamma(\mathcal{A}(q))]$, where $\Gamma(\mathcal{A}(p)) = \overline{K}_{q-1}$ and $\Gamma(\mathcal{A}(q)) = \overline{K}_{p-1}$. Using Theorem 6.3.8, the distance matrix of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ is given by

$$D(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$

Note that $n = pq$ has no proper divisor d such that n/d^2. Also it is obvious that a square matrix M has eigenvalue -1 if and only if the matrix $M + I$ has nullity at least one. Thus -1 is not an eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ fro any primes $p < q$. Thus using Theorem 6.3.9, we see that -2 is an eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ with multiplicity $p + q - 4$ and the other distance eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ are determined by its vertex weighted distance matrix,

$$T_D(\Upsilon_{pq}) = \begin{bmatrix} 2(q-2) & p-1 \\ q-1 & 2(p-2) \end{bmatrix}$$

Thus, the remaining two distance eigenvalues of this graph are determined by the polynomial, $Q(\lambda) = \lambda^2 - 2\lambda(p + q - 4) + 3pq - 7(p + q) + 15$. Thus,

$$SpecD(\mathit{\Gamma}(\mathbb{Z}_{pq})) =$$
$$\left\{ \begin{array}{ccc} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{array} \right\}$$

Remark 6.3.12. *Note that for $n = 8, pq$, where $p < q$ are distinct primes, -1 is not a distance-eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_n)$. Also, for $n = p^2$, -2 is not a distance* *eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_n)$. For all other values of n, both -1 and -2 are distance eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_n)$.*

6.3.3 Distance spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$, $k \geqslant 3$

The proper divisors of p^k are $p, p^2, ..., p^{k-1}$ and $\{\mathcal{A}(p), \mathcal{A}(p^2), ..., \mathcal{A}(p^{k-1})\}$ forms an equitable partition for $V(\mathit{\Gamma}(\mathbb{Z}_{p^k}))$. The order of the graph $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ is $p^{k-1} - 1$.

$$\Gamma(\mathcal{A}(p^j)) = \begin{cases} \overline{K}_{\phi(p^{k-j})} & \text{if } j < \lceil \frac{k}{2} \rceil \\ K_{\phi(p^{k-j})} & \text{if } j \geqslant \lceil \frac{k}{2} \rceil \end{cases}$$

where $\overline{K}_{\phi(p^{k-j})}$ is a null graph which is 0- regular and $K_{\phi(p^{k-j})}$ is a complete graph which is $\phi(p^{k-j}) - 1$ - regular. In the proper divisor graph Υ_{p^k}, $p^i \sim p^j$ for distinct i and j, if and only if $i + j \geqslant k$. Also, the vertex p^{k-1} is adjacent to every other vertices of Υ_{p^k}. Thus, Υ_{p^k} is a connected graph with diameter 2 [49]. Also, for $i \in \{1, 2, ..., k-1\}, j \in \{i+1, ..., k\}$, $i \neq j$,

$$d_{i,j} = \begin{cases} 1 & \text{if } i + j \geqslant k \\ 2 & \text{otherwise.} \end{cases}$$

The zero-divisor graph,

$$\mathit{\Gamma}(\mathbb{Z}_{p^k}) = \Upsilon_{p^k}[\Gamma(\mathcal{A}(p)), \Gamma(\mathcal{A}(p^2)), ..., \Gamma(\mathcal{A}(p^{k-1}))].$$

For $j = 1, 2, ...k - 1$, as in Theorem 6.3.1, let

$$M_j = \begin{cases} 2(J - I)_{\phi(p^{k-j})} & \text{if } j < \lceil \frac{k}{2} \rceil \\ (J - I)_{\phi(p^{k-j})} & \text{if } j \geqslant \lceil \frac{k}{2} \rceil \end{cases}. \tag{6.12}$$

Then, it is obvious that, if $j < \lceil \frac{k}{2} \rceil$

$$Spec(M_j) = \left\{ \begin{array}{cc} -2 & 2\left((\phi(p^{k-j}) - 1\right) \\ \phi(p^{k-j}) - 1 & 1 \end{array} \right\} \tag{6.13}$$

and if $j \geqslant \lceil \frac{k}{2} \rceil$,

$$Spec(M_j) = \left\{ \begin{array}{cc} -1 & \phi(p^{k-j}) - 1 \\ \phi(p^{k-j}) - 1 & 1 \end{array} \right\}. \tag{6.14}$$

Taking the union of the eigenvalues of $M_j, j = 1, 2, ...k-1$, from equations (6.12) and (6.13), the number of times, -2 is counted as an eigenvalue, is

$$\Sigma_{j<\lceil \frac{k}{2} \rceil} \left(\phi(p^{k-j}) - 1\right) = \Sigma_{j=1}^{\lfloor \frac{k}{2} \rfloor} \left(\phi(p^{k-j}) - 1\right) = p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil.$$

similarly from equations (6.12) and (6.14), the eigenvalue -1 is counted $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ times while taking the union of $M_j, j = 1, 2, ...k-1$. Thus, applying Theorem 6.3.2, the next Theorem follows.

Theorem 6.3.13. *For $k \geqslant 3$, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^k})$ has distance eigenvalue -2 and -1 with multiplicities $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil$ and $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ and the remaining distance eigenvalues are the eigenvalues of the vertex weighted distance matrix,*

$$T_D(\Upsilon_{p^k}) = \begin{bmatrix} t_1 & d_{1,2}\phi(p^{k-2}) & \dots & d_{1,k-1}\phi(p) \\ d_{1,2}\phi(p^{k-1}) & t_2 & \dots & d_{2,k-1}\phi(p) \\ \vdots & \vdots & \ddots & \vdots \\ d_{1,k-1}\phi(p^{k-1}) & d_{2,k-2}\phi(p^{k-2}) & \dots & t_{k-1} \end{bmatrix},$$

where, $d_{i,j} = \begin{cases} 1 & \text{if } i+j \geqslant k \\ 2 & \text{otherwise} \end{cases}$ *and* $t_j = \begin{cases} 2\left((\phi(p^{k-j}) - 1\right) & \text{if } j < \lceil \frac{k}{2} \rceil \\ \phi(p^{k-j}) - 1 & j \geqslant \lceil \frac{k}{2} \rceil \end{cases}$

Example 6.3.14. The number of non-zero zero divisors of $\mathbb{Z}_{p^3}$ is p^2-1. The proper divisors of p^3 are p and p^2 and the compressed zero-divisor graph Υ_{p^3} is isomorphic to K_2. The distance matrix of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$, for $p \neq 2$ is given by,

$$D(\mathit{\Gamma}(\mathbb{Z}_{p^3})) = \left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

The distance eigenvalues of $D(\mathit{\Gamma}(\mathbb{Z}_{p^3}))$ are -2 and -1 with multiplicities p^2-p-1 and $p-2$ respectively and the remaining two distance eigenvalues are the eigenvalues of the vertex weighted distance matrix $T_D(\Upsilon_{p^3})$ given by ,

$$T_D(\Upsilon_{p^3}) = \begin{bmatrix} 2\,(p^2-p-1) & p-1 \\ p^2-p & p-2 \end{bmatrix}$$

Thus the distance spectrum of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ is given by,

$$SpecD(\mathit{\Gamma}(\mathbb{Z}_{p^3})) = \left\{ \begin{array}{cccc} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{array} \right\}.$$

6.4 Distance Laplacian spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$

Let $G = \mathit{\Gamma}(\mathbb{Z}_n)$. Let $d_1, d_2, ..., d_k$ be the proper divisors of n such that $\Gamma(\mathcal{A}(d_1))$, $\Gamma(\mathcal{A}(d_2)), ..., \Gamma(\mathcal{A}(d_k))$ are null graphs and let $\Gamma(\mathcal{A}(d_{k+1})), ..., \Gamma(\mathcal{A}(d_{\xi(n)}))$ be complete subgraphs of $\mathit{\Gamma}(\mathbb{Z}_n)$. Then, since $\mathit{\Gamma}(\mathbb{Z}_n)$ is a generalised join of its induced subgraphs which are either null (corresponding to the divisors $d_1, d_2, ..., d_k$) or complete (corresponding to the divisors $d_{k+1}, ..., d_{\xi(n)}$); the distance between any two distinct vertices of $\mathcal{A}(d_j)$ for fixed $j \in \{1, 2, ..., k\}$ is 2 and the distance between any two distinct vertices of $\mathcal{A}(d_J)$, for fixed $j \in \{k+1, ..., \xi(n)\}$ is 1. Let M_j be the matrix as described in Section 6.3. Since M_j is non-negative, symmetric and irreducible, it has a Perron eigenvalue and let it be denoted by $\lambda_1(M_j)$, for $j = 1, 2, ..., \xi(n)$. It is easy to see that,

$$\lambda_1(M_j) = \begin{cases} 2(\phi(\frac{n}{d_j}) - 1) & \text{if } j \leqslant k \\ \phi(\frac{n}{d_j}) - 1 & j \geqslant k+1 \end{cases}$$

Then, for any vertex $v_{d_1} \in \mathcal{A}(d_1)$, the transmission degree of v_{d_1} is given by

$$Tr(v_{d_1}) = \Sigma_{u\in V(G)} d_G(u, v_{d_1}) = 2(\phi(\frac{n}{d_1})-1) + d_{1,2}\phi(\frac{n}{d_2}) + d_{1,3}\phi(\frac{n}{d_3}) + ...d_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}})$$

$$= \lambda_1(M_1) + \Sigma_{j\neq 1} d_{1,j}\phi(\frac{n}{d_j}).$$

Similar results hold for $Tr(v_{d_2}), ..., Tr(v_{d_k})$ and for any vertex $v_i \in \mathcal{A}(d_i)$,

$$Tr(v_{d_i}) = \lambda_1(M_i) + \Sigma_{j\neq i} d_{i,j}\phi(\frac{n}{d_j}), i = 1, 2, ...k.$$

Since, $\mathcal{A}(d_{k+1}), ..., \mathcal{A}(d_{\xi(n)})$ induce complete subgraphs, the transmission degree of any vertex $v_{d_i} \in \mathcal{A}(d_i), i = k+1, ..., \xi(n)$ is given by

$$Tr(v_{d_i}) = \phi(\frac{n}{d_i}) - 1 + \Sigma_{j\neq i} d_{i,j}\phi(\frac{n}{d_j}) = \lambda_1(M_i) + \Sigma_{j\neq i} d_{i,j}\phi(\frac{n}{d_j}), i = k+1, ..., \xi(n).$$

Thus, the diagonal matrix of vertex transmission of G is given by,

$$Tr(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} T_1 & O & \dots & O \\ O & T_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & T_{\xi(n)} \end{bmatrix}$$

where the diagonal blocks T_i are given by

$$T_i = (\lambda_1(M_i) + \tau_i)\, I_{\phi(\frac{n}{d_i})}$$

where $\tau_i = \Sigma_{j\neq i} d_{i,j}\phi(\frac{n}{d_j}), i = 1, 2, ..., \xi(n)$.

Clearly, $\lambda_1(M_i) + \tau_i$ is the Perron eigenvalue of T_i such that

$$T_i \mathbf{1}_{\phi(\frac{n}{d_i})} = (\lambda_1(M_i) + \tau_i)\, \mathbf{1}_{\phi(\frac{n}{d_i})}. \tag{6.15}$$

Since the distance Laplacian matrix of any connected graph G is $Tr(G) - D(G)$, it follows that,

$$D^L(\mathit{\Gamma}(\mathbb{Z}_n)) = \begin{bmatrix} L_1 & -d_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \dots & -d_{1,\xi(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} \\ -d_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & L_2 & \dots & -d_{2,\xi(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ -d_{1,\xi(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} & -d_{2,\xi(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} & \dots & L_{\xi(n)} \end{bmatrix}$$

where the diagonal blocks L_is are given by,

$$L_i = T_i - M_i \quad = (\lambda_1(M_i) + \tau_i)\, I_{\phi(\frac{n}{d_i})} - M_i, \quad i = 1, 2, ..., \xi(n). \tag{6.16}$$

Since T_i and M_i commute each other, for $i = 1, 2, ..., \xi(n)$, it can be easily seen from equation (6.16) that, each eigenvalue $\lambda(L_i)$ is given by

$$\lambda(L_i) = \lambda(T_i) - \lambda(M_i), \quad i = 1, 2, ..., \xi(n).$$

Thus, from (6.15), τ_i is an eigenvalue of L_i for $i = 1, 2, ...\xi(n)$, such that

$$L_i\mathbf{1}_{\phi(\frac{n}{d_i})} = \tau_i\mathbf{1}_{\phi(\frac{n}{d_i})}.$$

Thus applying Corollary 6.3.4, the next Theorem follows.

Theorem 6.4.1. *The distance Laplacian spectrum of $G = \mathit{\Gamma}(\mathbb{Z}_n)$ is given by*

$$Spec_{D^L}(G) = \bigcup_{i=1}^{\xi(n)} \{Spec(L_i)\backslash\tau_i\} \cup Spec(\widetilde{B})$$

where $\widetilde{B}$ is the vertex weighted distance Laplacian matrix of the proper divisor graph Υ_n, given by

$$\widetilde{B} = \begin{bmatrix} \tau_1 & -d_{1,2}\phi(\frac{n}{d_2}) & \dots & -d_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ -d_{1,2}\phi(\frac{n}{d_1}) & \tau_2 & \dots & -d_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ -d_{1,\xi(n)}\phi(\frac{n}{d_1}) & -d_{2,\xi(n)}\phi(\frac{n}{d_2}) & \dots & \tau_{\xi(n)} \end{bmatrix}.$$

For example, $\mathit{\Gamma}(\mathbb{Z}_{pq}) = \Upsilon_{pq}[\Gamma(\mathcal{A}(p)), \Gamma(\mathcal{A}(q))]$, where $\Upsilon_{pq} \equiv K_2$. In this case, $T_1 = (p+2q-5)I_{q-1}, M_1 = 2(J-I)_{q-1}, T_2 = (2p+q-5)I_{p-1}, M_2 = 2(J-I)_{p-1}$ and thus, the distance Laplacian matrix of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ is given by

$$D^L(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} (p+2q-5)I - 2(J-I)_{(q-1)\times(q-1)} & -J_{(q-1)\times(p-1)} \\ \hline -J_{(p-1)\times(q-1)} & (2p+q-5)I - 2(J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

Here

$$L_1 = (p+2q-3)I_{q-i} - 2J_{q-1}$$

$$L_2 = (2p+q-3)I_{p-i} - 2J_{p-1}$$

Clearly, $\tau_1 = \lambda_1(L_1) = p-1$ and $\tau_2 = \lambda_1(L_2) = q-1$.

$$Spec(L_1) = \left\{\begin{array}{cc} p-1 & p+2q-3 \\ 1 & q-2 \end{array}\right\} \quad \text{and} \quad Spec(L_2) = \left\{\begin{array}{cc} q-1 & 2p+q-3 \\ 1 & p-2 \end{array}\right\}$$

Thus, it can be seen that $p+2q-3$ and $2p+q-3$ are distance Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ with multiplicities $q-2$ and $p-2$ respectively and the remaining distance Laplacian eigenvalues are the eigenvalues of the vertex weighted matrix of Υ_{pq} given by

$$T_D^L(\Upsilon_{pq}) = \begin{bmatrix} p-1 & -(p-1) \\ -(q-1) & q-1 \end{bmatrix}.$$

The characteristic polynomial of the above matrix is $\lambda^2 - (p+q-2)\lambda$ and has eigenvalues 0 and $p+q-2$. Thus, the distance Laplacian spectrum of this graph is obtained as

$$SpecD^L(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left\{\begin{array}{cccc} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{array}\right\}.$$

6.4.1 Distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$

For any connected graph G with n vertices, the distance Laplacian matrix $D^L(G)$ is positive semi definite and the least distance Laplacain eigenvalue is 0 with multiplicity 1. ie $\partial_1^L \geqslant \partial_2^L \geqslant ... > \partial_n^L = 0$ [12]. The distance Laplacian eigenvalues of a graph of diameter at most 2 can be expressed in terms of its Laplacian eigenvalues as in the following Theorem.

Theorem 6.4.2. [28] *Let G be a connected graph on n vertices with diameter $d \leqslant 2$. Let $\mu_1(G) \geqslant \mu_2(G) \geqslant ... \geqslant \mu_n(G) = 0$, be the Laplacian eigenvalues of G. Then, the distance Laplacian eigenvalues of G are*

$$2n - \mu_{n-1}(G) \geqslant 2n - \mu_{n-2}(G) \geqslant ... \geqslant 2n - \mu_1(G) > \partial_n^L = 0$$

The next Theorem explores the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$

Theorem 6.4.3. [25] *Consider $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$. Then the following hold*

(i) If $k = 2m$ for some $m \geqslant 2$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\left\{ \begin{array}{cccccccc} p^{2m-1}-1 & p^{2m-2}-1 & \cdots & p^{m+1}-1 & p^m-1 & p^{m-1}-1 & \cdots & p-1 & 0 \\ \phi(p) & \phi(p^2) & \cdots & \phi(p^{m-1}) & \phi(p^m)-1 & \phi(p^{m+1}) & \cdots & \phi(p^{2m-1}) & 1 \end{array} \right\}$$

(ii) If $k = 2m+1$ for some $m \geqslant 1$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k})$ is given by

$$\left\{ \begin{array}{ccccccccc} p^{2m}-1 & p^{2m-1}-1 & \cdots & p^{m+1}-1 & p^m-1 & p^{m-1}-1 & \cdots & p-1 & 0 \\ \phi(p) & \phi(p^2) & \cdots & \phi(p^m) & \phi(p^{m+1})-1 & \phi(p^{m+2}) & \cdots & \phi(p^{2m}) & 1 \end{array} \right\}$$

Thus, the following Theorem can be easily proved.

Theorem 6.4.4. *Let $k \geqslant 3$.*

(i) If $k = 2m$ for some $m \geqslant 2$, then the distance Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ are $2p^{2m-1}-p-1$, $2p^{2m-1}-p^2-1, \cdots, 2p^{2m-1}-p^{m-1}-1$, $2p^{2m-1}-p^m-1$, $2p^{2m-1}-p^{m+1}-1, \cdots, p^{2m-1}-1$, and 0, with multiplicities $\phi(p^{2m-1})$, $\phi(p^{2m-2}), \cdots,$ $\phi(p^{m+1})$, $\phi(p^m)-1$, $\phi(p^{m-1}), \cdots, \phi(p)$, and 1 respectively.

(ii) If $k = 2m+1$ for some $m \geqslant 1$, then the distance Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ are $2p^{2m}-p-1$, $2p^{2m}-p^2-1, \cdots, 2p^{2m}-p^{m-1}-1$, $2p^{2m}-p^m-1$, $2p^{2m}-p^{m+1}-1, \cdots, p^{2m}-1$, and 0, with multiplicities $\phi(p^{2m})$, $\phi(p^{2m-1}), \cdots,$ $\phi(p^{m+2})$, $\phi(p^{m+1})-1$, $\phi(p^m), \cdots, \phi(p)$, and 1 respectively.

Proof. The compressed zero divisor graph Υ_{p^k} is a connected graph of diameter 2 for $k \geqslant 4$, since $p^{k-1} \sim p^j$, $\forall j = 1, 2, ...k-2$. And for $k = 3$, $\Upsilon_{p^k} \equiv K_2$ and hence it is of diameter 1. As described in Section 6.3, it follows that $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ is of diameter at most 2 for $k \geqslant 3$. The number of vertices in $\mathit{\Gamma}(\mathbb{Z}_{p^k})$ is $p^{k-1}-1$. Thus, the conclusion follows from the above two Theorems. □

Chapter 7

Seidel spectrum of the zero divisor graph

In the Section 2 of this Chapter, the Seidel spectrum of the generalized join of regular graphs is investigated. The Seidel spectrum of the zero-divisor graph of $\mathbb{Z}_n$, is thereby computed in terms of the spectrum of the vertex weighted combinatorial matrix of the proper divisor graph of n. In the third and fourth Sections, this investigation is repeated for Seidel Laplacian and Seidel signless Laplacian spectrum respectively.

7.1 Introduction

Van Lint and Seidel [84] introduced the Seidel matrix of G, defined as $S(G) = [s_{i,j}]$ where $s_{i,j} = -1$ if the vertices v_i and v_j are adjacent and $s_{i,j} = 1$ if the vertices v_i and v_j are not adjacent and $s_{i,j} = 0$ if $i = j$. Thus, $S(G)$ is a $(-1, 0, 1)$

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adjacency matrix of a graph G and $S(G) = J - I - 2A(G)$. Clearly, if $\overline{G}$ denotes the complement of a graph G, then $S(\overline{G}) = -S(G)$. The collection of Seidel eigenvalues of G, together with their multiplicities, is known as Seidel spectrum of G, denoted by $Spec_S(G)$. In [43], Haemers has defined the Seidel energy of G as $E_S(G)) = \sum_{i=1}^{n} |\theta_i|$, where $\theta_1, \theta_2, ..., \theta_n$ are the Seidel eigenvalues of G. This parameter finds various chemical applications.

For a complete graph K_n,

$$Spec_S(K_n) = \left\{ \begin{array}{cc} 1-n & 1 \\ 1 & n-1 \end{array} \right\}.$$

For a null graph on n vertices, $\overline{K_n}$,

$$Spec_S(\overline{K_n}) = \left\{ \begin{array}{cc} n-1 & -1 \\ 1 & n-1 \end{array} \right\}.$$

Recall Definition 2.3.6 for the Seidel Laplacian matrix and Seidel signless Laplacian matrix of a graph G. The Seidel Laplacian eigenvalues of a graph G of order n sre denoted and $\theta_1^L, \theta_2^L, ..., \theta_n^L$ and the Seidel signless Laplacian eigenvalues by $\theta_1^Q, \theta_2^Q, ..., \theta_n^Q$.

For a complete graph K_n, $S^L(K_n) = J_n - nI_n$ and $S^Q(K_n) = (2-n)I_n - J_n$. For a null graph on n vertices, $\overline{K_n}$, (the complement of a complete graph), $S^L(\overline{K_n}) = nI_n - J_n$ and $S^Q(\overline{K_n}) = (n-2)I_n + J_n$.

7.2 Seidel spectrum of join of regular graphs

The following result can also be seen in [44], which demands lengthy computation using the method of equitable partitions is used.

Theorem 7.2.1. *Let G_i be r_i- regular of order n_i, for $i = 1, 2$. Then each of the Seidel eigenvalues of G_1 and G_2 other than $n_1 - 2r_1 - 1, n_2 - 2r_2 - 1$ is a Seidel eigenvalue of $G_1 \bigtriangledown G_2$ and the remaining Seidel eigenvalues of $G_1 \bigtriangledown G_2$ are the zeroes of the polynomial $\lambda^2 - [n_1 + n_2 - 2r_1 - 2r_2 - 2]\lambda + [(n_1 - 2r_1 - 1)(n_2 - 2r_2 - 1) - n_1 n_2]$.*

Proof. Let S_1 and S_2 denote the Seidel matrices of G_1 and G_2 respectively. Since G_1 and G_2 are regular, it can be seen that, the row sums of S_1 and S_2 are the constants $n_1 - 2r_1 - 1$ and $n_2 - 2r_2 - 1$ respectively and in fact, these two values are the Seidel eigenvalues of G_1 and G_2 with corresponding eigenvectors $\mathbf{1}_{n_1}$ and $\mathbf{1}_{n_2}$ respectively. Also, the Seidel matrix of the join of the graphs G_1 and G_2 is given by

$$S(G_1 \bigtriangledown G_2) = \begin{bmatrix} S_1 & -J_{n_1 \times n_2} \\ -J^T_{n_1 \times n_2} & S_2 \end{bmatrix}.$$

Applying Lemma 4.2.4, and Lemma 3.3.6, it can be seen that

$$det(\lambda I - S(G_1 \bigtriangledown G_2)) = det(\lambda I - S_1) \cdot det(\lambda I - S_2) \cdot (1 - \Gamma_{S_1}(\lambda)\Gamma_{S_2}(\lambda)). \quad (7.1)$$

where $\Gamma_{S_1}(\lambda) = \dfrac{n_1}{\lambda - (n_1 - 2r_1 - 1)}$ and $\Gamma_{S_2}(\lambda) = \dfrac{n_2}{\lambda - (n_2 - 2r_2 - 1)}$.
Simplifying equation (7.1),

$$det(\lambda I - S(G_1 \bigtriangledown G_2)) = \frac{det(\lambda I - S_1)}{(\lambda - (n_1 - 2r_1 - 1))} \cdot \frac{det(\lambda I - S_2)}{(\lambda - (n_2 - 2r_2 - 1))}\Phi(\lambda),$$

where $\Phi(\lambda) = \lambda^2 - [n_1 + n_2 - 2r_1 - 2r_2 - 2]\lambda + [(n_1 - 2r_1 - 1)(n_2 - 2r_2 - 1) - n_1 n_2]$. $\square$

Example 7.2.2. *Consider $G = K_5 \bigtriangledown C_4$. The Seidel spectrum of K_5 and C_4 are given as follows. $Spec_S(K_5) = \left\{ \begin{array}{cc} -4 & 1 \\ 1 & 4 \end{array} \right\}$ and $Spec_S(C_4) = \left\{ \begin{array}{cc} -1 & 3 \\ 3 & 1 \end{array} \right\}$.*

Thus, the Seidel eigenvalues of G are $-1, 1, 3$ with multiplicities $2, 4$ and 1 respectively together with the zeroes of the polynomial $\lambda^2 + 5\lambda - 16$. Thus Seidel spectrum of $K_5 \bigtriangledown C_4$ is $\left\{ \begin{array}{ccccc} -1 & 1 & 3 & \frac{-5+\sqrt{89}}{2} & \frac{-5-\sqrt{89}}{2} \\ 2 & 4 & 1 & 1 & 1 \end{array} \right\}$.

Since the Seidel matrix loses non negativity, the Perron- Frobeniuos Theorem may not hold good for such matrices. However, it is of great relevance that, for a r-regular connected graph on n vertices, $n - 2r - 1$ is a Seidel eigenvalue with $\mathbf{1}_n$ as the associated Seidel eigenvector. In the following section, Theorem 7.2.1 is extended to the genrealised join of regular graphs using Fiedler's Lemma and the Seidel spectrum of $G[H_1, H_2, ..., H_k]$ is explored, where G_i is r_i-regular for $i = 1, 2, ..., k$, using the adjacency spectrum of $H_1, H_2, ..., H_k$ and a combinatorial vertex weighted Seidel matrix of G.

7.2.1 Seidel spectrum of the joined union of regular graphs

Consider $G[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ with the Seidel matrix $S(G) = [s_{i,j}]$ where $s_{i,j} = -1$ if the vertices i and j are adjacent and $s_{i,j} = 1$ if the vertices i and j are not adjacent and $s_{i,j} = 0$ for the diagonal entries. Let H_j be r_j-regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let S_j denote the Seidel matrix of $H_j, j = 1, 2, ..., k$.
It can be easily seen that, the Seidel matrix of the G- union of $H_1, H_2, ..., H_k$ is given by

$S(G\,[H_1, H_2, ..., H_k])=$

$$\begin{bmatrix} S_1 & s_{1,2}J_{n_1\times n_2} & s_{1,3}J_{n_1\times n_3} & \dots & s_{1,k}J_{n_1\times n_k} \\ s_{1,2}J^T_{n_1\times n_2} & S_2 & s_{2,3}J_{n_2\times n_3} & \dots & s_{2,k}J_{n_2\times n_k} \\ s_{1,3}J^T_{n_1\times n_3} & s_{2,3}J^T_{n_2\times n_3} & S_3 & \dots & s_{3,k}J_{n_3\times n_k} \\ \vdots & \vdots & & \ddots & \vdots \\ s_{1,k-1}J^T_{n_1\times n_{k-1}} & s_{2,k-1}J^T_{n_2\times n_{k-1}} & \dots & S_{k-1} & s_{k-1,k}J_{n_{k-1}\times n_k} \\ s_{1,k}J^T_{n_1\times n_k} & s_{2,k}J^T_{n_2\times n_k} & \dots & \dots & S_k \end{bmatrix}. \tag{7.2}$$

Since the each row of S_j sums to the constant $n_j - 2r_j - 1$, for $j = 1, 2, ..., k$, one of the Seidel eigenvalues of S_j is $n_j - 2r_j - 1$ with the associated Seidel eigenvector $\mathbf{1}_{n_j}$, for $j = 1, 2, ..., k$.
Thus, as in equation (6.3), taking

$$M_j = S_j, \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = \left(n_j - 2r_j - 1 \ , \ \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j}\right)$$

and the real numbers

$$\rho_{l,q} = s_{l,q}\sqrt{n_l n_q}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, and applying Theorem 6.2.2, the Theorem follows.

Theorem 7.2.3. *Consider $G\,[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let S_j denote the Seidel adjacency matrix of H_j. Then the Seidel spectrum of the G-join of the graphs $H_1, H_2, ..., H_k$ is given by,*

$$Spec_S(G\,[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} Spec(S_j)\backslash\{n_j - 2r_j - 1\}\right) \cup Spec(\widetilde{S}),$$

where

$$\widetilde{S} = \begin{bmatrix} n_1 - 2r_1 - 1 & \rho_{1,2} & \dots & \rho_{1,k} \\ \rho_{1,2} & n_2 - 2r_2 - 1 & \dots & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,k} & \rho_{2,k} & \dots & n_k - 2r_k - 1 \end{bmatrix}.$$

and

$$\rho_{l,q} = s_{l,q}\sqrt{n_l n_q} = \begin{cases} -\sqrt{n_l n_q} & \textit{if } lq \in E(G) \\ \sqrt{n_l n_q} & \textit{if } lq \notin E(G) \end{cases}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$.

Let $T_S(G)$ be the combinatorial vertex weighted Seidel matrix of G, given by

$$T_S(G) = \begin{bmatrix} n_1 - 2r_1 - 1 & s_{1,2}n_2 & \dots & s_{1,k}n_k \\ s_{1,2}n_1 & n_2 - 2r_2 - 1 & \dots & s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k}n_1 & s_{2,k}n_2 & \dots & n_k - 2r_k - 1 \end{bmatrix}.$$

It is easy to verify that $T_S(G) = W^{-\frac{1}{2}}\widetilde{S}W^{\frac{1}{2}}$, where W is a diagonal matrix of vertex weights as defined in Section 6.3. Thus $\widetilde{S}$ and $T_S(G)$ are similar and hence $Spec(\widetilde{S}) = Spec(T_S(G))$.

Lemma 7.2.4. [9] *Let G be k-regular graph of order n. If $k, \lambda_2, ..., \lambda_n$ are the adjacency eigenvalues and $\theta_1, \theta_2, ..., \theta_n$ are the Seidel eigenvalues of G, then* $\theta_1 = n - 2k - 1, \theta_2 = -1 - 2\lambda_2, \theta_3 = -1 - 2\lambda_3, ..., \theta_n = -1 - 2\lambda_n$.

As an immediate consequence of Theorem 7.2.3 and Lemma 7.2.4, the Seidel spectrum of $G[H_1, H_2, ..., H_k]$ is found in terms of the adjacency spectrum of $H_1, H_2, ..., H_k$.

Corollary 7.2.5. *If $\lambda_{1j} = r_j, \lambda_{2j}, ..., \lambda_{n_j j}$ are the adjacency eigenvalues of H_j for $j = 1, 2, ..., k$, then*

$$Spec_S(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} \bigcup_{i=2}^{n_j} \{-1 - 2\lambda_{ij}\} \right) \cup Spec(T_S(G)),$$

where

$$T_S(G) = \begin{bmatrix} n_1 - 2r_1 - 1 & s_{1,2}n_2 & \dots & s_{1,k}n_k \\ s_{1,2}n_1 & n_2 - 2r_2 - 1 & \dots & s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k}n_1 & s_{2,k}n_2 & \dots & n_k - 2r_k - 1 \end{bmatrix}.$$

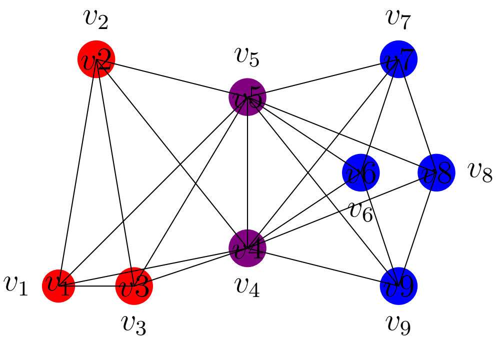

Figure 7.1: $P_3[K_3, K_2, C_4]$

Example 7.2.6. Consider the graph $G = P_3[K_3, K_2, C_4]$, the P_3-union of the complete graphs K_3, K_2 and the cycle C_4. Note that $G = P_3, H_1 = K_3$, $H_2 = K_2, H_3 = C_4$ and $n_1 = 3, n_2 = 2, n_3 = 4$ and $r_1 = 2, r_2 = 1, r_3 = 2$. The adjacency spectrum and the Seidel spectrum of the graphs K_3, K_2 and C_4 are given below.

$$Spec(K_3) = \begin{Bmatrix} 2 & -1 \\ 1 & 2 \end{Bmatrix}, \quad Spec(K_2) = \begin{Bmatrix} 1 & -1 \\ 1 & 1 \end{Bmatrix}, \quad Spec(C_4) = \begin{Bmatrix} 2 & 0 & -2 \\ 1 & 2 & 1 \end{Bmatrix}.$$

$$Spec_S(K_3) = \left\{ \begin{array}{cc} -2 & 1 \\ 1 & 2 \end{array} \right\}, \quad Spec_S(K_2) = \left\{ \begin{array}{cc} -1 & 1 \\ 1 & 1 \end{array} \right\}, \quad Spec_S(C_4) = \left\{ \begin{array}{cc} -1 & 3 \\ 3 & 1 \end{array} \right\}.$$

Also, the Seidel matrix of the joining graph P_3 and the combinatorial vertex weighted Seidel matrix of G are given by

$$S(P_3) = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}, \quad T_S(G) = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & -4 \\ 3 & -2 & -1 \end{bmatrix}.$$

Using Corollary 7.2.5, the Seidel spectrum $P_3[K_3, K_2, C_4]$ is $\left\{ \begin{array}{ccc} 3 & 1 & -1 \\ 1 & 3 & 2 \end{array} \right\}$, together with the spectrum of $T_S(P_3) = \begin{bmatrix} -2 & -2 & 4 \\ -3 & -1 & -4 \\ 3 & -2 & -1 \end{bmatrix}$, which is found to be $\{-5, \frac{1\pm\sqrt{65}}{2}\}$. Thus, the Seidel spectrum of this graph is obtained as below.

$$Spec_S(P_3[K_3, K_2, C_4]) = \left\{ \begin{array}{cccccc} 3 & 1 & -1 & -5 & \frac{1+\sqrt{65}}{2} & \frac{1-\sqrt{65}}{2} \\ 1 & 3 & 2 & 1 & 1 & 1 \end{array} \right\}.$$

7.2.2 Seidel matrix and Seidel spectrum of $\Gamma(\mathbb{Z}_n)$

The following Theorem completely determines the Seidel spectrum of $\Gamma(\mathbb{Z}_n)$ in terms of the Seidel eigenvalues of its induced subgraphs and the proper divisor graph of n.

Theorem 7.2.7. *If $d_1, d_2, ..., d_{\xi(n)}$ are the proper divisors of n, then the Seidel matrix of $\Gamma(\mathbb{Z}_n)$ is given by*

$$S(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} S_1 & s_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \cdots & s_{1,\xi(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} \\ s_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & S_2 & \cdots & s_{2,\xi(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,\xi(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} & s_{2,\xi(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} & \cdots & S_{\xi(n)} \end{bmatrix},$$

where,

$$S_j = \begin{cases} (J-I)_{\phi(\frac{n}{d_j})} & \textit{if } n \nmid d_j^2 \\ (I-J)_{\phi(\frac{n}{d_j})} & \textit{if } n \mid d_j^2 \end{cases}$$

and for $i \in \{1., 2, ..., \xi(n)-1\}$ *and* $j \in \{i+1, ..., \xi(n)\}$, $i \neq j$,

$$s_{i,j} = \begin{cases} -1 & \textit{if } n \mid d_i d_j \\ 1 & \textit{if } n \nmid d_i d_j \end{cases}.$$

Proof. $\Gamma(\mathbb{Z}_n)$ is the Υ_n - join of $\Gamma(\mathcal{A}(d_1)), \Gamma(\mathcal{A}(d_2)), ..., \mathcal{A}(d_{\xi(n)}))$. The induced subgraphs $\Gamma(\mathcal{A}(d_j))$ are either a complete graph or the complement of a complete graph, by Lemma 4.3.2.

Hence, the Seidel matrix of the induced subgraphs $\Gamma(\mathcal{A}(d_j))$, denoted by S_j are given by,

$$S_j = \begin{cases} (J-I)_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ (I-J)_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases},$$

and the spectrum of S_j is known in each case. Also, any two vertices d_i and d_j in the joining graph Υ_n are adjacent if and only if $n \mid d_i d_j$ and accordingly the ij-th entry in the Seidel matrix of Υ_n is -1 if $n \mid d_i d_j$ and 1 if $n \nmid d_i d_j$, and 0 on the diagonal. Thus the Theorem follows, taking $G = \Upsilon_n$ and $H_j = \Gamma(\mathcal{A}(d_j))$, as in equation (7.2). □

Theorem 7.2.8. *Let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors of any positive integer n. Then, the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ has Seidel eigenvalues -1 and 1 with multiplicities $\sum_{d \mid n, n \nmid d^2}(\phi(\frac{n}{d}) - 1)$ and $\sum_{d \mid n, n \mid d^2}(\phi(\frac{n}{d}) - 1)$ respectively and the remaining Seidel eigenvalues are the eigenvalues of the vertex weighted Seidel matrix of Υ_n, given below.*

$$T_S(\Upsilon_n) = \begin{bmatrix} t_1 & s_{1,2}\phi(\frac{n}{d_2}) & \dots & s_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ s_{1,2}\phi(\frac{n}{d_1}) & t_2 & \dots & s_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,\xi(n)}\phi(\frac{n}{d_1}) & s_{2,\xi(n)}\phi(\frac{n}{d_2}) & \dots & t_{\xi(n)} \end{bmatrix},$$

where,

$$t_j = \begin{cases} \phi(\frac{n}{d_j}) - 1 & \text{if } n \nmid d_j^2 \\ 1 - \phi(\frac{n}{d_j}) & \text{if } n \mid d_j^2 \end{cases}$$

and for $i \in \{1, 2, ..., \xi(n)-1\}, j \in \{i+1, ..., \xi(n)\}, i \neq j,$ $\quad s_{i,j} = \begin{cases} -1 & \text{if } n \mid d_i d_j \\ 1 & \text{if } n \nmid d_i d_j \end{cases}.$

Corollary 7.2.9. *$\Gamma(\mathbb{Z}_n)$ is Seidel integral if and only if $T_S(\Upsilon_n)$ is integral.*

Corollary 7.2.10. *For distinct primes p and q, $p < q$, $\Gamma(\mathbb{Z}_{pq})$ has only two distinct Seidel eigenvalues -1 and $p + q - 3$.*

Proof. Consider $\Gamma(\mathbb{Z}_{pq})$, where $p < q$ are distinct primes. It can be easily seen that the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$, is the K_2 join of $\Gamma(\mathcal{A}(p))$ and $\Gamma(\mathcal{A}(q))$, where $\Gamma(\mathcal{A}(p)) = \overline{K}_{q-1}$ and $\Gamma(\mathcal{A}(q)) = \overline{K}_{p-1}$. Note that the proper divisor graph of pq is $\Upsilon_{pq} = K_2$.

That is $\Gamma(\mathbb{Z}_{pq}) = K_2[\overline{K}_{q-1}, \overline{K}_{p-1}]$. Using Theorem 7.2.7, the Seidel matrix of

$\mathit{\Gamma}(\mathbb{Z}_{pq})$ is given by

$$S(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left[\begin{array}{c|c} (J-I)_{(q-1)\times(q-1)} & (-1)J_{(q-1)\times(p-1)} \\ \hline (-1)J_{(p-1)\times(q-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right].$$

And by Theorem 7.2.8, since both $\Gamma(\mathcal{A}(p))$ and $\Gamma(\mathcal{A}(q))$ are null graphs, -1 is a Seidel eigenvalue of $\mathit{\Gamma}(\mathbb{Z}_{pq})$ with multiplicity $p+q-4$ and the other two Seidel eigenvalues are the eigenvalues of its vertex weighted Seidel matrix,

$$T_S(\Upsilon_{pq}) = \begin{bmatrix} q-2 & -(p-1) \\ -(q-1) & p-2 \end{bmatrix}.$$

Thus, the remaining two Seidel eigenvalues of this graph are -1 and $p+q-3$. Thus

$$Spec_S(\mathit{\Gamma}(\mathbb{Z}_{pq})) = \left\{\begin{matrix} -1 & p+q-3 \\ p+q-3 & 1 \end{matrix}\right\}.$$

□

Corollary 7.2.11. *For any prime p, -1 and 1 are the Seidel eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ with multiplicities p^2-p-1 and $p-2$ respectively and the remaining Seidel eigenvalues are* $\dfrac{(p-1)^2 \pm \sqrt{p^4+4p^3-14p^2+4p+9}}{2}$.

Proof. The proper divisors of p^3 are p and p^2 and the proper divisor graph of p^3 is $\Upsilon_{p^3} = K_2$. Also, $\mathit{\Gamma}(\mathbb{Z}_{p^3}) = K_2[\overline{K}_{p(p-1)}, K_{p-1}]$. Thus, applying Theorem 7.2.8, -1 and 1 are the Seidel eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ with multiplicities p^2-p-1 and $p-2$ respectively and the remaining two Seidel eigenvalues are the eigenvalues of the vertex weighted Seidel matrix,

$$T_S(\Upsilon_{p^3}) = \begin{bmatrix} p^2-p-1 & -(p-1) \\ -(p^2-p) & 2-p \end{bmatrix}.$$

□

Corollary 7.2.12. *For distinct primes $p < q$, -1 and 1 are the Seidel eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$ with multiplicities $p^2 + pq - 2p - 3$ and $p - 2$ respectively and the remaining Seidel eigenvalues are the zeroes of the polynomial*
$\lambda^4 + \lambda^3(1 - p^2 + 3p - pq) + \lambda^2(5p^2 - 2p^3 - 2p^2q + p + pq - 3) + \lambda(4p^4q - 4p^4 + 12p^3 + 4p^3q^2 - 20p^3q - 7p^2 - 8p^2q^2 + 24p^2q + p + 4pq^2 - 7pq - 5) + (8p^4q^2 - 12p^4q + 4p^4 - 28p^3q^2 + 44p^3q - 18p^3 + 32p^2q^2 - 54p^2q + 27p^2 - 12pq^2 + 23pq - 13p - 2)$.

Proof. The proper divisors of p^2q are p, q, p^2, pq. and the proper divisor graph of p^2q is the path P_4, where $p \sim pq \sim p^2 \sim q$. Hence, it can be seen that, the zero-divisor graph

$$\Gamma(\mathbb{Z}_{p^2q}) = P_4[\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{(q-1)}, K_{p-1}].$$

Applying Theorem 7.2.8, -1 and 1 are the Seidel eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$ with multiplicities $p^2 + pq - 2p - 3$ and $p - 2$ respectively and the remaining four Seidel eigenvalues are the eigenvalues of the vertex weighted Seidel matrix,

$$T_S(\Upsilon_{p^2q}) = \begin{bmatrix} (p-1)(q-1)-1 & p^2-p & q-1 & -(p-1) \\ (p-1)(q-1) & p^2-p-1 & -(q-1) & p-1 \\ (p-1)(q-1) & p-p^2 & q-2 & -(p-1) \\ -(p-1)(q-1) & p^2-p & -(q-1) & 2-p \end{bmatrix}.$$

□

7.2.3 Seidel spectrum of $\Gamma(\mathbb{Z}_{p^k})$, $k \geqslant 3$

The proper divisors of p^k are $p, p^2, ..., p^{k-1}$ and $\{\mathcal{A}(p), \mathcal{A}(p^2), ..., \mathcal{A}(p^{k-1})\}$ forms an equitable partition for $V(\Gamma(\mathbb{Z}_{p^k}))$. The proper divisor graph Υ_{p^k} is a simple connected graph of diameter 2 in which the vertices are labeled as $p, p^2, ...p^{k-1}$ and for distinct i and j, $p^i \sim p^j$ if and only if $i + j \geqslant k$. Also,

$$\Gamma(\mathbb{Z}_{p^k}) = \Upsilon_{p^k}[\overline{K}_{\phi(p^{k-1})}, \overline{K}_{\phi(p^{k-2})}, ..., \overline{K}_{\phi(p^{\lfloor \frac{k}{2} \rfloor+1})}, K_{\phi(p^{\lfloor \frac{k}{2} \rfloor})}, ..., K_{\phi(p)}].$$

Using the properties of the Euler-totient function ϕ, it can be seen that

$$\begin{aligned}\Sigma_{j<\lceil \frac{k}{2} \rceil} \big(\phi(p^{k-j}) - 1\big) &= \Sigma_{j=1}^{\lceil \frac{k}{2} \rceil -1} \big(\phi(p^{k-j}) - 1\big) \\ &= p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1,\end{aligned}$$

and

$$\Sigma_{j \geqslant \lceil \frac{k}{2} \rceil} \big(\phi(p^{k-j}) - 1\big) = p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1.$$

Thus, on account of Theorem 7.2.8, the next Theorem follows.

Theorem 7.2.13. *For $k \geqslant 3$, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^k})$ has Seidel eigenvalue -1 and 1 with multiplicities $p^{k-1} - p^{\lfloor \frac{k}{2} \rfloor} - \lceil \frac{k}{2} \rceil + 1$ and $p^{\lfloor \frac{k}{2} \rfloor} - \lfloor \frac{k}{2} \rfloor - 1$ and the remaining Seidel eigenvalues are the eigenvalues of the vertex weighted Seidel matrix,*

$$T_S(\Upsilon_{p^k}) = \begin{bmatrix} \zeta_1 & s_{1,2}\phi(p^{k-2}) & \dots & s_{1,k-1}\phi(p) \\ s_{1,2}\phi(p^{k-1}) & \zeta_2 & \dots & s_{2,k-1}\phi(p) \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k-1}\phi(p^{k-1}) & s_{2,k-1}\phi(p^{k-2}) & \dots & \zeta_{k-1} \end{bmatrix},$$

where, $s_{i,j} = \begin{cases} -1 & \text{if } i + j \geqslant k \\ 1 & \text{otherwise} \end{cases}$ *and* $\zeta_j = \begin{cases} \phi(p^{k-j}) - 1 & \text{if } j < \lceil \frac{k}{2} \rceil \\ 1 - \phi(p^{k-j}) & j \geqslant \lceil \frac{k}{2} \rceil \end{cases}$.

Corollary 7.2.14. *For any prime p, the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ has Seidel eigenvalue -1 and 1 with multiplicities $p^3 - p^2 - 1$ and $p^2 - 3$ respectively and the remaining Seidel eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$ are the zeroes of the polynomial $\lambda^3 - \lambda^2(p^3 - 2p^2 + 2) - \lambda(2p^5 - 2p^4 - 4p^3 + 4p^2 + 1) - (4p^6 - 14p^5 + 14p^4 - p^3 - 2p^2 - 2)$.*

Proof. The divisors of p^4 are p, p^2, p^3 and the proper divisor graph of p^4 is the path P_3, where $p \sim p^3 \sim p^2$. It can be seen that

$$\Gamma(\mathbb{Z}_{p^4}) = P_3[\overline{K}_{p^2(p-1)}, K_{p(p-1)}, K_{p-1}].$$

Again using Theorem 7.2.8, it is found that -1 and 1 are the Seidel eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$ with multiplicities $p^3 - p^2 - 1$ and $p^2 - 3$ respectively and the remaining Seidel eigenvalues of $\Gamma(\mathbb{Z}_{p^4})$ are the eigenvalues of the vertex weighted Seidel matrix,

$$T_S(\Upsilon_{p^4}) = \begin{bmatrix} p^3 - p^2 - 1 & p^2 - p & -(p-1) \\ p^3 - p^2 & p - p^2 + 1 & -(p-1) \\ -(p^3 - p^2) & p - p^2 & 2 - p \end{bmatrix}.$$

□

7.3 Seidel Laplacian spectrum of the join of regular graphs

In [73], H.S.Ramane et.al express the Seidel Laplacian polynomial of the join of two regular graphs. In this Section, the same is found in a fairly shorter method, by applying the well known Fiedler's Lemma and the result is extended to the join of more than two regular graphs.

Theorem 7.3.1. *Let G_i be r_i regular of order n_i, for $i = 1, 2$. Then the Seidel Laplacian polynomial of $G_1 \bigtriangledown G_2$ is given by*

$$\Phi_{S^L}(G_1 \bigtriangledown G_2; \lambda) = \frac{\lambda(\lambda + n_1 + n_2)}{(\lambda + n_1)(\lambda + n_2)} \cdot \Phi_{S^L}(G_1; \lambda + n_2) \cdot \Phi_{S^L}(G_2; \lambda + n_1).$$

Proof. Let $S(G_1)$ and $S(G_2)$ denote the Seidel adjacency matrices of G_1 and G_2 respectively. Then, the Seidel Laplacian matrices of G_1 and G_2 are given as follows.

$$S^L(G_1) = (n_1 - 2r_1 - 1)I_{n_1} - S(G_1),$$

$$S^L(G_2) = (n_2 - 2r_2 - 1)I_{n_2} - S(G_2).$$

Since G_1 is regular, by Lemma 7.2.4, $n_1 - 2r_1 - 1$ is an eigenvalue of $S(G_1)$ with corresponding eigenvector $\mathbf{1}_{n_1}$. Also, 1 is an eigenvalue of I_{n_1} with eigenvector $\mathbf{1}_{n_1}$. Thus it can be seen that, 0 is an eigenvalue of $S^L(G_1)$ with corresponding eigenvector $\mathbf{1}_{n_1}$. Similarly, 0 is an eigenvalue of $S^L(G_2)$ with corresponding eigenvector $\mathbf{1}_{n_2}$. Clearly, $S^L(G_1 \bigtriangledown G_2)$

$$= \begin{bmatrix} (n_1 + n_2 - 2(r_1 + n_2) - 1)I_{n_1} - S_1 & J_{n_1 \times n_2} \\ J^T_{n_1 \times n_2} & (n_1 + n_2 - 2(r_2 + n_1) - 1)I_{n_2} - S_2 \end{bmatrix}$$

$$= \begin{bmatrix} S^L(G_1) - n_2 I_{n_1} & J_{n_1 \times n_2} \\ J^T_{n_1 \times n_2} & S^L(G_2) - n_1 I_{n_2} \end{bmatrix}.$$

Taking $A = S^L(G_1) - n_2 I_{n_1}$, $B = S^L(G_2) - n_1 I_{n_2}$ $\alpha_1 = -n_2$, $\beta_1 = -n_1$, $\mathbf{u}_1 = \frac{1}{\sqrt{n_1}}\mathbf{1}_{n_1}$, $\mathbf{u}_2 = \frac{1}{\sqrt{n_2}}\mathbf{1}_{n_2}$, $\rho = \sqrt{n_1 n_2}$ and applying Fiedler's Lemma, it can be seen that $Spec_{S^L}(G_1 \bigtriangledown G_2)$

$= Spec\left(S^L(G_1) - n_2 I_{n_1}\right) \backslash \{-n_2\} \bigcup Spec\left(S^L(G_2) - n_1 I_{n_2}\right) \backslash \{-n_1\} \bigcup Spec\hat{F}$, where the matrix $\hat{F} = \begin{bmatrix} -n_2 & \sqrt{n_1 n_2} \\ \sqrt{n_1 n_2} & -n_1 \end{bmatrix}$. Clearly, $Spec(\hat{F}) = \begin{Bmatrix} 0 & -n_1 - n_2 \\ 1 & 1 \end{Bmatrix}$.

Thus,

$$\Phi_{S^L}(G_1 \bigtriangledown G_2; \lambda) = \frac{\lambda(\lambda + n_1 + n_2)}{(\lambda + n_1)(\lambda + n_2)} \cdot \Phi_{S^L}(G_1; \lambda + n_2) \cdot \Phi_{S^L}(G_2; \lambda + n_1).$$

□

7.3.1 Seidel Laplacian spectrum of the joined union of regular graphs

Consider $G[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ with the Seidel matrix $S(G) = [s_{i,j}]$ where $s_{i,j} = -1$ if the vertices i and j are adjacent and $s_{i,j} = 1$ if the vertices i and j are not adjacent and $s_{i,j} = 0$ for the diagonal entries. Let H_j be r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let S_j and $S^L(H_j)$ denote the Seidel matrix and Seidel Laplacian matrix of $H_j, j = 1, 2, ..., k$. The degree of each vertex of H_j, in the joined graph $G[H_1, H_2, ..., H_k]$ is $r_j + \sum_{i\sim j} n_i$. Hence, if S_j^L denotes the j^{th} diagonal block in the Seidel Laplacian matrix of $G[H_1, H_2, ..., H_k]$, then,

$$\begin{aligned} S_j^L &= \left(\sum_{i=1}^{k} n_i - 2(r_j + \sum_{j\sim i} n_i) - 1\right) I_{n_j} - S_j \\ &= (n_j - 2r_j - 1)I_{n_j} - S_j + \left(\sum_{j\nsim i} n_i - \sum_{j\sim i} n_i\right) I_{n_j} \\ &= S^L(H_j) + \left(\sum_{i=1}^{k} s_{ij} n_i\right) I_{n_j} \\ &= S^L(H_j) + \tau_j I_{n_j}. \end{aligned}$$

where $\tau_j = \sum_{i=1}^{k} s_{i,j} n_i$. Thus, it can be easily seen that, the Seidel Laplacian matrix of the G- union of $H_1, H_2, ..., H_k$ is given by

$$S^L(G[H_1, H_2, ..., H_k]) = \begin{bmatrix} S^L(H_1) + \tau_1 I_{n_1} & -s_{1,2}J_{n_1\times n_2} & \dots & -s_{1,k}J_{n_1\times n_k} \\ -s_{1,2}J^T_{n_1\times n_2} & S^L(H_2) + \tau_2 I_{n_2} & \dots & -s_{2,k}J_{n_2\times n_k} \\ \vdots & & \ddots & \vdots \\ -s_{1,k}J^T_{n_1\times n_k} & -s_{2,k}J^T_{n_2\times n_k} & \dots & S^L(H_k) + \tau_k I_{n_k} \end{bmatrix} \tag{7.3}$$

where $\tau_j = \sum_{i=1}^{k} s_{i,j}n_i$.

Remark 7.3.2. *Let G be a k-regular graph of order n. Since, $S^L(G) = (n - 2k - 1)I_n - S(G)$, from Lemma 7.2.4 it follows that, $S^L(G)$ has an eigenvalue 0 with multiplicity at least 1. For example, the Seidel Laplacian spectrum of the cycle C_4, which is 2-regular is $spec_{S^L}(C_4) = \begin{Bmatrix} 0 & -4 \\ 3 & 1 \end{Bmatrix}$. However, the Seidel Laplacian matrix of complete graphs and null graphs (complement of complete graphs) has an eigenvalue 0 with multiplicity 1.*

Theorem 7.3.3. *Consider $G[H_1, H_2, ..., H_k]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ and and $S = [s_{i,j}]_{k\times k}$ is the Seidel matrix of G and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $\{\theta^L_{j1} = 0, \theta^L_{j2}, ..., \theta^L_{jn_j}\}$ be the Seidel Laplacian eigenvalues of H_j, for $j = 1, 2, ..., k$. Then, the Seidel Laplacian spectrum of the G-join of the graphs $H_1, H_2, ..., H_k$ is given by,*

$$Spec_{S^L}(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\theta^L_{ji} + \tau_j)\right) \cup Spec(T_{S^L}(G)),$$

where $\tau_j = \sum_{i=1}^{k} s_{i,j} n_i$ *and*

$$T_{S^L}(G) = \begin{bmatrix} \tau_1 & -s_{1,2}n_2 & \dots & -s_{1,k}n_k \\ -s_{1,2}n_1 & \tau_2 & \dots & -s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,k}n_1 & -s_{2,k}n_2 & \dots & \tau_k \end{bmatrix}.$$

Proof. Since H_j is regular, 0 is a Seidel Laplacian eigenvalue of H_j with corresponding eigenvector $\mathbf{1}_{n_j}$, for every j. Thus from equation (7.3), it is evident that each of the diagonal blocks $S^L(H_j) + \tau_j I_{n_j}$ is a symmetric matrix which has τ_j as an eigenvalue with corresponding eigenvector $\mathbf{1}_{n_j}$, for $j = 1, 2, ..., k$. Thus, as in (6.3), taking

$$M_j = S^L(H_j) + \tau_j I_{n_j}, \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = \left(\tau_j \ , \ \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j}\right)$$

and the real numbers

$$\rho_{l,q} = -s_{l,q}\sqrt{n_l n_q}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, and applying Theorem 6.2.2, the Seidel Laplacian spectrum of $G[H_1, H_2, ..., H_k]$ is obtained as,

$$Spec_{S^L}(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k} Spec\left(S^L(H_j) + \tau_j I_{n_j}\right) \backslash \{\tau_j\}\right) \cup Spec((\widetilde{L})), \tag{7.4}$$

where $\tau_j = \sum_{i=1}^{k} s_{i,j} n_i$ and $\widetilde{L} = \begin{bmatrix} \tau_1 & \rho_{1,2} & \dots & \rho_{1,k} \\ \rho_{1,2} & \tau_2 & \dots & \rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1,k} & \rho_{2,k} & \dots & \tau_k \end{bmatrix}$.

Obviously, $\rho_{l,q} = -s_{l,q}\sqrt{n_l n_q} = \begin{cases} \sqrt{n_l n_q} & \text{if } lq \in \text{E(G)} \\ -\sqrt{n_l n_q} & \text{if } lq \notin \text{E(G)} \end{cases}$ for $l \in \{1, 2, ..., k-$

$1\}$, $q \in \{l+1, ..., k\}$. Let $T_{S^L}(G)$ be the combinatorial vertex weighted Seidel Laplacian matrix of G, given by

$$T_{S^L}(G) = \begin{bmatrix} \tau_1 & -s_{1,2}n_2 & \dots & -s_{1,k}n_k \\ -s_{1,2}n_1 & \tau_2 & \dots & -s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,k}n_1 & -s_{2,k}n_2 & \dots & \tau_k \end{bmatrix}.$$

It is easy to verify that, $T_{S^L}(G) = W^{-\frac{1}{2}}\widetilde{L}W^{\frac{1}{2}}$, where W is given as in Section:6.3. Thus, $\widetilde{L}$ and $T_{S^L}(G)$ are similar and hence $Spec(\widetilde{L}) = Spec(T_{S^L}(G))$. Hence, from equation(7.4),

$$Spec_{S^L}(G[H_1, H_2, ..., H_k]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\theta_{ji}^L + \tau_j)\right) \cup Spec(T_{S^L}(G)). \tag{7.5}$$

$\square$

7.3.2 Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$

Theorem 7.3.4. *If $d_1, d_2, ..., d_{\xi(n)}$ are the proper divisors of n, then the Seidel Laplacian matrix of $\Gamma(\mathbb{Z}_n)$ is given by,*

$$S^L(\Gamma(\mathbb{Z}_n)) = \begin{bmatrix} S_1^L & -s_{1,2}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & \dots & -s_{1,\xi(n)}J_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} \\ -s_{1,2}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_2})} & S_2^L & \dots & -s_{2,\xi(n)}J_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,\xi(n)}J^T_{\phi(\frac{n}{d_1})\times\phi(\frac{n}{d_{\xi(n)}})} & -s_{2,\xi(n)}J^T_{\phi(\frac{n}{d_2})\times\phi(\frac{n}{d_{\xi(n)}})} & \dots & S^L_{\xi(n)} \end{bmatrix}$$

where,

$$S_j^L = \begin{cases} \left(\phi(\frac{n}{d_j}) + \sum_{i=1}^{\xi(n)} s_{ij}\phi(\frac{n}{d_i})\right) I_{\phi(\frac{n}{d_j})} - J_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ J_{\phi(\frac{n}{d_j})} + \left(\sum_{i=1}^{\xi(n)} s_{ij}\phi(\frac{n}{d_i}) - \phi(\frac{n}{d_j})\right) I_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases}$$

and

$$s_{i,j} = \begin{cases} -1 & \textit{if } n \mid d_i d_j \\ 1 & \textit{if } n \nmid d_i d_j \end{cases}.$$

Proof. In the proper divisor graph Υ_n, two vertices d_i and d_j are adjacent if and only if $n \mid d_i d_j$. Thus if $S = [s_{i,j}]_{\xi(n)\times\xi(n)}$ denotes the Seidel adjacency matrix of Υ_n, then

$$s_{i,j} = \begin{cases} -1 & \text{if } n \mid d_i d_j \\ 1 & \text{if } n \nmid d_i d_j \end{cases}.$$

By Lemma 4.3.4, $\mathit{\Gamma}(\mathbb{Z}_n)$ is the Υ_n - join of $\Gamma(\mathcal{A}(d_1)), \Gamma(\mathcal{A}(d_2)), ..., \Gamma(\mathcal{A}(d_{\xi(n)}))$, where the induced subgraphs $\Gamma(\mathcal{A}(d_j))$ are given by

$$\Gamma(\mathcal{A}(d_j)) = \begin{cases} \overline{K}_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ K_{\phi(\frac{n}{d_j})} & \text{if } n \mid d_j^2 \end{cases},$$

by Lemma 4.3.2. Also, the Seidel Laplacian matrix of K_n and $\overline{K_n}$ are given by

$$S^L(K_n) = J_n - nI_n,$$
$$S^L(\overline{K_n}) = nI_n - J_n.$$

Thus, if S_j^L denotes the j^{th} diagonal block in the Seidel Laplacian matrix of $\mathit{\Gamma}(\mathbb{Z}_n)$, which corresponds to the vertices of $\Gamma(\mathcal{A}(d_j))$, then, $S_j^L = S^L(\Gamma(\mathcal{A}(d_j))) + \tau_j I_{\phi(\frac{n}{d_j})}$ where, $\tau_j = \sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i})$. Hence,

$$S_j^L = \begin{cases} \phi(\frac{n}{d_j}) I_{\phi(\frac{n}{d_j})} - J_{\phi(\frac{n}{d_j})} + \tau_j I_{\phi(\frac{n}{d_j})} & \text{if } n \nmid d_j^2 \\ J_{\phi(\frac{n}{d_j})} - \phi(\frac{n}{d_j}) I_{\phi(\frac{n}{d_j})} + \tau_j I_{\phi(\frac{n}{d_j})} & \text{if } n/d_j^2 \end{cases}.$$

Thus the result follows equation (7.3), taking $G = \Upsilon_n$ and $H_j = \Gamma(\mathcal{A}(d_j))$. $\square$

Applying Theorem 7.3.3 and Theorem 7.3.4, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is determined in the following Corollary.

Corollary 7.3.5. *Let $d_1, d_2, ..., d_{\xi(n)}$ be the proper divisors of n. Then, $\Gamma(\mathbb{Z}_n)$ has Seidel Laplacian eigenvalues $\phi(\frac{n}{d_j})+\sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i})$ with multiplicity $\phi(\frac{n}{d_j})-1$ corresponding to the divisors d_j such that $n \nmid d_j^2$ and $\sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i}) - \phi(\frac{n}{d_j})$ with multiplicity $\phi(\frac{n}{d_j}) - 1$ corresponding to the divisors d_j such that $n \mid d_j^2$ and the remaining Seidel Laplacian eigenvalues are the eigenvalues of*

$$T_{S^L}(G) = \begin{bmatrix} \eta_1 & -s_{1,2}\phi(\frac{n}{d_2}) & \dots & -s_{1,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ -s_{1,2}\phi(\frac{n}{d_1}) & \eta_2 & \dots & -s_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}}) \\ \vdots & \vdots & \ddots & \vdots \\ -s_{1,\xi(n)}\phi(\frac{n}{d_1}) & -s_{2,k}\phi(\frac{n}{d_2}) & \dots & \eta_{\xi(n)} \end{bmatrix} \tag{7.6}$$

where $\eta_j = \sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i}), j = 1, 2, ..., \xi(n)$.

Theorem 7.3.6. 0 *ia a simple Seidel Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$, for any n.*

Proof. It is first shown that 0 is an eigenvalue of the matrix $T_{S^L}(G)$ of multiplicity 1. Arrange the proper divisors of n in the ascending order, $d_1 < d_2 < ... < d_{\xi(n)}$. It is obvious that $\phi(\frac{n}{d_1}) > \phi(\frac{n}{d_i}), i = 2, 3, ..., \xi(n)$. Note that, $T_{S^L}(G)$ is a square matrix of size $\xi(n)$. Since, $\eta_j = \sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i})$,

$$\eta_j + \sum_{i\neq j} -s_{i,j}\phi(\frac{n}{d_i}) = 0.$$

Hence it follows from equation (7.6) that, the sum of each row of $T_{S^L}(G)$ is zero. The first column of $T_{S^L}(G)$ can be transformed to the zero column on adding $2^{nd}, 3^{rd}, ..., \xi(n)^{th}$ columns to it. Hence, $T_{SL}(G)$ is singular which implies that 0 is an eigenvalue of $T_{S^L}(G)$. To prove that the multiplicity is 1, it suffices to prove that the rank of $T_{S^L}(G)$ is $\xi(n)-1$. For this, consider the matrix T obtained from $T_{S^L}(G)$ by deleting the first row and first column of $T_{S^L}(G)$. Since

$$s_{i,j}=\pm 1, \quad \phi(\frac{n}{d_1})>\phi(\frac{n}{d_i}), i=2,3,...,\xi(n) \quad \Rightarrow \quad |\eta_j|>\sum_{i\neq j, i\neq 1}|s_{i,j}\phi(\frac{n}{d_i})|.$$

Thus, T is strictly diagonally dominant. For example, consider the first row of T, say

$[\eta_2, -s_{2,3}\phi(\frac{n}{d_3}), -s_{2,4}\phi(\frac{n}{d_4}), ..., -s_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}})]$. Since, $\eta_2 = s_{1,2}\phi(\frac{n}{d_1})+s_{2,3}\phi(\frac{n}{d_3})+ .... + s_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}})$, it follows that $|\eta_2|>|s_{2,3}\phi(\frac{n}{d_3})|+...+|s_{2,\xi(n)}\phi(\frac{n}{d_{\xi(n)}})|$. Hence by Levy- Desplanques Theorem, T is non-singular, and hence the rank of $T_{S^L}(G)$ is $\xi(n)-1$.

By Corollary 7.3.5, the remaining Seidel eigenvalues of $\Gamma(\mathbb{Z}_n)$ are $\phi(\frac{n}{d_j})+\sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i})$ and $\sum_{i=1}^{\xi(n)} s_{i,j}\phi(\frac{n}{d_i})-\phi(\frac{n}{d_j})$, neither of which is zero. This proves the Theorem. □

Theorem 7.3.7. *For distinct primes p and q, $p<q$, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by*

$$Spec_{S^L}(\Gamma(\mathbb{Z}_{pq}))=\left\{\begin{array}{llll} 0 & -(p+q-2) & p-q & q-p \\ 1 & 1 & p-2 & q-2 \end{array}\right\}.$$

Proof. The proper divisors of pq are p and q. By Lemma 4.3.2 and Lemma 4.3.4, the zero-divisor graph $\Gamma(\mathbb{Z}_{pq})$ is the join of $\Gamma(\mathcal{A}(p))$ and $\Gamma(\mathcal{A}(q))$, where $\Gamma(\mathcal{A}(p))=\overline{K}_{q-1}$ and $\Gamma(\mathcal{A}(q))=\overline{K}_{p-1}$. That is

$$\Gamma(\mathbb{Z}_{pq})=\overline{K}_{q-1}\bigtriangledown \overline{K}_{p-1}.$$

Clearly $Spec_{S^L}(\overline{K}_{q-1}) = \left\{\begin{array}{ll} 0 & q-1 \\ 1 & q-2 \end{array}\right\}$ and $Spec_{S^L}(\overline{K}_{p-1}) = \left\{\begin{array}{ll} 0 & p-1 \\ 1 & p-2 \end{array}\right\}$. And the result follows from Theorem 7.3.1. □

Theorem 7.3.8. *For any prime p, the Seidel Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is*

$$Spec_{SL}(\Gamma(\mathbb{Z}_{p^3})) = \left\{ \begin{array}{ccc} 0 & 1-p^2 & p^2-2p+1 \\ 1 & p-1 & p^2-p-1 \end{array} \right\}.$$

Proof. The proper divisors of p^3 are p and p^2. By lemma 4.3.2 it can be seen that, the subgraphs of $\Gamma(\mathbb{Z}_{p^3})$, induced by $\mathcal{A}(p)$ and $\mathcal{A}(p^2)$ are $\overline{K}_{p(p-1)}$ and K_{p-1} respectively. Also by Lemma 4.3.4,

$$\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \bigtriangledown K_{p-1}.$$

$$Spec_{SL}(\overline{K}_{p(p-1)}) = \left\{ \begin{array}{cc} 0 & p(p-1) \\ 1 & p^2-p-1 \end{array} \right\},$$

$$Spec_{SL}(K_{p-1}) = \left\{ \begin{array}{cc} 0 & 1-p \\ 1 & p-2 \end{array} \right\}.$$

Hence the result follows from Theorem 7.3.1. □

Theorem 7.3.9. *For any prime p, the Seidel Laplacian spectrum of the zero-divisor graph $\Gamma(\mathbb{Z}_{p^4})$ is*

$$Spec_{SL}(\Gamma(\mathbb{Z}_{p^4})) = \left\{ \begin{array}{cccc} 0 & 1-p^3 & p^3-2p^2+1 & p^3-2p+1 \\ 1 & p-1 & p^2-p-1 & p^3-p^2 \end{array} \right\}.$$

Proof. The divisors of p^4 are p, p^2, p^3. The proper divisor graph of p^4 is the path P_3, in which $p \sim p^3 \sim p^2$. The subgraph induced by $\mathcal{A}(p)$ is the null graph $\overline{K}_{p^2(p-1)}$, whereas the subgraphs induced by $\mathcal{A}(p^2)$ and $\mathcal{A}(p^3)$ are the complete graphs $K_{p(p-1)}$ and K_{p-1} respectively. It can be seen that,

$$\Gamma(\mathbb{Z}_{p^4}) = P_3[\overline{K}_{p^2(p-1)}, K_{p(p-1)}, K_{p-1}].$$

Applying Corollary 7.3.5, $p^3 - 2p + 1, p^3 - 2p^2 + 1, 1 - p^3$ are Seidel Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^4})$ with multiplicities $p^3 - p^2 - 1, p^2 - p - 1$ and $p - 2$ respectively. And the remaining Seidel Laplacian eigenvalues of $\mathit{\Gamma}(\mathbb{Z}_{p^4})$ are the eigenvalues of the matrix,

$$T_{S^L}(\Upsilon_{p^4}) = \begin{bmatrix} p^2 - 2p + 1 & p - p^2 & p - 1 \\ p^2 - p^3 & p^3 - p^2 - p + 1 & p - 1 \\ p^3 - p^2 & p^2 - p & p - p^3 \end{bmatrix}.$$

It can be seen that the above matrix has three eigen values, $\lambda_1 = 0, \lambda_2 = 1 - p^3, \lambda_3 = p^3 - 2p + 1$. $\square$

7.4 Seidel signless Laplacian spectrum of the join of regular graphs

Note that, each diagonal block in the Seidel signless Laplacian matrix of the join of two regular graphs is a symmetric matrix which bears an eigenvalue with all-one vector as the corresponding eigenvector, which facilitates the use of Fiedler's Lemma in the investigation of its spectrum. Since the main theorems of this section are in the same frame work of Fiedlers Lemma, repetition of proofs is avoided, except in Theorem 7.4.1, where the concepts of Coronal of a square matrix and the Schur complement are incorporated.

Theorem 7.4.1. *Let G_i be r_i-regular graph of order n_i, for $i = 1, 2$. The Seidel signless Laplacian polynomial of $G_1 \bigtriangledown G_2$ is*

$$\Phi_{SQ}(G_1 \bigtriangledown G_2; \lambda) = \frac{(\lambda - \kappa_1)(\lambda - \kappa_2) - n_1 n_2}{(\lambda - \kappa_1)(\lambda - \kappa_2)} \cdot \Phi_{SQ}(G_1; \lambda + n_2) \cdot \Phi_{SQ}(G_2; \lambda + n_1)$$

where, $\kappa_1 = 2n_1 - n_2 - 4r_1 - 2$ *and* $\kappa_2 = 2n_2 - n_1 - 4r_2 - 2$.

Proof. Let $S^Q(G_1)$ and $S^Q(G_2)$ denote the Seidel signless Laplacian matrices of G_1 and G_2 respectively. Both $S^Q(G_1)$ and $S^Q(G_2)$ are symmetric and it can be seen that, their row sums are the constants $2(n_1 - 2r_1 - 1)$ and $2(n_2 - 2r_2 - 1)$ respectively. Also,

$$\begin{aligned} S^Q(G_1 \bigtriangledown G_2) &= \begin{bmatrix} S^Q(G_1) - n_2 I_{n_1} & -J_{n_1 \times n_2} \\ -J^T_{n_1 \times n_2} & S^Q(G_2) - n_1 I_{n_2} \end{bmatrix} \\ &= \begin{bmatrix} S_1^Q & -J_{n_1 \times n_2} \\ -J^T_{n_1 \times n_2} & S_2^Q \end{bmatrix}, \end{aligned}$$

where $S_1^Q = S^Q(G_1) - n_2 I_{n_1}$ and $S_2^Q = S^Q(G_2) - n_1 I_{n_2}$. Both S_1^Q and S_2^Q are symmetric and it can be easily seen that their row sums are the constants $\kappa_1 = 2(n_1 - 2r_1 - 1) - n_2$ and $\kappa_2 = 2(n_2 - 2r_2 - 1) - n_1$ respectively. Thus, the coronal of S_1^Q and S_2^Q are found to be $\Gamma_{S_1^Q}(\lambda) = \dfrac{n_1}{\lambda - \kappa_1}$ and $\Gamma_{S_2^Q}(\lambda) = \dfrac{n_2}{\lambda - \kappa_2}$ respectively. Applying Lemma 4.2.4, and Lemma 3.3.6, it can be seen that

$$det(\lambda I - S^Q(G_1 \bigtriangledown G_2)) = det(\lambda I - S_1^Q) \cdot det(\lambda I - S_2^Q) \cdot \Big(1 - \Gamma_{S_1^Q}(\lambda)\Gamma_{S_2^Q}(\lambda)\Big). \quad (7.7)$$

where $\Gamma_{S_1^Q}(\lambda) = \dfrac{n_1}{\lambda - (\kappa_1)}$ and $\Gamma_{S_2^Q}(\lambda) = \dfrac{n_2}{\lambda - (\kappa_2)}$.
Simplifying equation (7.7), the result is obtained. □

Generalizing the above theorem to the join of regular graphs, $G[H_1, H_2, ..., H_k]$, the next Theorem follows.

$$S^Q(G\left[H_1, H_2, ..., H_k\right]) = \begin{bmatrix} S^Q(H_1)+\nu_1 I_{n_1} & s_{1,2}J_{n_1\times n_2} & \dots & s_{1,k}J_{n_1\times n_k} \\ s_{1,2}J^T_{n_1\times n_2} & S^Q(H_2)+\nu_2 I_{n_2} & \dots & s_{2,k}J_{n_2\times n_k} \\ \vdots & & \ddots & \vdots \\ s_{1,k}J^T_{n_1\times n_k} & s_{2,k}J^T_{n_2\times n_k} & \dots & S^Q(H_k)+\nu_k I_{n_k} \end{bmatrix} \tag{7.8}$$

where $\nu_j = \sum_{i=1}^{k} s_{i,j}n_i$.

Theorem 7.4.2. *Consider $G\left[H_1, H_2, ..., H_k\right]$, where G is a simple connected graph with vertices labeled as $1, 2, ..., k$ and $S = [s_{i,j}]_{k\times k}$ is the Seidel matrix of G and H_j is r_j- regular and $|V(H_j)| = n_j$, for every $j = 1, 2, ..., k$. Let $\{\theta^Q_{j1} = 2(n_j - 2r_j - 1), \theta^Q_{j2}, ..., \theta^Q_{jn_j}\}$ be the Seidel signless Laplacian eigenvalues of H_j, for $j = 1, 2, ..., k$. Then, the Seidel signless Laplacian spectrum of the G-join of the graphs $H_1, H_2, ..., H_k$ is given by,*

$$Spec_{S^Q}(G\left[H_1, H_2, ..., H_k\right]) = \left(\bigcup_{j=1}^{k}\bigcup_{i=2}^{n_j}(\theta^Q_{ji} + \nu_j)\right) \cup Spec(T_{S^Q}(G))$$

where $\nu_j = \sum_{i=1}^{k} s_{i,j}n_i$ and

$$T_{S^Q}(G) = \begin{bmatrix} 2(n_1-2r_1-1)+\nu_1 & s_{1,2}n_2 & \dots & s_{1,k}n_k \\ s_{1,2}n_1 & 2(n_2-2r_2-1)+\nu_2 & \dots & s_{2,k}n_k \\ \vdots & \vdots & \ddots & \vdots \\ s_{1,k}n_1 & s_{2,k}n_2 & \dots & 2(n_k-2r_k-1)+\nu_k \end{bmatrix}$$

Proof. Since H_j is r_j-regular, $S^Q(H_j) = (n_j - 2r_j - 1)I_{n_j} + (S(H_j))$, where $S(H_j)$ is the Seidel matrix of H_j. By lemma 7.2.4, $n_j - 2r_j - 1$ is a Seidel eigenvalue of H_j with corresponding eigenvector $\mathbf{1}_{n_j}$. Thus, $S^Q(H_j)$ has eigenvalue $2(n_j - 2r_j - 1)$ with corresponding eigenvector $\mathbf{1}_{n_j}$, for every $j = 1, 2, ..., k$. Hence, as evident from equation (7.8), the j^{th} diagonal block of the Seidel signless Laplacian matrix of $G[H_1, H_2, ..., H_k]$ is the symmetric matrix $S^Q(H_j) + \nu_j I_{n_j}$ which has an eigenvalue $2(n_j - 2r_j - 1) + \nu_j$ with eigenvector $\mathbf{1}_{n_j}$, for $j = 1, 2, ..., k$. Thus, taking

$$M_j = S^Q(H_j) + \nu_j I_{n_j}, \quad (\alpha_{i_j,j}, \mathbf{u}_{i_j,j}) = \left(2(n_j - 2r_j - 1) + \nu_j \ , \ \frac{1}{\sqrt{n_j}}\mathbf{1}_{n_j}\right)$$

and the real numbers

$$\rho_{l,q} = s_{l,q}\sqrt{n_l n_q}$$

for $l \in \{1, 2, ..., k-1\}$, $q \in \{l+1, ..., k\}$, the result follows from Theorem 6.2.2. $\square$

Example 7.4.3. Consider the zero-divisor graph $\Gamma(\mathbb{Z}_{p^2q})$, where $p < q$ are distinct primes. The proper divisors of p^2q are $d_1 = p$, $d_2 = q$, $d_3 = p^2$, $d_4 = pq$. The proper divisor graph of p^2q is the path P_4, where $p \sim pq \sim p^2 \sim q$. The subgraphs induced by $\mathcal{A}(p), \mathcal{A}(q), \mathcal{A}(p^2), \mathcal{A}(pq)$ are $\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{(q-1)}$ and K_{p-1} respectively. Hence,

$$\Gamma(\mathbb{Z}_{p^2q}) = P_4[\overline{K}_{(p-1)(q-1)}, \overline{K}_{p(p-1)}, \overline{K}_{(q-1)}, K_{p-1}].$$

Applying Theorem 7.4.2, the Seidel signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^2q})$ is the multi set

$$\begin{Bmatrix} p^2 + pq - 3p - 1 & p^2 + pq - p - 2q - 1 & pq - p^2 - p - 1 & p^2 - pq - p + 3 \\ pq - p - q & p^2 - p - 1 & q - 2 & 3 - p \end{Bmatrix}$$

together with the spectrum of the matrix,

$T_{S^Q}(\Upsilon_{p^2q})=$

$$\begin{bmatrix} p^2+2pq-4p-q & p^2-p & q-1 & 1-p \\ (p-1)(q-1) & 2p^2+pq-2p-2q-1 & 1-q & p-1 \\ (p-1)(q-1) & p-p^2 & pq-p^2-p+q-2 & 1-p \\ -(p-1)(q-1) & p^2-p & 1-q & p^2-pq-2p+4 \end{bmatrix}.$$

Chapter 8

Conclusion And Further Scope Of Research

8.1 Summary of the Thesis

The first Chapter is the introductory Chapter which includes the review of literature of recent works on this area.

In the second Chapter, the preliminaries from Graph Theory and Matrix Theory are provided with due focus given to Spectral Graph Theory.

In the third Chapter, the characteristic polynomial of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ is found for $n = p^2q, p^2q^2$, where p and q are distinct primes. In the second Section, the adjacency matrix of $\Gamma(\mathbb{Z}_{p^2q^2})$ is explored. Also, the girth, diameter, stability number and clique number of this graph are traced. In Section 3, the characteristic polynomial of this graph along with the multiplicities of the eigenvalues 0 and -1 are found by direct computation using matrix tools. In Section 4, the adjacency matrix and eigenvalues of $\Gamma(\mathbb{Z}_{p^2q})$ are investigated.

Chapter 4 focuses on the generalisation of the results in Chapter 3. The spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^3, p^4$ where p is any prime, is found in the second Section of this Chapter. The third Section contains the analysis of the adjacency matrix of $\Gamma(\mathbb{Z}_{p^k})$ for any $k \geqslant 3$ and the investigation of some graph parameters of this graph. Also, the characteristic polynomial and two eigenvalues of the adjacency matrix of this graph, namely 0 and -1 are explored with multiplicities. In Section 4, a general method is proposed to compute the eigenvalues of $\Gamma(\mathbb{Z}_n)$ for any n, using the quotient matrix of equitable partition of its vertex set.

In Chapter 5, the distance matrix, distance Laplacian matrix and distance signless Laplacian matrix of the zero divisor graphs $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{p^3})$, for $p < q$, are found. Also the spectra of these matrices are found using the direct computation method. Some results from Linear Algebra and Matrix Theory are made use for this purpose.

Chapter 6 is the generalisation of the results of Chapter 5 from particular values of n to general. The role of Fiedler's Lemma and its generalization, to the computation of the distance spectrum of the generalized join of regular graphs is described in the second Section. The third Section contains the investigation of the distance spectrum of $\Gamma(\mathbb{Z}_n)$ for any n and in particular for $n = p^k$ for any prime p and $k \geqslant 3$ using Fiedler's Lemma. Also, the distance eigenvalues -1 and -2 are explored with multiplicities for the zero divisor graphs $\Gamma(\mathbb{Z}_n)$, for sny n. In Section 4, the computation of the distance Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$, for any n is described and the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^k}), k \geqslant 3$ is completely determined.

Chapter 7 contains the computation of the Seidel spectrum of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for any n. In Section 2, the Seidel spectrum of the generalized join of regular graphs is investigated. The Seidel spectrum of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$, is thereby computed in terms of the spectrum of the vertex weighted combinatorial matrix of the proper divisor graph of n. Sections 3 and 4 focuses on the investigation of Seidel Laplacian and Seidel signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ respectively.

8.2 Further Scope of Research

The following are listed as the topics for further research.

1. Study the connection between the spectral properties and the graph parameters of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for any n.
2. Explore the spectral radius and spectral gap of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for any n.
3. Investigate the bound for the largest and smallest eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$.
4. Study the relation between the spectra of $\Gamma(\mathbb{Z}_n)$ and the spectra of its proper divisor graph for any n.
5. Investigate the spectrum of the zero divisor graph on other commutative rings.
6. Extend the study of spectrum of zero divisor graph to non commutative rings.
7. Characterise rings on the spectral properties of the zero divisor graph.

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APPENDIX

List of Publications

1. P.M. Magi, Sr. Magie Jose, Anjaly Kishore, Adjacency matrix and eigenvalues of the zero divisor graph $\Gamma(\mathbb{Z}_n)$, J. Math. Comput. Sci., 10 (2020), 1285-1297.

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5. P.M. Magi, Sr. Magie Jose, Anjaly Kishore, Seidel spectrum of the zero divisor graph on the ring of integers modulo n, Advances and applications in discrete mathematics. Volume 28, No. 1, (2021), 145-167.

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