Ph.D. THESIS MATHEMATICS

A STUDY ON CAYLEY FUZZY GRAPHS AND CAYLEY FUZZY GRAPH STRUCTURES

Thesis submitted to the

University of Calicut

for the award of the degree of

DOCTOR OF PHILOSOPHY

in Mathematics
under the Faculty of Science

by

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NOVEMBER 2022

DEPARTMENT OF MATHEMATICS
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CERTIFICATE

I hereby certify that the thesis entitled “A STUDY ON CAYLEY FUZZY GRAPHS AND CAYLEY FUZZY GRAPH STRUCTURES” is a bonafide work carried out by **Ms. NEETHU K. T.,** under my guidance for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

Anil Kumar V.
(Research Supervisor)

DECLARATION

I hereby declare that the thesis, entitled "**A STUDY ON CAYLEY FUZZY GRAPHS AND CAYLEY FUZZY GRAPH STRUCTURES**" is based on the original work done by me under the supervision of **Dr. Anil Kumar V.**, Professor, Department of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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ACKNOWLEDGEMENT

I would like to place on record my heartfelt thanks for those who helped me, in numerous ways, in the completion of this dissertation.

With deep respect, I wish to express my sincere gratitude to my Research Supervisor, Dr. Anil Kumar V., Professor and Head, Department of Mathematics, University of Calicut for reviewing the entire draft with great care to details and more for his invaluable corrective commentary. To him I am much grateful more than what I can mention.

I like to extend my sincere gratitude to Dr. Ramachandran P. T., Retd. Professor and Head, Department of Mathematics, University of Calicut for the affectionate support that he has given me. I would like to thank Dr. Raji Pilakkat, Professor (Former Head), Department of Mathematics for extending necessary help at different stages of my work. I am also grateful to Dr. Preethi Kuttipulackal (Former Head) and Dr. Sini P. for their kind words and encouragement.

It is my previlage to thank my fellow researchers Dr. Chithra, Ms. Noushida, Dr. Dhanya, Dr. Kavitha, Dr. Vandana, Dr. Jisna, Ms. Sruthy, Dr. Seena, Dr. Shikhi, Dr. Jalsiya, Mr. Ajeesh, Mr. Midhun, Mr. Fasil, Mr. Saleel, Ms. Naheeda, Ms. Saira, Dr. Latheeshkumar, Dr. Vineesh and Dr. Libeesh for their cheerful companionship and discussions they had with me. I like to thank Babeesh, Beenechi and Praveen, the non teaching staff of Department of Mathematics, University of Calicut. Further, thanks to M. Phil and M. Sc. students of the department for their vibrant presence during the course of study.

A special thanks to my hostel mates, specially Dr. Neethu and Dr. Rahanas for supporting and helping me during my hard times.

It is my pleasure to acknowledge the support given by the management and Principal of Little Flower College, Guruvayoor. I am extremely grateful to the former Principal Sr. Philomena C A (Retd.)and Sr. Shirly T A (Retd. Associate Professor, Former Head, Department of Mathematics), Little Flower College for being so generous and supportive. I would like to thank all my colleagues in the Department of Mathematics for their support and prayers.

I also acknowledge my thanks to CSIR for the research fellowship they granted me.

A very special thanks to my parents, Mini K P and Sethumadhavan K T for their blessings on me. With immense pleasure I thank my husband Mr. Sameer K P for his support, encouragement and great patience at all times. He stood by me through the good times and bad. I gratefully acknowledge the love and support of my husband.

University of Calicut,
29 November 2022. Neethu K. T.

Contents

List of Symbols

A_α	α - cut
A_α^+	strong α - cut
$< A >$	the subloop generated by the bipolar fuzzy subset A
$Cay(V, A)$	Cayley graph induced by V and A a subset of V
$CayF(V, \nu)$	Cayley fuzzy graph induced by the loop V with respect to a fuzzy subset ν
$CayF_I(V, A)$	Cayley intuitionistic fuzzy graph induced by the loop V with respect to fuzzy subset A.
$CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$	Cayley fuzzy digraph structure of a group V with respect to $\nu_1, \nu_2, \ldots, \nu_n$.
$CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$	Cayley fuzzy digraph structure induced by the loop V with respect to $\nu_1, \nu_2, \ldots, \nu_n$.
$CayF_B(V, A)$	Cayley bipolar fuzzy graph induced by the loop $(V, *)$ with respect to A.
$CONN_G(x, y)$	strength of connectedness between x and y.
$d(u, v)$	distance between two vertices u nd v
$diam(G)$	diameter of a graph G
FR	fuzzy relation

$< \nu >$	fuzzy sub quasigroup generated by ν
$ind(u)$	in-degree of a vertex u
$outd(u)$	out- degree of a vertex u
$\mathcal{P}_G(x, y)$	set of all paths in $G = CayF(V, \nu)$ from x to y.
$S(G)$	strength connectivity of $G = CayF(V, \nu)$.
$strength(P)$	strength of a path P in $CayF(V, \nu)$
$supp(A)$	support of a fuzzy subset A
$S(\mu_R^P)$	μ_R^P -strength of a path.
$S(\mu_R^N)$	μ_R^N -strength of a path.

Chapter 0

Introduction

This thesis mainly deals with the study of Cayley fuzzy graphs and digraph structures induced by some algebraic structures. This thesis mainly focus on the study of Cayley fuzzy graphs, Cayley bipolar fuzzy graphs and Cayley intuitionistic fuzzy graphs induced by loops. Moreover, we study Cayley fuzzy digraph structure and Cayley bipolar fuzzy digraph structure induced by groups and loops.

In our day to day life, we have to deal with many problems, of which most of them are imprecise or vague concepts. If we represent such problems using the idea of classical set theory, the solution obtained may be far away from the reality. This reveals the importance of a set theory which can help fix this. Zadeh, in 1965, introduced the concept of a fuzzy set [14]. In his paper, a fuzzy set is defined as a class of objects with a continuium of grades of memberships. To each object a grade of membership ranging between zero and one is assigned by a membership function. Such set can completely be characterised by this membership function. The membership function coincides with the characteristic function in case of an ordinary set, where an object may belong to this set with membership value one or may not belong to this set with membership value zero. Hence we can view fuzzy sets as generalisation of ordinary sets.

As most of real life problems deals with vague or imprecise concepts, this theory of fuzzy sets has a wider application. A few among the real world problems where this theory has application are pattern recognition, information processing, increasing the efficiency of a system and multivalued decision processing.

As we already mentioned, the fuzzy set is a generalisation of fundamental mathematical concept of a set. Most of the mathematical theories can be extended using the concepts of a fuzzy set and fuzzy logic. Thus fuzzy mathematics has wider application than that of classical theories.

Rosenfeld first introduced the fuzzy graph theory as a generalisation of Euler's graph theory [1]. The fuzzy relation between fuzzy sets were first considered by him and also he developed the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts.

The concepts of Cayley fuzzy graphs were first introduced and studied by Namboothiri et. al. [15]. A natural question is: Does weaker algebraic structure induce fuzzy Cayley graphs?

This thesis comprises six chapters. The first chapter contains the preliminary definitions and results that were used in the remaining chapters. There are five sections in this chapter. The first section contains the preliminaries of graphs, second section includes that of fuzzy graphs, third of groups, fourth and fifth comprises respectively the preliminaries of Cayley graphs and Cayley Fuzzy graphs.

In second chapter, we introduced the concept of Cayley fuzzy graphs induced by loops and studied graph theoretic properties in terms of algebraic properties. In the first section of this chapter we defined Cayley fuzzy graphs induced by loops and proved that they are vertex transitive and also in-regular and out-regular. We also investigate the necessary and sufficient condition for Cayley fuzzy graphs to be symmetric, reflexive, transitive, complete, linear order, a Hasse diagram, etc.

In second section we discussed different types of connectedness including weakly connectedness, semi-connectedness etc., by investigating necessary and sufficient condition for these connectedness.

In the next section we discussed strength of connectedness and necessary and sufficient condition for different types of α- connectedness.

In the third chapter we introduced Cayley bipolar fuzzy graphs induced by loops and studied many basic properties. In this chapter we extend the results carried out in chapter 1. Here we have four sections, were we discussed basic results analogous to that of second chapter, connectedness and strength of connectedness.

In fourth chapter Cayley intuitionistic fuzzy graph is defined. The four sections of this chapter deals with many of the basic properties and connectedness of Cayley intuitionistic fuzzy graphs induced by loops.

In the next two chapters, fifth and sixth, we extend our studies to digraph structures induced by groups and also to those induced by loops.

Cayley fuzzy digraph structure induced by groups and loops are defined in the fifth chapter and its properties are studied in the two sections of this chapter.

In the sixth chapter we defined Cayley bipolar fuzzy digraph structure induced by groups and loops and studied its properties and we obtained results analogues to that in fifth chapter.

Chapter 1

Preliminaries

In this chapter we list the basic definitions and results used in the upcoming chapters. There are five sections in this chapter. Definitions of basic terminologies related to graph is included in section one. The second section deals with fuzzy graphs. We follow [12] for standard terminology and notations in fuzzy set theory. The third section is devoted to Cayley graphs. In the fourth section we discuss Cayley fuzzy graphs. In the last section of this chapter group and its weaker structure viz; loops are discussed.

1.1 Graphs

A *graph* $G = (V(G), E(G))$ is an ordered pair, where $V(G)$ is a non-empty set and $E(G)$ is a binary relation on $V(G)$. The elements of $V(G)$ are called *vertices* and elements of $E(G)$ are called *edges*. If there is no ambiguity we simply write $G = (V, E)$ or just G instead of $G = (V(G), E(G))$ and if $e = \{u, v\}$, where $e \in E$ and $u, v \in V$, we simply write $e = uv$. An edge of the form $(x, x) \in V(G) \times V(G)$ is called a loop.

If $e = uv$ is an edge of G then the two vertices u and v are said to be *adjacent* to each other and the edge e is said to incident with (incident to or incident at)

u and v [13]. The vertices u and v are called the end vertices of the edge e.

The indegree $ind(u)$ of a vertex u in G is the number of arcs with head u; the outdegree $outd(u)$ of u is the number of arcs with tail u.

A *walk* in G is a finite non-null sequence $W = v_o e_1 v_1 e_2 v_2 \ldots e_k v_k$, whose terms are alternately vertices and edges, such that, for $1 \leq i \leq k$, the end of e_i are v_{i-1} and v_i. We say that W is a walk from v_0 to v_k or a $(v_0, v_k) - walk$ [11]. The vertices v_0 and v_k are called the origin and terminals of W, respectively, and $v_1, v_2, \ldots, v_{k-1}$, its internal vertices. The integer k is the length of W.

If the edges $e_1, e_2, \ldots, e_k$ of a walk W are distinct, W is called a *trail* [11]. If, in addition, the vertices $v_0, v_1, \ldots, v_k$ are distinct, W is called a *path*. A walk is closed if it has positive length and its origin and terminals are the same. A closed trail whose origin and internal vertices are distinct is a *cycle*.

A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph* [11]. Two vertices u and v of G are said to be *connected* if there is a $(u, v) - path$ in G. A graph G is said to be connected if two vertices of G are linked by a path in G. Otherwise, it is a disconnected graph [28].

1.2 Fuzzy Graphs

A fuzzy subset A of any set X is a function $A : X \rightarrow [0, 1]$. For a fuzzy subset A of X and for $\alpha \in [0, 1]$, $\{x : A(x) \geq \alpha\}$ is called *α-cut* of A and $\{x : A(x) > \alpha\}$ is called the *strong α-cut* of A. They are, respectively, denoted by A_α and A_α^+. For a fuzzy subset A of X, the *support of* A is the set $\{x \in X : A(x) > 0\}$ and is denoted by $supp(A)$. It can be noted that $supp(A) = A_\circ^+$. Let S and T be two sets and let μ and ν be fuzzy subsets of S and T, respectively. Then a fuzzy relation ρ from μ to ν is a fuzzy subset of $S \times T$ such that $\rho(x, y) \leq \mu(x) \wedge \nu(y)$ for all $x \in S$ and $y \in T$. If $S = T$ and $\mu(x) = \nu(x) = 1$ for all $x \in X$, then ρ is said to be a fuzzy relation on S. Let ρ be a fuzzy relation from a fuzzy subset μ of S into a fuzzy subset ν of T . Then ρ^2 is defined by $\rho^2(x,z) = \vee\{\rho(x,y) \wedge \omega(y,z) : y \in S\}$ for all $x \in S$.

Let ρ be a fuzzy relation on a fuzzy subset μ of S. Then ρ is said to be reflexive if $\rho(x,x) = \mu(x)$ for all $x \in S$; symmetric if $\rho(x,y) = \rho(y,x)$ for all $x, y \in S$; antisymmetric if $\rho(x,y) = \rho(y,x)$ if and only if $x = y$; transitive if $\rho^2 \leq \rho$; fuzzy partial order if it is reflexive, antisymmetric and transitive; a fuzzy equivalence relation if it is reflexive, symmetric and transitive; fuzzy linear order if it is a partial order and $(\rho \vee \rho^{-1})(x,y) > 0$ for all $x, y \in S$.

A *fuzzy directed graph (fuzzy digraph)* G is a triplet (V, μ, ρ), where V is a non-empty set, μ is a fuzzy subset of V and ρ is a fuzzy relation on μ. In case $\mu = \chi_V$, where χ_V is the characteristic function on V, then the fuzzy digraph (V, μ, ρ) is simply denoted by $G = (V, \rho)$. Furthermore, G is said to be a *fuzzy graph* if the fuzzy relation is symmetric. In this paper, we consider fuzzy digraphs of the form $G = (V, \rho)$. Let $G = (V, \rho)$ and $G' = (V', \rho')$ be two fuzzy digraphs. Then G is said to be *isomorphic* to G' if there is a bijection $f : V \to V'$ such that for all $u, v \in V$, $\rho(u,v) = \rho'(f(u), f(v))$. The function f is called an *isomorphism* from G into G'. An isomorphism from a digraph G into itself is called an *automorphism*. Observe that, if $(V, *)$ is a group and ν is a fuzzy subset of V, then $R : V \times V \to [0,1]$ defined by $R(x,y) = \nu(x^{-1}y)$ for all $x, y \in V$ is a fuzzy relation on V.
Let $G = (V, \rho)$ be a fuzzy digraph. If $u \in V$, then the in-degree of u, denoted by $ind(u)$, is defined by $\text{ind}(u) = \sum\limits_{v \in V} \rho(v,u)$. Similarly, the out-degree of u, denoted by $outd(u)$, is defined by $\text{outd}(u) = \sum\limits_{v \in V} \rho(u,v)$.

A fuzzy digraph in which each vertex has same out-degree r is called an *out-regular digraph* with index of out-regularity r. In-regular digraphs are similarly defined. Let $G = (V, R)$ be a fuzzy digraph. Let k and k' be two positive numbers. Then G is said to be (k,k')-*regular* if $ind(u) = k$ and $outd(u) = k'$ for all $u \in V$. A fuzzy digraph is said to be regular if it is (k,k)-*regular* for some positive number k. Let $G = (V, R)$ be a fuzzy digraph. Then a path (directed path) of length n in G from a vertex x to a vertex y is a sequence of distinct vertices $x = x_\circ, x_1, \ldots, x_n = y$ such that $R(x_{i-1}, x_i) > 0$ for $1 \leq i \leq n$. A fuzzy digraph $G = (V, R)$ is said to be: (i)*connected (strongly connected)* if for all $x, y \in V$, there is a directed path from x to y, (ii)*weakly connected* if $G' = (V, R \vee R^{-1})$ is connected, (iii)*semi-connected* if for all $x, y \in V$, there is a directed path from x to y or a directed path from y to x in G, (iv)*locally connected*, if for any $x, y \in V$, there is a directed path from x to y whenever there is a directed path from y to x in G, (v)*quasi-connected (strongly quasi-connected)* if for every pair $x, y \in V$, there is some $z \in V$ such that there is a directed path from z to x and there is a directed path from z to y, (vi)*Hasse diagram*, if G is connected and for any path $x_\circ, x_1, \ldots, x_n,\ n \geq 2$ from $x_\circ$ to x_n in G, $R(x_\circ, x_n) = 0$ and (vii)*complete* if $R(x, y) = 1$ for all distinct $x, y \in V$. A vertex x in G is said to be a *source* in G if there is a directed path from x to every other vertex in G. Let $G = (V, R)$ be a fuzzy graph. The distance between two points u and v in G, $d(u, v)$, is the length of the shortest path from u to v. If there is no path from u to v, then we define $d(u, v) = \infty$. The diameter of a fuzzy graph $G = (V, R)$, denoted by $diam(G)$, is defined as $\text{diam}(G) = sup\{d(u, v) : u, v \in V\}$. Observe that a finite digraph has a source if and only if it is quasi-connected [13].

1.2.1 Bipolar fuzzy graphs

Let V be a nonempty set. A *bipolar fuzzy set* B in V is an object of the form $B = \{(x, \mu_B^P(x), \mu_B^N(x)) : x \in V\}$, where μ_B^P and μ_B^N are respectively the functions, $\mu_B^P : V \to [0, 1]$ and $\mu_B^N : V \to [-1, 0]$ [30]. A *bipolar fuzzy relation* $R = (\mu_R^P(x, y), \mu_R^N(x, y))$ in a universe $X \times Y$ is a bipolar fuzzy set of the form $R = \{((x, y), \mu_R^P(x, y), \mu_R^N(x, y)) : (x, y) \in X \times Y\}$, where $\mu_R^P : X \times Y \to [0, 1]$ and $\mu_R^N : X \times Y \to [-1, 0]$ [30]. A *bipolar fuzzy digraph* of a digraph (V, E) is a pair (A, B) where $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy set in V and $B = (\mu_B^P, \mu_B^N)$ is a bipolar fuzzy relation on $E \subseteq V \times V$ such that $\mu_B^P(x, y) \leq min(\mu_A^P(x), \mu_A^P(y)),\ \mu_B^N(x, y) \geq max(\mu_A^N(x), \mu_A^N(y))$ for all $x, y \in V$ [17].

1.2.2 Intuitionistic fuzzy graphs

An *intuitionistic fuzzy set* (IFS, for short) on a universe X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in V\}$, where $\mu_A(x) \in [0,1]$ is called degree of membership of x in A and $\nu_A(x) \in [0,1]$ is called degree of nonmembership of x in A, and $\mu_A,\ \nu_A$ satisfies the following condition for all $x \in X,\ \mu_A(x)+\nu_A(x) \leq 1$ [19]. Let X be an intuitionistic fuzzy set. For any subset A and for any $\alpha \in [0,1]$, $\{\mu_A(x) \geq \alpha, \nu_A(x) \leq \alpha\}$ is called α-cut of A. It is denoted by A_α. And $\{\mu_A(x) > \alpha, \nu_A < \alpha\}$ is called *strong* α-cut of A. It is denoted by A^+_α. An *intuitionistic fuzzy relation* $R = (\mu_R, \nu_R)$ in $X \times X$ is an intuitionistic fuzzy set of the form $R(x,y) = \{\langle (x,y), \mu_R(x,y), \nu_R(x,y)\rangle : (x,y) \in X\}$, where $\mu_R : X \times X \to [0,1]$ and $\nu_R : X \times X \to [0,1]$ [19]. The intuitionistic fuzzy relation R satisfies $\mu_R(x,y)+\nu_R(x,y) \leq 1$ for all $x, y \in X$. An intuitionistic fuzzy relation R on X is called an *intuitionistic fuzzy equivalence relation* on X if it is (a) reflexive, i.e.,$R(x,x) = (1,0)$, (b) symmetric, i.e., $R(x,y) = R(y,x)$, and (c) transitive, i.e., $R(x,y) \geq \bigvee_{z\in X}(R(x,z) \wedge R(z,y))$, for all $x, y \in X$. R is called an *intuitionistic fuzzy partial order relation* if it is (a) reflexive, (b) antisymmetric, i.e., for all distinct $x, y \in X$ $R(x,y) \neq R(y,x)$, and (c) transitive. R is called an *intuitionistic fuzzy linear order relation* if it is an intuitionistic partial order relation and for all $x, y \in X$, $(R \vee R^{-1})(x,y) > 0$. An *intuitionistic fuzzy digraph* of a digraph (V, E) is a pair $G = (A, B)$, where $A = \langle V, \mu_A, \nu_A\rangle$ is an intuitionistic fuzzy set in V and $B = \langle V \times V, \mu_B, \nu_B\rangle$ is an intuitionistic fuzzy relation on V such that $\mu_B(x,y) \leq min(\mu_A(x), \mu_A(y))$ and $\nu_B(x,y) \leq max(\nu_A(x), \nu_A(y))$, $0 \leq \mu_B(x,y) + \nu_B(x,y) \leq 1$ for all $x, y \in V$ [19].

1.3 Group

A groupoid V is called a *quasigroup*, if for every $a, b \in V$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in V [7]. This implies both left and right cancelation laws. A quasigroup with an identity element is called a *loop* [7].

Let G be a set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say G is a *group* under this operation if the following three properties are satisfied [10].

1. The operation is associative; that is, $(ab)c = a(bc)$ for all $a, b, c \in G$.
2. There is an element e (called the identity) in G such that $ae = ea = a$ for all $a \in G$.
3. For each element $a \in G$, there is an element $b \in G$ (called an inverse of a) such that $ab = ba = e$.

If a subset H of a group G is itself a group under the operation of G, we say that H is a subgroup of G [10].

Example 1.3.1. *Under ordinary addition, the set of integers, the set of rationals and the set of real numbers are all groups.*

Example 1.3.2. *The set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ for $n \geq 1$ is a group under addition modulo n.*

1.4 Cayley Graphs

Let V be a group and A be any subset of V. The *Cayley graph* $\mathrm{Cay}(V, A)$ is the digraph with vertex set V, and the vertex x is adjacent to the vertex y if and only if $x^{-1}y \in A$. Cayley graph $\mathrm{Cay}(V, A)$ has as its vertex-set and edge-set, respectively, V and $R = \{(x, y) : x^{-1}y \in A\}$ [15].

Example 1.4.1. *Let us consider the group $\mathbb{Z}_5$ and let $A = \{3, 4\}$. Then*

$$R = \{(0,3), (0,4), (1,0), (1,4), (2,0), (2,1), (3,1), (3,2), (4,2), (4,3)\}.$$

A diagramatic representation of $Cay(\mathbb{Z}_5, \{3, 4\})$ is shown in Figure 1.1

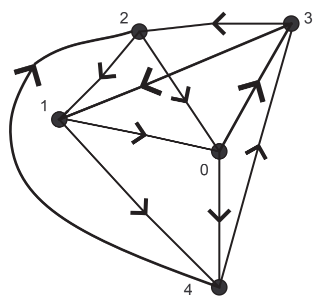

Figure 1.1: The Cayley graph $Cay(\mathbb{Z}_5, \{3,4\})$

1.5 Cayley Fuzzy Graphs

Let V be a group and ν be a fuzzy subset of V. Then the fuzzy relation R defined on V by $R(x,y) = \nu(x^{-1}y)$ for all $x, y \in V$ induces a fuzzy graph $G = (V, R)$ called the Cayley fuzzy graph induced by the pair (V, ν) [15].

Example 1.5.1. Let us consider the group $\mathbb{Z}_3 = 0, 1, 2$ and take $V = \mathbb{Z}_3$. Define $\nu : V \to [0,1]$ *by* $\nu(0) = 1, \nu(1) = \frac{1}{2}, \nu(2) = 0$. Then the Cayley fuzzy graph (V, R) induced by $(\mathbb{Z}_3, \nu)$ is given by the following table and figure *1.5.1.*

a	*0*	*0*	*0*	*1*	*1*	*1*	*2*	*2*	*2*
b	*0*	*1*	*2*	*0*	*1*	*2*	*0*	*1*	*2*
$a^{-1}b$	*0*	*1*	*2*	*2*	*0*	*1*	*1*	*2*	*0*
$\nu(a^{-1}b)$	*1*	$\frac{1}{2}$	*0*	*0*	*1*	$\frac{1}{2}$	$\frac{1}{2}$	*0*	*1*

Let $(V, *)$ be a group and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V. Then the bipolar fuzzy relation R on V defined by

$$R(x,y) = \left\{ \left(\mu_A^P(x^{-1}y), \mu_A^N(x^{-1}y) \right) \ \forall\ x, y \in V \right\}$$

induces a bipolar fuzzy digraph $G = (V, R)$, called the *Cayley bipolar fuzzy graph* induced by the triplet $(V, *, A)$ [24].

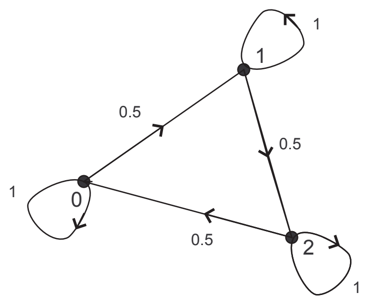

Figure 1.2: Ex.1.5.1

The *Cayley intuitionistic fuzzy graph* $G = (V, R)$ in [19] introduced by M. Akram et al. is an intuitionistic fuzzy graph with the vertex set V and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of V. The intuitionistic fuzzy relation on V is defined by

$$R(x, y) = \{(\mu_A(x^{-1}y), \nu_A(x^{-1}y)) : x, y \in V \text{ and } x^{-1}y \in S\}.$$

Chapter 2

Cayley Fuzzy Graphs Induced by Loops

In this chapter we introduce Cayley fuzzy graphs induced by loops and prove that all Cayley fuzzy graphs induced by loops are vertex-transitive and hence regular. We prove that a bigger class of Cayley fuzzy graphs could be induced by loops, a weaker algebraic structure than a group. Contents of this chapter is published in Far East Journal of Mathematical Sciences [20].

2.1 Cayley fuzzy graphs

Madhavan Namboothiri N.M. et al. [15] in 2013 introduced a class of Cayley fuzzy graphs induced by groups and studied the properties of Cayley fuzzy graph in terms of algebraic properties. We generalise the results of Madhavan Namboothiri N.M. et al. to loops, a weaker algebraic structure than groups.

Definition 2.1.1. *Let V be a loop and A be any subset of V. Then the Cayley graph $Cay(V, A)$ is the graph with vertex set V and edge set $R = \{(x, y) : y/x \in A\}$, where y/x denotes the solution of the equation $y = xz$.*

Definition 2.1.2. *Let V be a loop. A fuzzy subset ν on V is called a scaled fuzzy* *subset of* V *if* $\nu(y/x) = \nu(zy/zx)$*, for all* $x, y, z \in V$*.*

Definition 2.1.3. *Let* V *be a loop,* ν *be a scaled fuzzy subset of* V*. Define a fuzzy relation* FR *on* V *by* $FR(x, y) = \nu(y/x)$*. Then* $G = (V, FR)$ *is a fuzzy digraph called Cayley fuzzy graph induced by the loop* V*. This graph is denoted by* $CayF(V, \nu)$*.*

Definition 2.1.4. *Let* V *be a loop and* A *be a subset of* V*. Then* A *is called a right associative subset of* V*, if* $(xy)A = x(yA)$ *for all* $x, y \in V$*. This means, if* $x, y \in V$ *and* $a \in A$*, then* $(xy)a = x(ya')$ *for some* $a' \in A$*.*

Lemma 2.1.5. *Let* ν *be a scaled fuzzy subset of a loop* V*. Then supp*$(\nu) = \nu_\circ^+$ *is a right associative subset of* V*.*

Proof. Let $x, y \in V$, and $z \in \nu_\circ^+$. We have $(xy)z = x(yz')$, for some $z' \in V$. Observe that

$$\begin{aligned} z \in \nu_\circ^+ &\Leftrightarrow \nu(z) > 0 \\ &\Leftrightarrow \nu((xy)z/xy) > 0 \\ &\Leftrightarrow \nu(x(yz')/xy) > 0 \\ &\Leftrightarrow \nu(yz'/y) > 0 \\ &\Leftrightarrow \nu(z') > 0 \\ &\Leftrightarrow z' \in \nu_\circ^+. \end{aligned}$$

Therefore, $(xy)z = x(yz')$ for some $z' \in \nu_\circ^+$. This implies that $\nu_\circ^+$ is right associative. □

Theorem 2.1.6. $CayF(V, \nu)$ *is vertex transitive.*

Proof. Let $a, b \in V$ and $b = z_\circ a$, for some $z_\circ \in V$. Define $\Psi : V \to V$ by $\psi(x) = z_\circ x$. Clearly, ψ is one-to-one and onto. Also, $\psi(a) = z_\circ a = b$.
Furthermore, we have, for each $x, y \in V$, $FR(\psi(x), \psi(y)) = FR(z_\circ x, z_\circ y) = \nu(z_\circ y/z_\circ x) = \nu(y/x) = FR(x, y)$.

This implies that for $a, b \in V$, ψ is an automorphism on G such that $\psi(a) = b$. This implies that $CayF(V, \nu)$ is vertex-transitive. □

Theorem 2.1.7. *$CayF(V, \nu)$ is in-regular and out-regular.*

Proof. Let $u_1, u_2 \in V$. Since $CayF(V, \nu)$ is vertex transitive, there exist an automorphism ψ on V such that $\psi(u_1) = u_2$.
Then,

$$\begin{aligned} ind(u_1) &= \sum_{v \in V} FR(v, u_1) \\ &= \sum_{v \in V} FR(\psi(v), \psi(u_1)) \quad \text{(since vertex transitive)} \\ &= \sum_{\psi(v) \in V} FR(\psi(v), u_2) \\ &= \sum_{u \in V} FR(u, u_2) \\ &= ind(u_2). \end{aligned}$$

Similar proof holds for out-regularity □

Theorem 2.1.8. *$CayF(V, \nu)$ is symmetric if and only if $\nu(x) = \nu(1/x)$, for all $x \in V$.*

Proof. Suppose that FR is symmetric. Then, $FR(x, y) = FR(y, x)$, for all $x, y \in V$. Therefore, $\nu(x) = FR(1, x) = FR(x, 1) = \nu(1/x)$.

Conversely, suppose that $\nu(x) = \nu(1/x)$ for all $x \in V$. Let $x, y \in V$ and $y = xt_\circ$, for some $t_\circ \in V$. Then, $FR(x, y) = \nu(y/x) = \nu(t_\circ) = \nu(1/t_\circ) = \nu(x/xt_\circ) = \nu(x/y) = FR(y, x)$.

This implies that FR is symmetric. □

Theorem 2.1.9. *$CayF(V, \nu)$ is reflexive if and only if $\nu(1) = 1$.*

Proof. First suppose that FR is reflexive. Then $FR(x,x) = \mu(x) = 1$, since $\mu = \chi_v$. Also by definition, $FR(x,x) = \nu(x/x) = \nu(1)$. Therefore, $\nu(1) = 1$.

Now, let $\nu(1) = 1$. Then, $FR(x,x) = \nu(x/x) = \nu(1) = 1$, for all $x \in V$. Therefore, $FR(x,x) = 1 = \mu(x)$, since $\mu = \chi_v$. This implies that FR is reflexive. Hence the proof. □

Theorem 2.1.10. *$CayF(V,\nu)$ is antisymmetric if and only if*

$$\{x : \nu(x) = \nu(1/x)\} = \{1\}.$$

Proof. First suppose that FR is antisymmetric. Let $x \in \{x : \nu(x) = \nu(1/x)\}$. Then $\nu(x) = \nu(1/x)$, which implies $FR(1,x) = FR(x,1)$ and thus $x = 1$. Hence, $\{x : \nu(x) = \nu(1/x)\} = \{1\}$. Conversely, suppose that $\{x : \nu(x) = \nu(1/x)\} = \{1\}$. Then, for any $x, y \in V$,

$$\begin{aligned} FR(x,y) = FR(y,x) &\Leftrightarrow \nu(y/x) = \nu(x/y) \\ &\Leftrightarrow \nu(xt_\circ/x) = \nu(x/xt_\circ), \text{ where } y = xt_\circ \\ &\Leftrightarrow \nu(t_\circ) = \nu(1/t_\circ) \\ &\Leftrightarrow t_\circ = 1 \quad \text{(by assumption)} \\ &\Leftrightarrow y = x. \end{aligned}$$

Therefore, FR is antisymmetric. This completes the proof. □

Definition 2.1.11. *Let V be a loop. Let A be a fuzzy subset of V. Then A is said to be a fuzzy sub quasigroup of V if, for all $a, b \in V$, $A(ab) \geq A(a) \wedge A(b)$.*

Theorem 2.1.12. *$CayF(V,\nu)$ is transitive if and only if ν is a fuzzy sub quasigroup of V.*

Proof. First assume that FR is transitive. We have,

$$\nu(x) \wedge \nu(y) \leq \vee\{\nu(z) \wedge \nu(xy/z) : z \in V\}$$

$$= \vee\{FR(1,z) \wedge FR(z,xy) : z \in V\}$$
$$= FR^2(1,xy) \leq FR(1,xy) = \nu(xy).$$

Therefore, $\nu(x) \wedge \nu(y) \leq \nu(xy)$, which implies that ν is a fuzzy sub quasigroup of V.

Conversely, let $\nu(x) \wedge \nu(y) \leq \nu(xy)$, for all $x, y \in V$. For any $x, y \in V$, choose an arbitrary $z \in V$. Then there exist unique $t_\circ, t_1, t \in V$ such that $z = xt_\circ,\ y = zt_1,\ y = xt$.

Then,

$$\begin{aligned}
FR(x,y) &= \nu(y/x) \\
&= \nu((xt_\circ)t_1/x) \\
&= \nu(x(t_\circ t_1')/x) \text{for some } t_1' \in \text{V} \\
&= \nu(t_\circ t_1') \\
&\geq \nu(t_\circ) \wedge \nu(t_1') \text{(by assumption)} \\
&= \nu(z/x) \wedge \nu(t_\circ t_1'/t_\circ) \\
&= \nu(z/x) \wedge \nu(x(t_\circ t_1')/xt_\circ) \\
&= \nu(z/x) \wedge \nu((xt_\circ)t_1/xt_\circ) \\
&= \nu(z/x) \wedge \nu(y/z).
\end{aligned}$$

That is, $FR(x,y) \geq \nu(z/x) \wedge \nu(y/z)$ for any $z \in V$. Therefore, $FR(x,y) \geq \vee\{\nu(z/x) \wedge \nu(y/z) : z \in V\}$, which implies that $FR(x,y) \geq \vee\{FR(x,z) \wedge FR(z,y) : z \in V\} = FR^2(x,y)$. Hence FR is transitive. □

Theorem 2.1.13. *The fuzzy relation FR is a partial order if and only if ν is a fuzzy sub quasigroup of V satisfying :*

(i) $\nu(1) = 1$, *and*

(ii) $\{x : \nu(x) = \nu(1/x)\} = \{1\}$.

Theorem 2.1.14. *The fuzzy relation FR is a linear order if and only if ν is a fuzzy sub quasigroup of V satisfying :*

$$\begin{aligned} &(i) \quad \nu(1) = 1, \\ &(ii) \ \{x : \nu(x) = \nu(1/x)\} = \{1\}, \ \textit{and} \\ &(iii) \ \{x : \nu(x) \vee \nu(1/x) > 0\} = V. \end{aligned}$$

Proof. Suppose that FR is a linear order. Then by Theorem 2.1.13 (i) and (ii) are satisfied. Also, $(FR \vee FR^{-1})(x, y) > 0$ for $x, y \in V$. Then, for $x \in V$,

$$\begin{aligned} \nu(x) \vee \nu(1/x) &= FR(1, x) \vee FR(x, 1) \\ &= FR(1, x) \vee FR^{-1}(1, x) \\ &= (FR \vee FR^{-1})(1, x) > 0. \end{aligned}$$

This implies that $x \in \{x : \nu(x) \vee \nu(1/x) > 0\}$. Therefore, $V \subseteq \{x : \nu(x) \vee \nu(1/x) > 0\}$. Also, we have $\{x : \nu(x) \vee \nu(1/x) > 0\} \subseteq V$. Hence $\{x : \nu(x) \vee \nu(1/x) > 0\} = V$.

Conversely, suppose that the conditions (i), (ii) and (iii) hold. Then, for $x, y \in V$,

$$\begin{aligned} FR \vee FR^{-1}(x, y) &= FR(x, y) \vee FR^{-1}(x, y) \\ &= \nu(y/x) \vee \nu(x/y) \\ &= \nu(xt/x) \vee \nu(x/xt), \text{where} t = y/x \\ &= \nu(t) \vee \nu(1/t) \\ &> 0, \text{by assumption}(iii). \end{aligned}$$

That is, $FR \vee FR^{-1}(x, y) > 0$. Hence FR is a linear order. □

Theorem 2.1.15. *The fuzzy relation FR is an equivalence relation if and only if ν is a fuzzy sub quasigroup of V satisfying :*

$$(i) \ \nu(1) = 1 \ \textit{and}$$

$$(ii)\ \nu(x) = \nu(1/x)\ for\ all\ x \in V.$$

Theorem 2.1.16. *The Cayley fuzzy graph $CayF(V, \nu)$ is regular.*

Proof. Observe that, for $u \in V$

$$\begin{aligned}
ind(u) &= \sum_{v \in V} FR(v, u) \\
&= \sum_{v \in V} \nu(u/v) \\
&= \sum_{z_u \in V} \nu(z_u), \qquad \text{where,} z_u = u/v \\
&= \sum_{z \in V} \nu(z) \\
&= \sum_{z_v \in V} \nu(z_v), \qquad \text{where} z_v = v/u \\
&= \sum_{v \in V} \nu(v/u) \\
&= \sum_{v \in V} FR(u, v) \\
&= outd(u).
\end{aligned}$$

This completes the proof, since $CayF(V, \nu)$ are in-regular and out-regular. □

Theorem 2.1.17. *$CayF(V, \nu)$ is complete if and only if $\nu \geq \chi_{V-\{1\}}$, where $\chi_{V-\{1\}}$ is the characteristic function of $V - \{1\}$.*

Proof. Suppose $CayF(V, \nu)$ is complete. Then for any two distinct $x, y \in V$, $FR(x, y) = 1$. For any $z \in V - \{1\}$, there exist some $x, y \in V$ such that $y = xz$. Therefore, since $FR(x, y) = 1$, $\nu(z) = \nu(y/x) = FR(x, y) = 1$. That is, $\nu(z) = 1$ for every $z \in V - \{1\}$, which implies that $\nu \geq \chi_{V-\{1\}}$.

Conversely, suppose that $\nu \geq \chi_{V-\{1\}}$. Then, $\nu(z) = 1$ for every $z \in V - \{1\}$. Also, for any two distinct $x, y \in V$ there exist some $t \in V - \{1\}$ such that

$y = xt$. Therefore, $FR(x, y) = \nu(y/x) = \nu(t) = 1$, since $\nu \geq \chi_{V-\{1\}}$. That is, $FR(x, y) = 1$ for any two distinct $x, y \in V$. Hence $CayF(V, \nu)$ is complete. $\square$

Theorem 2.1.18. *$CayF(V, \nu)$ is a Hasse diagram if and only if it is connected and $\nu\left((\ldots((x_1x_2)x_3)\ldots)x_n\right) = 0$ for any collection $x_1, x_2, \ldots, x_n$ of vertices in V with $n \geq 2$ and $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$.*

Proof. Suppose $CayF(V, \nu)$ is a Hasse diagram and let $x_1, x_2, \ldots, x_n$ be vertices in V with $n \geq 2$ and $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$. Then, for $i = 1, 2, \ldots, n$ $FR\left((\ldots((x_1x_2)x_3)\ldots)x_{i-1}, (\ldots((x_1x_2)x_3)\ldots)x_i\right) = \nu(x_i) > 0$. Thus, $1, x_1, x_1x_2, \ldots, (\ldots((x_1x_2)x_3)\ldots)x_n$ is a path from 1 to $(\ldots((x_1x_2)x_3)\ldots)x_n$. Since $CayF(V, \nu)$ is a Hasse diagram, $FR\left(1, (\ldots((x_1x_2)x_3)\ldots)x_n\right) = 0$. This implies that $\nu\left((\ldots((x_1x_2)x_3)\ldots)x_n\right) = 0$. Also, since $CayF(V, \nu)$ is a Hasse diagram, $CayF(V, \nu)$ is connected.

Conversely, let $CayF(V, \nu)$ is connected and for any collection $x_1, x_2, \ldots, x_n$ of vertices in V with $n \geq 2$ and $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$, we have $\nu\left((\ldots((x_1x_2)x_3)\ldots)x_n\right) = 0$. Let $x_\circ, x_1, \ldots, x_n$ be a path in $CayF(V, \nu)$ from $x_\circ$ to x_n with $n \geq 2$. Then, $FR(x_{i-1}, x_i) > 0$ for $i = 1, 2, \ldots, n$. Therefore, $\nu(x_i/x_{i-1}) > 0$ for $i = 1, 2, \ldots, n$. Let $x_1 = x_\circ t_1,\ x_2 = x_1t_2, \ldots, x_n = x_{n-1}t_n$. Then, $\nu(t_i) = \nu(x_i/x_{i-1}) > 0$ for $i = 1, 2, \ldots, n$. We have, $x_n = x_{n-1}t_n = (x_{n-2}t_{n-1})t_n \ldots = (\ldots((x_\circ t_1)t_2)\ldots)t_n = (\ldots(x_\circ(t_1t_2'))\ldots)t_n \ldots = x_\circ(\ldots((t_1t_2')t_3')\ldots)t_n')$. Therefore,

$$FR(x_\circ, x_n) = \nu(x_n/x_\circ) = \nu\left((\ldots((t_1t_2')t_3')\ldots)t_n'\right). \tag{2.1}$$

We have,
$\nu(t_1) > 0$ and $\nu(t_2') = \nu(t_1t_2'/t_1) = \nu\left(x_\circ(t_1t_2')/x_\circ t_1\right) = \nu((x_\circ t_1)t_2/x_\circ t_1) = \nu(t_2) > 0$ . . . . In general, $\nu(t_i') = \nu(t_i) > 0$ for $i = 2, 3, \ldots, n$. Therefore, since $t_1, t_i' \in V, i = 2, 3, \ldots, n, n \geq 2$ and $\nu(t_1) > 0, \nu(t_i') > 0$, for $i = 2, 3, \ldots, n$, we have, $\nu\left((\ldots((t_1t_2')t_3')\ldots)t_n'\right) = 0$. Therefore, (2.1) implies that $FR(x_\circ, x_n) = 0$. Hence $CayF(V, \nu)$ is a Hasse diagram. $\square$

Theorem 2.1.19. *Let $CayF(V,\nu)$ be finite and connected. Then $diam(G)$ is the least positive integer n such that for any $x \in V$, there exist $x_1, x_2, \ldots, x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, ., n$ and*

$$x = (\ldots((x_1x_2)x_3)\ldots)x_n.$$

Proof. Suppose n be the least positive integer such that for any $x \in V$ there exist elements $x_1, x_2, \ldots, x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$ and $x = (\ldots((x_1x_2)x_3)\ldots)x_n$. Note that, for any $x, y \in V$, we have $z \in V$ such that $y = xz$. Then, by assumption there exist $x_1, x_2, \ldots, x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$ and $z = (\ldots((x_1x_2)x_3)\ldots)x_n$. Therefore, $y = x((\ldots(x_1x_2)\ldots)x_n) = (x\ldots((x_1x_2)x_3)\ldots)x'_n = \ldots = (\ldots((xx_1)x'_2)\ldots)x'_n$.
Also,

$$\begin{aligned}
\nu(x_2) &= \nu(x_1x_2/x_1) \\
&= \nu(x(x_1x_2)/xx_1) \\
&= \nu((xx_1)x'_2/xx_1) \\
&= \nu(x'_2).
\end{aligned}$$

Therefore, we have $\nu(x'_2) > 0$. And

$$\begin{aligned}
\nu(x_3) &= \nu((x_1x_2)x_3/x_1x_2) \\
&= \nu(x((x_1x_2)x_3)/x(x_1x_2)) \\
&= \nu((x(x_1x_2))x'_3/(x(x_1x_2))) \\
&= \nu(x'_3).
\end{aligned}$$

Therefore, $\nu(x'_3) > 0$. In general, $\nu(x'_i) = \nu(x_i) > 0$ for $i = 2, 3, \ldots, n$. Also, $\nu(x_1) > 0$. Therefore, we have, $y = (\ldots((xx_1)x'_2)\ldots)x'_n$ and $\nu(x_1) > 0$, $\nu(x_i) >$

$0, \ i = 1, 2, \ldots, n$. Therefore,

$$FR((\ldots((xx_1)x'_2)\ldots)x'_{i-1}, \ (\ldots((xx_1)x'_2)\ldots)x'_i) = \nu(x'_i) > 0.$$

Hence, $x, xx_1, (xx_1)x'_2, \ldots, (\ldots((xx_1)x'_2)\ldots)x'_n$ is a path from x to y of length n. Since x and y are arbitrary,

$$diam(G) \leq n. \tag{2.2}$$

Since n is the least positive integer such that for any $x \in V$, $\exists x_1, x_2, \ldots, x_n \in V$ with $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n$ and $x = (\ldots(x_1x_2)x_3)\ldots)x_n$. Thus, there exists an $x \in V$ such that for any collection of $n-1$ elements, say $x_1, x_2, \ldots, x_{n-1}$ with $\nu(x_i) > 0$ for $i = 1, 2, \ldots, n-1$, we have

$$x \neq (\ldots((x_1x_2)x_3)\ldots)x_{n-1}. \tag{2.3}$$

Suppose that $diam(G) \leq n-1$. Then, there is a path $1, x_1, x_2, \ldots, x_m$ from 1 to x of length m, where $m \leq n-1$. Then, $FR(x_{i-1}, x_i) > 0$ for $i = 1, 2, \ldots, m$. Let $x_2 = x_1t_2, \ x_3 = x_2t_3, \ldots, x_m = x_{m-1}t_m$. Therefore, $\nu(t_i) = \nu(x_i/x_{i-1}) = FR(x_{i-1}, x_i) > 0$ and $x = x_m = x_{m-1}t_m = (x_{m-2}t_{m-1})t_m = (\ldots((x_1t_2)t_3)\ldots t_{m-1})t_m$. Therefore,
$x = (\ldots(((x_1t_2)t_3)\ldots)t_m)t_{m+1}\ldots)t_{n-1}$, where $t_{m+1} = t_{m+2} = \ldots = t_{n-1} = 1$, which is a contradiction to (2.3). Thus,

$$diam(G) > n-1. \tag{2.4}$$

Hence (2.2) and (2.4) gives $diam(G) = n$. □

Definition 2.1.20. *Let V be a loop and let ν be a fuzzy subset of V. Then the fuzzy sub quasigroup generated by ν is the smallest fuzzy sub quasigroup of V which contains ν and is denoted by $\langle \nu \rangle$.*

Remark 2.1.21. *Let V be a loop and let ν be a fuzzy subset of V. Then the fuzzy sub quasigroup generated by ν is the meet of all fuzzy sub quasigroups of V*

which contains ν.

Theorem 2.1.22. *Let V be a loop and ν be a fuzzy subset of V. Then the fuzzy subset $\langle \nu \rangle$ is precisely given by $\langle \nu \rangle(x) = \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \ldots \wedge \nu(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n,\ x_i \in \nu_\circ^+, n \in \mathbb{N}$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$.*

Proof. Let ν' be the fuzzy subset of V defined by $\nu'(x) = \vee\{\nu(x_1 \wedge \nu(x_2) \wedge \ldots \wedge \nu(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n,\ x_i \in \nu_\circ^+, n \in \mathbb{N}$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. If $y \in V$, by definition of ν', it is clear that $\nu'(y) \geq \nu(y)$. Thus, we have $\nu \leq \nu'$. Let $x, y \in V$. If $\nu(x) = 0$ or $\nu(y) = 0$, then $\nu(x) \wedge \nu(y) = 0$. Then, $\nu'(xy) \geq \nu(x) \wedge \nu(y)$. Again, if $\nu(x) \neq 0$ and $\nu(y) \neq 0$, then by definition of ν', we have $\nu'(xy) \geq \nu(x) \wedge \nu(y)$. Hence ν' is a fuzzy sub quasigroup of V containing ν. Now let A be any fuzzy sub quasigroup of V containing ν. Then, for any $x \in V$ with $x = (\ldots((x_1x_2)x_3)\ldots)x_n,\ x_i \in \nu_\circ^+, n \in \mathbb{N}$ for $i = 1, 2, \ldots, n$, we have $A(x) \geq A(x_1) \wedge A(x_2) \wedge \ldots \wedge A(x_n) \geq \nu(x_1) \wedge \nu(x_2) \wedge \ldots \wedge \nu(x_n)$, which implies that $A(x) \geq \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \ldots \wedge \nu(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n,\ x_i \in \nu_\circ^+, n \in \mathbb{N}$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. Therefore, $A(x) \geq \nu'(x)$ for all $x \in V$. Thus, $\nu' = \langle \nu \rangle$. That is, $\langle \nu \rangle(x) = \vee\{\nu(x_1) \wedge \nu(x_2) \wedge \ldots \wedge \nu(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n,\ x_i \in \nu_\circ^+, n \in \mathbb{N}$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. □

Theorem 2.1.23. *Let V be a loop and ν be a fuzzy subset of V. Then, for any $\alpha \in [0, 1]$, $\langle \nu_\alpha \rangle = \langle \nu \rangle_\alpha$ and $\langle \nu_\alpha^+ \rangle = \langle \nu \rangle_\alpha^+$, where $\langle \nu_\alpha \rangle$ denotes the fuzzy sub quasigroup generated by ν_α and $\langle \nu \rangle$ denotes the fuzzy sub quasigroup generated by ν.*

Proof. Observe that

$$\begin{aligned} x \in \langle \nu_\alpha \rangle &\Leftrightarrow \exists\ x_1, x_2, \ldots, x_n \in \nu_\alpha \ni x = (\ldots(x_1x_2)\ldots)x_n \\ &\Leftrightarrow \exists\ x_1, x_2, \ldots, x_n \in V \ni \nu(x_i) \geq \alpha, \text{ for all i=1,2, \ldots,n} \\ &\text{and } x = (\ldots((x_1x_2)x_3)\ldots)x_n \\ &\Leftrightarrow \langle \nu \rangle(x) \geq \alpha \\ &\Leftrightarrow x \in \langle \nu \rangle_\alpha. \end{aligned}$$

Therefore, $\langle \nu_\alpha \rangle = \langle \nu \rangle_\alpha$. Similarly, we can prove that $\langle \nu_\alpha^+ \rangle = \langle \nu \rangle_\alpha^+$. □

Remark 2.1.24. *If $\alpha = 0$, theorem 2.1.23 implies that $\langle \nu_\circ^+ \rangle = \langle \nu \rangle_\circ^+$. That is, $\langle supp(\nu) \rangle = supp(\langle \nu \rangle)$.*

2.2 Connectedness in Cayley fuzzy graphs induced by loops

Theorem 2.2.1. *$G = CayF(V, \nu)$ is connected if and only if $V - \{1\} \subseteq supp\langle \nu \rangle$.*

Proof. Suppose G is connected and let $x \in V - \{1\}$. Since G is connected, there exist a path from 1 to x, say, $1, x_1, x_2, \ldots, x_n = x$. This implies that, there exist $t_1, t_2, \ldots, t_n \in supp(\nu)$ such that $x = x_n = x_{n-1}t_n,\ x_{n-1} = x_{n-2}t_{n-1}, \ldots,\ x_1 = 1t_1$. Therefore, $x = x_n = x_{n-1}t_n = (x_{n-2}t_{n-1})t_n = \ldots = (\ldots((t_1t_2)t_3)\ldots)t_n,\ t_i \in supp(\nu),\ i = 1, 2, \ldots, n$, which implies that $x \in \langle supp(\nu) \rangle$. Therefore, $V - \{1\} \subseteq \langle supp(\nu) \rangle = supp\langle \nu \rangle$.

Conversely, let $V - \{1\} \subseteq supp\langle \nu \rangle$. Let x, y be two distinct elements in V. Then, there exist $z \neq 1 \in V$ such that $y = xz$. Since $z \neq 1,\ z \in V - \{1\} \subseteq \langle supp(\nu) \rangle$. Then, there exist $z_1, z_2, \ldots, z_m \in supp(\nu)$ such that $z = z_1 z_2 \ldots z_m$. Clearly, $1, z_1, z_1z_2, \ldots, (\ldots((z_1z_2)z_3)\ldots)z_m = z$ is a path from 1 to z. Then $x, xz_1, x(z_1z_2), x((z_1z_2)z_3), \ldots, x((\ldots((z_1z_2)z_3)\ \ldots)z_m) = xz = y$ is a path from x to y, since $supp(\nu)$ is right associative. Therefore, G is connected. □

Theorem 2.2.2. *$G = CayF(V, \nu)$ is weakly connected if and only if $V - \{1\} \subseteq supp(\langle \nu \vee \nu_\ell \rangle)$ where $\nu_\ell(x) = \nu(1/x)$.*

Proof. Suppose that G is weakly connected. Then $CayF(V, \nu \vee \nu_\ell)$ is connected. Thus by Theorem 2.2.1, we have $V - \{1\} \subseteq supp(\langle \nu \vee \nu_\ell \rangle)$. This completes the proof. □

Theorem 2.2.3. *$G = CayF(V, \nu)$ is semi-connected if and only if*

$$supp(\langle \nu \rangle \vee \langle \nu \rangle_\ell) = V - \{1\}.$$

Proof. First assume that G is semi-connected. Let $x \in V - \{1\}$. Since G is semi-connected, there exist a path from x to 1 or a path from 1 to x. Suppose there exist a path $1, x_1, x_2, \ldots, x_n, x$ from 1 to x. Then, there exist $t_1, t_2, \ldots, t_{n+1} \in supp(\nu)$ such that $x_1 = 1t_1,\ x_2 = x_1t_2, \ldots, x = x_nt_{n+1}$. Then, $x = x_nt_{n+1} = (x_{n-1}t_n)t_{n+1} = \ldots = (\ldots((t_1t_2)t_3)\ldots)t_{n+1},\ t_i \in supp(\nu),\ i = 1, 2, \ldots, n+1$, which implies that $x \in \langle supp(\nu) \rangle$. Or suppose there exist a path $x, y_1, y_2, \ldots, y_m, 1$ from x to 1. Then, there exist $k_i \in supp(\nu)$ for $i = 0, 1, \ldots, m$ such that $y_1 = xk_\circ,\ y_2 = y_1k_1, \ldots,\ 1 = y_mk_m$.
Then,

$$\begin{aligned} 1 &= (\ldots((xk_\circ)k_1)\ldots)k_m \\ &= (\ldots(x(k_\circ k_1'))\ldots)k_m \\ &\vdots \\ &= x((\ldots(k_\circ k_1')\ldots)k_m'). \end{aligned}$$

Here, $k_i' \in supp(\nu)$, for $i = 0, 1, \ldots, m$, since $supp(\nu)$ is right associative. This implies $1 = xk$, where $k = (\ldots((k_\circ k_1')k_2')\ldots)k_m' \in \langle supp(\nu) \rangle$. Hence $x \in \langle supp(\nu) \rangle_\ell$. Therefore, G is semi-connected implies

$$\langle supp(\nu) \rangle \cup \langle supp(\nu) \rangle_\ell = V - \{1\}.$$

Conversely, assume that $\langle supp(\nu) \rangle \cup \langle supp(\nu) \rangle_\ell = V - \{1\}$. Let $x, y \in G$. Then $y = xz$ for some $z \in G$. Then, $z \in \langle supp(\nu) \rangle \cup \langle supp(\nu) \rangle_\ell$.
If $z \in \langle supp(\nu) \rangle$, $\exists t_1, t_2, \ldots, t_n \in supp(\nu)$ such that $z = (\ldots((t_1t_2)t_3)\ldots)t_n$. Clearly $1, t_1, t_1t_2, \ldots, (\ldots((t_1t_2)t_3)\ldots)t_n = z$ is a path from 1 to z. Then, $x, xt_1, \ldots, x((\ldots((t_1t_2)t_3)\ldots)t_n)$ is a path from x to y. If $z \in \langle supp(\nu) \rangle_\ell$, $1 =$

zt for some $t \in \langle supp(\nu)\rangle$, which implies there exist $p_1, p_2, \ldots, p_m \in supp(\nu)$ such that $t = (\ldots(p_1p_2)\ldots)p_m$. Then, $1 = zt = z((\ldots((p_1p_2)p_3)\ldots)p_m) = (z\ldots((p_1p_2)p_3)\ldots)p'_m = \ldots = (\ldots((zp_1)p'_2)\ldots)p'_m,\ p_i \in supp(\nu)$, since $supp(\nu)$ is right associative and $p_i \in supp(\nu)$. Let $k_1 = zp_1$. $k_2 = k_1p'_2,\ k_3 = k_2p'_3, \ldots, k_m = k_{m-1}p'_m$. Then, $k_m = k_{m-1}p'_m = \ldots = (\ldots((zp_1)p'_2)\ldots)p'_m = 1$. Clearly, $z, k_1, k_2, \ldots, k_m = 1$ is a path from z to 1. Then, $xz, xk_1, xk_2, \ldots, xk_m = x$ is a path from y to x. Thus, for any $x, y \in V$ there exist a path from x to y or a path from y to x, which implies that G is semi-connected. This completes the proof. □

Theorem 2.2.4. *$G = CayF(V,\nu)$ is locally connected if and only if $supp\langle\nu\rangle = supp\langle\nu\rangle_\ell$.*

Proof. First suppose that G is locally connected. Let $x \in \langle supp(\nu)\rangle$. Then, there exist $x_1, x_2, \ldots, x_n \in supp(\nu)$ such that $x = (\ldots((x_1x_2)x_3)\ \ldots)x_n$. Therefore $1, x_1, x_1x_2, \ldots, (\ldots((x_1x_2)x_3)\ldots)x_n$ is a path from 1 to x. Then, since G is locally connected, there exist a path from x to 1. Let $x, z_1, z_2, \ldots, z_{n-1}$ be a path from x to 1 which implies there exist $a_1, a_2, \ldots, a_n \in supp(\nu)$ such that $z_1 = xa_1,\ z_2 = z_1a_2, \ldots, z_{n-1} = z_{n-2}a_{n-1},\ 1 = z_{n-1}a_n$. Then, $1 = z_{n-1}a_n = (z_{n-2}a_{n-1})a_n = \ldots = (\ldots((xa_1)a_2)\ldots)a_n$. Then, we have, $1 = (\ldots(x(a_1a'_2))\ldots)a_n = \ldots = x(\ldots((a_1a'_2)a'_3)\ldots)a'_n, a'_i \in supp(\nu)$. That is, $1 = xt,\ t \in \langle supp(\nu)\rangle$, implies $x \in \langle supp(\nu)\rangle_\ell$. Thus,

$$\langle supp(\nu)\rangle \subseteq \langle supp(\nu)\rangle_\ell. \tag{2.5}$$

Let $x_\ell \in \langle supp(\nu)\rangle_\ell$, which implies there exist an $x \in \langle supp(\nu)\rangle$ such that $1 = x_\ell x$. Then, $1, x_1, x_1x_2, \ldots, (\ldots((x_1x_2)x_3)\ldots)x_m$ is a path from 1 to x. Thus, since G is locally connected, there exist a path from x to 1. We have,

$$\begin{aligned} 1 &= x_\ell x \\ &= x_\ell((\ldots((x_1x_2)x_3)\ldots)x_m) \\ &= x_\ell((\ldots(x_1(x_2x'_3))\ldots)x_m) \end{aligned}$$

$$\vdots$$
$$= (\dots((x_\ell x_1)x_2')\dots)x - m'\ x_1 \in supp(\nu),$$

and here $x_i' \in supp(\nu),\ i = 1, 2, \dots, m$, since $supp(\nu)$ is right associative. Now, let $t_1 = x_\ell x_1,\ t_2 = t_1 x_2', \dots, 1 = t_m = t_{m-1}x_m'$. Then, $x_\ell, t_1, t_2, \dots, t_{m-1}, t_m = 1$ is a path from x_ℓ to 1. This implies that there exist a path from 1 to x_ℓ, since G is locally connected. Let $1, k_1, k_2, \dots, k_r = x_\ell$ be a path from 1 to x_ℓ. Then, there exist $p_i \in supp(\nu),\ i = 1, 2, \dots, r$ such that $k_1 = 1.p_1,\ k_2 = k_1p_2, \dots, k_r = k_{r-1}p_r$. Thus, $x_\ell = k_r = k_{r-1}p_r = (k_{r-2}p_{r-1})p_r = \dots = (\dots((p_1p_2)p_3)\dots)p_r$, which implies that $x_\ell \in \langle supp(\nu)\rangle$. Hence,

$$\langle supp(\nu)\rangle_\ell \subseteq \langle supp(\nu)\rangle. \tag{2.6}$$

Therefore, from equations (6) and (7) we get $\langle supp(\nu)\rangle = \langle supp(\nu)\rangle_\ell$. That is $supp\langle\nu\rangle = supp\langle\nu\rangle_\ell$.

Conversely, suppose $supp\langle\nu\rangle = supp\langle\nu\rangle_\ell$. Let $x, y \in V$ and there exist a path from x to y say $x, x_1, x_2, \dots, x_{n-1}, y$. Then, there exist $a_i \in supp(\nu)$ for $i = 1, 2, \dots, n$ such that $x_1 = xa_1,\ x_2 = x_1a_2, \dots, x_{n-1} = x_{n-2}a_{n-1},\ y = x_{n-1}a_n$. Thus, $y = x_{n-1}a_n = (x_{n-2}a_{n-1}a_n = \dots = (\dots((xa_1)a_2)\dots)a_n$. Therefore, $y = x((\dots((a_1a_2')a_3')\dots)a_n')$, where $a_i' \in supp(\nu)$, since $supp(\nu)$ is right associative, which implies $y/x \in \langle supp(\nu)\rangle$. Then, $x/y \in \langle supp(\nu)\rangle_\ell$. Now, since $k = x/y \in \langle supp(\nu)\rangle_\ell$ and $\langle supp(\nu)\rangle = \langle supp(\nu)\rangle_\ell$, $k = x/y \in \langle supp(\nu)\rangle$. Then, there exist $k_1, k_2, \dots, k_p \in supp(\nu)$ such that $k = (\dots((k_1k_2)k_3)\dots)k_p$. Clearly, $1, k_1, k_1k_2, \dots, (\dots((k_1k_2)k_3)\dots)k_p = k$ is a path from 1 to k. Then, $y, yk_1, y(k_1k_2), \dots, y((\dots((k_1k_2)k_3)\dots)k_p) = yk = x$ is a path from y to x. Hence G is locally connected. This completes the proof. □

Theorem 2.2.5. *Let V be a finite loop then, $G = CayF(V, \nu)$ is quasi-connected if and only if it is connected.*

Proof. Every connected graphs are quasi-connected. Therefore, if G is connected then G is quasi-connected. Now, suppose that G is quasi-connected. Note that G is finite. Since we have a result stating a finite digraph has a source if and only if it is quasi-connected, G has a source, say z. Then for any $x \in V$ with $x \neq z$, there is a directed path from z to x. Let $z, z_1, z_2, \ldots, z_n, x$ be a path from z to x. Then, there exist $k_i \in supp(\nu)$ for $i = 0, 1, \ldots n$ such that $z_1 = zk_\circ,\ z_2 = z_1k_1, \ldots,\ z_n = z_{n-1}k_{n-1},\ x = z_nk_n$. Then, $x = (\ldots((zk_\circ)k_1)\ldots)k_n = (\ldots(z(k_\circ k_1'))\ldots)k_n = \ldots = z((\ldots(k_\circ k_1')\ldots)k_n')$. Here, $k_i' \in supp(\nu)$, for $i = 0, 1, \ldots, n$, since $supp(\nu)$ is right associative. Therefore, $x/z = (\ldots(k_\circ k_1')\ldots)k_n') \in \langle supp(\nu)\rangle$. Thus it is clear that $x/z \in \langle supp(\nu)\rangle = supp\langle\nu\rangle$ for every $x \in V$ with $x \neq z$. Hence $supp\langle\nu\rangle \supseteq V - \{1\}$. Hence, by Theorem 2.2.1, G is connected. □

2.3 Strength of connectedness in Cayley fuzzy graphs induced by loops

Definition 2.3.1. *Let $P = (x_\circ, x_1, \ldots, x_n)$ be a path in a fuzzy graph $G = (V, \rho)$. Then the strength of the path P in $CayF(V, \nu)$, denoted $strength(P)$, is defined as*

$$strength(P) = \wedge_{i=1}^{n} \rho(x_{i-1}, x_i).$$

Definition 2.3.2. *Let $G = (V, \rho)$ be a fuzzy graph and let $\alpha \in (0, 1]$. Then $CayF(V, \nu)$ is said to be: (i) α-connected if for every pair of vertices $x, y \in G$, there is a path P from x to y such that $strength(P) \geq \alpha$, (ii) weakly α-connected if the fuzzy graph $(V, FR \vee FR^{-1})$ is α-connected, (iii) semi α-connected if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in $CayF(V, \nu)$ (iv) locally α-connected if for every pair of vertices x and y, there is a path P of strength greater than or equal to α from x to y whenever there is a path P' of strength greater than or equal to α from y to x, (v) quasi α-connected if for every pair $x, y \in V$, there is some $z \in V$ such* *that there is a directed path from z to x of strength greater than or equal to α and there is a directed path from z to y of strength greater than or equal to α.*

Observe that if $\alpha, \beta \in (0, 1]$, $\alpha < \beta$ and $CayF(V, \nu)$ is β-connected, then $CayF(V, \nu)$ is also α-connected. Thus, a finite graph $CayF(V, \nu)$ is connected if it is α-connected for some $\alpha \in (0, 1]$. But for infinite fuzzy graphs, this is not true. For example, consider the graph $G = (\mathbb{N}, R)$, where $R(m, n) = 1/n$ if $n - m = 1$, $R(m, n) = 1$ if $m = n$ and $R(m, n) = 0$ otherwise and $\mathbb{N}$ is the set of all natural numbers. Then $CayF(V, \nu)$ is not α-connected for any $\alpha \in (0, 1]$ but it is connected. A fuzzy graph $G = (V, R)$ is said to be α-*complete* if $R(x, y) \geq \alpha$ for all $x, y \in V$. Observe that any complete fuzzy graph is α-complete for all $\alpha \in [0, 1]$.

Definition 2.3.3. *Let $G = (V, \rho)$ be a fuzzy graph and let $x, y \in V$. Let $\mathcal{P}_G(x, y)$ denote the set of all paths in $CayF(V, \nu)$ from x to y. Then the strength of connectedness between x and y, denoted $CONN_G(x, y)$, is defined as*

$$CONN_G(x, y) = \bigvee_{P \in \mathcal{P}_G(x,y)} strength(P).$$

We define the strength connectivity of $CayF(V, \nu)$ as

$$SC(G) = \bigwedge_{x,y \in V} CONN_G(x, y).$$

2.3.1 Different types of α-connectedness in Cayley fuzzy graphs

In this subsection, we prove the following theorems based on different types of α-connectedness.

Theorem 2.3.4. *$CayF(V, \nu)$ is α-connected if and only if $\langle \nu \rangle_\alpha \supseteq V - \{1\}$.*

Proof. Suppose G is α - connected and let $x \in V - \{1\}$. Since G is α - connected, there exist a path P from 1 to x, say, $1, x_1, x_2, \ldots, x_n = x$. with

$strength(P) \geq \alpha$. This implies that, there exist $t_1, t_2, \ldots, t_n \in \langle \nu_\alpha \rangle$ such that $x = x_n = x_{n-1}t_n,\ x_{n-1} = x_{n-2}t_{n-1}, \ldots,\ x_1 = 1t_1$. Therefore, $x = x_n = x_{n-1}t_n = (x_{n-2}t_{n-1})t_n = \ldots = (\ldots((t_1t_2)t_3)\ldots)t_n,\ t_i \in \langle \nu_\alpha \rangle,\ i = 1, 2, \ldots, n$, which implies that $x \in \langle \nu_\alpha \rangle$. Therefore, $V - \{1\} \subseteq \langle \nu_\alpha \rangle = \langle \nu \rangle_\alpha$.

Conversely, let $V - \{1\} \subseteq \langle \nu \rangle_\alpha$. Let x, y be two distinct elements in V. Then, there exist $z \neq 1 \in V$ such that $y = xz$. Since $z \neq 1,\ z \in V - \{1\} \subseteq \langle \nu \rangle_\alpha = \langle \nu_\alpha \rangle$. Then, there exist $z_1, z_2, \ldots, z_m \in \langle \nu_\alpha \rangle$ such that $z = z_1z_2\ldots z_m$. Clearly, $1, z_1, z_1z_2, \ldots, (\ldots((z_1z_2)z_3)\ldots)z_m = z$ is a path from 1 to z with strength greater than α. Then $x, xz_1, x(z_1z_2), x((z_1z_2)z_3), \ldots, x((\ldots((z_1z_2)z_3)\ \ldots)z_m) = xz = y$ is a path from x to y, with strength greater than α. Therefore, G is α - connected. □

Theorem 2.3.5. *$CayF(V, \nu)$ is weakly α-connected if and only if*

$$\langle \nu \vee \nu_\ell \rangle_\alpha \supseteq V - \{1\}.$$

Proof. Suppose that $CayF(V, \nu)$ is weakly α- connected. Then the fuzzy graph $(V, FR \vee FR^{-1})$ is α-connected. Hence by theorem 2.3.4, $\langle \nu \vee \nu_\ell \rangle_\alpha \supseteq V - \{1\}$. □

Theorem 2.3.6. *$CayF(V, \nu)$ is semi α-connected if and only if*

$$\langle \nu \rangle_\alpha \cup \langle \nu \rangle_{\alpha\ell} \supseteq V - \{1\}.$$

Proof. $CayF(V, \nu)$ is semi α-connected if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in $CayF(V, \nu)$. The definition of semi α - connectedness together with the theorem 2.3.4 makes it clear that $CayF(V, \nu)$ is semi α-connected if and only if

$$\langle \nu \rangle_\alpha \cup \langle \nu \rangle_{\alpha\ell} \supseteq V - \{1\}.$$

□

Theorem 2.3.7. *$CayF(V, \nu)$ is locally α-connected if and only if $\langle \nu \rangle_\alpha = \langle \nu \rangle_{\alpha\ell}$.*

Proof. $CayF(V,\nu)$ is locally α-connected if and only if (V, FR_α) is locally connected. That is if and only if $\langle\nu_\alpha\rangle = \langle\nu_{\alpha\ell}\rangle$, that is if and only if $\langle\nu\rangle_\alpha = \langle\nu\rangle_{\alpha\ell}$. $\square$

Theorem 2.3.8. *If $CayF(V,\nu)$ is finite then it is quasi α-connected if and only if it is α- connected.*

Proof. The proof is similar to that of Theorem 2.3.7 $\square$

Theorem 2.3.9. *Let x and y be any two vertices of the Cayley fuzzy graph $CayF(V,\nu)$. Then $CONN_G(x,y) = \langle\nu\rangle(y/x)$.*

Proof. Let $\alpha \in (0,1]$. Suppose that $CONN_G(x,y) = \alpha$. Then for any $\epsilon > 0$, there exist a path, say $P = (x, x_1, x_2, \ldots, x_n, y)$ from x to y such that $strength(P) > \alpha - \epsilon$. This implies that

$$FR(x_{i-1}, x_i) > \alpha - \epsilon \; for \; all \; i = 1, 2, \ldots, n+1,$$

where $x_\circ = x,\ x_{n+1} = y$. This implies that $\nu(x_i/x_{i-1}) > \alpha - \epsilon$ for all $i = 1, 2, \ldots, n+1$. Let $x_i = x_{i-1}t_i$. Therefore, $\nu(x_i/x_{i-1}) = \nu(t_i) > \alpha - \epsilon$. That is, $x_1 = x_\circ t_1,\ x_2 = x_1 t_2, \ldots, x_n = x_{n-1}t_n,\ x_{n+1} = x_n t_{n+1}$, which gives, $y = x_{n+1} = x_n t_{n+1} = (x_{n-1}t_n)t_{n+1}\ldots = (\ldots((x_\circ t_1)t_2)\ldots)t_{n+1} = (\ldots(x_\circ(t_1 t'_2))\ldots)t_n \ldots = x_\circ(\ldots((t_1 t'_2)t'_3)\ldots)t'_{n+1})$.
Thus, $y/x = (\ldots((t_1 t'_2)t'_3)\ldots)t'_{n+1}$. Since ν is a scaled fuzzy subset of V, $\nu(t'_i) = \nu(t_i)$. Therefore, $\nu(t_i{}') > \alpha - \epsilon > 0$, for $i = 1, 2, \ldots, n+1$. Thus, $\langle\nu\rangle(y/x) \geq \nu(t_1) \wedge \nu(t'_2) \wedge \ldots \wedge \nu(t'_{n+1}) > \alpha - \epsilon$. Since ϵ is arbitrary,

$$\langle\nu\rangle(y/x) \geq \alpha = CONN_G(x,y). \tag{2.7}$$

Now, since $y/x = \in V$, by Lemma 2.1.22, there exist $z_1, z_2, \ldots, z_m \in V$ such that $y/x = (\ldots(z_1 z_2)\ldots)z_m$ and $\langle\nu\rangle(y/x) = \nu(z_1) \wedge \nu(z_2) \wedge \ldots \wedge \nu(z_m)$. Consider the sequence, $y_\circ = x,\ y_1 = xz_1,\ y_3 = (xz_1)z_2, \ldots, y_m = (\ldots((xz_1)z_2)\ldots)z_m$. Then, $\underset{i=1}{\overset{m}{\wedge}} FR(y_{i-1}, y_i) = \nu(z_1) \wedge \nu(z_2) \wedge \ldots \wedge \nu(z_m) = \langle\nu\rangle(y/x) \neq 0$. This implies that $P' = (y_\circ, y_1, \ldots, y_m)$ is a path from x to y. Note that

$$CONN_G(x,y) = \bigvee_{P \in \mathcal{P}_G(x,y)} strength(P) \geq strength(P') = \langle \nu \rangle (y/x). \qquad (2.8)$$

Therefore, (2.7) and (2.8) gives $CONN_G(x,y) = \langle \nu \rangle (y/x)$. □

Theorem 2.3.10. *$CayF(V, \nu)$ is α-complete if and only if $\nu_\alpha \supseteq V - \{1\}$.*

Proof. Suppose that $CayF(V, \nu)$ is α-complete. Then, for every $x, y \in V$ with $x \neq y$, $FR(x,y) \geq \alpha$. In particular, $\nu(x) = FR(1,x) \geq \alpha$ for all $x \neq 1$. This implies that $x \in \nu_\alpha$ for all $x \neq 1$. Therefore, since x is arbitrary, $\nu_\alpha \supseteq V - \{1\}$.

Conversely, suppose that $\nu_\alpha \supseteq V - \{1\}$. Then, for each distinct $x, y \in V$, $y/x \in V$. This implies that $y/x \in \nu_\alpha$ for $x \neq y$. That is, $FR(x,y) = \nu(y/x) \geq \alpha$ for all $x \neq y$. Hence, $CayF(V, \nu)$ is α-complete. □

Theorem 2.3.11. *If $SC(G) = \alpha$, then $\langle \nu \rangle_\alpha \supseteq V - \{1\}$, that is,*

$$\langle \nu \rangle_{SC(G)} \supseteq V - \{1\}.$$

Proof. Assume that $\alpha \neq 0$. From the definition of $SC(G)$, it is obvious that $CONN_G(x,y) \geq \alpha$ for all $x, y \in V$. In particular, $CONN_G(1,x) \geq \alpha$ for all $x \in V$. This implies that there exists a path, say P from 1 to x such that $strength(P) \geq \alpha$. Then it can be easily verified that for $x \neq 1$, $\langle \nu \rangle (x) \geq \alpha$. This implies that $x \in \langle \nu \rangle_\alpha$ for all but $x = 1$. Consequently, $\langle \nu \rangle_\alpha \supseteq V - \{1\}$. □

Theorem 2.3.12. *For any Cayley fuzzy graph $G(V, R)$ induced by loop,*

$$SC(G) = \bigwedge_{\alpha \in [0,1]} \{\alpha : \langle \nu \rangle_\alpha \subsetneq V\}.$$

Proof. Let $\alpha \in [0,1]$. If $\langle \nu \rangle_\alpha \supsetneq V$, then there exist $x, y \in V$ such that every path from x to y has strength less than α. This implies that $CONN_G(x,y) < \alpha$.

Consequently, $SC(G) < \alpha$. Hence

$$SC(G) \leq \bigwedge_{\alpha \in [0,1]} \{\alpha : \langle \nu \rangle_\alpha \neq V\}.$$

Suppose that there is a β such that $SC(G) < \beta < \bigwedge_{\alpha \in [0,1]} \{\alpha : \langle \nu \rangle_\alpha \neq V\}$. This implies that $\langle \nu \rangle_\beta = V$,

$$\langle \nu \rangle(x) \geq \beta \; for \; all \; x \in V. \tag{2.9}$$

Let x and y be two elements in V. Then, by equation (2.9), we have $\langle \nu \rangle(y/x) \geq \beta$. This implies that there exists a path from x to y of strength greater than or equal to β. That is, $CONN_G(x, y) \geq \beta$ for all $x, y \in V$. In other words, $SC(G) \geq \beta$. This contradiction completes the proof. □

Theorem 2.3.13. *If ν is a sub-loop and $\alpha, \beta \in [0, 1]$ such that $CayF(V, \nu)$ is α-complete and not β-complete, then either $\alpha < SC(G) \leq \beta$ or $\alpha \leq SC(G) < \beta$.*

Proof. Since $CayF(V, \nu)$ is α-complete and not β-complete, it is clear that $\alpha < \beta$. Now ν is a sub-loop implies that $\nu = \langle \nu \rangle$. Then, by Theorem 2.3.12,

$$\begin{aligned} SC(G) &= \bigwedge_{\gamma \in [0,1]} \{\gamma : \langle \nu \rangle_\gamma \subsetneq V - \{1\} \\ &= \bigwedge_{\gamma \in [0,1]} \{\gamma : \nu_\gamma \subsetneq V - \{1\} \\ &= \bigwedge_{\gamma \in [0,1]} \{\gamma : G \; is \; not \; \gamma - complete\}. \end{aligned}$$

Thus, since $CayF(V, \nu)$ is not β-complete, we have $SC(G) \leq \beta$. Also, note that since $CayF(V, \nu)$ is α-complete, $CONN_G(x, y) \geq \alpha$ for all $x, y \in V$. Hence $SC(G) \geq \alpha$.

From these arguments, it is clear that either $\alpha < SC(G) \leq \beta$ or $\alpha \leq SC(G) < \beta$. This completes the proof. □

Chapter 3

Cayley Bipolar Fuzzy Graphs Induced by Loops

In this chapter, we define Cayley bipolar fuzzy graphs induced by loops and study its properties in terms of algebraic properties. First section is about basic definitions and some basic results. The second and third sections discuss about the connectedness in Cayley bipolar fuzzy graphs induced by loops Contents of this chapter is published in Global Journal of Pure and Applied Mathematics [21].

3.1 Cayley bipolar fuzzy graphs

In [24] N. O. Alshehri and M. Akram introduced Cayley bipolar fuzzy graphs induced by groups and discussed its properties in terms of algebraic properties.

In this section we introduce a class of Cayley Bipolar Fuzzy graphs induced by Loops and discuss some of its basic properties.

Definition 3.1.1. *Let $(V, *)$ be a loop and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V satisfying the condition, $\mu_A^P(y/x) = \mu_A^P(ay/ax)$ and $\mu_A^N(y/x) = \mu_A^N(ay/ax)$, for all $a, x, y \in V$, where y/x denote the solution of $y = xt$ in*

V. Then this bipolar fuzzy subset of V is called scaled bipolar fuzzy subset of the loop $(V, *)$.

Definition 3.1.2. *Let $(V, *)$ be a loop and let $A = (\mu_A^P, \mu_A^N)$ be a scaled bipolar fuzzy subset of V. Define a bipolar fuzzy relation $R = \left(\mu_R^P, \mu_R^N\right)$ by $R(x, y) = \{\left(\mu_A^P(y/x), \mu_A^N(y/x)\right), \forall\, x, y \in V\}$. That is, $\mu_R^P(x, y) = \mu_A^P(y/x)$ and $\mu_R^N(x, y) = \mu_A^N(y/x)$. Then $G = (V, R)$ is called the Cayley bipolar fuzzy graph induced by the loop $(V, *)$ and is denoted by $CayF_B(V, A)$.*

Definition 3.1.3. *Let $(V, *)$ be a loop and let $\alpha \in [0, 1]$. For any bipolar fuzzy subset $A = (\mu_A^P, \mu_A^N)$ of V, $\{x : \mu_A^P(x) \geq \alpha,$ and $\mu_A^N(x) \leq \alpha\}$ is called α-cut of A and $\{x : \mu_A^P(x) > \alpha,$ and $\mu_A^N(x) < \alpha\}$ is called strong α-cut of A and are denoted respectively by A_α and A_α^+.*

Theorem 3.1.4. *$CayF_B(V, A)$ is vertex transitive.*

Proof. Let $a, b \in V$ and $b = z_\circ a,\ z_\circ \in V$. Define $\Psi : V \to V$ by $\Psi(x) = z_\circ x$. Clearly, Ψ is a bijective map. For each $x, y \in V$,

$$\begin{aligned}
R(\Psi(x), \Psi(y)) &= \left(\mu_R^P(\Psi(x), \Psi(y)), \mu_R^N(\Psi(x), \Psi(y))\right) \\
&= \left(\mu_R^P(z_\circ x, z_\circ y), \mu_R^N(z_\circ x, z_\circ y)\right) \\
&= \left(\mu_A^P(z_\circ y/z_\circ x), \mu_A^N(z_\circ y/z_\circ x)\right) \\
&= \left(\mu_A^P(y/x), \mu_A^N(y/x)\right) \\
&= R(x, y).
\end{aligned}$$

Therefore, $R(\Psi(x), \Psi(y)) = R(x, y)$. Hence Ψ is an automorphism on $CayF_B(V, A)$. Also $\Psi(a) = b$. Hence $CayF_B(V, A)$ is vertex transitive. □

Theorem 3.1.5. *$CayF_B(V, A)$ is regular.*

Proof. Every vertex transitive bipolar fuzzy graphs are regular and $CayF_B(V, A)$ is vertex transitive. Hence $CayF_B(V, A)$ is regular. □

3.1.1 Basic Results

Theorem 3.1.6. *$CayF_B(V, A)$ is reflexive if and only if*

$$\mu_A^P(1) = 1 \text{ and } \mu_A^N(1) = -1.$$

Proof. R is reflexive if and only if $R(x, x) = (1, -1)$ for all $x \in V$. Now, $R(x, x) = \left(\mu_A^P(x/x), \mu_A^N(x/x)\right) = \left(\mu_A^P(1), \mu_A^N(1)\right)$. Therefore, R is reflexive implies $\mu_A^P(1) = 1$ and $\mu_A^N(1) = -1$. Hence R is reflexive if and only if $\mu_A^P(1) = 1$ and $\mu_A^N(1) = -1$. □

Theorem 3.1.7. *$CayF_B(V, A)$ is symmetric if and only if*

$$\left(\mu_A^P(x), \mu_A^N(x)\right) = \left(\mu_A^p(1/x), \mu_A^N(1/x)\right), \text{ for all } x \in V.$$

Proof. Suppose that $CayF_B(V, A)$ is symmetric. Then for any $x \in V$,

$$\begin{aligned}
\left(\mu_A^P(x), \mu_A^N(x)\right) &= \left(\mu_A^P(x^2/x), \mu_A^N(x^2/x)\right) \\
&= R(x, x^2) \\
&= R(x^2, x) \\
&= \left(\mu_A^P(x/x^2), \mu_A^N(x/x^2)\right) \\
&= (\mu_A^P(1/x), \mu_A^N(1/x)).
\end{aligned}$$

Conversely, suppose that $\left(\mu_A^P(x), \mu_A^N(x)\right) = \left(\mu_A^P(1/x), \mu_A^N(1/x)\right)$ for all $x \in V$. Then,

$$\begin{aligned}
R(x, y) &= \left(\mu_A^P(y/x), \mu_A^N(y/x)\right) \\
&= \left(\mu_A^P(xt/x), \mu_A^N(xt/x)\right) \\
&= \left(\mu_A^P(t), \mu_A^N(t)\right) \\
&= \left(\mu_A^P(1/t), \mu_A^N(1/t)\right) \\
&= \left(\mu_A^P(x/xt), \mu_A^N(x/xt)\right)
\end{aligned}$$

$$= \big(\mu_A^P(x/y), \mu_A^N(x/y)\big)$$
$$= R(y, x).$$

Hence $CayF_B(V, A)$ is symmetric. □

Theorem 3.1.8. *$CayF_B(V, A)$ is antisymmetric if and only if*

$$\big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\} = \{1\}.$$

Proof. First, assume that R is antisymmetric.
Let $x \in \big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\}$. Then $\mu_A^P(x) = \mu_A^P(1/x)$ and $\mu_A^N(x) = \mu_A^N(1/x)$. Therefore, $R(1, x) = R(x, 1)$ which implies $x = 1$. Hence,

$$\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\} = \{1\}.$$

Conversely, let $\big\{x : \big(\mu_A^P(x), \mu_A^N(x)\big) = \big(\mu_A^P(1/x), \mu_A^N(1/x)\big)\big\} = \{1\}$. Then,

$$\begin{aligned}
R(x, y) = R(y, x) &\Leftrightarrow \big(\mu_A^P(y/x), \mu_A^N(y/x)\big) = \big(\mu_A^P(x/y), \mu_A^N(x/y)\big) \\
&\Leftrightarrow \big(\mu_A^P(xt/x), \mu_A^N(xt/x)\big) = \big(\mu_A^P(x/xt), \mu_A^N(x/xt)\big)\,,\ t = y/x \\
&\Leftrightarrow \big(\mu_A^P(t), \mu_A^N(t)\big) = \big(\mu_A^P(1/t), \mu_A^N(1/t)\big) \\
&\Leftrightarrow t = 1 \\
&\Leftrightarrow y = x.
\end{aligned}$$

Therefore, R is antisymmetric. Hence the proof. □

Definition 3.1.9. *Let $(V, *)$ be a loop. Let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V. Then A is said to be a bipolar fuzzy sub quasigroup of V if for all $x, y \in V$, $\mu_A^P(xy) \geq \mu_A^P(x) \wedge \mu_A^P(y)$ and $\mu_A^N(xy) \leq \mu_A^N(x) \vee \mu_A^N(y)$.*

Theorem 3.1.10. *$CayF_B(V, A)$ is transitive if and only if A is a bipolar fuzzy sub quasigroup of V.*

Proof. Suppose R is transitive and let $x, y \in V$. Then $R^2 \leq R$. That is $\mu_{R^2}^P \leq \mu_R^P$

and $\mu^N_{R^2} \geq \mu^N_R$.
Now,

$$\begin{aligned}
\mu^P_A(x) \wedge \mu^P_A(y) &\leq \vee\{\mu^P_A(z) \wedge \mu^P_A(xy/z) : z \in V\} \\
&= \vee\{\mu^P_R(1,z) \wedge \mu^P_R(z,xy) : z \in V\} \\
&= \mu^P_{R^2}(1,xy) \\
&\leq \mu^P_R(1,xy) \\
&= \mu^P_A(xy)
\end{aligned}$$

and

$$\begin{aligned}
\mu^N_A(x) \vee \mu^N_A(y) &\geq \wedge\{\mu^N_A(z) \vee \mu^N_A(xy/z) : z \in V\} \\
&= \wedge\{\mu^N_R(1,z) \wedge \mu^N_R(z,xy) : z \in V\} \\
&= \mu^N_{R^2}(1,xy) \\
&\geq \mu^N_R(1,xy) \\
&= \mu^N_A(xy).
\end{aligned}$$

Therefore, $\mu^P_A(xy) \geq \mu^P_A(x) \wedge \mu^P_A(y)$ and $\mu^N_A(xy) \leq \mu^N_A(x) \vee \mu^N_A(y)$. Hence A is a bipolar fuzzy sub quasigroup of $(V,*)$.

Conversely, suppose that A is a bipolar fuzzy sub quasigroup of $(V,*)$. For $x,y \in V$, choose an arbitrary $z \in V$. Then there exist some $t, t_\circ, t_1 \in V$ such that $y = xt, z = xt_\circ, y = zt_1$. Then,

$$\begin{aligned}
\mu^P_R(x,y) &= \mu^P_A(y/x) \\
&= \mu^P_A((xt_\circ)t_1/x) \\
&= \mu^P_A(t_\circ t'_1) \\
&\geq \mu^P_A(t_\circ) \wedge \mu^P_A(t'_1) \\
&= \mu^P_A(z/x) \wedge \mu^P_A(t_\circ t'_1/t_\circ) \\
&= \mu^P_A(z/x) \wedge \mu^P_A((xt_\circ)t_1/xt_\circ)
\end{aligned}$$

$$= \mu_A^P(z/x) \wedge \mu_A^P(y/z).$$

Therefore, $\mu_R^P(x,y) \geq \mu_A^P(z/x) \wedge \mu_A^P(y/z)$, for any $z \in V$. That is,

$$\begin{aligned}
\mu_R^P(x,y) &\geq \vee\{\mu_A^P(z/x) \wedge \mu_A^P(y/z) : z \in V\} \\
&= \vee\{\mu_R^P(x.z) \wedge \mu_R^P(z,y) : z \in V\} \\
&= \mu_{R^2}^P(x,y).
\end{aligned}$$

Therefore, $\mu_{R^2}^P(x,y) \leq \mu_R^P(x,y)$. Similarly, $\mu_{R^2}^N(x,y) \geq \mu_R^N(x,y)$. Hence $R = (\mu_R^P, \mu_R^N)$ is transitive. $\square$

Theorem 3.1.11. *R is a partial order if and only if $A = (\mu_A^P, \mu_A^N)$ is a bipolar fuzzy sub quasigroup of $(V,*)$ satisfying:*

(i) $\mu_A^P(1) = 1$ *and* $\mu_A^N(1) = -1$,

(ii) $\{x : (\mu_A^P(x), \mu_A^N(x)) = (\mu_A^P(1/x), \mu_A^N(1/x))\} = \{1\}$.

Theorem 3.1.12. *R is a linear order if and only if (μ_A^P, μ_A^N) is a bipolar fuzzy sub quasigroup of $(V,*)$ satisfying:*

(i) $\mu_A^P(1) = 1$ *and* $\mu_A^N(1) = -1$,

(ii) $\{x : (\mu_A^P(x), \mu_A^N(x)) = (\mu_A^P(1/x), \mu_A^N(1/x))\} = \{1\}$, *and*

(iii) $\{x : \mu_A^P(x) \vee \mu_A^P(1/x) > 0,\ \mu_A^N(x) \wedge \mu_A^N(1/x) < 0\} = V$.

Proof. First, suppose that R is a linear order. Then the conditions (i) and (ii) are satisfied and (μ_A^N, μ_A^N) is a bipolar fuzzy sub quasigroup of $(V,*)$. For $x \in V$, $(\mu_R^P \vee \mu_{R-1}^P)(1,x) > 0$ and $(\mu_R^N \wedge \mu_{R-1}^N)(1,x) < 0$, which implies, $\mu_R^P(1,x) \vee \mu_R^P(x,1) > 0$, and $\mu_R^N(1,x) \wedge \mu_R^N(x,1) < 0$. Then $\mu_A^P(x) \vee \mu_A^P(1/x) > 0$ and $\mu_A^N(x) \wedge \mu_A^N(1/x) < 0$. That is $x \in \{x : \mu_A^P(x) \vee \mu_A^P(1/x) > 0,\ \mu_A^N(x) \wedge \mu_A^N(1/x) < 0\}$. Hence, condition (iii) is satisfied.

Conversely, suppose that the conditions $(i), (ii), (iii)$ hold and (μ_A^P, μ_A^N) is a

bipolar fuzzy sub quasigroup of $(V,*)$. Then, (μ_A^P,μ_A^N) is a bipolar fuzzy sub quasigroup and the conditions (i) and (ii) together implies that R is a partial order. For $x,y\in V,\ t=y/x\in V$. Then by assumption (iii), $\mu_A^P(t)\vee\mu_A^P(1/t)>0$ and $\mu_A^N(t)\wedge\mu_A^N(1/t)<0$. But we have, $\mu_A^P(1/t)=\mu_A^P(x/xt)=\mu_A^P(x/y)$. Therefore, $\mu_A^P(y/x)=\mu_A^P(x/y)>0$ and $\mu_A^N(y/x)\wedge\mu_A^N(x/y)<0$, which implies, $\mu_R^P(x,y)\vee\mu_R^P(y,x)>0$ and $\mu_R^N(x,y)\wedge\mu_R^N(y,x)<0$. That is, $(\mu_R\vee\mu_{R-1}^p)(x,y)>0$ and $(\mu_R^N\wedge\mu_{R-1}^N)(x,y)<0$. Thus, R is a linear order.
Hence the proof. $\square$

Theorem 3.1.13. *R is an equivalence relation if and only if (μ_A^P,μ_A^N) is a bipolar fuzzy sub quasigroup of $(V,*)$ satisfying:*

$$(i)\ \mu_A^P(1)=1,\ \mu_A^N(1)=-1,\ \textit{and}$$
$$(ii)\ \big(\mu_A^P(x),\mu_A^N(x)\big)=\big(\mu_A^P(1/x),\mu_A^N(1/x)\big)\ \textit{for all}\ x\in V.$$

Theorem 3.1.14. *$CayF_B(V,A)$ is a Hasse diagram if and only if it is connected and for any collection $x_1,x_2,\ldots,x_n$ of vertices in V with $n\geq 2$ and $\mu_A^P(x_i)>0$, $\mu_A^N(x_i)<0$, for $i=1,2,\ldots,n$ we have, $\mu_A^P\left((\cdots(x_1x_2)\cdots)x_n\right)=0$ and $\mu_A^N\left((\cdots(x_1x_2)\cdots)x_n\right)=0$.*

Proof. Suppose $CayF_B(V,A)$ is a Hasse diagram and let $x_1,x_2,\ldots,x_n$ be vertices in V with $n\geq 2$ and $\mu_A^P(x_i)>0,\ \mu_A^N(x_i)<0$, for $i=1,2,\ldots,n$. Then $R\left((\cdots(x_1x_2)\cdots)x_{i-1}\right)=\big(\mu_A^P(x_i),\mu_A^N(x_i)\big)$, implies

$$1,x_1,x_1x_2,\ldots,(\cdots(x_1x_2)\cdots)x_n$$

is a path from 1 to $(\cdots(x_1x_2)\cdots)x_n$. Since $CayF_B(V,A)$ is a Hasse diagram, we have, $R\left(1,(\cdots(x_1x_2)\cdots)x_n\right)=0$. Therefore, $\mu_A^P\left((\cdots(x_1x_2)\cdots)x_n\right)=0$ and $\mu_A^N\left((\cdots(x_1x_2)\cdots)x_n\right)=0$.

Conversely, suppose that $CayF_B(V,A)$ is connected and for any collection $x_1,x_2,\ldots,x_n$ of vertices in V with $n\geq 2$ and $\mu_A^P(x_i)>0,\ \mu_A^N(x_i)<0$, for $i=$

$1, 2, \ldots, n$ we have, $\mu_A^P((\cdots(x_1x_2)\cdots)x_n) = 0$ and $\mu_A^N((\cdots(x_1x_2)\cdots)x_n) = 0$. Let $(x_\circ, x_1, \ldots, x_n)$ be a path in $CayF_B(V, A)$ from $x_\circ$ to x_n with $n \geq 2$. Then, $\mu_A^P(x_i) > 0$ and $\mu_A^N(x_i) < 0$ for $i = 1, 2, \ldots, n$. Let $x_1 = x_\circ t_1,\ x_2 = x_1 t_2, \ldots, x_n = x_{n-1} t_n$. Then, $\mu_A^P(t_i) = \mu_A^P(x_i/x_{i-1}) > 0$ for $i = 1, 2, \ldots, n$.
We have, $x_n = x_{n-1}t_n = \ldots = (\cdots((x_\circ t_1)t_2)\cdots)t_n = (\cdots(x_\circ(t_1t_2')\ )\cdots)t_n = \ldots = x_\circ(\cdots((t_1t_2')t_3')\cdots)t_n')$.
Therefore, $\mu_R^P(x_\circ, x_n) = \mu_A^P(x_n/x_\circ) = \mu_A^P((\cdots((t_1t_2')t_3')\cdots)t_n')$. We have, $\mu_A^P(t_1) > 0$ and $\mu_A^P(t_2') = \mu_A^P(t_1t_2'/t_1) = \mu_A^P(x_\circ(t_1t_2')/x_\circ t_1) = \mu_A^P((x_\circ t_1)t_2/x_\circ t_1) = \mu_A^P(t_2) > 0$.

In general, $\mu_A^P(t_i') = \mu_A^P(t_i) > 0$ for $i = 2, 3, \ldots, n$. Therefore, since $t_1, t_i' \in V, i = 2, 3, \ldots, n, n \geq 2$ and $\mu_A^P(t_1) > 0, \mu_A^P(t_i') > 0$, for $i = 2, 3, \ldots, n$, we have,$\mu_R^P(x_\circ, x_n) = \mu_A^P((\cdots((t_1t_2')t_3')\cdots)t_n') = 0$. Similarly, we can prove that $\mu_R^N(x_\circ, x_n) = \mu_A^N((\cdots((t_1t_2')t_3')\cdots)t_n') = 0$, both together gives $R(x_\circ, x_n) = 0$.
Hence $CayF_B(V, A)$ is a Hasse diagram. □

Definition 3.1.15. *Let $(V, *)$ be a loop and let $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset induced by V. Then the sub-loop generated by A is the meeting of all bipolar fuzzy sub-loops of V which contains A. It is denoted by $\langle A \rangle$.*

Theorem 3.1.16. *Let $(V, *)$ be a loop and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V. Then the fuzzy subset $\langle A \rangle$ is precisely given by $\langle \mu_A^P \rangle(x) = \vee\{\mu_A^P(x_1) \wedge \mu_A^P(x_2) \wedge \cdots \wedge \mu_A^P(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A^P(x_i) > 0$ for $i = 1, 2, \ldots, n\}$, $\langle \mu_A^N \rangle(x) = \wedge\{\mu_A^N(x_1) \vee \mu_A^N(x_2) \vee \cdots \vee \mu_A^N(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A^N(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$.*

Proof. Let $A' = (\mu_A^{P'}, \mu_A^{N'})$ be the bipolar fuzzy subset of V defined by $\mu_A^{P'}(x) = \vee\{\mu_A^P(x_1 \wedge \mu_A^P(x_2) \wedge \cdots \wedge \mu_A^P(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A^P(x_i) > 0$ for $i = 1, 2, \ldots, n\}$, $\mu_A^{N'}(x) = \wedge\{\mu_A^N(x_1) \vee \mu_A^N(x_2) \vee \cdots \vee \mu_A^N(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A^N(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. If $y \in V$, by definition of $\mu_A^{P'}$ and $\mu_A^{N'}$, it is clear that $\mu_A^{P'}(y) \geq \mu_A^P(y)$ and $\mu_A^{N'}(y) \leq \mu_A^N(y)$.

Thus, we have $\mu_A^P \le \mu_A^{P'}$ and $\mu_A^N \ge \mu_A^{N'}$. Let $x, y \in V$. If $\mu_A^P(x) = 0$ or $\mu_A^P(y) = 0$, $\mu_A^P(x) \wedge \mu_A^P(y) = 0$ and if $\mu_A^N(x) = 0$ or $\mu_A^N(y) = 0$, $\mu_A^N(x) \vee \mu_A^N(y) = 0$. Then, $\mu_A^{P'}(xy) \ge \mu_A^P(x) \wedge \mu_A^P(y)$ and $\mu_A^{N'}(xy) \le \mu_A^N(x) \wedge \mu_A^N(y)$. Again, if $\mu_A^P(x) \ne 0$ and $\mu_A^P(y) \ne 0$, then by definition of $\mu_A^{P'}$, we have $\mu_A^{P'}(xy) \ge \mu_A^P(x) \wedge \mu_A^P(y)$ and if $\mu_A^N(x) \ne 0$ and $\mu_A^N(y) \ne 0$, by definition of $\mu_A^{N'}$, we have $\mu_A^{N'}(xy) \le \mu_A^N(x) \wedge \mu_A^N(y)$. Hence A' is a bipolar fuzzy sub-loop of V containing A.

Now let L be any fuzzy sub-loop of V containing A. Then, for any $x \in V$ with $x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A^P(x_i) > 0$, $\mu_A^N(x_i) < 0$ for $i = 1, 2, \ldots, n$, we have $\mu_L^P(x) \ge \mu_L^P(x_1) \wedge \mu_L^P(x_2) \wedge \cdots \wedge \mu_L^P(x_n) \ge \mu_A^P(x_1) \wedge \mu_A^P(x_2) \wedge \cdots \wedge \mu_A^P(x_n)$ and $\mu_L^N(x) \le \mu_L^N(x_1) \vee \mu_L^N(x_2) \vee \cdots \vee \mu_L^N(x_n) \le \mu_A^N(x_1) \vee \mu_A^N(x_2) \vee \cdots \vee \mu_A^N(x_n)$, which implies that $\mu_L^P(x) \ge \vee\{\mu_A^P(x_1) \wedge \mu_A^P(x_2) \wedge \cdots \wedge \mu_A^P(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A^P(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ and $\mu_A^N(x) \le \wedge\{\mu_A^N(x_1) \vee \mu_A^N(x_2) \vee \cdots \vee \mu_A^N(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A^N(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. Therefore, $\mu_L^P(x) \ge \mu_A^{P'}(x)$ and $\mu_L^N(x) \le \mu_A^{N'}$ for all $x \in V$. That is, $\langle\mu_A^P\rangle(x) = \vee\{\mu_A^P(x_1) \wedge \mu_A^P(x_2) \wedge \cdots \wedge \mu_A^P(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A^P(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ and $\langle\mu_A^N\rangle(x) = \wedge\{\mu_A^N(x_1) \vee \mu_A^N(x_2) \vee \cdots \vee \mu_A^N(x_n) : x = (\cdots((x_1x_2)x_3)\cdots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A^N(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. Thus, $A' = (\mu_A^{P'}, \mu_A^{N'}) = \langle(\mu_A^P, \mu_A^N)\rangle = \langle A\rangle$. Hence the proof. $\square$

3.2 Connectedness in Cayley bipolar fuzzy graphs induced by loops

Theorem 3.2.1. *Let $(V, *)$ be a loop and $A = (\mu_A^P, \mu_A^N)$ be a bipolar fuzzy subset of V. Then, for any $\alpha \in [-1, 1]$,*
$(\langle\mu_{A\alpha}^P\rangle, \langle\mu_{A\,\alpha}^N\rangle) = (\langle\mu_A^P\rangle_\alpha, \langle\mu_A^N\rangle_\alpha)$ and $(\langle\mu_{A\alpha}^{P+}\rangle, \langle\mu_{A\,\alpha}^{N+}\rangle) = (\langle\mu_A^P\rangle_\alpha^+, \langle\mu_A^N\rangle_\alpha^+)$, where

Chapter 4

Cayley Intuitionistic Fuzzy Graphs Induced by Loops

In this chapter we introduce Cayley intuitionistic fuzzy graphs induced by loops and study some of its properties in terms of algebraic properties. We also discuss connectedness and α-connectedness in these graphs. These graphs can be considered as generalisation of those in [19].Contents of this chapter is published in Far East Journal of Mathematical Sciences [22].

4.1 Cayley Intuitionistic Fuzzy Graphs Induced by Loops

Definition 4.1.1. *Let $(V, *)$ be a loop and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of V satisfying the condition, $\mu_A(y/x) = \mu_A(ay/ax)$ and $\nu_A(y/x) = \nu_A(ay/ax)$, for all $x, y, a \in V$, where y/x denote the solution of $y = xt$. Then this intuitionistic fuzzy subset of V is called scaled intuitionistic fuzzy subset of the loop $(V, *)$.*

Definition 4.1.2. *Let $(V, *)$ be a loop and let $A = (\mu_A, \nu_A)$ be a scaled intuitionistic fuzzy subset of V. Let the intuitionistic fuzzy relation $R = (\mu_R, \nu_R)$ on V be defined by $R(x, y) = (\mu_A(y/x), \nu_A(y/x))$. Then the intuitionistic fuzzy graph $G = (V, R)$ induced by the triplet $(V, *, A)$ is called the Cayley intuitionistic fuzzy graph induced by the loop V and is denoted as $CayF_I(V, A)$.*

Let $(V, *)$ be a loop. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of V. Then A is said to be an *intuitionistic fuzzy sub quasigroup* of V if for all $x, y \in V$ $\mu_A(xy) \geq min(\mu_A(x), \mu_A(y))$ and $\nu_A(xy) \leq max(\nu_A(x), \nu_A(y))$.

4.1.1 Basic Results

Theorem 4.1.3. *$CayF_I(V, A)$ is vertex transitive.*

Proof. Let $a, b \in V$ and $b = z_\circ a,\ z_\circ \in V$. Define $\Psi : V \to V$ by $\Psi(x) = z_\circ x$. Clearly, Ψ is a bijective map and it can be easily varified that $R(\Psi(x), \Psi(y)) = R(x, y)$. Hence Ψ is an automorphism on $CayF_I(V, A)$. Also $\Psi(a) = b$. Hence G is vertex transitive. □

Theorem 4.1.4. *[19] Every vertex transitive intuitionistic fuzzy graphs are regular.*

Remark 4.1.5.

(a) By Theorems 4.1.3, 4.1.4, we have, $CayF_I(V, A)$ is in-regular and out-regular.

(b) Also it can be easily seen that $CayF_I(V, A)$ is regular.

Theorem 4.1.6. *The intuitionistic fuzzy relation R defined above is :*

(i) reflexive if and only if $R(1, 1) = (1, 0)$,

(ii) symmetric if and only if $A(x) = (\mu_A(x), \nu(x)) = (\mu_A(1/x), \nu_A(1/x) = A(1/x)$,

(iii) antisymmetric if and only if
$\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(1/x), \nu_A(1/x))\} = 1,$

(iv) transitive if and only if $A = (\mu_A, \nu_A)$ *is an intuitionistic fuzzy subloop of* $(V, *)$.

Theorem 4.1.7. *R is a partial order if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subloop of $(V, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,

(ii) $\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(1/x), \nu_A(1/x))\} = \{1\}$.

Theorem 4.1.8. *R is a linear order if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subloop of $(V, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,

(ii) $\{x : (\mu_A(x), \nu_A(x)) = (\mu_A(1/x), \nu_A(1/x))\} = \{1\}$,

(iii) $R^2 \leq R$, *that is,* $\mu_R(x, y) \geq \mu_{R^2}(x, y)$ *and* $\nu_R(x, y) \leq \nu_{R^2}(x, y)$, *for all* $x, y \in V$,

(iv) $\{x : \mu_A(x) \vee \mu_A(1/x) > 0, \nu_A(x) \wedge \nu_A(1/x) < 0\} = V$.

Theorem 4.1.9. *R is an equivalence relation if and only if $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy subloop of $(V, *)$ satisfying*

(i) $\mu_A(1) = 1$ *and* $\nu_A(1) = 0$,

(ii) $(\mu_A(x), \nu_A(x)) = (\mu_A(1/x), \nu_A(1/x))$ *for all* $x \in V$.

Definition 4.1.10. *Let $(V, *)$ be a loop and let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset induced by V. Then the sub-loop generated by A is the meeting of all intuitionistic fuzzy subloops of V which contains A. It is denoted by $\langle A \rangle$.*

Theorem 4.1.11. *Let $(V,*)$ be a loop and $A = (\mu_A, \nu_A)$ be a intuitionistic fuzzy subset of V. Then the fuzzy subset $\langle A\rangle$ is precisely given by $\langle\mu_A\rangle(x) = \vee\{\mu_A(x_1)\wedge\mu_A(x_2)\wedge\ldots\wedge\mu_A(x_n) : x = (\ldots((x_1x_2)\ x_3)\ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$, $\langle\nu_A\rangle(x) = \wedge\{\nu_A(x_1) \vee \nu_A(x_2) \vee \ldots \vee \nu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\nu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$ for any $x \in V$.*

Proof. Let $A' = (\mu'_A, \nu'_A)$ be the intuitionistic fuzzy subset of V defined by $\mu'_A(x) = \vee\{\mu_A(x_1 \wedge \mu_A(x_2) \wedge \ldots \wedge \mu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ \ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$, $\nu'_A(x) = \wedge\{\nu_A(x_1) \vee \nu_A(x_2) \vee \ldots \vee \nu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\nu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$ for any $x \in V$. If $y \in V$, by definition of μ'_A and ν'_A, it is clear that $\mu'_A(y) \geq \mu_A(y)$ and $\nu'_A(y) \leq \nu_A(y)$. Thus, we have $\mu_A \leq \mu'_A$ and $\nu_A \geq \nu'_A$. Let $x, y \in V$. If $\mu_A(x) = 0$ or $\mu_A(y) = 0$, $\mu_A(x) \wedge \mu_A(y) = 0$ and if $\nu_A(x) = 0$ or $\nu_A(y) = 0$, $\nu_A(x) \vee \nu_A(y) = 0$. Then, $\mu'_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu'_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$. Again, if $\mu_A(x) \neq 0$ and $\mu_A(y) \neq 0$, then by definition of μ'_A, we have $\mu'_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and if $\nu_A(x) \neq 0$ and $\nu_A(y) \neq 0$, by definition of ν'_A, we have $\nu'_A(xy) \leq \nu_A(x) \wedge \nu_A(y)$. Hence A' is a intuitionistic fuzzy sub-loop of V containing A. Now let L be any intuitionistic fuzzy sub-loop of V containing A. Then, for any $x \in V$ with $x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\mu_A(x_i) > 0,\ \nu_A(x_i) < 0$ for $i = 1,2,\ldots,n$, we have $\mu_L(x) \geq \mu_L(x_1)\wedge\mu_L(x_2)\wedge\ldots\wedge\mu_L(x_n) \geq \mu_A(x_1)\wedge\mu_A(x_2)\wedge\ldots\wedge\mu_A(x_n)$ and $\nu_L(x) \leq \nu_L(x_1) \vee \nu_L(x_2) \vee \ldots \vee \nu_L(x_n) \leq \nu_A(x_1) \vee \nu_A(x_2) \vee \ldots \vee \nu_A(x_n)$, which implies that $\mu_L(x) \geq \vee\{\mu_A(x_1) \wedge \mu_A(x_2) \wedge \ldots \wedge \mu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$with a finite positive integer $n,\ x_i \in V$and $\mu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$ and $\nu_A(x) \leq \wedge\{\nu_A(x_1) \vee \nu_A(x_2) \vee \ldots \vee \nu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer $n,\ x_i \in V$ and $\nu_A(x_i) > 0$ for $i = 1,2,\ldots,n\}$ for any $x \in V$. Therefore, $\mu_L(x) \geq \mu'_A(x)$ and $\nu_L(x) \leq \nu'_A$ for all $x \in V$. That is, $\langle\mu_A\rangle(x) = \vee\{\mu_A(x_1) \wedge \mu_A(x_2) \wedge \ldots \wedge \mu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer n, $x_i \in V$ and $\mu_A(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ and $\langle \nu_A \rangle(x) = \wedge\{\nu_A(x_1) \vee \nu_A(x_2) \vee \ldots \vee \nu_A(x_n) : x = (\ldots((x_1x_2)x_3)\ldots)x_n$ with a finite positive integer n, $x_i \in V$ and $\nu_A(x_i) > 0$ for $i = 1, 2, \ldots, n\}$ for any $x \in V$. Thus, $A' = (\mu'_A, \nu'_A) = \langle(\mu_A, \nu_A)\rangle = \langle A \rangle$. Hence the proof. □

4.2 Connectedness in Cayley Intuitionistic Fuzzy Graphs Induced by Loops

Theorem 4.2.1. *Let $(V, *)$ be a loop and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of V. Then, for any $\alpha \in [0, 1]$,*
$(\langle \mu_{A\alpha} \rangle, \langle \nu_{A\alpha} \rangle) = (\langle \mu_A \rangle_\alpha, \langle \nu_A \rangle_\alpha)$ and $(\langle \mu_{A^+_\alpha} \rangle, \langle \nu_{A^+_\alpha} \rangle) = (\langle \mu_A \rangle^+_\alpha, \langle \nu_A \rangle^+_\alpha)$, where $(\langle \mu_{A\alpha} \rangle, \langle \nu_{A\alpha} \rangle)$ and $(\langle \mu_A \rangle, \langle \nu_A \rangle)$ denotes respectively the intuitionistic fuzzy sub-loop generated by $(\mu_{A\alpha}, \nu_{A\alpha})$ and (μ_A, ν_A).

Proof. Observe that

$$\begin{aligned} x \in (\langle \mu_{A\alpha} \rangle, \langle \nu_{A\alpha} \rangle) &\Leftrightarrow \exists x_1, x_2, \ldots, x_n \text{ in } A_\alpha \ni x = (\ldots(x_1x_2)\ldots)x_n \\ &\Leftrightarrow \exists 1, x_2, \ldots, x_n \text{ in } V \ni \mu_A(x_i) \geq \alpha, \nu_A(x_i) \\ &\leq \alpha \forall i = 1, 2, \ldots, n \text{ and } x = (\ldots(x_1x_2)\ldots)x_n \\ &\Leftrightarrow \langle \mu_A(x) \rangle \geq \alpha, \langle \nu_A(x) \rangle \leq \alpha \\ &\Leftrightarrow x \in \langle \mu_A \rangle_\alpha, x \in \langle \nu_A \rangle_\alpha. \end{aligned}$$

Therefore, $(\langle \mu_{A\alpha} \rangle, \langle \nu_{A\alpha} \rangle) = (\langle \mu_A \rangle_\alpha, \langle \nu_A \rangle_\alpha)$.
Similarly, we can prove that $(\langle \mu_{A^+_\alpha} \rangle, \langle \nu_{A^+_\alpha} \rangle) = (\langle \mu_A \rangle^+_\alpha, \langle \nu_A \rangle^+_\alpha)$. □

Remark 4.2.2. *Let $(V, *)$ be a loop and $A = (\mu^P_A, \mu^N_A)$ be an intuitionistic fuzzy subset of V. Then by Theorem 4.2.1, we have $\langle supp(A) \rangle = supp(\langle A \rangle)$.*

Theorem 4.2.3. *$CayF_I(V, A)$ is connected if and only if $supp\langle A \rangle \supseteq V - \{1\}$.*

Proof. Suppose $CayF_I(V, A)$ is connected. Let $x \in V - \{1\}$. Since $CayF_I(V, A)$ is connected, there exist a path from 1 to x, say, $1, x_1, x_2, \ldots, x_n = x$. This implies that, there exist $t_1, t_2, \ldots, t_n \in supp(A)$ such that $x = x_n = x_{n-1}t_n$, $x_{n-1} = x_{n-2}t_{n-1}, \ldots, x_2 = x_1t_2$, $x_1 = 1.t_1$. Therefore, $x = x_n = x_{n-1}t_n = (x_{n-2}t_{n-1})t_n = \ldots = (\ldots((t_1t_2)t_3)\ldots)t_n$, $t_i \in supp(A)$, $i = 1, 2, \ldots, n$, which implies that $x \in \langle supp(A)\rangle$. Therefore, $V - \{1\} \subseteq \langle supp(A)\rangle = supp\langle A\rangle$.

Conversely, let $V - \{1\} \subseteq supp\langle A\rangle$. Let x, y be two distinct elements in V. Then, there exist $z \neq 1 \in V$ such that $y = xz$. Since $z \neq 1$, $z \in V - \{1\} \subseteq \langle supp(A)\rangle$. Then, there exist $z_1, z_2, \ldots, z_m \in supp(A)$ such that $z = z_1z_2\ldots z_m$. Clearly, $1, z_1, z_1z_2, \ldots, (\ldots((z_1z_2)z_3)\ldots)z_m = z$ is a path from 1 to z. Then $x, xz_1, x(z_1z_2), x((z_1z_2)z_3), \ldots, x((\ldots((z_1z_2)z_3)\ \ldots)\ z_m) = xz = y$ is a path from x to y. Therefore, $CayF_I(V, A)$ is connected. □

Theorem 4.2.4. *$CayF_I(V, A)$ is weakly connected if and only if $supp(\langle A \vee A_\ell\rangle) \supseteq V - \{1\}$ where $A_\ell(x_\ell) = A(x)$, $1 = x_\ell x$.*

Proof. Suppose that $CayF_I(V, A)$ is weakly connected.
Then $CayF_I(V, A \vee A_\ell)$ is connected. Thus by Theorem 4.2.3, we have $V - \{1\} \subseteq supp(\langle A \vee A_\ell\rangle)$. This completes the proof. □

Theorem 4.2.5. *$CayF_I(V, A)$ is semi-connected if and only if*

$$\langle supp(A)\rangle \cup \langle supp(A)\rangle_\ell \supseteq V - \{1\}.$$

Proof. First assume that $CayF_I(V, A)$ is semi-connected. Let $x \in V - \{1\}$. Since $CayF_I(V, A)$ is semi-connected, there exist a path from x to 1 or a path from 1 to x. Suppose there exist a path $1, x_1, x_2, \ldots, x_n, x$ from 1 to x. Then, there exist $t_1, t_2, \ldots, t_{n+1} \in supp(A)$ such that $x_1 = 1t_1$, $x_2 = x_1t_2, \ldots, x = x_nt_{n+1}$. Then, $x = x_nt_{n+1} = (x_{n-1}t_n)t_{n+1} = \ldots = (\ldots((t_1t_2)t_3)\ldots)t_{n+1}$, $t_i \in supp(A)$, $i = 1, 2, \ldots, n+1$, which implies that $x \in \langle supp(A)\rangle$. Or suppose there exist a path $x, y_1, y_2, \ldots, y_m, 1$ from x to 1. Then, there exist $k_i \in supp(A)$ for $i = 0, 1, \ldots, m$ such that $y_1 = xk_\circ$, $y_2 = y_1k_1, \ldots,$ $1 = y_mk_m$. Then, $1 = (\ldots((xk_\circ)k_1)\ldots)k_m = (\ldots(x(k_\circ k'_1))\ldots)k_m = \ldots = x((\ldots(k_\circ k'_1)\ldots)k'_m)$. Here,

$k'_i \in supp(A)$, for $i = 0, 1, \ldots, m$, since $supp(A)$ is right associative. This implies $1 = xk$, where $k = (\ldots((k_\circ k'_1)k'_2)\ldots)k'_m \in \langle supp(A)\rangle$. Hence $x \in \langle supp(A)\rangle_\ell$. Therefore, $CayF_I(V, A)$ is semi-connected implies $\langle supp(A)\rangle \cup \langle supp(A)\rangle_\ell \supseteq V - \{1\}$.

Conversely, assume that $\langle supp(A)\rangle \cup \langle supp(A)\rangle_\ell \supseteq V - \{1\}$. Let $x, y \in CayF_I(V, A)$ be two distinct elements. Then $y = xz$ for some $z \in CayF_I(V, A)$. Then, $z \in \langle supp(A)\rangle \cup \langle supp(A)\rangle_\ell$.

If $z \in \langle supp(A)\rangle$, then there exist $t_1, t_2, \ldots, t_n \in supp(A)$ such that $z = (\ldots((t_1t_2)t_3)\ldots)t_n$. Clearly $1, t_1, t_1t_2, \ldots, (\ldots((t_1t_2)t_3)\ldots)t_n = z$ is a path from 1 to z. Then, $x, xt_1, \ldots, x((\ldots((t_1t_2)t_3)\ldots)t_n)$ is a path from x to y. Or else, if $z \in \langle supp(A)\rangle_\ell$, $1 = zt$ for some $t \in \langle supp(A)\rangle$, which implies there exist $p_1, p_2, \ldots, p_m \in supp(A)$ such that $t = p_1p_2\ldots p_m$.
Then,

$$\begin{aligned}
1 &= zt \\
&= z((\ldots((p_1p_2)p_3)\ldots)p_m) \\
&= (z\ldots((p_1p_2)p_3)\ldots)p'_m \\
&\vdots \\
&= (\ldots((zp_1)p'_2)\ldots)p'_m,\ p_i \\
&\in supp(A), \text{since } supp(A) \text{ is right associative and } p_i \in supp(A).
\end{aligned}$$

Let $k_1 = zp_1,\ k_2 = k_1p'_2,\ k_3 = k_2p'_3, \ldots, k_m = k_{m-1}p'_m$. Then, $k_m = k_{m-1}p'_m = \ldots = (\ldots((zp_1)p'_2)\ldots)p'_m = 1$. Clearly, $z, k_1, k_2, \ldots, k_m = 1$ is a path from z to 1. Then, $xz, xk_1, xk_2, \ldots, xk_m = x$ is a path from y to x. Thus, for any $x, y \in V$ there exist a path from x to y or a path from y to x, which implies that $CayF_I(V, A)$ is semi-connected.
This completes the proof. $\square$

Theorem 4.2.6. *$CayF_I(V, A)$ is locally connected if and only if*

$$supp\langle A\rangle = supp\langle A\rangle_\ell.$$

Proof. First suppose that $CayF_I(V, A)$ is locally connected. Let $x \in \langle supp(A)\rangle$. Then, there exist $x_1, x_2, \ldots, x_n \in supp(A)$ such that $x = (\ldots((x_1x_2)x_3)\ldots)x_n$. Therefore $1, x_1, x_1x_2, \ldots, (\ldots((x_1x_2)x_3)\ldots)x_n$ is a path from 1 to x. Then, since G is locally connected, there exist a path from x to 1. Let $x, z_1, z_2, \ldots, z_{n-1}$ be a path from x to 1 which implies there exist $a_1, a_2, \ldots, a_n \in supp(A)$ such that $z_1 = xa_1,\ z_2 = z_1a_2, \ldots, z_{n-1} = z_{n-2}a_{n-1},\ 1 = z_{n-1}a_n$.
Then, $1 = z_{n-1}a_n = \ (z_{n-2}a_{n-1})a_n = \ldots = (\ldots((xa_1)a_2)\ldots)a_n$. Then, we have, $1 = (\ldots(x(a_1a_2'))\ldots)a_n = \ldots = x(\ldots((a_1a_2')a_3')\ldots)a_n',\ a_i' \in supp(A)$. That is, $1 = xt,\ t \in \langle supp(A)\rangle$, implies $x \in \langle supp(A)\rangle_\ell$. Thus,

$$\langle supp(A)\rangle \subseteq \langle supp(A)\rangle_\ell. \tag{4.1}$$

Let $x_\ell \in \langle supp(A)\rangle_\ell$, which implies there exist an $x \in \langle supp(A)\rangle$ such that $1 = x_\ell x$. Then, $1, x_1, x_1x_2, \ldots, (\ldots((x_1x_2)x_3)\ldots)x_m$ is a path from 1 to x. Thus, since $CayF_I(V, A)$ is locally connected, there exist a path from x to 1. We have, $1 = x_\ell x = x_\ell((\ldots((x_1x_2)x_3)\ldots)x_m) = x_\ell((\ldots(x_1(x_2x_3'))\ldots)x_m) = \ldots = (\ldots((x_\ell x_1)x_2')\ldots)x - m'\ x_1 \in supp(A)$, and here $x_i' \in supp(A),\ i = 1, 2, \ldots, m$, since $supp(A)$ is right associative. Now, let $t_1 = x_\ell x_1,\ t_2 = t_1x_2', \ldots, 1 = t_m = t_{m-1}x_m'$. Then, $x_\ell, t_1, t_2, \ldots, t_{m-1}, t_m = 1$ is a path from x_ℓ to 1. This implies that there exist a path from 1 to x_ℓ, since $CayF_I(V, A)$ is locally connected. Let $1, k_1, k_2, \ldots, k_r = x_\ell$ be a path from 1 to x_ℓ. Then, there exist $p_i \in supp(A),\ i = 1, 2, \ldots, r$ such that $k_1 = 1.p_1,\ k_2 = k_1p_2, \ldots, k_r = k_{r-1}p_r$. Thus, $x_\ell = k_r = k_{r-1}p_r = (k_{r-2}p_{r-1})p_r = \ldots = (\ldots((p_1p_2)p_3)\ldots)p_r$, which implies that $x_\ell \in \langle supp(A)\rangle$. Hence,

$$\langle supp(A)\rangle_\ell \subseteq \langle supp(A)\rangle. \tag{4.2}$$

Therefore, from equations (4.1) and (4.2) we get $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$.

Conversely, suppose $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$. Let $x, y \in V$ and there exist a path from x to y say $x, x_1, x_2, \ldots, x_{n-1}, y$. Then, there exist $a_i \in supp(A)$ for $i = 1, 2, \ldots, n$ such that $x_1 = xa_1,\ x_2 = x_1a_2, \ldots, x_{n-1} = x_{n-2}a_{n-1},\ y = x_{n-1}a_n$. Thus, $y = x_{n-1}a_n = (x_{n-2}a_{n-1}a_n = \ldots = (\ldots((xa_1)a_2)\ldots)a_n$. Therefore, $y = x((\ldots((a_1a_2')a_3')\ldots)a_n')$, where $a_i' \in supp(A)$, since $supp(A)$ is right associative, which implies $y/x \in \langle supp(A)\rangle$. Then, $x/y \in \langle supp(A)\rangle_\ell$. Now, since $k = x/y \in \langle supp(A)\rangle_\ell$ and $\langle supp(A)\rangle = \langle supp(A)\rangle_\ell$, $k = x/y \in \langle supp(A)\rangle$. Then, there exist $k_1, k_2, \ldots, k_p \in supp(A)$ such that $k = (\ldots((k_1k_2)k_3)\ldots)k_p$. Clearly, $1, k_1, k_1k_2, \ldots, (\ldots((k_1k_2)k_3)\ldots)k_p = k$ is a path from 1 to k. Then, $y, yk_1, y(k_1k_2), \ldots, y((\ldots((k_1k_2)k_3)\ldots)k_p) = yk = x$ is a path from y to x. Hence G' is locally connected. □

Theorem 4.2.7. *A finite Cayley intuitionistic fuzzy graph $CayF_I(V, A)$, where $(V, *)$ is a finite loop, is quasi-connected if and only if it is connected.*

Proof. Every connected graphs are quasi-connected.
Therefore, if $CayF_I(V, A)$ is connected then $CayF_I(V, A)$ is quasi-connected. Now, suppose that $CayF_I(V, A)$ is quasi-connected. Note that $CayF_I(V, A)$ is finite. Thus, $CayF_I(V, A)$ has a source, say z. Then for any $x \in V$ with $x \neq z$, there is a directed path from z to x. Let $z, z_1, z_2, \ldots, z_n, x$ be a path from z to x. Then, there exist $k_i \in supp(A)$ for $i = 1, 2, \ldots, n+1$ such that $z_1 = zk_1,\ z_2 = z_1k_2, \ldots,\ z_n = z_{n-1}k_n,\ x = z_nk_{n+1}$. Then, $x = (\ldots((zk_1)k_2)\ldots)k_{n+1} = (\ldots(z(k_1k_2'))\ldots)k_{n+1} = \ldots = z((\ldots(k_1k_2')\ldots)k_{n+1}')$. Here, $k_i' \in supp(A)$, for $i = 1, 2, \ldots, n+1$, since $supp(A)$ is right associative.
Therefore, $x/z = (\ldots(k_1k_2')\ldots)k_{n+1}') \in \langle supp(A)\rangle$. Thus it is clear that $x/z \in \langle supp(A)\rangle$ for every $x \in V$ with $x \neq z$. Hence $\langle supp(A)\rangle \supseteq V - \{1\}$. Hence, by Theorem 4.2.3, $CayF_I(V, A)$ is connected. □

4.3 Strength of Connectedness in Cayley Intuitionistic Fuzzy Graphs Induced by Loops

In this subsection, we prove the following theorems based on different types of α-connectedness. For any $\alpha \in [-1, 1]$, let A_α be the α-cut of A. Then

$$R_\alpha = (\mu_{R_\alpha}, \nu_{R_\alpha}) = (\mu_{A_\alpha}, \nu_{A_\alpha}) = \{(x, y) \in V \times V : y/x \in A_\alpha\}.$$

Definition 4.3.1. *The μ_R-strength of a path $x_\circ, x_1, \ldots, x_n$ in $CayF_I(V, A)$ is defined as $min(\mu_R(x_{i-1}, x_i))$ for $i = 1, 2, \ldots, n$ and is denoted as $S(\mu_R)$. The ν_R-strength of this path is defined as $max(\nu_R(x_{i-1}, x_i))$ for $i = 1, 2, \ldots, n$ and is denoted as $S(\nu_R)$.*

Definition 4.3.2. *Strength of a path P' in $CayF_I(V, A)$, denoted by $strength(P')$ is defined to be $strength(P') = (S(\mu_R), S(\nu_R))$ and is said to be greater than or equal to α if $S(\mu_R) \geq \alpha$ and $S(\nu_R) \leq \alpha$.*

Definition 4.3.3. *Let $G = (V, \rho)$ be an intuitionistic fuzzy graph. Then G is said to be: (i) α-connected if for every pair of vertices $x, y \in G$, there is a path P from x to y such that $strength(P) \geq \alpha$, (ii) weakly α-connected if the fuzzy graph $(V, R \vee R^{-1})$ is α-connected, (iii) semi α-connected if for every $x, y \in V$, there is a path of strength greater than or equal to α from x to y or from y to x in G and (iv) locally α-connected if for every pair of vertices x and y, there is a path P of strength greater than or equal to α from x to y whenever there is a path P' of strength greater than or equal to α from y to x. (v) quasi α-connected if for every pair $x, y \in V$, there is some $z \in V$ such that there is a directed path from z to x of strength greater than or equal to α and there is a directed path from z to y of strength greater than or equal to α. (vi) α-complete if $R(x, y) \geq \alpha$ for all $x, y \in V$.*

Theorem 4.3.4. *$G = CayF_I(V, A)$ is*

(i) α-connected if and only if $\langle A\rangle_\alpha \supseteq V - \{1\}$, where $\langle A\rangle_\alpha = (\langle \mu_A\rangle_\alpha), \langle \nu_A\rangle_\alpha)$.

(ii) weakly α*-connected if and only if* $\langle A \cup A_\ell \rangle_\alpha \supseteq V - \{1\}$.

(iii) semi α*-connected if and only if* $\langle A \rangle_\alpha \cup \langle A \rangle_{\alpha\ell} \supseteq V - \{1\}$.

(iv) locally α*-connected if and only if* $\langle A \rangle_\alpha = \langle A \rangle_{\alpha\ell}$.

(v) is α*-complete if and only if* $A_\alpha \supseteq V - \{1\}$.

Theorem 4.3.5. *If* $CayF_I(V, A)$ *is finite then it is quasi* α*-connected if and only if it is* α*-connected.*

Chapter 5

Cayley Fuzzy Digraph Structure Induced by Groups and Loops

In the first section of this chapter we introduced Cayley Fuzzy Digraph Structure induced by groups and studied the properties of Cayley fuzzy digraph structure in terms of algebraic properties. Contents of this section is published in Mathematical Combinatorics [23]. In second section we introduce a class of Cayley fuzzy digraph structure induced by loops and we generalise the results and prove that a bigger class of Cayley fuzzy digraph structure could be induced by loops, a weaker algebraic structure than a group.

5.1 Basic definitions

Definition 5.1.1. *Let V be a non-empty set and $S_1, S_2, \ldots, S_k$ are relations on V which are mutually disjoint, then $G' = (V, S_1, S_2, \ldots,\ S_n)$ is a digraph structure. In addition, if $S_1, S_2, \ldots, S_k$ are symmetric and irreflexive, then $G' = (V, S_1, S_2, \ldots, S_k)$ is a graph structure [6].*

Let G be a group and $S_1, S_2, \ldots, S_n$ be mutually disjoint subsets of G. Then the *Cayley digraph structure* of G with respect to $S_1, S_2, \ldots, S_n$ is defined as the graph structure $X = (G; E_1, E_2, \ldots, E_n)$, where $E_i = \{(x, y) : x^{-1}y \in S_i\}$ [4]. In case, a digraph structure with only one connection set is the usual Cayley digraph. So a Cayley digraph structure is a generalization of the Cayley digraph.

Definition 5.1.2. *Let $G' = (V, S_1, S_2, \ldots, S_k)$ be a graph (digraph) structure and $\mu, \rho_1, \rho_2, \ldots, \rho_k$ be fuzzy subsets of $V, S_1, S_2, \ldots, S_k$ respectively such that $\rho_i(x, y) \leq \mu(x) \wedge \mu(y)$, for all $x, y \in V$ and $i = 1, 2, \ldots, k$.*
Then $G = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy graph (digraph) structure of G' [29].

Let V be a non-empty set, μ be fuzzy subset of V and $R_1, R_2, \ldots, R_n$ be mutually disjoint fuzzy relations on μ. Then $G = (\mu, R_1, R_2, \ldots, R_n)$ is a fuzzy digraph structure on V. In case $\mu = \chi_V$, where χ_V is the characteristic function on V, then the fuzzy digraph structure $(\mu, R_1, R_2, \ldots, R_n)$ is simply denoted by $G = (V; R_1, R_2, \ldots, R_n)$.

A fuzzy digraph structure $G = (V; R_1, R_2, ..., R_n)$ is called (i) trivial if $R_i \equiv 0$ for all i, (ii) reflexive if for all $x \in V, R_i(x, x) = 1$ for some i, (iii) symmetric if $R_i = R_i^{-1}$ for all i, (iv) transitive if for every i and j, $R_i \wedge R_j \leq R_k$ for some k, (v) a hasse diagram if for every positive integer $m \geq 2$ and for every $x_1, x_2, \ldots, x_m$ of V with $R_i(x_j, x_{j+1}) > 0$ for all $j = 0, 1, 2, ..., m-1$, implies $R_i(x_0, x_m) = 0$ for all i, and (vi) complete if for any $x, y \in V, R_i(x, y) > 0$, for some $i = 1, 2, \ldots, n$. A walk of length k in a digraph structure is an alternating sequence $W = x_0, e_0, x_1, \ldots, e_{k-1}, x_k$, where $e_j = (x_j, x_{j+1})$ and $R_i(e_j) > 0$ for some i. A walk W is called a path if all the vertices are distinct. We use notation $x_0, x_1, x_2, \ldots, x_k$ for the walk W. A walk is called a circuit if its first and last vertices are the same, but no other vertex is repeated. A weak path is a sequence $x_1, x_2, \ldots, x_m$ of distinct vertices of V such that for $j = 1, 2, \ldots, m-1$, $R_i \vee R_i^{-1}(x_j, x_{j+1}) > 0$ for some $i = 1, 2, \ldots, n$. Distance between to vertices x and y in G is the length of the shortest path from x to y and is denoted by $d(x, y)$. Diameter of the fuzzy digraph structure G, denoted by $d(G)$, is defined by $d(G) = max_{x,y \in G} d(x, y)$. A fuzzy digraph structure $G = (V; R_1, R_2, ..., R_n)$ is called (i) connected (strongly connected) if y is connected to x for all $x, y \in V$, and (ii) weakly connected if any two vertices can be joined by a weak path, that is, the fuzzy digraph structure $G' = (V; R_1 \vee R_1^{-1}, R_2 \vee R_2^{-1}, \ldots, R_n \vee R_n^{-1})$ is connected. A weakly connected fuzzy digraph structure $G = (V; R_1, R_2, ..., R_n)$ with out any circuits is called a tree.

The present work is a generalisation of the work in [15] in which Madhavan Namboothiri N.M. et al. introduced a class of Cayley fuzzy graphs induced by groups.

5.2 Cayley fuzzy digraph structure

Definition 5.2.1. *Let V be a group and $\nu_1, \nu_2, \ldots, \nu_n$ be mutually disjoint fuzzy subsets of V. Then Cayley Fuzzy Digraph Structure with respect to $\nu_1, \nu_2, \ldots, \nu_n$ of V is defined as $(V; R_1, R_2, \ldots, R_n)$ where $R_i(x, y) = \nu_i(x^{-1}y)$ and is denoted by $CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$. The subsets $\nu_1, \nu_2, \ldots, \nu_n$ are called connection fuzzy subsets of $CayF_D(V; \nu_1\ , \nu_2, \ldots, \nu_n)$. In case, a Cayley fuzzy digraph structure with only one connection set is usual Cayley fuzzy graph.*

Theorem 5.2.2. $G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ *is vertex-transitive.*

Proof. Let a and b be any two arbitrary elements in G.
Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x$ for all $x \in V$. Clearly, ψ is a bijection onto itself.
Furthermore, we have, for each $x, y \in V$,

$$\begin{aligned} R_i(\psi(x), \psi(y)) &= R_i(ba^{-1}x, ba^{-1}y) \\ &= \nu_i((ba^{-1}x)^{-1}(ba^{-1}y)) \\ &= \nu_i(x^{-1}y) \end{aligned}$$

$$= R_i(x,y).$$

Hence the proof. $\square$

Theorem 5.2.3. *Cayley fuzzy digraph structures are regular*

Proof. Let $G = CayF_D(V;\nu_1,\nu_2,\ldots,\nu_n)$ be a Cayley fuzzy digraph structure. Let $u, v \in V$. Since Cayley fuzzy digraph structures are vertex transitive, there exist an automorphism say, f on G such that, $f(u) = v$ and $R_i(f(x), f(y)) = R_i(x,y)$ for any $x, y \in V$ and $i = 1, 2, \ldots, n$.
Then the in-degree of u, $ind(u) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} R_i(x,u) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} R_i(f(x), f(u)) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} R_i(f(x), v) = \sum\limits_{f(x)\in V}\sum\limits_{i=1}^{n} R_i(f(x), v) = \sum\limits_{y\in V}\sum\limits_{i=1}^{n} R_i(y,v) = ind(v)$. Similarly, we can prove that $outd(u) = outd(v)$. Therefore, G is in-regular and out-regular. Now to prove that G is regular we just need to show that $ind(1) = outd(1)$. We have, $ind(1) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} R_i(x,1) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} \nu_i(x^{-1}) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} \nu_i(x) = \sum\limits_{x\in V}\sum\limits_{i=1}^{n} R_i(1,x) = outd(1)$. Therefore, G is regular. $\square$

Theorem 5.2.4. $G = CayF_D(V;\nu_1,\nu_2,\ldots,\nu_n)$ *is a trivial graph if and only if* $\nu_i \equiv 0$ *for all* i.

Proof. By definition, G is trivial if and only if $R_i \equiv 0$ for all i. This implies that $\nu_i \equiv 0$ for all i. $\square$

Theorem 5.2.5. $G = CayF_D(V;\nu_1,\nu_2,\ldots,\nu_n)$ *is reflexive if and only if* $\nu_i(1) = 1$ *for some* i.

Proof. Assume that $G = CayF_D(V;\nu_1,\nu_2,\ldots,\nu_n)$ is reflexive. Then for every $x \in V$, $R_i(x,x) = 1$ for some i. This implies that $\nu_i(x^{-1}x) = \nu_i(1) = 1$ for some i.
Conversely, let $\nu_i(1) = 1$ for some i, say $i = k$. This implies that for each $x \in V$, $R_k(x,x) = \nu_k(x^{-1}x) = \nu_k(1) = 1$. That is G is reflexive. $\square$

Theorem 5.2.6. *$G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ is symmetric if and only if $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V,\ i = 1, 2, \ldots, n$.*

Proof. Suppose that G is symmetric. Then for any $x \in V$,

$\nu_i(x) = \nu(x^{-1}x^2) = R_i(x, x^2) = R_i{}^{-1}(x, x^2) = R_i(x^2, x) = \nu_i(x^{-1}x^{-1}x) = \nu_i(x^{-1})$. Therefore, $\nu_i(x) = \nu_i(x^{-1})$.

Conversely, suppose that $\nu_i(x) = \nu_i(x^{-1})$ for all $x \in V$. Then for any $x, y \in V$, $R_i(x, y) = \nu_i(x^{-1}y) = \nu_i((x^{-1}y)^{-1}) = \nu_i(y^{-1}x) = R_i(y, x)$. This implies that, R is symmetric. Hence the proof. □

Theorem 5.2.7. *$G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ is transitive if and only if for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.*

Proof. First assume that G is transitive. That is, for every i, j, $R_i \circ R_j \leq R_k$ for some k. For $x, y \in V$,

$$\begin{aligned}
\nu_i(x) \wedge \nu_j(y) &\leq \vee\{\nu_i(z) \wedge \nu_j(z^{-1}(xy)) : z \in V\} \\
&= \vee\{R_i(1, z) \wedge R_j(z, xy) : z \in V\} \\
&= R_i \circ R_j(1, xy) \\
&\leq R_k(1, xy) \\
&= \nu_k(xy).
\end{aligned}$$

That is, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

Now let for any $x, y \in V$ and i, j, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k. Then,

$$\begin{aligned}
(R_i \circ R_j)(x, y) &= \vee\{R_i(x, z) \wedge R_j(z, y) : z \in V\} \\
&= \vee\{\nu_i(x^{-1}z) \wedge \nu_j(z^{-1}y) : z \in V\} \\
&\leq \vee\{\nu_k((x^{-1}z)(z^{-1}y)) : z \in V\} \\
&= \nu_k(x^{-1}y) = R_k(x, y).
\end{aligned}$$

Thus, $R_i \circ R_j \leq R_k$ for some k. This completes the proof. $\square$

Theorem 5.2.8. *$G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ is complete if and only if $\cup {\nu_{i_\circ}}^+ = V$.*

Proof. First assume that G is complete. That is $\cup R_{i_\circ}^+ = V \times V$. Clearly, $\cup \nu_{i_\circ}^+ \subseteq V$. Now let $x \in V$. Then $(1, x) \in {R_{i_\circ}}^+$ for some i. That is, $R_i(1, x) \geq 0$, which implies, $\nu_i(x) \geq 0$. Thus, $x \in \cup {\nu_{i_\circ}}^+$. Therefore, $V \subseteq \cup {\nu_{i_\circ}}^+$. That is, $\cup {\nu_{i_\circ}}^+ = V$.

Conversely, assume $\cup {\nu_{i_\circ}}^+ = V$. Let $(x, y) \in V \times V$. Then $x, y \in V \Rightarrow x^{-1}y \in V \Rightarrow x^{-1}y \in \cup {\nu_{i_\circ}}^+ \Rightarrow x^{-1}y \in {\nu_{i_\circ}}^+$ for some i. Then, $\nu_i(x^{-1}y) \geq 0$. That is, $R_i(x, y) \geq 0$ which implies $(x, y) \in {R_{i_\circ}}^+$. Hence, $V \times V \subseteq \cup {R_{i_\circ}}^+$. Therefore, $\cup {R_{i_\circ}}^+ = V \times V$. This completes the proof. $\square$

Let A_k be the set of all elements $x \in V$ of the form $x = x_1 x_2 \ldots x_k$, where $x_j \in \nu_{i0}^+$ for some $i = 1, 2, \ldots, n$. Then $[\vartheta]$ is defined as $[\vartheta] = \cup_{k=1}^n A_k$.

Let B_k be the set of all elements $y \in V$ of the form $y = y_1 y_2 \ldots y_k$, where $y_j \in (\nu_i \wedge \nu_i^{-1})_0^+$ for some $i = 1, 2, \ldots, n$. Then $[[\vartheta]]$ is defined as $[[\vartheta]] = \cup_{k=1}^n B_k$.

Theorem 5.2.9. *$G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ is connected if and only if $V = [\vartheta]$.*

Proof. First assume that $G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ is connected.

Clearly, $[\vartheta] \subseteq V$. Now let $x \in V$. Then there exists a path from 1 *to* x say, $(1, y_1, y_2, \ldots, y_k = x)$. Then, for some i, $R_{i_1}(1, y_1) > 0$, that is, $y_1 \in \nu_{i_1 0}^+$. Also, $y_{j-1}^{-1} y_j \in \nu_{i_j 0}^+$,for $j = 2, 3, \ldots, k$. This implies that $x \in A_k$, since, $x = (1.y_1)(y_1^{-1}y_2)(y_2^{-1}y_3) \ldots (y_{k-1}^{-1}y_k)$. Therefore, $x \in \bigcup_{k=1}^{n} A_k = [\vartheta]$. Hence, $V = [\vartheta]$.
Conversely, assume that $V = [\vartheta]$.
Let $x, y \in V$. Then $z = x^{-1}y \in V$, implies, $z \in [\vartheta] = \bigcup_{k=1}^{n} A_k$. Then $z = z_1 z_2 \ldots z_k$. Then $1, z_1, z_1 z_2, \ldots, z_1 z_2 \ldots z_k = z$ is a path from 1 to z. Then

$x, xz_1, xz_1z_2, \dots, xz_1z_2\dots z_k = xz = y$ is a path from x to y, implies G is connected. This completes the proof. $\square$

Theorem 5.2.10. *$G = CayF_D(V; \nu_1, \nu_2, \dots, \nu_n)$ is weakly connected if and only if $V = [[\vartheta]]$.*

Proof. Assume G be weakly connected. Clearly, $[[\vartheta]] \subseteq V$. Let $x \in V$. Then there exist a weak path say, $1, x_1, x_2, \dots, x_k = x$ from 1 to x. Then, $1x_1 \in (\nu_{i_1} \vee \nu_{i_1}^{-1})_0^+$, $x_1^{-1}x_2 \in (\nu_{i_2} \vee \nu_{i_2}^{-1})_0^+, \dots,$ $x_{k-1}^{-1}x_k \in (\nu_{i_k} \vee \nu_{i_k})_0^+$, which clearly implies that $x \in \bigcup_k B_k = [[\vartheta]]$. Hence, $V \in [[\vartheta]]$.

Conversely, assume that $V = [[\vartheta]]$. Let $x, y \in V$, implies $z = x^{-1}y \in V$. Therefore, $z \in [[\vartheta]]$. Then there exist elements $z_j \in (\nu_{i_j} \vee \nu_{i_j}^{-1})_0^+$, $j = 1, 2, \dots, k$, such that $z = z_1z_2\dots z_k$, for some $k \in \{1, 2, \dots, n\}$.
Then $1, z_1, z_1z_2, \dots, z_1z_2\dots z_k = z$ is a weak path from 1 to z and hence $x, xz_1, xz_1z_2, \dots, xz_1z_2\dots z_k = xz = y$ is a weak path from x to y. Therefore, G is weakly connected. This completes the proof. $\square$

Theorem 5.2.11. *$G = CayF_D(V; \nu_1, \nu_2, \dots, \nu_n)$ is partially ordered if and only if*

(i) $\nu_i(1) = 1$ for some i.

(ii) for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

(iii) $\{x : \nu(x) = \nu(x^{-1})\} = \{1\}$ for all $i = 1, 2, \dots, n$.

Theorem 5.2.12. *$G = CayF_D(V; \nu_1, \nu_2, \dots, \nu_n)$ is quasi-ordered if and only if*

(i) $\nu_i(1) = 1$ for some i.

(ii) for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

Theorem 5.2.13. *$G = CayF_D(V; \nu_1, \nu_2, \dots, \nu_n)$ is a hasse diagram if and only if G is connected and $\nu_k(x_1x_2\dots x_m) = 0$, $k = 1, 2, \dots, n$, for any collection $x_1, x_2, \dots, x_m$ of vertices in V with $m \geq 2$ and $\nu_{i_j}(x_j) > 0$ for $j = 1, 2, \dots, m$.*

Proof. Suppose G is a hasse diagram. Since $\nu_{i_j}(x_j) > 0$, for $j = 1, 2, \ldots, m$, $(1, x_1, x_1x_2, \ldots, x_1x_2 \ldots x_m)$ is a path from 1 to $x_1x_2 \ldots x_m$. Now since G is a hasse diagram, $R_k(1, x_1x_2 \ldots x_m) = 0$ for all k. Therefore $\nu_k(x_1x_2 \ldots x_m) = 0$ for all $k = 1, 2, \ldots, n$.

Conversely suppose, G is connected and $\nu_k(x_1x_2 \ldots x_m) = 0,\ k = 1, 2, \ldots, n$, for any collection $x_1, x_2, \ldots, x_m$ of vertices in V with $m \geq 2$ and $\nu_{i_j}(x_j) > 0$ for $j = 1, 2, \ldots, m$. Let $(x_0, x_1, \ldots, x_m)$ be a path in G from x_1 to $x_m,\ m \geq 2$. Then $R_{i_1}(x_0, x_1) > 0,\ R_{i_2}(x_1, x_2) > 0,\ \ldots, R_{i_m}(x_{m-1}, x_m) > 0$ which implies, $\nu_{i_1}(x_0^{-1}x_1) > 0,\ \nu_{i_2}(x_1^{-1}x_2) > 0,\ \ldots, \nu_{i_m}(x_{m-1}^{-1}x_m) > 0$. Thus, by assumption, $\nu_k(x_0^{-1}x_1x_1^{-1}x_2 \ldots x_{m-1}^{-1}\ x_m) = \nu_k(x_0^{-1}x_m) = 0$. Therefore, $R_k(x_0, x_m) = 0$ for all $k = 1, 2, \ldots, n$. Hence, G is a hasse diagram. This completes the proof. □

Theorem 5.2.14. *For $k = 1, 2, \ldots, n$,let A_k be the set of all products of the form $\nu_{i_1}\nu_{i_2} \ldots \nu_{i_k} = \{x_1x_2 \ldots x_k : x_j \in \nu_{i_j 0}^{+}, j = 1, 2, \ldots, k\}$. If Cayley Fuzzy Digraph Structure of V, $G = CayF_D(V; \nu_1, \nu_2, \ldots, \nu_n)$ has finite diameter, then the diameter of G is the least positive integer m such that*

$$G = \bigcup_{A \in A_m} A.$$

Proof. Let m be the least integer such that $G = \bigcup_{A \in A_m} A$.
For any $x, y \in G,\ y = xz$, for some $z \in G$. Since $z \in G$, $z \in A$ for some $A \in A_m$. Then there exist $z_1, z_2, \ldots, z_m$ such that $\nu_{i_j 0}^{+}(z_j) > 0$ for $j = 1, 2, \ldots, m$ and $z = z_1z_2 \ldots z_m$. Then $x, xz_1, xz_1z_2, \ldots, xz_1z_2 \ldots z_m = y$ is a path of length m from x to y.Then we have $d(G) \leq m$.
If possible, let $d(G) = d < m$. Since $d < m$, there exist an x in G such that for any $x_1, x_2, \ldots, x_d$ with $\nu_{i_j 0}^{+}(x_j) > 0$ for $j = 1, 2, \ldots, d,\ x \neq x_1x_2 \ldots x_d$. But since $d(G) = d$, there exist a path $1, x_1, x_2, \ldots, x_d = x$ from 1 to x. Then $y_1 = x_1, y_2 = x_1^{-1}x_2, y_3 = x_2^{-1}x_3, \ldots, y_d = x_{d-1}^{-1}x_d$ are such that $x = y_1y_2y_3 \ldots y_d$ and $\nu_{i_j 0}^{+}(y_j) > 0$ for $j = 1, 2, \ldots, d$. This is a contradiction. Hence $d(G) = m$. □

Definition 5.2.15. *Let $(S, *)$ be a semigroup. Let A be a fuzzy subset of S. Then*

A is said to be fuzzy sub-semigroup of S if for all $a, b \in S$, $A(ab) \geq A(a) \wedge A(b)$ [15].

Definition 5.2.16. *Let $(S, *)$ be a semigroup and let $\nu_1, \nu_2, \ldots, \nu_n$ be mutually disjoint fuzzy subsets of S. The fuzzy sub-semigroup generated by $\nu_1, \nu_2, \ldots, \nu_n$ is the smallest fuzzy sub-semigroup of S which contains $\nu_1, \nu_2, \ldots, \nu_n$. Let us denote it by $\langle \nu_{(123\ldots n)} \rangle$.*

Lemma 5.2.17. *Let $(S, *)$ be a semigroup and let $\nu_1, \nu_2, \ldots, \nu_n$ be mutually disjoint fuzzy subsets of S. Then the fuzzy subset $\langle \nu_{(123\ldots n)} \rangle$ is precisely given by*

$$\begin{aligned}\langle \nu_{(123\ldots n)} \rangle(x) = \vee\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \nu_{jk}(x_k) : x = x_1 x_2 \ldots x_k \\ \textit{with a finite positive integer } k, x_i \in S \textit{ and} \\ \nu_{j_i}(x_i) > 0 \textit{ for some } j_i = 1, 2, \ldots, n\} \textit{ for any } x \in S.\end{aligned}$$

Proof. Let ν' be the fuzzy subset of V defined by

$$\begin{aligned}\nu'(x) = \vee\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1 x_2 x_3 \ldots x_m,\ x j_i \in \nu^+_{j_{i_\circ}}, \\ m \in \{1, 2, 3, \ldots, n\}\}, \text{ for any } x \in V.\end{aligned}$$

If $y \in V$, by definition of ν', it is clear that $\nu'(y) \geq \nu_{j_k}(y)$ where $j_k \in \{1, 2, \ldots, n\}$ and $\nu_{j_k}(y) \geq 0$. Thus, we have $\nu_{j_k} \leq \nu'$ for all j_i. This implies that ν' contains $\nu_1, \nu_2, \ldots, \nu_n$. Let $x, y \in V$. If $\nu_{j_i}(x) = 0$ or $\nu_{j_i}(y) = 0$, then $\nu_{j_i}(x) \wedge \nu_{j_i}(y) = 0$. Then, $\nu'(xy) \geq \nu_{j_i}(x) \wedge \nu_{j_i}(y)$. Again, if $\nu_{j_i}(x) \neq 0$ and $\nu_{j_i}(y) \neq 0$, then by definition of ν', we have $\nu'(xy) \geq \nu_{j_i}(x) \wedge \nu_{j_i}(y)$. Hence ν' is a fuzzy sub semigroup of V containing $\nu_i, i \in \{1, 2, \ldots, n\}$. Now let A be any fuzzy sub semigroup of V containing $\nu_i, i \in \{1, 2, \ldots, n\}$. Then, for any $x \in V$ with $x = x_1 x_2 x_3 \ldots x_m,\ x_i \in \nu^+_{j_{i_\circ}}$, for $i = 1, 2, \ldots, n, m \in \{1, 2, 3, \ldots, n\}$ we have $A(x) \geq A(x_1) \wedge A(x_2) \wedge \ldots \wedge A(x_m) \geq \nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m)$, which implies that $A(x) \geq \vee\{\nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1 x_2 x_3 \ldots x_m,\ x_{j_i} \in \nu^+_{j_{i_\circ}}, m \in \{1, 2, 3, \ldots, n\}$ for $j_i \in \{1, 2, \ldots, n\}$ for any $x \in V$. Therefore, $A(x) \geq \nu'(x)$ for all $x \in V$. Thus, $\nu' = \langle \nu_{(123\ldots n)} \rangle$. That is,

$$\langle \nu_{(123\ldots n)} \rangle(x) = \vee \{ \nu_{j_1}(x_1) \wedge \nu_{j_2}(x_2) \wedge \ldots \wedge \nu_{j_m}(x_m) : x = x_1 x_2 x_3 \ldots x_m, \\ x_{j_i} \in \nu^{+}_{j_{i_\circ}}, m \in \{1, 2, 3, \ldots, n\} \} \text{ for any } x \in V.$$

□

5.3 Cayley Fuzzy Digraph Structure Induced by Loops

In this section we introduce Cayley fuzzy digraph structure induced by loops and prove that these graphs are vertex-transitive and hence regular.

Definition 5.3.1. *Let V be a loop. A fuzzy subset ν on V is called a scaled fuzzy subset of V if $\nu(y/x) = \nu(zy/zx)$, where y/x denotes the solution of the equation $y = xz$, for all $x, y, z \in V$ [23].*

Definition 5.3.2. *Let V be a loop and $\nu_1, \nu_2, \ldots, \nu_n$ be mutually disjoint scaled fuzzy subsets of V. Then Cayley Fuzzy Digraph Structure of V with respect to $\nu_1, \nu_2, \ldots, \nu_n$ is defined as $(V; R_1, R_2, \ldots, R_n)$ where $R_i(x, y) = \nu_i(y/x)$ and is denoted by $CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$. The subsets $\nu_1, \nu_2, \ldots, \nu_n$ are called connection fuzzy subsets of $CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$. In case, a Cayley fuzzy digraph structure induced by loops with only one connection set is usual Cayley fuzzy graph induced by loops.*

Theorem 5.3.3. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is vertex-transitive.*

Proof. Let $a, b \in V$ and $b = z_\circ a$, $z_\circ \in V$. Define $\Psi : V \to V$ by $\psi(x) = z_\circ x$. Clearly, ψ is one-to-one and onto. Also, $\psi(a) = z_\circ a = b$.
Furthermore, we have, for each $x, y \in V$,

$$R_i(\psi(x), \psi(y)) = R_i(z_\circ x, z_\circ y)$$

$$
\begin{aligned}
&= \nu(z_\circ y/z_\circ x) \\
&= \nu(y/x) \\
&= \nu_i(y/x) \\
&= R_i(x, y).
\end{aligned}
$$

Hence the proof. □

Theorem 5.3.4. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is regular*

Proof. Let $u, v \in V$. Since $G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ are vertex transitive, there exist an automorphism say, f on G such that, $f(u) = v$ and $R_i(f(x), f(y)) = R_i(x, y)$ for any $x, y \in V$ and $i = 1, 2, \ldots, n$.Then the in-degree of u, $ind(u) = \sum_{x\in V}\sum_{i=1}^{n} R_i(x, u) = \sum_{x\in V}\sum_{i=1}^{n} R(f(x), f(u)) = \sum_{x\in V}\sum_{i=1}^{n} R_i(f(x), v) = \sum_{f(x)\in V}\sum_{i=1}^{n} R_i(f(x), v) = \sum_{y\in V}\sum_{i=1}^{n} R_i(y, v) = ind(v)$. Similarly, we can prove that $outd(u) = outd(v)$. Therefore, G is in-regular and out-regular. Now to prove that G is regular we just need to show that $ind(1) = outd(1)$. We have,
$ind(1) = \sum_{x\in V}\sum_{i=1}^{n} R_i(x, 1) = \sum_{x\in V}\sum_{i=1}^{n} \nu_i(1/x) = \sum_{y\in V}\sum_{i=1}^{n} \nu_i(y) = \sum_{y\in V}\sum_{i=1}^{n} R_i(1, y) = outd(1)$. Therefore, G is regular. □

Theorem 5.3.5. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is a trivial graph if and only if $\nu_i \equiv 0$ for all i.*

Proof. By definition, G is trivial if and only if $R_i \equiv 0$ for all i. This implies that $\nu_i \equiv 0$ for all i. □

Theorem 5.3.6. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is reflexive if and only if $\nu_i(1) = 1$ for some i.*

Proof. Assume that $G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is reflexive. Then for every $x \in V$, $R_i(x, x) = 1$ for some i. This implies that $\nu_i(x/x) = \nu_i(1) = 1$ for some i.

Conversely, let $\nu_i(1) = 1$ for some i, say $i = k$. This implies that for each $x \in V$, $R_k(x,x) = \nu_k(x/x) = \nu_k(1) = 1$. That is G is reflexive. $\square$

Theorem 5.3.7. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \dots, \nu_n)$ is symmetric if and only if $\nu_i(x) = \nu_i(1/x)$ for all $x \in V$, $i = 1, 2, \dots, n$.*

Proof. Suppose that G is symmetric. Then for any $x \in V$, $\nu_i(x) = R_i(1,x) = R_i(x,1) = \nu_i(1/x)$. Therefore, $\nu_i(x) = \nu_i(1/x)$.

Conversely, suppose that $\nu_i(x) = \nu_i(1/x)$ for all $x \in V$. Then for any $x, y \in V$, $y = xt$ for some $t \in V$. Then $R_i(x,y) = \nu_i(y/x) = \nu_i(t) = \nu_i(1/t) = \nu_i(x/xt) = \nu_i(x/y) = R_i(y,x)$. This implies that, R is symmetric. Hence the proof. $\square$

Theorem 5.3.8. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \dots, \nu_n)$ is transitive if and only if for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.*

Proof. First assume that G is transitive. That is, for every i, j, $R_i \circ R_j \leq R_k$ for some k. For $x, y \in V$,

$$\begin{aligned}
\nu_i(x) \wedge \nu_j(y) &\leq \vee\{\nu_i(z) \wedge \nu_j((xy)/z) : z \in V\} \\
&= \vee\{R_i(1,z) \wedge R_j(z,xy) : z \in V\} \\
&= R_i \circ R_j(1,xy) \\
&\leq R_k(1,xy) \\
&= \nu_k(xy).
\end{aligned}$$

That is, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

Now let for any $x, y \in V$ and i, j, $\nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k. Then,

$$\begin{aligned}
(R_i \circ R_j)(x,y) &= \vee\{R_i(x,z) \wedge R_j(z,y) : z \in V\} \\
&= \vee\{\nu_i(z/x) \wedge \nu_j(y/z) : z \in V\} \\
&\leq \vee\{\nu_k((z/x)(y/z)) : z \in V\} \\
&= \nu_k(y/x) = R_k(x,y).
\end{aligned}$$

Thus, $R_i \circ R_j \leq R_k$ for some k. This completes the proof. $\square$

Theorem 5.3.9. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is complete if and only if*

$$\cup {\nu_{i\circ}}^{+} = V.$$

Proof. First assume that G is complete. That is $\cup R_{i\circ}^{+} = V \times V$. Clearly, $\cup \nu_{i\circ}^{+} \subseteq V$. Now let $x \in V$. Then $(1, x) \in {R_{i\circ}}^{+}$ for some i. That is, $R_i(1, x) \geq 0$, which implies, $\nu_i(x) \geq 0$. Thus, $x \in \cup {\nu_{i\circ}}^{+}$. Therefore, $V \subseteq \cup {\nu_{i\circ}}^{+}$. That is, $\cup {\nu_{i\circ}}^{+} = V$.

Conversely, assume $\cup {\nu_{i\circ}}^{+} = V$. Let $(x, y) \in V \times V$. Then $x, y \in V \Rightarrow x^{-1}y \in V \Rightarrow y/x \in \cup {\nu_{i\circ}}^{+} \Rightarrow y/x \in {\nu_{i\circ}}^{+}$ for some i. Then, $\nu_i(y/x) \geq 0$. That is, $R_i(x, y) \geq 0$ which implies $(x, y) \in {R_{i\circ}}^{+}$. Hence, $V \times V \subseteq \cup {R_{i\circ}}^{+}$. Therefore, $\cup {R_{i\circ}}^{+} = V \times V$. This completes the proof. $\square$

Let A_k be the set of all elements $x \in V$ of the form $x = x_1 x_2 \ldots x_k$, where $x_j \in \nu_{i0}^{+}$ for some $i = 1, 2, \ldots, n$. Then $[\vartheta]$ is defined as $[\vartheta] = \bigcup\limits_{k=1}^{n} A_k$.

Let B_k be the set of all elements $y \in V$ of the form $y = y_1 y_2 \ldots y_k$, where $y_j \in (\nu_i \wedge \nu_i^{-1})_0^{+}$ for some $i = 1, 2, \ldots, n$, $\nu_i^{-1}(x) = \nu_i(1/x)$. Then $[[\vartheta]]$ is defined as $[[\vartheta]] = \bigcup\limits_{k=1}^{n} B_k$.

Theorem 5.3.10. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is connected if and only if $V = [\vartheta]$.*

Proof. First assume that G is connected.

Clearly, $[\vartheta] \subseteq V$. Now let $x \in V$. Then there exists a path from 1 *to* x say, $(1, y_1, y_2, \ldots, y_k = x)$. Then, for some $i_1 \in \{1, 2, \ldots, n\}$, $R_{i_1}(1, y_1) > 0$, that is, $y_1 \in \nu_{i_1 0}^{+}$. Also, there exist $t_j \in V$ such that $y_j/y_{j-1} = t_j \in \nu_{i_j 0}^{+}$,for $j = 2, 3, \ldots, k$, $i_j \in \{1, 2, \ldots, n\}$. This implies that $x \in A_k$, since, $x = y_k = y_{k-1}t_k = y_{k-2}t_{k-1}t_k = \ldots = y_1 t_2 t_3 \ldots t_k$. Therefore, $x \in \bigcup\limits_{k=1}^{n} A_k = [\vartheta]$. Hence,

$V = [\vartheta]$.

Conversely, assume that $V = [\vartheta]$.

Let $x, y \in V$. Then $z = y/x \in V$, implies, $z \in [\vartheta] = \bigcup_{k=1}^{n} A_k$. Then $z = z_1 z_2 \ldots z_k$. Then $1, z_1, z_1z_2, \ldots, z_1z_2\ldots z_k = z$ is a path from 1 to z. Then $x, xz_1, xz_1z_2, \ldots, xz_1z_2\ldots z_k = xz = y$ is a path from x to y, implies G is connected. This completes the proof.

□

Theorem 5.3.11. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is weakly connected if and only if $V = [[\vartheta]]$.*

Proof. Assume G be weakly connected. Clearly, $[[\vartheta]] \subseteq V$. Let $x \in V$. Then there exist a weak path say, $1, x_1, x_2, \ldots, x_k = x$ from 1 to x. Then, $1x_1 \in (\nu_{i_1} \vee \nu_{i_1}^{-1})_0^+$, $x_2/x_1 \in (\nu_{i_2} \vee \nu_{i_2}^{-1})_0^+, \ldots,$ $x_k/x_{k-1} \in (\nu_{i_k} \vee \nu_{i_k}^{-1})_0^+$, which clearly implies that $x \in \bigcup_k B_k = [[\vartheta]]$. Hence, $V \in [[\vartheta]]$.

Conversely, assume that $V = [[\vartheta]]$. Let $x, y \in V$, implies $z = y/x \in V$. Therefore, $z \in [[\vartheta]]$. Then there exist elements $z_j \in (\nu_{i_j} \vee \nu_{i_j}^{-1})_0^+$, $j = 1, 2, \ldots, k$, such that $z = z_1z_2\ldots z_k$, for some $k \in \{1, 2, \ldots, n\}$. Then $1, z_1, z_1z_2, \ldots, z_1z_2\ldots z_k = z$ is a weak path from 1 to z and hence $x, xz_1, xz_1z_2, \ldots, xz_1z_2\ldots z_k = xz = y$ is a weak path from x to y. Therefore, G is weakly connected. This completes the proof. □

Theorem 5.3.12. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is partially ordered if and only if*

(i) $\nu_i(1) = 1$ for some i.

(ii) for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

(iii) $\{x : \nu(x) = \nu(1/x)\} = \{1\}$ for all $i = 1, 2, \ldots, n$.

Theorem 5.3.13. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ is quasi-ordered if and only if*

(i) $\nu_i(1) = 1$ for some i.

(ii) for every i, j and for any $x, y \in V, \nu_i(x) \wedge \nu_j(y) \leq \nu_k(xy)$ for some k.

Theorem 5.3.14. *$G = CayF_{D_L}(V; \nu_1, \nu_2, \dots, \nu_n)$ is a hasse diagram if and only if G is connected and $\nu_k(x_1x_2 \dots x_m) = 0,\ k = 1, 2, \dots, n$, for any collection $x_1, x_2, \dots, x_m$ of vertices in V with $m \geq 2$ and $\nu_{i_j}(x_j) > 0$ for $j = 1, 2, \dots, m$.*

Proof. Suppose G is a hasse diagram. Since $\nu_{i_j}(x_j) > 0$, for $j = 1, 2, \dots, m$, $(1, x_1, x_1x_2, \dots, x_1x_2 \dots x_m)$ is a path from 1 to $x_1x_2 \dots x_m$. Now since G is a hasse diagram, $R_k(1, x_1x_2 \dots x_m) = 0$ for all k. Therefore $\nu_k(x_1x_2 \dots x_m) = 0$ for all $k = 1, 2, \dots, n$.

Conversely suppose, G is connected and $\nu_k(x_1x_2 \dots x_m) = 0,\ k = 1, 2, \dots, n$, for any collection $x_1, x_2, \dots, x_m$ of vertices in V with $m \geq 2$ and $\nu_{i_j}(x_j) > 0$ for $j = 1, 2, \dots, m$. Let $(x_0, x_1, \dots, x_m)$ be a path in G from x_1 to $x_m,\ m \geq 2$. Then $R_{i_1}(x_0, x_1) > 0,\ R_{i_2}(x_1, x_2) > 0,\ \dots, R_{i_m}(x_{m-1}, x_m) > 0$ which implies, $\nu_{i_1}(x_1/x_0) > 0,\ \nu_{i_2}(x_2/x_1) > 0,\ \dots, \nu_{i_m}(x_m/x_{m-1}) > 0$. Let $x_1 = x_\circ t_1,\ x_2 = x_1t_2, \dots, x_m = x_{m-1}t_m$. Then, $\nu_{i_j}(t_j) = \nu_{i_j}(x_j/x_{j-1}) > 0$ for $j = 1, 2, \dots, m$. We have, $x_m = x_{m-1}t_m = (x_{m-2}t_{m-1})t_m \dots = (\dots((x_\circ t_1)t_2)\dots)t_m = (\dots(x_\circ(t_1t'_2))\dots)t_m \dots = x_\circ(\dots((t_1t'_2)t'_3)\dots)t'_m)$. Therefore, for $k \in \{1, 2, \dots, n\}$,

$$R_k(x_\circ, x_m) = \nu_k(x_m/x_\circ) = \nu_k\left((\dots((t_1t'_2)t'_3)\dots)t'_m\right). \tag{5.1}$$

We have, $\nu_{i_1}(t_1) > 0$ and

$$\begin{aligned}
\nu_{i_2}(t'_2) &= \nu_{i_2}(t_1t'_2/t_1) \\
&= \nu_{i_2}\left(x_\circ(t_1t'_2)/x_\circ t_1\right) \\
&= \nu_{i_2}((x_\circ t_1)t_2/x_\circ t_1) \\
&= \nu_{i_2}(t_2) > 0.
\end{aligned}$$

In general, $\nu_{i_j}(t'_j) = \nu_{i_j}(t_j) > 0$ for $j = 2, 3, \dots, m$.

Therefore, since $t_1, t'_j \in V, j = 2, 3, \dots, m, m \geq 2$ and $\nu_{i_1}(t_1) > 0, \nu_{i_j}(t'_j) > 0$, for $j = 2, 3, \dots, m$, we have, $\nu_k\left((\dots((t_1t'_2)t'_3)\dots)t'_m\right) = 0$. Therefore, (5.1) implies that $R_k(x_\circ, x_m) = 0$. Hence, G is a hasse diagram. This completes the proof. □

Theorem 5.3.15. *For $k = 1, 2, \ldots, n$,let A_k be the set of all products of the form $\nu_{i_1}\nu_{i_2}\ldots\nu_{i_k} = \{x_1x_2\ldots x_k : x_j \in {\nu_{i_j}}_0^+, j = 1, 2, \ldots, k\}$. If the Cayley fuzzy digraph structure induced by the loop V, $G = CayF_{D_L}(V; \nu_1, \nu_2, \ldots, \nu_n)$ has finite diameter, then the diameter of G is the least positive integer m such that*

$$G = \bigcup_{A \in A_m} A.$$

Proof. The proof of this theorem is analogous to that of theorem 5.2.14 □

Chapter 6

Cayley Bipolar Fuzzy Digraph Structure Induced by Groups and Loops

This chapter comprises of two main sections. In the first section we introduced Cayley bipolar fuzzy digraph structure induced by groups and a study in terms of algebraic properties is carried out. In the second section we introduced and studied Cayley bipolar fuzzy digraph structure induced by loops. This can be considered as a generalisation of the work done in chapter three and in [24], by N. O. Alshehri and M. Akram.

6.1 Definitions

Definition 6.1.1. *Let* V *be a non-empty set and* $S_1, S_2, \ldots, S_k$ *are relations on* V *which are mutually disjoint, then* $G' = (V, S_1, S_2, \ldots, S_n)$ *is a digraph structure. In addition, if* $S_1, S_2, \ldots, S_k$ *are symmetric and irreflexive, then* $G' = (V, S_1, S_2, \ldots, S_k)$ *is a graph structure [6].*

Definition 6.1.2. *Let* $G' = (V, S_1, S_2, \ldots, S_k)$ *be a graph (digraph) structure*

and $\mu, \rho_1, \rho_2, \ldots, \rho_k$ be fuzzy subsets of $V, S_1, S_2, \ldots, S_k$ respectively such that $\rho_i(x,y) \leq \mu(x) \wedge \mu(y)$, for all $x, y \in V$ and $i = 1, 2, \ldots, k$.
Then $G = (\mu, \rho_1, \rho_2, \ldots, \rho_k)$ is a fuzzy graph (digraph) structure of G' [29].

Definition 6.1.3. *$F_{BD} = (M, R_1, R_2, \ldots, R_n)$ is called a bipolar fuzzy graph structure of a graph structure $G' = (V, S_1, S_2, \ldots, S_n)$ if $M = (\mu_M^P, \mu_M^N)$ is a bipolar fuzzy set on V and for each $i = 1, 2, \ldots, n, R_i = (\mu_{R_i}^P, \mu_{R_i}^N)$ is a bipolar fuzzy set on S_i such that $\mu_{R_i}^P(x,y) \leq \mu_M^P(x) \wedge \mu_M^P(y)$, $\mu_{R_i}^N(x,y) \geq \mu_M^N(x) \vee \mu_M^N(y)$. While V and $S_i, (i = 1, 2, \ldots, n)$ are called underlying vertex set and underlying i-edge sets of F_{BD}, respectively [25].*

In case $M = (\chi_V, -\chi_V)$, where χ_V is the characteristic function on V, then the bipolar fuzzy digraph structure $(M, R_1, R_2, \ldots, R_n)$ is simply denoted by $F_{BD} = (V; R_1, R_2, \ldots, R_n)$.

6.2 Cayley Bipolar Fuzzy Digraph Structure

Definition 6.2.1. *Let V be a group and $A_1, A_2, \ldots, A_n$ be bipolar fuzzy sets on V. Then the bipolar fuzzy digraph structure $(V; R_1, R_2, \ldots, R_n)$ defined by $R_i(x,y) = (\mu_{R_i}^P(x,y), \mu_{R_i}^N(x,y)) = \big((\mu_{A_i}^P(x^{-1}y), \mu_{A_i}^N(x^{-1}y)\big)$ for all $x, y \in V$, $i = 1, 2, \ldots, n$ is called the Cayley Bipolar Fuzzy Digraph Structure and is denoted by $CayF_{BD}(V; A_1, A_2, \ldots, A_n)$.*

Theorem 6.2.2. *$CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ is vertex transitive.*

Proof. Let a and b be any two arbitrary elements in V. Define $\psi : V \to V$ by $\psi(x) = ba^{-1}x$ for all $x \in V$. Clearly, ψ is a bijection onto itself and $\psi(a) = b$. Furthermore, we have, for each $x, y \in V$, $i = 1, 2, \ldots, n$,

$$\begin{aligned} R_i(\psi(x), \psi(y)) &= R_i(ba^{-1}x, ba^{-1}y) \\ &= \big(\mu_{R_i}^P(ba^{-1}x, ba^{-1}y), \mu_{R_i}^N(ba^{-1}x, ba^{-1}y)\big) \end{aligned}$$

$$= \big(\mu^P_{A_i}((ba^{-1}x)^{-1}(ba^{-1}y)), \mu^N_{A_i}((ba^{-1}x)^{-1}(ba^{-1}y))\big)$$
$$= \big(\mu^P_{A_i}(x^{-1}y), \mu^N_{A_i}(x^{-1}y)\big)$$
$$= R_i(x, y).$$

Hence the proof. □

Theorem 6.2.3. *Every vertex transitive bipolar fuzzy digraph structure is in-regular and out-regular.*

Proof. Let $G = (V; R_i, R_i, \ldots, R_n)$ be a vertex transitive bipolar fuzzy digraph structure. Let $u, v \in V$ and $i \in I = \{1, 2, \ldots, n\}$. Then there is an automorphism f on G such that $f(u) = v$. Then,

$$\begin{aligned}
ind(u) &= \sum_{i\in I}\sum_{x\in V} R_i(x, u) \\
&= \sum_{i\in I}\sum_{x\in V} \big(\mu^P_{R_i}(x, u), \mu^N_{R_i}(x, u)\big) \\
&= \sum_{i\in I}\sum_{x\in V} \big(\mu^P_{R_i}(f(x), f(u)), \mu^N_{R_i}(f(x), f(u))\big) \\
&= \sum_{i\in I}\sum_{f(x)\in V} \big(\mu^P_{R_i}(f(x), v), \mu^N_{R_i}(f(x), v)\big) \\
&= \sum_{i\in I}\sum_{y\in V} \big(\mu^P_{R_i}(y, v), \mu^N_{R_i}(y, v)\big) \\
&= \sum_{i\in I}\sum_{y\in V} R_i(y, v) \\
&= ind(v)
\end{aligned}$$

Hence G is in-regular.
Similarly it can be proved that G is out-regular. □

Theorem 6.2.4. $G = CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ *is regular.*

Proof. From theorems (6.2.2) and (6.2.3) it can easily be seen that G is regular. □

Theorem 6.2.5. *$G = CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ is reflexive if and only if $A_i(1) = (1, -1)$ for some $i = 1, 2, \ldots, n$.*

Proof. G is reflexive if and only if for every $x \in V$ $R_i(x, x) = (1, -1)$ for some i say, k. Then

$$\begin{aligned} A_k(1) &= \big(\mu^P_{A_k}(1), \mu^N_{A_k}(1)\big) \\ &= \big(\mu^P_{A_k}(x^{-1}x), \mu^N_{A_k}(x^{-1}x)\big) \\ &= \big(\mu^P_{R_k}(x, x), \mu^N_{R_k}(x, x)\big) \\ &= R_k(x, x) \\ &= (1, -1). \end{aligned}$$

Hence $A_i(1) = (1, -1)$ for some $i = 1, 2, \ldots, n$.

Conversely let $A_i(1) = (1, -1)$ for some i, say $i = k$.
Then for any $x \in V$,

$$\begin{aligned} R_k(x, x) &= \big(\mu^P_{R_i}(x, x), \mu^N_{R_i}(x, x)\big) \\ &= \big(\mu^P_{A_i}(x^{-1}x), \mu^N_{A_i}(x^{-1}x)\big) \\ &= \big(\mu^P_{A_i}(1), \mu^N_{A_i}(1)\big) \\ &= A_i(1) = (1, -1). \end{aligned}$$

That is, for every $x \in V$, $R_i(x, x) = (1, -1)$ for some $i = 1, 2, \ldots, n$.
Hence the proof. □

Theorem 6.2.6. *$G = CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ is symmetric if and only if $A_i(x) = A_i(x^{-1})$.*

Proof. G is symmetric if and only if for every $x \in V$ $R_i(x, y) = R_i(y, x)$. Then

$$\begin{aligned} A_i(x) &= A_i(x^{-1}xx) \\ &= A_i(x^{-1}x^2) \end{aligned}$$

$$
\begin{aligned}
&= \big(\mu^P_{A_i}(x^{-1}x^2), \mu^N_{A_i}(x^{-1}x^2)\big)\\
&= \big(\mu^P_{R_i}(x, x^2), \mu^N_{R_i}(x, x^2)\big)\\
&= R_i(x, x^2)\\
&= R_i(x^2, x), \qquad\qquad (\textit{since symmetric})\\
&= \Big(\mu^P_{A_i}((x^2)^{-1}x), \mu^N_{A_i}((x^2)^{-1}x)\Big)\\
&= \big(\mu_{A_i}(x^{-1}), \mu^N_{A_i}(x^{-1})\big)\\
&= A_i(x^{-1}).
\end{aligned}
$$

Conversely let $A_i(x) = A_i(x^{-1})$. Then,

$$
\begin{aligned}
R_i(x, y) &= \big(\mu^P_{R_i}(x, y), \mu^N_{R_i}(x, y)\big)\\
&= \big(\mu^P_{A_i}(x^{-1}y), \mu^N_{A_i}(x^{-1}y)\big)\\
&= A_i(x^{-1}y)\\
&= A_i\left((x^{-1}y)^{-1}\right)\\
&= \big(\mu^P_{A_i}(y^{-1}x), \mu^N_{A_i}(y^{-1}x)\big)\\
&= \big(\mu^P_{R_i}(y, x), \mu^N_{R_i}(y, x)\big)\\
&= R_i(y, x).
\end{aligned}
$$

Hence the proof. □

Theorem 6.2.7. *$G = CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ is antisymmetric if and only if $\{x : A_i(x) = A_i(x^{-1})\} = \{1\}$.*

Proof. First assume that G is antisymmetric. Then,

$$
\begin{aligned}
A_i(x) = A_(x^{-1}) &\Leftrightarrow \big(\mu^P_{A_i}(x), \mu^N_{A_i}(x)\big) = \big(\mu^P_{A_i}(x^{-1}), \mu^N_{A_i}(x^{-1})\big)\\
&\Leftrightarrow \big(\mu^P_{R_i}(1, x), \mu^N_{R_i}(1, x)\big) = \big(\mu^P_{R_i}(x, 1), \mu^N_{R_i}(x, 1)\big)\\
&\Leftrightarrow R_i(1, x) = R_i(x, 1)\\
&\Leftrightarrow x = 1. \ \ (\textit{Since antisymmetric})
\end{aligned}
$$

Therefore, $\{x : A_i(x) = A_i(x^{-1})\} = \{1\}$.

Conversely let $\{x : A_i(x) = A_i(x^{-1})\} = \{1\}$. Then,

$$\begin{aligned} R_i(x,y) = R_i(y,x) &\Leftrightarrow \left(\mu^P_{R_i}(x,y), \mu^N_{R_i}(x,y)\right) = \left(\mu^P_{R_i}(y,x), \mu^N_{R_i}(y,x)\right) \\ &\Leftrightarrow \left(\mu^P_{A_i}(x^{-1}y), \mu^N_{A_i}(x^{-1}y)\right) = \left(\mu^P_{A_i}(y^{-1}x), \mu^N_{A_i}(y^{-1}x)\right) \\ &\Leftrightarrow A_i(x^{-1}y) = A_(y^{-1}x) \\ &\Leftrightarrow A_i(x^{-1}y) = A_i((x^{-1}y)^{-1}) \\ &\Leftrightarrow x^{-1}y = 1 \\ &\Leftrightarrow x = y. \end{aligned}$$

Hence the proof. $\square$

Theorem 6.2.8. *$G = CayF_{BD}(V; A_1, A_2, \ldots, A_n)$ is transitive if and only if for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $\mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) \leq \mu^P_{A_k}(xy)$ and $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$ for some $k \in I$.*

Proof. Assume that G is transitive. Then for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $R_i \circ R_j(x,y) \leq R_k$ for some $k \in I$. Then,

$$\begin{aligned} \mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) &\leq \vee \left\{\mu^P_{A_i}(z) \wedge \mu^P_{A_j}(z^{-1}(xy)) : z \in V\right\} \\ &\leq \vee \left\{\mu^P_{R_i}(1,z) \wedge \mu^P_{R_j}(z, xy) : z \in V\right\} \\ &\leq \mu^P_{R_i} \circ \mu^P_{R_j}(1, xy) \\ &\leq \mu^P_{R_k}(1, xy) \\ &\leq \mu^P_{A_k}(xy). \end{aligned}$$

Similarly we obtain $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$.

Conversely let for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $\mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) \leq \mu^P_{A_k}(xy)$ and $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$ for some $k \in I$. Then,

$$\left(\mu^P_{R_i} \circ \mu^P_{R_j}\right)(x,y) = \vee \left\{\mu^P_{R_i}(x,z) \wedge \mu^P_{R_j}(z,y) : z \in V\right\}$$

$$= \vee \left\{ \mu_{A_i}^P(x^{-1}z) \wedge \mu_{A_j}^P(z^{-1}y) : z \in V \right\}$$
$$\leq \mu_{A_k}(x^{-1}y)$$
$$\leq \mu_{R_k}^P(x,y).$$

In a similar way we can obtain the inequality related to the negative membership degree.

Hence the proof. □

6.3 Cayley Bipolar Fuzzy Digraph Structure Induced by Loops

Definition 6.3.1. *Let V be a group and $A_1, A_2, \ldots, A_n$ be scaled bipolar fuzzy sets on V. Then the bipolar fuzzy digraph structure $(V; R_1, R_2, \ldots, R_n)$ defined by $R_i(x,y) = (\mu_{R_i}^P(x,y), \mu_{R_i}^N(x,y)) = \big((\mu_{A_i}^P(y/x), \mu_{A_i}^N(y/x)\big)$ for all $x, y \in V$, $i = 1, 2, \ldots, n$ is called the Cayley Bipolar Fuzzy Digraph Structure induced by loops and is denoted by $CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$.*

Theorem 6.3.2. *$CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ is vertex transitive.*

Proof. Let a and b be any two arbitrary elements in V and $b = z_o a$, $z_o \in V$. Define $\psi : V \to V$ by $\psi(x) = z_o x$ for all $x \in V$. Clearly, ψ is a bijection onto itself and $\psi(a) = b$.

Furthermore, we have, for each $x, y \in V$, $i = 1, 2, \ldots, n$,

$$\begin{aligned} R_i(\psi(x), \psi(y)) &= R_i(z_o x, z_o y) \\ &= \big(\mu_{R_i}^P(z_o x, z_o y), \mu_{R_i}^N(z_o x, z_o y)\big) \\ &= \big(\mu_{A_i}^P(z_o y / z_o x), \mu_{A_i}^N(z_o y / z_o x)\big) \\ &= \big(\mu_{A_i}^P(y/x), \mu_{A_i}^N(y/x)\big) \\ &= R_i(x,y). \end{aligned}$$

Hence the proof. □

Theorem 6.3.3. *Every vertex transitive bipolar fuzzy digraph structure is in-regular and out-regular.*

Proof. Let $G = (V; R_i, R_i, \ldots, R_n)$ be a vertex transitive bipolar fuzzy digraph structure. Let $u, v \in V$ and $i \in I = \{1, 2, \ldots, n\}$. Then there is an automorphism f on G such that $f(u) = v$. Then,

$$\begin{aligned}
ind(u) &= \sum_{i\in I}\sum_{x\in V} R_i(x,u) \\
&= \sum_{i\in I}\sum_{x\in V} \left(\mu^P_{R_i}(x,u), \mu^N_{R_i}(x,u)\right) \\
&= \sum_{i\in I}\sum_{x\in V} \left(\mu^P_{R_i}(f(x),f(u)), \mu^N_{R_i}(f(x),f(u))\right) \\
&= \sum_{i\in I}\sum_{f(x)\in V} \left(\mu^P_{R_i}(f(x),v), \mu^N_{R_i}(f(x),v)\right) \\
&= \sum_{i\in I}\sum_{y\in V} \left(\mu^P_{R_i}(y,v), \mu^N_{R_i}(y,v)\right) \\
&= \sum_{i\in I}\sum_{y\in V} R_i(y,v) \\
&= ind(v)
\end{aligned}$$

Hence G is in-regular.
Similarly it can be proved that G is out-regular. □

Theorem 6.3.4. $G = CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ *is regular.*

Proof. From theorems (6.3.2) and (6.3.3) it can easily be seen that G is regular. □

Theorem 6.3.5. $G = CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ *is reflexive if and only if* $A_i(1) = (1, -1)$ *for some* $i = 1, 2, \ldots, n$.

Proof. G is reflexive if and only if for every $x \in V$ $R_i(x,x) = (1,-1)$ for some i say, k. Then

$$\begin{aligned} A_k(1) &= \big(\mu^P_{A_k}(1), \mu^N_{A_k}(1)\big) \\ &= \big(\mu^P_{A_k}(x/x), \mu^N_{A_k}(x/x)\big) \\ &= \big(\mu^P_{R_k}(x,x), \mu^N_{R_k}(x,x)\big) \\ &= R_k(x,x) \\ &= (1,-1). \end{aligned}$$

Hence $A_i(1) = (1,-1)$ for some $i = 1, 2, \ldots, n$.
Conversely let $A_i(1) = (1,-1)$ for some i, say $i = k$.
Then for any $x \in V$,

$$\begin{aligned} R_k(x,x) &= \big(\mu^P_{R_i}(x,x), \mu^N_{R_i}(x,x)\big) \\ &= \big(\mu^P_{A_i}(x/x), \mu^N_{A_i}(x/x)\big) \\ &= \big(\mu^P_{A_i}(1), \mu^N_{A_i}(1)\big) \\ &= A_i(1) = (1,-1). \end{aligned}$$

That is, for every $x \in V$, $R_i(x,x) = (1,-1)$ for some $i = 1, 2, \ldots, n$.
Hence the proof. □

Theorem 6.3.6. *$G = CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ is symmetric if and only if $A_i(x) = A_i(1/x)$.*

Proof. G is symmetric if and only if for every $x \in V$ $R_i(x,y) = R_i(y,x)$. Then

$$\begin{aligned} A_i(x) &= A_i((\frac{x}{x})x) \\ &= A_i(x^2/x) \\ &= \big(\mu^P_{A_i}(x^2/x), \mu^N_{A_i}(x^2/x)\big) \\ &= \big(\mu^P_{R_i}(x,x^2), \mu^N_{R_i}(x,x^2)\big) \end{aligned}$$

$$\begin{aligned}
&= R_i(x, x^2)\\
&= R_i(x^2, x), && (since\ symmetric)\\
&= \big(\mu^P_{A_i}(x/x^2), \mu^N_{A_i}(x/x^2)\big)\\
&= \big(\mu_{A_i}(1/x), \mu^N_{A_i}(1/x)\big)\\
&= A_i(1/x).
\end{aligned}$$

Conversely let $A_i(x) = A_i(1/x)$. Then,

$$\begin{aligned}
R_i(x, y) &= \big(\mu^P_{R_i}(x, y), \mu^N_{R_i}(x, y)\big)\\
&= \big(\mu^P_{A_i}(y/x), \mu^N_{A_i}(y/x)\big)\\
&= A_i(y/x)\\
&= A_i\left(1/(y/x)\right)\\
&= \big(\mu^P_{A_i}(x/y), \mu^N_{A_i}(x/y)\big)\\
&= \big(\mu^P_{R_i}(y, x), \mu^N_{R_i}(y, x)\big)\\
&= R_i(y, x).
\end{aligned}$$

Hence the proof. □

Theorem 6.3.7. *$G = CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ is antisymmetric if and only if $\{x : A_i(x) = A_i(1/x)\} = \{1\}$.*

Proof. First assume that G is antisymmetric. Then,

$$\begin{aligned}
A_i(x) = A_i(1/x) &\Leftrightarrow \big(\mu^P_{A_i}(x), \mu^N_{A_i}(x)\big) = \big(\mu^P_{A_i}(1/x), \mu^N_{A_i}(1/x)\big)\\
&\Leftrightarrow \big(\mu^P_{R_i}(1, x), \mu^N_{R_i}(1, x)\big) = \big(\mu^P_{R_i}(x, 1), \mu^N_{R_i}(x, 1)\big)\\
&\Leftrightarrow R_i(1, x) = R_i(x, 1)\\
&\Leftrightarrow x = 1. && (Since\ antisymmetric)
\end{aligned}$$

Therefore, $\{x : A_i(x) = A_i(1/x)\} = \{1\}$.

Conversely let $\{x : A_i(x) = A_i(1/x)\} = \{1\}$. Then,

$$\begin{aligned}
R_i(x,y) = R_i(y,x) &\Leftrightarrow \big(\mu^P_{R_i}(x,y), \mu^N_{R_i}(x,y)\big) = \big(\mu^P_{R_i}(y,x), \mu^N_{R_i}(y,x)\big) \\
&\Leftrightarrow \big(\mu^P_{A_i}(y/x), \mu^N_{A_i}(y/x)\big) = \big(\mu^P_{A_i}(x/y), \mu^N_{A_i}(x/y)\big) \\
&\Leftrightarrow A_i(y/x) = A_i(x/y) \\
&\Leftrightarrow A_i(y/x) = A_i(1/(y/x)) \\
&\Leftrightarrow y/x = 1 \\
&\Leftrightarrow x = y.
\end{aligned}$$

Hence the proof. □

Theorem 6.3.8. *$G = CayF_{BD_L}(V; A_1, A_2, \ldots, A_n)$ is transitive if and only if for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $\mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) \leq \mu^P_{A_k}(xy)$ and $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$ for some $k \in I$.*

Proof. Assume that G is transitive. Then for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $R_i \circ R_j(x,y) \leq R_k$ for some $k \in I$. Then,

$$\begin{aligned}
\mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) &\leq \vee \Big\{\mu^P_{A_i}(z) \wedge \mu^P_{A_j}((xy)/z) : z \in V\Big\} \\
&\leq \vee \Big\{\mu^P_{R_i}(1,z) \wedge \mu^P_{R_j}(z,xy) : z \in V\Big\} \\
&\leq \mu^P_{R_i} \circ \mu^P_{R_j}(1,xy) \\
&\leq \mu^P_{R_k}(1,xy) \\
&\leq \mu^P_{A_k}(xy).
\end{aligned}$$

Similarly we obtain $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$.

Conversely let for any $i, j \in I = \{1, 2, \ldots, n\}$ and for any $x, y \in V$, $\mu^P_{A_i}(x) \wedge \mu^P_{A_j}(y) \leq \mu^P_{A_k}(xy)$ and $\mu^N_{A_i}(x) \vee \mu^N_{A_j}(y) \geq \mu^N_{A_k}(xy)$ for some $k \in I$. Then,

$$\begin{aligned}
\left(\mu^P_{R_i} \circ \mu^P_{R_j}\right)(x,y) &= \vee \Big\{\mu^P_{R_i}(x,z) \wedge \mu^P_{R_j}(z,y) : z \in V\Big\} \\
&= \vee \Big\{\mu^P_{A_i}(z/x) \wedge \mu^P_{A_j}(y/z) : z \in V\Big\}
\end{aligned}$$

$$\leq \mu_{A_k}(y/x)$$
$$\leq \mu^P_{R_k}(x, y).$$

In a similar way we can obtain the inequality related to the negative membership degree.

Hence the proof. $\square$

Chapter 7

Conclusion and Further Scope of Research

7.1 Summary of the Thesis

In 2013, Madhavan Namboothiri N.M. et al.introduced a class of Cayley fuzzy graphs induced by groups and studied the properties of the Cayley fuzzy graphs in terms of algebraic properties. In the second chapter, We generalise the results of Madhavan Namboothiri N.M. et al. and prove that a bigger class of Fuzzy Cayley graphs could be induced by loops, weaker algebraic structure than groups. Moreover, we studied various graph properties in terms of algebraic properties.

Noura O. Alshehri and Muhammad Akram generalised the results of Madhavan Namboothiri N.M. et al. and introduced bipolar fuzzy Cayley graphs and derived graph properties in terms of algebraic properties. They also discussed connectedness in Cayley bipolar fuzzy graph. In the third chapter, we generalized the results of Noura O. Alshehri1and Muhammad Akram and investigated bipolar Cayley Fuzzy graphs induced by loops and analogous results are derived.

In the fourth chapter, we introduced the concept of Cayley Intuitionistic

Fuzzy Graphs induced by loops and derived many results.

A graph structure is a powerful tool for solving combinatorial problems in different areas of computer science and computational intelligence systems. In 2012, Anil Kumar V and Parameswarn Ashok Nair introduced the concept of Cayley digraph structures induced by groups and derived many interesting results. In chapter 5, we introduced the concept Cayley Fuzzy Digraph structure induced by groups and loops and also investigated several results in terms of algebraic properties. These results can be considered as a generalisation of those obtained in [15] and [20]. The last chapter is the generalisation of the work carried out in chapter 3 and the results obtained in "Cayley bipolar fuzzy graphs", the Scientific World Journal (2013).

7.2 Further Scope of Research

(i) We will study Cayley fuzzy graphs induced by a more weaker structure, called quasigroups in terms of algebraic properties.

(ii) Study Cayley bipolar fuzzy graphs induced by quasigroups.

(iii) Study Cayley intuitionistic fuzzy graphs induced by quasigroups.

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