Ph.D. THESIS **MATHEMATICS**

ČEBYŠEV SUBSPACES AND NONCOMMUTATIVE KOROVKIN-TYPE APPROXIMATION THEORY

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PRAMOD S.

under the guidance of

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CERTIFICATE

I hereby certify that the thesis entitled "**ČEBYŠEV SUBSPACES AND NONCOMMUTATIVE KOROVKIN-TYPE APPROXIMATION THEORY**" is a bonafide work carried out by **Mr. Pramod S.,** under my guidance for the award of Degree of Ph.D., in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**ČEBYŠEV SUBSPACES AND NONCOMMUTATIVE KOROVKIN-TYPE APPROXIMATION THEORY**" is based on the original work done by me under the supervision of Dr. A. K. Vijayarajan, Professor, Kerala School of Mathematics, University of Calicut and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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ABSTRACT

In this thesis, we introduced the noncommutative Haar condition and proved that it is equivalent to Haar condition in the classical case. We proved a theorem which asserts that for a unital C^*-algebra, if we consider a linearly independent subset of n elements which includes the identity of the C^*-algebra, the n-dimensional subspace spanned by the set being a Čebyšev subspace is equivalent to the set satisfying noncommutative Haar condition. We also proved that for a unital C^*-algebra which is generated by a two-dimensional Čebyšev subspace which contains the identity of the C^*-algebra, an irreducible representation of it under certain conditions will become a boundary representation.

Two results are also given to give clarity to the two conditions in the definition of noncommutative Haar condition.

We give the definition of a separating operator system. We prove that if S is an operator system generating a C^*-algebra, every irreducible representation is a boundary representation for S exactly when S is quasi hyperrigid separating operator system such that restriction of every irreducible representation to S is pure.

പ്രബന്ധ സംഗ്രഹം

ഈ പ്രബന്ധത്തിൽ, നോൺ കമ്മ്യൂട്ടേറ്റീവ് ഹാർ നിബന്ധന അവതരിപ്പിക്കുകയും അത് ക്ലാസിക്കൽ കേസിൽ ഹാർ നിബന്ധനയ്ക്ക് തുല്യമാണെന്ന് തെളിയിക്കുകയും ചെയ്യുന്നു. C*-ആൾജിബ്രയുടെ ഐഡന്റിറ്റി ഉൾപ്പെടുന്ന n അംഗങ്ങളുടെ ഒരു രേഖീയ സ്വതന്ത്ര ഉപഗണം പരിഗണിക്കുകയാണെങ്കിൽ ആ ഉപഗണം ഉത്പാദിപ്പിക്കുന്ന n മാനങ്ങളുള്ള സബ്സ്പേസ് ഒരു ചെബിഷേവ് സബ്സ്പേസ് ആകുന്നത് പ്രസ്തുത ഉപഗണം നോൺ കമ്മ്യൂട്ടേറ്റീവ് ഹാർ കണ്ടീഷൻ സാധൂകരിക്കുന്നതിന് തുല്യമാണ് എന്ന് തെളിയിക്കുന്നു . ഒരു യൂണിറ്റൽ C*-ആൾജിബ്രയുടെ ഐഡന്റിറ്റി അടങ്ങുന്ന ദ്വിമാന ചെബിഷേവ് സബ്സ്പെയ്സ് പ്രസ്തുത C*-ആൾജിബ്ര സൃഷ്ടിക്കുകയാണെങ്കിൽ ആ C*-ആൾജിബ്രയുടെ ലഘൂകരിക്കാനാവാത്ത ഒരു റെപ്രസന്റേഷൻ ചില വ്യവസ്ഥകൾക്കനുസൃതമായി ആ ചെബിഷേവ് സബ്സ്പേസിന്റെ ഒരു ബൗണ്ടറി റെപ്രസന്റേഷൻ ആയി മാറും. നോൺ കമ്മ്യൂട്ടേറ്റീവ് ഹാർ അവസ്ഥയുടെ നിർവചനത്തിലെ രണ്ട് വ്യവസ്ഥകൾക്കും വ്യക്തത നൽകുന്നതിന് രണ്ട് ഫലങ്ങൾ നൽകിയിട്ടുണ്ട്. സെപ്പറേറ്റിങ് ഓപ്പറേറ്റർ സിസ്റ്റത്തിന്റെ നിർവചനം നൽകുന്നു. തുടർന്ന് താഴെ പറയുന്ന ഫലം അവതരിപ്പിക്കുന്നു: S എന്നത് ഒരു C*-ആൾജിബ്ര സൃഷ്ടിക്കുന്ന ഓപ്പറേറ്റർ സിസ്റ്റമാണെങ്കിൽ, പ്രസ്തുത C*-ആൾജിബ്ര യുടെ ലഘൂകരിക്കാനാവാത്ത എല്ലാ റെപ്രസന്റേഷനുകളും S ന്റെ ബൗണ്ടറി റെപ്രസന്റേഷനുകൾ ആകുന്നതിനു സമാനമാണ് S ഒരു ക്വാസി -ഹൈപ്പർ റിജിഡ് ഓപ്പറേറ്റർ സിസ്റ്റം ആവുകയും ലഘൂകരിക്കാനാവാത്ത എല്ലാ റെപ്രസന്റേഷനുകളും S-ലേയ്ക്ക് നിയന്ത്രിക്കുമ്പോൾ പ്യൂവർ ആവുകയും ചെയ്യുന്ന അവസ്ഥ.

Contents

Chapter 1

Introduction

1.1 Motivation and Survey of Literature

Korovkin-type theorems refer to a particular class of results which provide simple but very useful tools to determine whether a given sequence of positive linear operators acting on some function space is an approximation process; i.e, converges strongly to the identity operator. In general, such theorems provide some 'test subsets' of functions which guarantee that the approximation property holds on the entire space provided it holds on them. The name derives from the result by P.P. Korovkin [25], a Russian mathematician, that the above-stated property holds for the functions 1, x, x^2 in the space $C[0,1]$ of all continuous functions on the interval $[0,1]$. This simple but very powerful result motivated many mathematicians around the world to extend it and to study it in various settings such as function spaces, Banach spaces, Banach algebras, abstract Banach lattices etc. Weirstrass approximation theorem is one of the major results in approximation theory and a beautiful application of Korovkin theorem gives a simple proof of it. In the celebrated Korovkin's theorem mentioned above, he proved a criterion to decide whether a given sequence $(\phi_n)_{n\in\mathbb{N}}$ of positive linear operators on the space of continuous functions $C([a,b])$ is an approximation process or not; that is, $\phi_n(f) \to f$ uniformly on $[a,b]$ for every $f \in C[a,b]$. In fact it is sufficient to verify that $\phi_n(f) \to f$ uniformly on $[a,b]$ only for $f \in \{1, x, x^2\}$. We may refer to the theorem as Type I Korovkin theorem,

The set $\{1, x, x^2\}$ is called as *Korovkin set* or *test set*.

Very strong and productive connection of this theory with many branches of mathematics viz. functional analysis, harmonic analysis, partial differential equations, measure theory and probability theory etc. also started to emerge. In the case of functional analysis, the main areas that got benefited were approximation problems in function algebras, abstract Choquet boundaries and convexity theory, uniqueness of extensions of positive linear forms, convergence of sequences of positive linear operators in Banach lattices, structure theory of Banach lattices, convergence of sequences of linear operators in Banach algebras and in C^*-algebras, structure theory of Banach algebras and approximation problems in function algebras. The main objectives of those who tried to extend the theorem to various directions include finding test sets other than $\{1, x, x^2\}$, establishing 'Korovkin-like' theorems in other spaces and for other classes of linear operators. Such developments in the ensuing decades resulted in the evolution of a separate theory which is referred to as Korovkin-type approximation theory. The concept of Korovkin closure (if A and B are unital C^*-algebras and $T : A \to B$ is a positive linear contraction, the Korovkin closure of a subset H of A is defined as $\{a \in A| \lim_\alpha \Phi_\alpha(a) = T(a)$ for every net of positive linear contractions $\{\Phi_\alpha\}_{\alpha\in\mathcal{I}}$ from A to B such that $\lim_\alpha \Phi_\alpha(h) = T(h)$

for every $h \in H$) turned out to be a very handy tool in solving many problems arising out of generalising Korovkin theorem. Majority of the Korovkin-type theorems that appeared during the early years deal with answering one of the following questions;

- when does the Korovkin closure have an algebraic structure?
- when does the Korovkin closure become the whole space?

In the setting of $C(X)$, where X is a compact Hausdorff space and function space (unital, separating, closed linear subspace of $C(X)$) there is a rich theory related to the concept of 'boundaries', specifically Choquet boundary and Shilov boundary. Bishop and de Leeuw [10] introduced the notion of Choquet boundary for a function space which is the set of all points in X with unique representing measure. Equivalently, we can see that a point in X belong to the Choquet boundary of a function space contained in $C(X)$ if the linear functional on the function space of evaluation at the point admits a unique completely positive extension to $C(X)$. Shilov boundary of a function space is the smallest closed subset of X on which every function in the function space attains its maximum modulus. It is known that Choquet boundary is dense in Shilov boundary ([36], Proposition 6.4). A major milestone in the development of the theory is geometric approach to Korovkin's theorems which has its origin in the paper of Saskin [43] in which he proved the very important theorem connecting Korovkin sets and Choquet boundary: a function space in $C(X)$ is a Korovkin set precisely when its Choquet boundary is the whole of X. A comprehensive account of these developments can be found in the survey article of Berens and Lorentz [9]. There are other works which documented these developments such as the monograph of Altomare and Campiti [2] and survey article of Altomare [3]. Priestley [37] in 1976 initiated the study of Korovkin's theorem in C^*-algebras. Priestley proved that for a C^*-algebra A with identity I, if $\{\phi_n\}_{n\in\mathbb{N}}$ is a sequence of positive linear maps from A into A satisfying $\phi_n(I) \leq I$ for all n, then

$$C = \{a \in A : a = a^*, \phi_n(a) \to a, \phi_n(a^2) \to a^2\}$$

is a norm-closed Jordan algebra of self-adjoint elements of A (J^*-algebra), that is a real linear subspace of A closed under the Jordan product $a \circ b = (ab + ba)/2$. Analogues of this theorem also holds in the weak operator topology and strong operator topology. Also, Priestley established above results in the trace norm convergence when $\{\phi_n\}$ acts on the trace class operators on $B(H)$.

Robertson [39] in 1977 generalized Priestley's results to (complex) C^*-algebras using ideas of Palmer [33] for large class of positive linear operators and obtained that the set C mentioned above is actually a C^*-algebra. Robertson proved that if $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of Schwarz maps for a C^*-algebra A such that $\phi_n(I) \leq I$ for all n, then the set

$$D = \{a \in A : ||\phi_n(x) - x|| \to 0 \text{ for } x = a, a^*a, aa^*\}$$

is a C^*-algebra. Meanwhile, Takahasi [50] improved Priestley's results in C^*-algebras considering norm convergence and without the assumption $a = a^*$. Limaye and Namboodiri [27] in 1982 obtained the generalization of the results of Priestly and Robertson as follows: Let A and B be complex C^*-algebras with identity, let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence of positive linear maps from A into B satisfying $\phi_n(I) \leq I$ for all n and ϕ is a

C^*-homomorphism from A to B. Then

$$E = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^* \circ a) \to \phi(a^* \circ a)\}$$

is a J^*-algebra where $\circ$ denotes the Jordan product. If all ϕ_n and ϕ are Schwarz maps, E is a C^*-subalgebra of A. The theorem holds for operator norm convergence, weak operator convergence and the strong operator convergence. There is also a slight modification of this theorem for the convergence. There are two more significant results due to P.P. Korovkin regarding test sets, namely

- (a) There is no test set for $C([a, b])$ consisting only of two functions. Thus the cardinality of a test set is at least 3.
- (b) A triple is a test set of $C([a, b])$ exactly when it is a Čebyšev system on $[a, b]$.

We may call theorems (a) and (b) as Korovkin's Type II and Type III theorems respectively. It may be mentioned here that while Type I Korovkin theorem attracted a lot of attention, the other two types still remain rather unexplored. Korovkin type approximation theory in $C(X)$ case was pursued and amplified by Wulbert [52], Berens & Lorenz, Bauer etc. Bauer in particular expanded investigation of Korovkin subspaces using suitable enveloping functions. It is almost certain that the first noncommutative Korovkin type theorem appeared in an unpublished work of Arveson [8]. He considered approximation of $*$-homomorphisms from $C(X)$, (X compact Hausdorff) to a general C^*-algebra. The theorem proves that under certain conditions, Korovkin closure of a certain subspace of $C(X)$ becomes the whole space. William Arveson, in his seminal work began the systematic noncommutative Choquet and Shilov theory by defining most of the terminology and techniques. Central to this theory is what Arveson termed as boundary representations-a family of 'special' representations of a C^*-algebra such that their restrictions to a subspace (closed, containing the identity of the C^*-algebra) have unique completely positive extension to the entire C^*-algebra. Arveson refers to the set of all unitary equivalence classes of boundary representations mentioned above as the noncommutative Choquet boundary for the subspace. Arveson identified that the noncommutative analogue of this fact is equivalent to the assertion that there exist *sufficiently many* boundary representations for an operator system(subspace of a unital C^*-algebra which is self-adjoint, closed, containing the identity of the C^*-algebra) where the operator system generates the C^*-algebra. The question whether an operator system has got sufficiently many boundary representations remained unsolved since 1969, and it was Arveson himself that proved it in the affirmative for a separable operator system-i.e, every separable operator system has sufficiently many boundary representations. The general case remained elusive, and Davidson and Kennedy settled it once and for all by proving that every operator system has got sufficiently many boundary representations to completely norm it. Arveson coined the term hyperrigid sets to denote the noncommutative version of Korovkin sets. He proved that a separable operator system is hyperrigid if and only if every representation of the C^*-algebra generated by the operator system has unique extension property relative to the operator system. It follows that when the operator system is hyperrigid, every irreducible representation has unique extension property. Arveson [7] conjectured that the converse is also true: i.e, if every irreducible representation of a C^*-algebra is a boundary representation for a separable operator system contained in it, then the operator system is hyperrigid. This is the celebrated Arveson's hyperrigidity conjecture and it remains unsolved till today for a general C^*-algebra, even though Arveson [7] himself could prove it for a C^*-algebra with countable spectrum and Craig Kleski [24] for Type-1 C^*-algebras. Recall that Type II Korovkin theorem in the classical case says that there is no test set in the case of $C[a, b]$ containing two elements. But when we consider the general case of a C^*-algebra generated by a hyperrigid set, an analogues result does not seem to exist. Arveson [7] has proved that if $x \in B(H)$ is self-adjoint with at least 3 distinct points in its spectrum, then $\{1, x, x^2\}$ is a hyperrigid set of generators for the C^*-algebra generated by x and the identity while $\{1, x\}$ will not be a hyperrigid generator. M.N.N.Namboodiri proved a similar result; for a normal operator $x \in B(H)$, $\{1, x, x^*x\}$ is a hyperrigid set of generators for the C^*-algebra generated by x and identity. Also if the spectrum of x contains three distinct points on a straight line, then $\{1, x\}$ will not be a hyperrigid generator for the C^*-algebra mentioned above. Now, coming to the Type-III Korovkin theorems, recall that a subspace of a Banach space is called a Čebyšev system if every point of the Banach space admits a unique closest point in the subspace. Haar [19] obtained a characterization for a finite dimensional subspace of $C(X)$, X compact Hausdorff: an n dimensional subspace is a Čebyšev subspace exactly when no non-zero element in the subspace has more than $n - 1$ zeros. Thus he gave an algebraic characterization of a property which is purely geometrical. The study of Čebyšev subspaces of C^*-algebras was carried out by Robertson, Yost and Pedersen. Attempts to prove the analogue of Haar's theorem [19] led to quite a few interesting results in the noncommutative C^*- algebra setting. Robertson and Yost [40] showed that in an infinite dimensional unital C^*- algebra, $\mathbb{C}1$ is the only finite dimensional *- subalgebra which is also a Čebyšev subspace. They also proved a remarkable result which says that every C^*- algebra in a certain large class contains an infinite dimensional Hilbert subspace with the property that each of its closed subspaces is a Čebyšev subspace. Robertson [38] provided the characterization of 1-dimensional Čebyšev subspaces of von Neumann algebras in terms of central projections. He also showed that a finite dimensional *-subalgebra of dimension greater than 1 of an infinite dimensional von Neumann algebra can't be a Čebyšev subspace. Berenz and Lorenz [9] proved that if X denotes a closed interval in $\mathbb{R}$ or unit circle, then each Čebyšev system contained in $C(X)$ is a Korovkin set.

1.2 Organisation of the Thesis

Čebyšev subspaces in the general C^*-algebra setting is studied and Chapter 1, we give a brief survey of the evolution of classical Korovkin's theory and its noncommutative counterpart. Important notions such as test sets, hyperrigidity, boundary representations, Choquet boundary etc are introduced.

In Chapter 2, we give the basics of C^*-algebras and along with various notions connected with classical and noncommutative Korovkin theory. In Section 2.1, as a prerequisite, we give the essentials of the theory of C^*-algebras etc. In section 2.2 the classical notion of Choquet boundary, and Shilov boundary are discussed. Some of the theorems related to Choquet boundary are explained. In Section 2.3, we describe the classical Korovkin's theorem, Korovkin set and Saskin's theorem relating the Korovkin set and Choquet boundary. In Section 2.4, completely positive maps on C^*-algebras which is essential for the study of hyperrigidity is introduced along with Stinespring's theorem for completely positive maps. In Section 2.5, We discuss non commutative versions of Choquet boundary and Shilov boundary. In Section 2.6, the notion of boundary representations is introduced and explained. . In Section 2.7, Arveson's hyperrigidity conjecture is discussed. Important theorems concerned and some special cases in which the conjecture has been proved are also given.

In Chapter 3, we study the notion of hyperrigidity. In Section 3.1, the concept is introduced and the characterization theorem for hyperrigid operator systems due to Arveson ([7], Theorem 2.1) is given. A few more results due to Arveson are also stated. In section 3.2, Arveson's hyperrigidity conjecure is stated and the special cases for which the conjecture has been proved are also discussed. Section 3.3 is about two other notions viz. weak hyperrigidity and strong hyperrrigidity as introduced by M.N.N. Namboodiri([31], [32]). In Chapter 4, we discus the concept of Čebyšev subspaces of general normed spaces as well as C^*-algebras. Section 4.1 is about the classical case where the underlying space is a normed space. Here we discuss the classical Haar condition. In section 4.2 Čebyšev subspaces of von Neumann algebras and general C^*-algebras are discussed. Section 4.3 deals with noncommutative Haar condition and its connection with Arveson's notion of boundary representations.

In Chapter 5 the notions of separating operator systems and quasi hyperrigid operator systems of C^*-algebras are discused. In section 5.1 we discuss Arveson's notion of separating subalgebras of C^*-algebras. In section 5.2 separating operator systems are introduced and establish its connection with the notion of quasi hyperrigidity.

In Chapter 6, we discuss some problems for further research may be possible. The problems are described briefly.

Chapter 2

Preliminaries

This chapter is devoted to briefly discuss the basics of C^*-algebras, and to introduce the terminologies and basic results in connection with the Korovkin theory-both classical and noncommutative, boundary representations, Čebyšev subspaces etc.

2.1 C*-algebra Preliminaries

A norm on a vector space V is a function $V \ni x \to \|x\| \in [0, \infty)$ satisfying the following conditions, for all $x, y \in V, \alpha \in \mathbb{C}$

i. $\|x\| = 0 \Leftrightarrow x = 0$.

ii. $\|\alpha x\| = |\alpha|\|x\|$.

iii. $\|x + y\| \leq \|x\| + \|y\|$.

A normed vector space is a pair $(V, \|\|)$ consisting of a vector space together with a norm on it. An algebra A is a vector space together with a bilinear map $V^2 \ni (x, y) \to xy$ such that $x(yz) = (xy)z, \quad x, y, z \in A$. A norm $\|.\|$ on an algebra A is called submultiplicative if

$$\|xy\| \leq \|x\|\|y\| \quad (x, y \in A).$$

A *Banach algebra* is an algebra which is complete w.r.t. a submultiplicative norm. A function $x \to x^*$ from A to A is said to be an involution (adjoint operation) if it is conjugate linear such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ $\forall x, y \in A$. The pair $(A, *)$ A is said to be a **-algebra*. An element $x \in A$ where A is a *-algebra is called self-adjoint if $x = x^*$. A Banach *-algebra A is a Banach algebra equipped with an involution. A Banach *-algebra is said to be a C^**-algebra* if it satisfies the C^*-identity; i.e., $||x^*x|| = ||x||^2$ $\forall x \in A$. If the C^*-algebra A admits a unit $1(1x = x1 = x \forall x \in A)$, then we say that A is a unital C^*-algebra. Now let us see some examples of C^*-algebras.

(i) The scalar field $\mathbb{C}$ is a C^*-algebra with the involution defined as complex conjugation $\lambda \to \bar{\lambda}$.

(ii) Let X denote a compact Hausdorff space and let

$$C(X) = \{f : X \to \mathbb{C} : f \text{ continuous } \}.$$

Then $C(X)$ is a C^*-algebra with involution defined by $f \to \bar{f}$.

(iii) If H is a complex Hilbert space then $B(H)$, which is the set of bounded linear operators on H is a C^*-algebra and in this case the involution is defined by $T \to T^*$

(T^* is the adjoint of the operator T).

If A is a unital C^*-algebra then the spectrum of an element $a \in A$ (denoted by $\sigma(a)$) is the set $\{\lambda \in \mathbb{C} | (a - \lambda.1) \text{is not invertible}\}$. An element $a \in A$ where A is a C^*-algebra is called positive if a is self-adjoint and $\sigma(a) \in [0.\infty)$. It can be proved that $a \in A$ is positive precisely when $a = b^*b$ for some $b \in A$.

A *representation* of a C^*-algebra A is a pair (π, H) where H is a Hilbert space and $\pi : A \to B(H)$ is a *-homomorphism. We also say that π is a representation of A on H. The representation π is *faithful* if π is one-to-one. For a given subspace M of A, let $[M]$ denote the closed linear span of elements of M in A. Now the representation π is called *non-degenerate* if $[\pi(A)H] = H$ and it is called *cyclic representation* if $\exists\ \eta \in H$ satisfying $[\pi(A)\eta] = H$. (In this case the vector η is called the *cyclic vector* for π). Let A be a C^*-algebra and (π, H) and (ρ, H') be two representations of A. We say that π is unitarily equivalent to ρ if there exists a unitary map $W : H \to H'$ such that $W\pi(y) = \rho(y)W$ for every $y \in A$. We write $\pi \sim \rho$ when π and ρ are unitarily equivlent. Let $\pi : A \to B(H)$ be a representation and let M be a subspace of H. M is said to be *invariant subspace* for $\pi(A)$, if $\pi(A)M \subseteq M$. If both M and $M^\perp$ are invariant for $\pi(A)$, then M is a *reducing subspace* for $\pi(A)$.

A representation $\rho : A \to B(H)$ is said to be *irreducible* if $\rho(A)$ does not commute with any non-trivial projection. This can again be proved to be equivalent to saying that other than 0 and H, $\rho(A)$ has no other closed invariant subspaces. Now let $[\pi] = \{\rho : \rho \sim \pi\}$. The *spectrum* $\hat{A}$ of A is defined as $\hat{A} = \{[\pi] : \pi \text{ is an irreducible representation of } A\}$.

A linear functional ϕ defined on a unital C^*-algebra A is called a state if it is positive; i.e, $\phi(a^*a) \geq 0 \ \ \forall a \in A$ and $\phi(1) = 1$. For every state ϕ on a unital C^*-algebra A, there exists a cyclic representation (π_ϕ, H_ϕ) and a cyclic vector ξ_ϕ such that $\phi(x) = \langle \pi_\phi(x)\xi_\phi, \xi_\phi \rangle \ \forall x \in A$. If S is a subspace of a C^*-algebra A, let $S^* = \{a : a^* \in S\}$. We say that S is self-adjoint when $S = S^*$. If the given C^*-algebra A is unital, then a self-adjoint subspace S containing the identity of A is called an *operator system*. An operator system $S \subseteq B(H)$, is known to be a concrete operator system. Choi and Effros [12] gave an abstract axiomatic definition of an operator system. They established a representation theorem so that every abstract operator system can be represented as a concrete operator system.

A unital subalgebra B of a C^*-algebra A is called an *operator algebra*. A concrete operator algebra is an operator algebra contained in $B(H)$. There is also an abstract notion of operator algebras which is due to Blecher, Ruan and Sinclair(BRS) [11]. The BRS theorem [11] asserts that all abstract operator algebras have realization as concrete operator algebras.

2.2 Choquet boundary and Shilov boundary-Classical Case

Let $M \subseteq C(X)$ and $1 \in M$. A boundary for M is a subset $Y \subseteq X$ such that for each $f \in M$, there is $y \in Y$ such that $\|f\| = f(y)$. In other words a boundary for a function system is a norm-attaining subset of X. A lot of work has taken place in this setting and the celebrated Krein-Milman theorem turned out to be crucial in the development of the theory of boundaries. The classical Krein Milmann theorem says that every compact convex subset of a locally convex space is the closed convex hull of its extreme points. Choquet generalised this theorem by replacing finite sums with integrals. If Y is a non-empty compact subset of a locally convex vector space X, and if μ is a probability measure on Y, a point $x \in X$ is said to be *represented* by μ if $f(x) = \int_Y f d\mu$ for every $f \in X^*$. A non-negative regular Borel measure μ defined on a compact Hausdorff space X is supported on a subset Y of X if Y is a Borel set and $\mu(X \setminus Y) = 0$. Bishop and de Leeuw introduced the concept of Choquet boundary for a function space. Let M be a function space in $C(X)$. Consider the set $K(M)$ of all linear functionals on M such that $\|\phi\| = 1 = \phi(1)$. Note that $K(M)$ is a weak*- compact and convex subset of M^*. But by Hahn- Banach extension theorem, we can extent each element of $K(M)$ to an element of $K(C(X))$. Also we have Riesz representation theorem by which we can identify each element of $K(C(X))$ with an element of $P(X)$ (the set of all probability measures on X). If $\phi \in K(C(X))$, there is a probability measure μ on X such that $\phi(f) = \int_X f d\mu$ for every $f \in C(X)$. Let $l_x \in K(C(X))$ denote the evaluation functional at the point x, i.e, $l_x(f) = f(x), f \in C(X)$.

Definition 2.2.1. *Let X be a compact Hausdorff space and let $C(X)$ be the set of continuous complex valued functions on X. A linear, closed, separating subspace of $C(X)$ that contains the constant functions is called a function space.*

Let us now give the definition of a Choquet boundary in the classical case.

Definition 2.2.2. *The Choquet boundary of a function space M in $C(X)$ where X is a compact Hausdorff space consists of all $x \in X$ such that l_x, the evaluation functional at the point x is an extreme point of $K(M)$.*

We have the following characterization of choquet boundary in terms of the representing measures.

Proposition 2.2.1. *Let M be a function space in $C(X)$ and $x \in X$ is in the Choquet boundary of M if and only if $\mu = \delta_x$ is the only probability measure on X such that $f(x) = \int_X f d\mu$ for every $f \in M$.*

It can be verified that the Choquet boundary of a function space is actually a boundary for the same.

Definition 2.2.3. *Let M be a function space in $C(X)$ where X is a compact Hausdorff space. If there is a smallest closed boundary (i.e, a closed boundary which is contained in every closed boundary)for M, it is called the Shilov boundary for M.*

Proposition 2.2.2. *Let M be a function space in $C(X)$ where X is a compact Hausdorff space. Then the closure of the Choquet boundary is the Shilov boundary for M.*

2.3 Classical Korovkin theorem and Saskin theorem

The classical Korovkin theorem [25] in 1953 gives conditions for uniform approximation of continuous functions on a compact metric space using sequences of positive linear operators. The theorem can be seen as generalization of the classical results of Weirstrass on uniform approximation of continuous functions using algebraic or trigonometric polynomials.

Theorem 2.3.1. *[9] **(Korovkin's Theorem)** For a sequence* $\{\varphi_m : m = 1, 2, 3, ...\}$ *of positive linear maps from* $C([0, 1])$ *to itself, and for each function* $p_l(t) = t^l, t \in [0, 1]$, $l = 0, 1, 2$*, if*

$$\lim_{m\to\infty} \varphi_m(p_l) = p_l \text{ uniformly on } [0, 1], l = 0, 1, 2,$$

then

$$\lim_{m\to\infty} \varphi_m(g) = g \text{ uniformly on } [0, 1], \ \forall\ g \text{ in } C[0, 1].$$

Definition 2.3.1. *An* $S \subseteq C([a, b])$ *is said to be a Korovkin set or test set, if for every sequence* $\{\varphi_m\} : C([a, b]) \to C([a, b],\ m = 1, 2, 3, ...$ *of positive linear maps* $\lim_{m\to\infty} \varphi_m(g) = g$ *uniformly on* $[a, b]$ $\forall g \in S$ *implies that* $\lim_{m\to\infty} \varphi_m(f) = f$ *uniformly on* $[a, b]$ $\forall f \in C([a, b])$.

By Korovkin theorem, $\{1, x, x^2\}$ is a Korovkin set for $C([0, 1])$.

The following beautiful theorem by Saskin in [43] showed the important connection between Korovkin sets and Choquet boundary.

Theorem 2.3.2. *[9] Let* S *be a function space in* $C(X)$*. Then the following are equivalent.*

1. S *is a Korovkin set.*

2. $\partial S = X$.

2.4 Completely Positive Maps

Completely positive maps are an important collection of morphisms between C^*-algebras. Apart from having a norm, a C^*-algebra has an order structure induced by the cone of 'positive' element. Positive elements play an important role in C^*-algebras. We use the notation $x \geq 0$ to denote that x is a positive element in a C^*-algebra A. The set A^+ of all positive elements in A forms a norm-closed, convex cone in A.

Let A be a C^*-algebra and $M_n(A)$ denote the set of $n \times n$ matrices with entries from A. An arbitrary element of $M_n(A)$ is represented as $[x_{ij}],\ x_{ij} \in A$. With respect to the following operations, $M_n(A)$ is a $*$-algebra. The adjoint operation is defined as $[x_{ij}]^* = [x_{ji}^*]$ and for $[x_{ij}]$ and $[y_{ij}]$ in $M_n(A)$, we define $[x_{ij}] \cdot [y_{ij}] = \left[\sum_{l=1}^{n} x_{il} y_{lj}\right]$. We have a natural way to make $M_n(A)$ into a C^*-algebra. For a Hilbert space H, the identification $M_n(B(H)) = B(H^{(n)})$ (where $H^{(n)} = H \oplus H \oplus \cdots \oplus H$, n times) induces a C^*-norm on $M_n(B(H))$. Assume that A acts faithfully on H. The $*$-algebra $M_n(A)$ can be realized as a $*$-subalgebra of $M_n(B(H))$. This makes $M_n(A)$ into a C^*-algebra. Note that the norm on $M_n(A)$ is independent of the choice of H as the norm on a C^*-algebra is unique.

Let B be another C^*-algebra. Let S be an operator system contained in A. We can endow $M_n(S)$ the norm and order structure that it inherits from $M_n(A)$. Let $\varphi : S \to B$ be a linear map. Then φ induces a map $\varphi_n : M_n(S) \to M_n(B)$, $n \in \mathbb{N}$ where $\varphi_n([s_{ij}]) = [\varphi(s_{ij})]$. Therefore each CP map ϕ on S gives rise to a class of maps $\{\phi_n\}_{n \in \mathbb{N}}$. The adjective *completely* signify that all the maps $\{\varphi_n\}$ exhibit a particular property. The map φ is called *n-positive* if φ_n is positive. The map φ is called *completely positive* (CP) if φ

is n-positive for all $n \geq 1$. The map φ is said to be *completely bounded* (CB) if $||\varphi||_{\text{CB}} = \sup_{n\geq 1} ||\varphi_n|| < \infty$. The map φ is said to be *completely contractive* (CC) if $||\varphi||_{CB} \leq 1$. In the same spirit we can define complete isometry. It is clear that if φ is n-positive, then φ is k-positive for every $k < n$. The map φ is *unital completely positive* (UCP) if φ is completely positive and $\varphi(1) = 1$. UCP maps are always completely contractive as $||\varphi||_{CB} = ||\varphi(1)||$ for CP maps.

Let $CP(A, H) = \{\phi : A \to B(H)/\phi \text{ is CP }\}$. The notation $UCP(A, H)$ is used to denote the subset of UCP maps. When dim$(H) = n < \infty$, elements of $UCP(A, H)$ are called matrix states.

W.F. Stinespring characterized completely positive maps in his famous theorem known as Stinespring's dilation theorem.

Theorem 2.4.1. *[34] **(Stinespring dilation theorem)** Given a unital C^*-algebra A and a CP map $\phi : A \to B(H)$, there exists a Hilbert space K, a representation $\pi : A \to B(K)$ and a bounded operator V from H to K such that*

$$\phi(x) = V^*\pi(x)V.$$

It is clear from the above theorem that when ϕ is unital, V is an isometry.

Let ϕ be a CP map and let $\phi(.) = V^*\pi(.)V$ be a Stinespring representation of ϕ. By taking $K_0 = [\pi(A)VH]$ one may define a new completely positive map by restricting π to K_0. Let $\pi_0 = \pi_{|K_0}$. Then π_0 also satisfies $\phi(a) = V^*\pi_0(a)V,\ a \in A$. Therefore without loss of generality we can assume that $[\pi(A)VH] = K$. With this condition,

$\phi(x) = V^*\pi_0(x)V,\ x \in A$ is called the minimal Stinespring's dilation of ϕ.

The space $UCP(A, H)$ can be equipped with a locally convex topology called bounded weak topology (BW-topology). A net $\{\phi_\alpha\}_{\alpha\in\wedge}$ in $CP(A, H)$ converges to ϕ in $CP(A, H)$ if $\phi_\alpha(x) \to \phi(x)$ in the weak operator topology for every $x \in A$. As an immediate consequence of a general theorem of Kadison [22] we get the following: The set $UCP(A, H)$ is compact in the BW-topology.

2.5 Choquet and Shilov boundary: noncommutative case

Arveson [4] through his landmark paper instigated the 'transplanting' of the sophisticated ideas available in the classical function space theory into general C^*-algebra setting. Arveson [4] proposed the existence of a special family of irreducible representations of a unital closed subalgebra of a C^*-algebra called *boundary representations* which have unique completely positive extension to the C^*-algebra generated by the subalgebra. The set of all boundary representations is the noncommutative analogue of Choquet boundary of a function algebra. Recall that the Choquet boundary for a function algebra is the set of all points which have unique representing measures.

Definition 2.5.1. *Let S be an operator system and $\psi : S \to B(H)$ be a unital completely positive map. Then ψ is said to have unique extension property if*

(i) ψ has a unique completely positive extension $\tilde{\psi} : C^*(S) \to B(H)$ and

(ii) $\tilde{\psi}$ is a representation of $C^*(S)$ on H.

Definition 2.5.2. *[6] Consider an operator system S and let A be a C^*-algebra where $A = C^*(S)$. An irreducible representation $\pi : A \to B(H)$ of A is called a boundary representation for S if $\pi_{|S}$ has a unique completely positive extension to A.*

Arveson introduced the noncommutative generalization of the classical Shilov boundary and the definition is as follows:

Definition 2.5.3. *Let A be a linear subspace of a C^*-algebra B such that A contains the identity and generates B as a C^*-algebra. Then a closed two-sided ideal $J \subseteq B$ is called a boundary ideal for A if the canonical quotient map $q : B \to B/J$ is completely isometric on A. A boundary ideal is called the Shilov boundary ideal if it contains every other boundary ideal.*

Now consider A and B as in the above setting. Arveson called A an $admissible$ subspace of B if the intersection of kernels of all boundary representations for A is a boundary ideal for A. We can prove that A is an admissible subspace of B if and only if it satisfies the condition: for every $n \geq 1$ and every $n \times n$ matrix $[a_{ij}] \in M_n(A)$,

$$||[a_{ij}]|| = \sup_{\pi \in \partial A} ||\pi([a_{ij}])||. \tag{2.1}$$

where the norm of $[a_{ij}]$ is inherited from $B \otimes M_n$. If the above condition is satisfied we say that A has sufficiently many boundary representations. In this connection Arveson proved the following:

Theorem 2.5.1. *Let A be a linear subspace of a C^*-algebra B such that A contains the* *identity and generates B as a C^*-algebra. Let A be an admissible subspace of B and K be the intersection of kernels of all boundary representations. Then K is the Shilov boundary ideal for A.*

It turns out from the above theorem that there are sufficiently many boundary representations for an operator system is equivalent to the condition that the Shilov boundary is the intersection of kernels all boundary representations. This is the noncommutative analogue of the classical theorem that for a function system $S \subseteq C(X)$ that separates points of a compact Hausdorff space, closure of Choquet boundary is the Shilov boundary. Arveson conjectured the existence of boundary representations in the above sense. The embedding $q(A) \subseteq B/J$ later came to be known as the C^*-envelop of A. The formal definition of the C^*-envelop in the setting of an operator system is as follows:

Definition 2.5.4. *[15] The C^*-envelope of an operator system S consists of a C^*-algebra $\mathcal{U}$ and a completely isometric imbedding $i : S \longrightarrow \mathcal{U}$ such that $U = C^*(i(S))$ with the following universal property: whenever $j : S \longrightarrow B = C^*(j(S))$ is a unital completely isometric map, then there is a $*$-homomorphism $\pi : B \longrightarrow U$ such that $i = \pi j$.*

Hamana's work [20] proved the existence of C^*-envelope without resorting to boundary representations. Later Dritschel and McCullough [15] proved the existence of the C^*-envelope using dilation theory. These developments prompted Arveson [6] to take up the question of existence of boundary representations once again and he was successful in proving it for separable operator systems.

Theorem 2.5.2. *[6] Let S be a separable operator system and $C^*(S)$ be the C^*-algebra* *generated by S. Then S has sufficiently many boundary representations.*

Craig Kleski [23] showed that the 'supremum' in equation 2.1 can be replaced by 'maximum' when the operator system under consideration is separable. Hence the Choquet boundary for a separable operator system is actually a boundary in the classical sense. The problem of existence of boundary representations has been settled once and for all by Davidson and Kennedy [13]. Inspired from the work of Arveson [4], and from that of Dritschel $\&$ McCullough [15] and using dilation theoretic arguments, Davidson and Kennedy proved that every operator system has sufficiently many boundary representations to generate the C^*-envelop.

Chapter 3

Hyperrigidity

3.1 Preliminaries

Arveson introduced the notions of noncommutative Choquet boundary and Shilov boundary and proved that every separable operator system has sufficiently many boundary representations, providing the noncommutative analogue of the classical result that the closure of Choquet boundary is the Shilov boundary. The central role played by Choquet and Shilov boundary in Classical approximation theory prompted Arveson to initiate the noncommutative approximation theory. He defined hyperrigid sets analogues to the Classical Korovkin sets.

Definition 3.1.1. *[7] Let S be the set that generates a separable C^*-algebra A. Then S is called hyperrigid if for every faithful representation $A \subseteq B(H)$ of A for some H and* *for all sequences of UCP maps* $\phi_m : B(H) \to B(H)$, $m = 1, 2, ...,$

$$\lim_{m\to\infty} ||\phi_m(s) - s|| = 0, \forall\, s \in S \Rightarrow \lim_{m\to\infty} ||\phi_m(x) - x|| = 0, \forall\, x \in A.$$

Note that in the above definition A is identified with its image $\pi(A)$ under a faithful nondegenerate representation $\pi \to B(H)$ on a Hilbert space H. The following is Arveson's theorem characterizing separable hyperrigid operator systems.

Theorem 3.1.1. *[7] Let* $S \subseteq A$ *be a separable operator system where* $A = C^*(S)$*. The following assertions are equivalent:*

i. S *is hyperrigid.*

ii. If $\rho : A \to B(H)$ *is a nondegenerate representation on a separable* H *such that for every sequence* $\varphi_m : A \to B(H)$ *of UCP maps,*

$$\lim_{m\to\infty} ||\varphi_m(g) - g|| = 0, \forall\, g \in S \Rightarrow \lim_{m\to\infty} ||\varphi_m(a) - a|| = 0, \forall\, a \in A.$$

iii. If $\rho : A \to B(H)$ *is a nondegenerate representation on a separable* H*, the map* $\rho_{|S}$ *has unique extension property.*

iv. If B *is any other* C^**-algebra with unit, and* $\theta : A \to B$ *is a* $*$*- homomorphism and* $\varphi : B \to B$ *is a UCP map such that*

$$\varphi(y) = y \ \ \forall\, y \in \theta(S), \text{ then } \varphi(y) = y \ \ \forall\, y \in \theta(A).$$

The following theorem by Arveson can be considered to be a noncommutative strengthening of the classical Korovkin theorem.

Theorem 3.1.2. *[7] Let $X \in B(H)$ be a self adjoint operator and let A be the C^*-algebra generated by X. Then $\{X, X^2\}$ is a hyperrigid generator for A.*

Arveson obtained a hyperrigidity result for a well known compact operator, namely the Volterra integration operator.

Theorem 3.1.3. *[7] Let H be the Hilbert space $H = L^2[0,1]$ and T be the Volterra operator on H,*

$$Tg(x) = \int_0^x g(t)dt, \quad g \in L^2[0,1].$$

T is irreducible and $C^*(T) = K(H)$ where $K(H)$ is the set of all compact operators on H. Then

(i) $G = \{T, T^2\}$ is hyperrigid; for all sequences of UCP maps $\varphi_n : B(H) \to B(H)$ for which

$$\lim_{n\to\infty} ||\varphi_n(T) - T|| = \lim_{n\to\infty} ||\varphi_n(T^2) - T^2|| = 0,$$

one has

$$\lim_{n\to\infty} ||\varphi_n(K) - K|| = 0$$

for every $K \in K(H)$.

(ii) The set $\{T\}$ is not hyperrigid in $K(H)$.

Theorem 3.1.4. *[7] Let $T_i \in B(H)$ for $i = 1, 2, ..., n$ be isometries and $A = C^*(T_1, T_2, ..., T_n)$. Then the set $\{T_1, ..., T_n, T_1T_1^*, ..., T_nT_n^*\}$ is a hyperrigid generator for A.*

3.2 Arveson's Hyperrigidity Conjecture

The following theorem by Arveson is about the necessary conditions for a separable operator system to be hyperrigid.

Theorem 3.2.1. *Let S be a separable operator system such that $A = C^*(S)$. If S is hyperrigid, then every irreducible representation of A is a boundary representation for S. In particular, the boundary ideal of a hyperrigid operator system must be $\{0\}$.*

Arveson coined the term 'obstructions to hyperrigidity' to denote the necessary conditions for a separable operator system to be hyperrigid. Essentially what Arveson's hyperrigidity conjecture says is that these are the only obstructions to hyperrigidity.

Conjecture 3.2.1. *[7] Let S be a separable operator system such that $A = C^*(S)$. If every irreducible representation of A is a boundary representation for S, then S is hyperrigid.*

What is remarkable is the fact that Arveson's hyperrigidity conjecture remains unsolved till this date even though it has been proved for certain special cases. In fact Arveson himself has proved if for C^*- algebras with countable spectrum.

Theorem 3.2.2. *[7] Let S by a separable operator system generating a C^*- algebra A such that $A = C^*(S)$. Assume that A has countable spectrum. If every irreducible representation of A is a boundary representation for S, then S is hyperrigid.*

It has also been verified in the cases where $C^*(S)$ is commutative [14]. Some partial results have also been obtained.

3.3 Weak hyperrigidity and strong hyperrigidity

M.N.N. Namboodiri [32] introduced the notion of Weak hyperrigidity and proved Korovkin-type theorem in the setting of W^* algebras. He also introduced strong hyperrigidity in the setting of completely contractive maps on C^*- algebras.

Definition 3.3.1. *[32] A subset S of a W^*-algebra A containing the identity 1_A is said to be weakly hyperrigid if*

(i) A equals the W^*-algebra generated by S.

(ii) For every faithful representation $A \subseteq B(H)$ of A where H separable and every net of contractive CP maps $\phi_\alpha : B(H) \to B(H), \lim_\alpha \phi_\alpha(s) = s$ weakly $\forall\ s \in S \implies \lim_\alpha \phi_\alpha(a) = a$ weakly $\forall\ a \in A$.

Now the characterization theorem for weakly hyperrigid sets.

Theorem 3.3.1. *[32] Let S be a separable operator system that generates the W^*-algebra A where $A = W^*(S)$. The following are equivalent:*

i. S is weakly hyperrigid.

ii. For every nondegenerate representation $\pi : A \to B(H)$ on a separable Hilbert space and every sequence $\phi_m : A \to B(H)$ of UCP maps,

$$\lim_{m\to\infty} ||\phi_m(s) - s|| = 0, \forall\, s \in S \Rightarrow \lim_{m\to\infty} ||\phi_m(a) - a|| = 0, \forall\, a \in A.$$

iii. For every nondegenerate representation $\pi : A \to B(H)$ on a separable Hilbert space, $\pi_{|S}$ has the unique extension property.

iv. For every unital W^*-algebra B, every unital $*$homomorphism $\theta : A \to B$ and every contractive completely positive map $\phi : B \to B$,

$$\phi(x) = x \;\; \forall\, x \in \theta(S) \Rightarrow \phi(x) = x \;\; \forall\, x \in \theta(A).$$

M.N.N.Namboodiri has also posed the following conjecture in line with Arveson's hyperrigidity conjecture.

Conjecture 3.3.1. *[32] Let A be a W^*-algebra and S be a separable operator system contained in A such that the C^*-algebra generated by S has countable spectrum. If every irreducible representation of A is a boundary representation for S, then S is weakly hyperrigid.*

M.N.N.Namboodiri [31] while examining the possibility of extending Arveson's charecterization theorem for hyperrigidity to linear contractions, defined strongly hyperrigid sets.

Definition 3.3.2. *[31] A finite or countably infinite set G of generators of a C^*-algebra A is said to be strongly hyperrigid if for every faithful representation π of A in $B(H)$ where H is a Hilbert space and every sequence of CC maps $\phi_m : B(H) \to B(H)$, $m = 1, 2, ...$,*

$$\lim_{m\to\infty} ||\phi_m((\pi(g)) - g|| = 0, \forall\, g \in G \Rightarrow \lim_{m\to\infty} ||\phi_m((\pi(a)) - a|| = 0, \forall\, a \in A.$$

M.N.N.Namboodiri [31] goes on to prove the characterization theorem for strongly hyperigid sets similar to that of Arveson's characterization theorem for hyperrigid sets with completely contractive maps replacing completely positive maps.

Chapter 4

Čebyšev Subspaces

The concept of Čebyšev subspaces and related ideas have been extensively used in approximation theory and in Banach spaces especially classical Banach spaces such as $C(X)$, the set of all complex valued continuous functions on a compact Hausdorff space X. The evolution of the theory of Čebyšev systems began with the work of Russian mathematician P.L. Čebyšev and his collaborators. The study of Čebyšev subspaces in the general C^*- algebra setting was initiated by A.G.Robertson [38].

4.1 Classical Case

The classical results are proved mainly using the lattice theoretic properties of scalar functions and the topology involved. But most of the pioneering results were proved using constructive hard analysis techniques. Excellent surveys are due to Ivanov Singer [44], Karl-Georg Steffens [46], H.Berens and G.G.Lorentz [9] to cite important ones. The classical concept of Čebyšev set in normed linear space is closely related to the more general theory of best approximation. We may now recall the notion of best approximation and a few basic results which are relevant to our discussion.

Definition 4.1.1. *Let (X, d) be a metric space, G be a subset of it with $x \in X$. An element g_0 in G is called* a point of best approximation of x *if*

$$d(x, g_0) = inf\{d(x, g) : g \in G\}.$$

For X and its subspace G as above, let

$$\mathcal{P}_G(x) = \{g_0 \in G : d(x, g_0) = inf d(x, g) : g \in G\}.$$

Some of the important theorems given below are stated as given in [44]. The first main theorem that characterizes best approximation in linear subspaces of normed linear spaces is as follows:

Theorem 4.1.1 ([44], Theorem 1.1)**.** *Let X be a normed linear space and G be a subspace of it, $x \in X \backslash \overline{G}$ and $g_0 \in G$. Then $g_0 \in \mathcal{P}_G(x)$ if and only if there exists $f \in X^*$ such that*

(i) $\|f\| = 1$,

(ii) $f(g) = 0 \quad (g \in G)$ *and*

(iii) $f(x - g_0) = \|x - g_0\|$.

The relevance of the theorem is that the functional f mentioned in the above theorem is 'maximal' in nature and it can be determined for many function spaces. The following theorem illustrates this.

For a compact Hausdorff space Ω, $\mathcal{C}(\Omega)$ (respectively $\mathcal{C}_{\mathcal{R}}(\Omega)$) will denote the set of all real or complex continuous functions (respectively continuous real functions) on Ω, with supremum norm.

Theorem 4.1.2 ([45], Theorem 1.2)**.** *Let G be a linear subspace of $C_R(\Omega)$, $x \in \mathcal{C}_{\mathcal{R}}(\Omega)\backslash\overline{G}$. We have $g_0 \in \mathcal{P}_G(x)$, if and only if there exist two disjoint closed sets $E_{g_{0+}}$ and $E_{g_{0-}}$ of Ω and a Radon measure μ on Ω such that*

(i) $\mid \mu \mid \Omega = 1,$

(ii) $\int_\Omega g(t)d\mu(t) = 0$, *for all g in G,*

(iii) $\mu \geq 0$ *on* $E_{g_{0+}}$ *and* $\mu \leq 0$ *on* $E_{g_{0-}}$ *and support* $\mu \subseteq E_{g_{0+}} \cup E_{g_{0-}}$ *and*

(iv) $$x(q) - g_0(q) = \begin{cases} \|x - g_0\| & \textit{for } q \textit{ in } E_{g_{0+}} \\ -\|x - g_0\| & \textit{for } q \textit{ in } E_{g_{0-}}. \end{cases}$$

A few more interesting results are there in this settings, but we restrict to the following one.

Theorem 4.1.3 ([45], Theorem 1.4)**.** *(a) For a positive measure space (Ω, ν), X=$\mathcal{L}^P(\Omega, \nu)$, $1 < p < \infty$, G be a linear subspace of X, $x \in X\backslash\overline{G}$ and $g_0 \in G$. We have $g_0 \in \mathcal{P}_G(x)$ if and only if*

$$\int_{\Omega} g(t)|x(t) - g_0(t)|^{p-1} sign[x(t) - g_0(t)] d\nu(t) = 0, \quad (g \in G).$$

(b) Let $\mathcal{H}$ be an inner product space, G be a linear subspace of $\mathcal{H}$. Let $x \in \mathcal{H} \backslash \overline{G}$ and $g_0 \in G$. We have $g_0 \in \mathcal{P}_G(x)$ if and only if

$$\langle g, x - g_0 \rangle = 0, \qquad (g \in G).$$

Now we define the notion of Čebyšev subspace of a normed linear space.

Definition 4.1.2. *A subspace G of a normed space X is called* a semi Čebyšev subspace *if each vector in X has at most one closest point in G. A subspace G of X is called a* Čebyšev subspace *if each vector in X admits a unique closest point in G.*

In the above definition, the subspace G is called *proximinal* if each vector in $\mathcal{A}$ has at least one closest point in G. Clearly, a Čebyšev subspace is the one which is both semi Čebyšev and proximinal.

Čebyšev sets were also called 'Haar sets' by some authors, e.g. by N. Efimov and S.B.Stečhkin [16].

Theorem 4.1.4 ([45], Theorem 3.1)**.** *A linear subspace G of a normed linear space X is a semi-Čebyšev subspace if and only if there do not exist f in X^*, x in X and g_0 in $G \setminus \{0\}$ such that*

$$\|f\| = 1, f(g) = 0, \qquad (g \in G), \qquad f(x) = \|x\| = \|x - g_0\|.$$

We state a couple of general theorems more before considering concrete cases. We use the following notations: For X and its subspace G as above,

$$\pi_G^{-1}(0) = \{x \in X; 0 \in \mathcal{P}_G(x)\}.$$

and for two sets A and B,

$$A - B = \{a - b : a \in A, b \in B\}.$$

Theorem 4.1.5 ([45], Proposition 3.1)**.** *For a closed linear subspace G of a normed linear space X, the following statements are equivalent.*

(i) G is a Čebyšev subspace.

(ii) $X = G \oplus \pi_G^{-1}(0)$, where $\oplus$ means that the sum decomposition of each element $x \in X$ is unique.

(iii) G is proximinal and $(\pi_G^{-1}(0) - \pi_G^{-1}(0)) \cap G = \{0\}$.

(iv) G is proximinal and the restriction $\omega_{G|\pi_G{}^{-1}(0)}$ of the canonical mapping $\omega_G : X \longrightarrow X/G$ to the set $\pi_G^{-1}(0)$ is one-to-one.

The next theorem characterizes finite dimensional Čebyšev subspaces of normed linear spaces.

Theorem 4.1.6 ([45], Theorem 3.3)**.** *An n-dimensional linear subspace G of a normed linear space X is a Čebyšev subspace if and only if there do not exist h extremal points $f_1, f_2, ..., f_h$ of S_{X^*}(unit sphere of X^*), where $1 \leq h \leq n$ if the scalars are real and $1 \leq h \leq 2n-1$ if the scalars are complex, h numbers $\lambda_1, \lambda_2, ..., \lambda_h > 0$ with $\sum_{j=1}^{h} \lambda_j = 1$ and $x \in X, g_0 \in G \setminus \{0\}$ such that we have $\sum_{j=1}^{h} \lambda_j f_j(x_h) = 0 (h = 1, 2, ..., n)$ and $f_j(x) = \|x\| = \|x - g_0\|, (j = 1, 2, ..., h)$.*

When $X = C(\Omega)$ (Ω compact), we get the celebrated theorem due to Haar which characterizes the n dimensional Čebyšev subspaces of $C(\Omega)$.

Theorem 4.1.7 ([19])**.** *Let G be an n dimensional linear subspace of $C(\Omega)$ spanned by the elements $x_1, x_2, ..., x_n$. Then G will be a Čebyšev subspace of $C(\Omega)$ if and only if $x_1, x_2, ..., x_n$ form a Čebyšev system (i.e, every $\sum_{i=1}^{n} \alpha_i x_i \neq 0$ has at most n-1 zeros on Ω).*

We will refer to the above equivalence condition for a Čebyšev subspace as the classical Haar condition. Thus Haar condition connects the geometrical and algebraic properties of functions. The characterization below of finite dimensional Čebyšev subspace of $C_{\mathbb{R}}[a, b]$ by Y. Ikebe [21] is also noteworthy.

Theorem 4.1.8 ([21])**.** *A finite dimensional subspace G of $C_{\mathbb{R}}[a, b]$ is a Čebyšev subspace if and only if*

$$\|g_0\| < 2\|x\|, (x \in C_{\mathbb{R}}[a, b] \backslash \{0\}, g_0 \in P_G(x)).$$

Remark 4.1.1. *It may be interesting to know whether a compact set Ω admits a Čebyšev system or not. The following theorem gives the answer. In spite of the nice Haar condition for a given finite dimensional subspace to be Čebyšev, it is not quite possible to find Čebyšev subspaces of arbitrary compact spaces. In fact the Mairhuber-Curtis theorem states that a compact space admits a Čebyšev system of order $n+1$ if and only if it is homeomorphic to a subset of the unit circle $\mathbb{T} = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 = 1\}$ in $\mathbb{R}^2$. Moreover, a compact space X can be homeomorphic to the unit circle if and only if n is even.*

It is clear that every normed linear space contains semi-Čebyšev subspaces. for example, if we consider a non-trivial linear subspace G which is dense in a normed linear space X, we have $\mathcal{P}_G(x) = \emptyset$, for every $x \in X \backslash G$, and therefore G is semi- Čebyšev. But for Banach spaces, the problem of existence of closed semi-Čebyšev subspaces though not trivial has got an affirmative answer, namely, every Banach space contains at least one semi-Čebyšev closed hyperplane [45]. In the case of Čebyšev subspace of Banach spaces, the situation is different. In fact, Garkavi [17] gives the example of a Banach space for which there are no Čebyšev subspaces. Here we quote equivalence conditions for the existence of Čebyšev subspaces of Banach spaces.

Theorem 4.1.9 ([44], Corollary 3.4)**.** *For a Banach space X, the following statements are equivalent.*

(i) All closed linear subspaces of X are Čebyšev subspaces.

(ii) All closed subspaces of X of a certain fixed finite co-dimension m where $1 \leq m \leq$

$dimX - 1$ *are Čebyšev subspaces.*

(iii) X *is reflexive and strictly convex.*

4.2 Čebyšev subspaces of C^*-algebras

The study of Čebyšev subspaces in the general operator algebra setting was initiated by A.G.Robertson [38] followed by Robertson and Yost [40] and then Pedersen [35].
In [38], Robertson gives a characterization of one dimensional Čebyšev subspaces of von Neumann algebras. The result is as follows:

Theorem 4.2.1 ([38], Theorem 1)**.** *Let* M *be a von Neumann algebra. Let* x *be an operator in* M*. Then the one-dimensional subspace* $\mathbb{C}x$ *spanned by* x *is a Čebyšev subspace of* M *if and only if* $\exists$ *a projection* p *in the centre of* M *such that* px *is left invertible in* pM *and* $(1-p)x$ *is right invertible in* $(1-p)M$*.*

The proof uses the existence of central projections in von Neumann algebras together with Hann-Banach and Krein-Milman theorems.
Another important result of Robertson is regarding the non existence of higher dimensional Čebyšev subspaces of infinite dimensional von Neumann algebras which are also $*$-subalgebras.

Theorem 4.2.2 ([38], Theorem 6)**.** *Let* M *be an infinite dimensional von Neumann algebra. Let* N *be a finite dimensional* **-subalgebra of* M *with dimension greater than one. Then* N *is not a Čebyšev subspace of* M*.*

For the proof, Robertson uses the rich structural properties of von Nueman algebras. Attempts to prove the analogue of Haar's theorem [19] led to quite a few interesting results in the noncommutative C^*-algebra setting. A result in that direction by Robertson and Yost is the following.

Theorem 4.2.3 ([40], Theorem 2.3)**.** *Let A be a norm-closed two sided ideal in a von Neuman algebra, $x \in A$. Then $\mathbb{C}x$ is a Čebyšev subspace in A, if and only if there is no irreducible representation π of A for which 0 is an eigenvalue of both $\pi(x)$ and $\pi(x^*)$. When this happens, $x^*x + xx^*$ is strictly positive.*

For the 'if' part, the existence of an extreme point of the unit ball of A^* satisfying certain properties, when $\mathbb{C}x$ is not a Čebyšev subspace of A is made use of. Assuming A to be acting on the Hilbert space H in its universal representation, one can write the above functional using a unit vector $\xi \in H$. A representation π of A is defined as the restriction of A to $A\xi$ which is irreducible. We can see that 0 is an eigenvalue for both $\pi(x)$ and $\pi(x^*)$.

The 'only if' part is proved using central projections and Kadison's irreducibility theorem. Robertson and Yost [40] also proved a remarkable result which says that every C^*-algebra in a certain large class contains an infinite dimensional Hilbert subspace with the property that each of its closed subspaces is a Čebyšev subspace.

Theorem 4.2.4 ([40], Theorem 2.8)**.** *Let M be a properly infinite von Neuman algebra, A a two-sided ideal in M. Suppose that A contains a strictly positive element (i.e, A has a one dimensional Čebyšev subspace). Then A contains an infinite dimensional Hilbert space V, which is Čebyšev in A. Moreover, each closed subspace of V is Čebyšev in A.*

So, A contains Čebyšev subspaces of all finite dimensions.

The existence of a sequence of orthogonal projections each equivalent to identity adding up to identity together with the strictly positive element enables one to define a orthonormal basis, the span of which is the Hilbert space. Proximinality of reflexive subspaces together with best approximation property assured by compactness with respect to ultra weak topology implies that all closed subspaces of the Hilbert space so obtained are Čebyšev in A.
It is to be noted that the above class of C^*-algebras includes $B(H)$ which means that it has got Čebyšev subspaces of all finite dimensions.

Remark 4.2.1. *The works of Robertson and Yost established that there exists no Čebyšev subspace of finite dimension greater than one if the space under consideration is any one of the following.*

(i) An infinite dimensional abelian von Neumann algebra.

(ii) An abelian non-separable C^*-algebra.

Theorem 4.2.4 tells us how different the situation is, in the noncommutative setting.

The following theorems [40] and the corollary establishes the dearth of Čebyšev subspaces of C^*-algebras which are $*$-subalgebras.

Theorem 4.2.5 ([40], Theorem 1.3)**.** *Let A be a C^*-algebra, B, a C^*-subalgebra. Suppose that one of A, B is unital, and that B is a Čebyšev subspace of A. Then A is unital* *and* $1 \in B$. *If* $B \neq \mathbb{C}1$, *then every maximal abelian* $*$*-subalgebra of* B *is maximal abelian in* A .

Corollary 4.2.1 ([40], Corollary 1.4)**.** *(1) Let* A *be an infinite dimensional* C^**-algebra,* B *a finite dimensional* $*$ *-subalgebra. If* B *is Čebyšev in* A*, then* A *is unital and* $B = \mathbb{C}1$.

(2) Let A *be a commutative* C^**-algebra,* B*, a finite dimensional subalgebra of* A*. If* B *is Čebyšev in* A*, then* A *is unital and* $B = \mathbb{C}1$.

Theorem 4.2.6 ([40], Theorem 1.5)**.** *Let* M *be a von Neuman algebra,* A *a proper* C^**-subalgebra of* M *with* $A \neq \mathbb{C}1$. *Suppose that* M *is not a factor of type II or III. If* A *is Čebyšev in* M*, then* M *is* $M_2(\mathbb{C})$*, with* A*, the algebra of diagonal matrices.*

Remark 4.2.2. *The above result establishes the fact that the only exception of a von Neuman algebra* A *having non-trivial Čebyšev subalgebra* $B(B \neq A, B \neq \mathbb{C}1)$ *is* $A = M_2(\mathbb{C})$ *for which the algebra of diagonal matrices is a Čebyšev subalgebra.*

Now we turn to the results of G.K. Pederen [35] who studied the finite dimensional Čebyšev subspaces of C^*-algebras quite extensively. Pedersen, in his attempt to extend the Haar's theorem to the noncommutative case, succeeds partially by giving a characterization of one dimensional and two dimensional Čebyšev subspaces of a C^*-algebras. Another result of him further extents the work initiated by Robertson and Yost to the case of C^*-algebras.

Theorem 4.2.7 ([35], Theorem 1)**.** *Let V be an n-dimensional subspace of a C^*-algebra $\mathcal{A}$ and assume that there is a unitary u in $M(\mathcal{A})$ and a non-zero element x_0 in V such that $\phi_i(x_0^*x_0) = \phi_i(ux_0x_0^*u^*) = 0$ for at least n orthogonal pure states $\phi_1, \phi_2, ..., \phi_n$ of $\mathcal{A}$. Then V is not a Čebyšev subspace of $\mathcal{A}$.*

Theorem 4.2.8 ([35], Theorem 2)**.** *Let V be an n-dimensional subspace of a C^*-algebra $\mathcal{A}$. The following conditions are equivalent.*

(i) V is not a Čebyšev subspace.

(ii) There is a unitary operator u in $\tilde{\mathcal{A}}$, a non-zero element x_0 in V and an atomic space ϕ, which is a convex combination of m orthogonal pure states, such that $\phi(x_0^*x_0) = \phi(ux_0x_0^*u^*) = 0$.
If $m < n$, we further have $\phi(uV) = 0$.

In the following two theorems Pedersen characterizes the one-dimensional and two-dimensional Čebyšev subspaces of C^*-algebras in terms of irreducible representations, their eigen values and eigen vectors. These results can also be seen as the generalization of Haar's theorem to the first two dimensions. Pedersen remarks in the context of the theorem above that it seems to be the best one can do in generalizing Haar's theorem (Theorem 4.1.7). Let $\mathcal{A}$ be a C^*-algebra with unit 1 and let $x_0 \in \mathcal{A}$ is not a multiple of 1. In this setting Pedersen [35] obtained the following results.

Theorem 4.2.9 ([35], Theorem 3)**.** *Let x_0 be a non-zero element in a C^*-algebra $\mathcal{A}$. The following conditions are equivalent.*

(i) $\mathbb{C}x_0$ *is a Čebyšev subspace of* $\mathcal{A}$.

(ii) $x_0^*x_0 + ux_0x_0^*$ *is stricly positive in* $\mathcal{A}$.

(iii) *In no irreducible representation* $(\pi, \mathcal{H})$ *of* $\mathcal{A}$ *do the operators* $\pi(x_0)$ *and* $\pi(x_0^*)$ *both have zero as an eigen value.*

Proposition 4.2.1 ([35], Proposition 1)**.** *Let* $x_0 \in \mathcal{A}$ *be as above. Then the following conditions are equivalent.*

(i) *The 2-dimensional subspace* $G = span\,(1, x_0)$ *is a Čebyšev subspace of* $\mathcal{A}$.

(ii) *For a given* $\lambda \in \mathbb{C}$*, there exists at most one irreducible representation* $(\pi, \mathcal{H})$ *of* $\mathcal{A}$ *(up-to equivalence) in which* x_0 *and* x_0^* *have the eigenvalues* λ *and* $\overline{\lambda}$ *respectively. Moreover, none of the multiplicities of* λ *and* $\overline{\lambda}$ *in* $\mathcal{H}$ *exceed 1 and the corresponding eigenvectors are not orthogonal.*

The following theorem establishes that there exists no non-trivial Čebyšev C^*-subalgebra of a non-unital C^*-algebra.

Theorem 4.2.10 ([35], Theorem 4)**.** *If* A *is a* C^**-algebra without unit and* B*, a Čebyšev* C^**-subalgebra of* A*, then* $B = A$.

Theorem 4.2.11 ([35], Theorem 5)**.** *If* A *is a* C^**-algebra with unit,* B*, a Čebyšev* C^**-subalgebra of* A*, then either* $B = A, B = \mathbb{C}1$*, or else* $A = M_2$ *and* B *is isomorphic to the algebra of diagonal matrices.*

Legg, Scranton and Waed [26] obtained some important results characterizing the semi-Čebyšev and Čebyšev subspaces of $K(\mathcal{H})$, the space of all compact operators on some Hilbert space $\mathcal{H}$. We quote a few of them:

Theorem 4.2.12 ([26], Theorem 3)**.** *Let $\mathcal{H}$ be a separable Hilbert space.Then $K(\mathcal{H})$ has N-dimensional Čebyšev subspace for each positive integer N.*

Theorem 4.2.13 ([26], Theorem 5)**.** *An N-dimensional subspace $\mathcal{V} \in K(\mathcal{H})$ is Čebyšev if and only if there does not exist a non-zero $C \in \mathcal{V}$, $C_j \in \mathcal{V}, j = 1, 2, ..., N-1$ and two sets A and B each consisting of m orthonormal elements so that*

(1) span $(C, C_1, ..., C_{N-1}) = \mathcal{V}$,

(2) $0 \neq A = \{v_1, v_2, ..., v_m\} \in$ ker C. $B = \{u_1, u_2, ..., u_m\} \in$ ker C^* and

(3) the $(N-1) \times m$ matrix $M = (\langle C_i v_j, v_j \rangle)_{i=1,2,...,N-1, j=1,2,...,m}$ has linearly independent columns.

As a consequence of the above theorem we get the following corollary:

Corollary 4.2.2 ([26], Corollary 3)**.** *If $\mathcal{H}$ is not separable, $K(\mathcal{H})$ has got no finite dimensional Čebyšev subspace.*

If H is separable, $K(\mathcal{H})$ belongs to the class mentioned in the theorem 4.2.4. In particular, $K(\mathcal{H})$ has an infinite dimensional Čebyšev subspace. This differs from the commutative theory, for c_0 has no infinite dimensional Čebyšev subspace [45].

4.3 Čebyšev subspaces and boundary representations

Further to the work by Pedersen in 1977 [35] in trying to extent the classical Haar condition to the noncommutative case, though with limited success (for dimensions one and two) nothing has been done in the last thirty to forty years. Here we extend the result of Pedersen to all finite dimensions. This work also establishes a still much to be explored relationship with Arveson' s notion of boundary representation.

We introduce the ***noncommutative Haar condition*** as follows.

Definition 4.3.1. *Let $\mathcal{A}$ be a C^*-algebra with unit $1_{\mathcal{A}}$. For $x_1, x_2, ..., x_{n-1} \in \mathcal{A}$, let $\mathcal{V} = \mathbb{C}1_{\mathcal{A}} + \mathbb{C}x_1 + ...\mathbb{C}x_{n-1}$ be n dimensional. Then $\{1_{\mathcal{A}}, x_1, ...x_{n-1}\}$ is said to satisfy the noncommutative Haar condition if the following conditions are satisfied:*

For a given $\lambda \in \mathbb{C}$,

(a) there are at most $n-1$ irreducible representations $(\pi_i, \mathcal{H}_i)$ (up to equivalence) and a non-zero vector $z_0 \in span\,(x_1, x_2..., x_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively ($i = 1, 2, ..., n-1$).

(b) Assume that there are $m \leq n-1$ irreducible representations $(\pi_i, \mathcal{H}_i)$ (up to equivalence) and a non-zero vector $z_0 \in span\,(x_1, x_2..., x_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively ($i = 1, 2, ..., n-1$). If n_i (respectively $\overline{n}_i$) $i = 1, 2, ..., m$ are the multiplicities of λ (respectively $\overline{\lambda}$) in $\mathcal{H}_i$, then $\sum_{i=1}^{m} n_i \leq n-1 \left(respectively \sum_{i=1}^{m} \overline{n_i} \leq n-1 \right)$. Moreover, at least one eigenvector

of $\tilde{\pi}_i(z_0^*)$ *of the form* $\tilde{\pi}_i(u)$ *for some unitary* $u \in \mathcal{A}$ *which is not in* $\mathcal{V}$ *corresponding to* $\overline{\lambda}$ *is not orthogonal to* $\tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}}_i$ *where* $\left(\tilde{\pi}_i, \tilde{\mathcal{H}}_i\right)$ *is the G.N.S representation corresponding to* $(\mathcal{A}, \phi_i)$*,* ϕ_i *is the pure state defined by* $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i\rangle\, ; i = 1, 2, ..., m, a \in \mathcal{A}$ *and* ξ_i *is an eigenvector of* $\pi_i(z_0)$ *corresponding to* λ*.*

We give the following result which shows that the noncommutative Haar condition is equivalent to the Haar condition in the classical case.

Proposition 4.3.1. *Let* $\mathcal{A} = C(X)$ *be a* C^**-algebra of all complex valued continuous functions on a compact Hausdorff space* X*. Let* $\mathcal{B} = \{1_{\mathcal{A}}, f_1, ..., f_{n-1}\} \subset \mathcal{A}$ *be a linearly independent set and let* $\mathcal{V} = span\mathcal{B}$*. Then* $\mathcal{B}$ *satisfies the noncommutative Haar condition if and only if it satisfies the classical Haar condition.*

Proof. Assume that $\mathcal{B}$ satisfies the *non-commutative Haar condition*. To show that $\mathcal{B}$ satisfies the classical *Haar condition.* Let $\widehat{f} = \lambda_1 1_{\mathcal{A}} + \lambda_2 f_1 + ... + \lambda_n f_{n-1}$ be a non-zero element in $\mathcal{V}$ with n distinct zeros $u_1, u_2, ..., u_n$ in X. Let φ_k denote the evaluation functional defined by $\varphi_k(f) = f(u_k), (k = 1, 2, ..., n)$ where $f \in C(X)$. Let$(\pi_k, \mathcal{H}_k)$ be the corresponding GNS representation of $C(X)$ by φ_k which is irreducible and hence one dimensional. Put $z_0 = \widehat{f} - \lambda_1 1_{\mathcal{A}}$. But $\pi_k(\widehat{f}) = 0$ for each k which means that $\pi_k(z_0) = -\lambda_1 I$. Thus $-\lambda_1$ and its conjugate are eigenvalues of $\pi_k(z_0)$ and $\pi_k(z_0^*)$ respectively for n non-equivalent irreducible representations π_k; $k = 1, 2, ..., n$. This is a contradiction. This shows that *non-commutative Haar condition* implies classical *Haar condition.*

Now assume that $\mathcal{B}$ satisfies the classical *Haar condition.* To show that $\mathcal{B}$ satisfies the *non-commutative Haar condition*.

Let $\widehat{f}$ be a non-zero element in $\mathbb{C}f_1 + ... + \mathbb{C}f_{n-1}$. Suppose that for the given $\lambda \in \mathbb{C}$, $\exists\ n$ evaluation functionals $\varphi_{x_k}, (k = 1, 2, ..., n)$ where $x_k \in X$ such that $\varphi_{x_k}(\widehat{f}) := \widehat{f}(x_k) = \lambda, (k = 1, 2, ..., n)$. Put $\widehat{g} = \widehat{f} - \lambda 1_{\mathcal{A}}$. Then $\widehat{g} \in \mathcal{V}$ such that $\widehat{g}(x_k) = 0$ for $(k = 1, 2, ..., n)$ which is a contradiction. Condition (b) follows trivially since all the irreducible representations of $C(X)$ are one dimensional. This proves the theorem. □

Now we state a general version of Proposition 4.2.1 for finite dimensional Čebyšev subspaces of $\mathcal{C}^*$- algebras.

Theorem 4.3.1. *Let $\mathcal{A}$ be a unital C^*-algebra. Consider a linearly independent set $\mathcal{B} = \{1_{\mathcal{A}}, x_1, x_2, ..., x_{n-1}\} \subseteq \mathcal{A}$ and define $\mathcal{V} = span\mathcal{B}$. Then the following are equivalent:*

(i) The subspace $\mathcal{V}$ is an n dimensional Čebyšev subspace of $\mathcal{A}$.

(ii) $\mathcal{B}$ satisfies the noncommutative Haar condition.

Proof. **(i)** ⇒ **(ii):** Suppose that for the given $\lambda \in \mathbb{C}$, $\exists$ irreducible representations $(\pi_i, \mathcal{H}_i)\,, i = 1, 2, ..., n$ such that $0 = \pi_i\left((z_0 - \lambda)\right)\xi_i = \pi_i\left((z_0^* - \overline{\lambda})\right)\eta_i$ for some unit vectors ξ_i, η_i in $\mathcal{H}_i$ and for some non-zero vector $z_0 \in span\,(x_1, x_2 ..., x_{n-1})$. By Kadison's transitivity theorem ([18], Corollary 7), $\exists$ unitary $u \in \mathcal{A}$ such that $\pi_i(u)\eta_i = \xi_i, i = 1, ..., n$. Define pure states ϕ_i, by $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i\rangle$; where $a \in \mathcal{A}, i = 1, 2, ..., n$.

Then, $\phi_i((z_0 - \lambda)^*(z_0 - \lambda) + u(z_0 - \lambda)(z_0 - \lambda)^* u^*) = 0$.

Therefore,

$$\phi_i((z_0-\lambda)^*(z_0-\lambda)) = 0 = \phi_i(u(z_0-\lambda)(z_0-\lambda)^*u^*). \tag{4.1}$$

Hence by Theorem 4.2.7, $\mathcal{V}$ is not Čebyšev.

Now assume that for a given $\lambda \in \mathbb{C}$, there exist m irreducible representations $(\pi_i, \mathcal{H}_i)$, $(i = 1, 2, ..., m)$ where $m \in \{1, 2, ..n-1\}$ (up to equivalence) and a vector $z_0 \in span\,(x_1, x_2, ..., x_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively, but $\sum_{i=1}^{m} n_i \geq n \left(or \sum_{i=1}^{m} \overline{n}_i \geq n \right)$ or all the eigenvectors corresponding to $\overline{\lambda}$ are orthogonal to $\tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}}_i$. Consider the case $\sum_{i=1}^{m} n_i \geq n$. This implies that there exist at least n orthogonal pure states $\phi_i, i = 1, 2, ..., n$, satisfying equations 2.1 above. Again this implies, by Theorem 4.2.7, that $\mathcal{V}$ is not Čebyšev. In the case where $\sum_{i=1}^{m} \overline{n}_i \geq n$, following similar steps, we arrive at the same conclusion.

Now consider the case where all the eigenvectors of $\tilde{\pi}_i(z_0^*)$ of the form $\tilde{\pi}_i(u)$ for some unitary $u \in \mathcal{A}$ which is not in $\mathcal{V}$ corresponding to $\overline{\lambda}$ are orthogonal to $\tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}}_i$. Assume that there exist m distinct irreducible representations $(\pi_i, \mathcal{H}_i)$ $(i = 1, 2, ..., m)$ where $m \in \{1, 2, ..n-1\}$ such that $\pi_i(z_0 - \lambda)\xi_i = 0 = \pi_i(z_0^* - \overline{\lambda})\eta_i$.

By Kadison's transitivity theorem, $\exists$ unitary $u \in \mathcal{A}$ such that $\pi_i(u)\eta_i = \xi_i$.

Let $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i\rangle$; where $a \in \mathcal{A}$. Then we have

$$\phi_i((z_0-\lambda)^*(z_0-\lambda)) = \langle \pi_i((z_0-\lambda)^*(z_0-\lambda))\xi_i, \xi_i\rangle = \|\pi_i(z_0-\lambda)\xi_i\|^2 = 0.$$

Similarly,

$$\phi_i(u(z_0-\lambda)(z_0-\lambda)^*u^*) = \langle \pi_i(u(z_0-\lambda)(z_0-\lambda)^*u^*)\xi_i, \xi_i\rangle = \|\pi_i(z_0^*-\overline{\lambda})\eta_i\|^2 = 0.$$

If $\left(\tilde{\pi}_i, \tilde{\mathcal{H}_i}\right)$ is the G.N.S corresponding to $(\mathcal{A}, \phi_i)$, we get $\tilde{\pi}_i(z_0^*-\overline{\lambda})(\tilde{u}^*)$ = 0. Hence by the assumption $\tilde{u}^*$ is orthogonal to $\tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}_i}$, we have $\phi_i(u\mathcal{V}) = \left\langle \tilde{\pi}_i(\mathcal{V})\tilde{\mathcal{H}_i}, \tilde{u}^*\right\rangle = 0$. Since pure states are atomic, ([41], page 237), each $\phi_i, i = 1, 2, .., m$ is an atomic state. Then by Theorem 4.2.8, $\mathcal{V}$ is not Čebyšev.

(ii) ⇒ (i): Assume that $\mathcal{V}$ is not Čebyšev. Then by Theorem 4.2.8, $\exists$ an atomic state $\phi = \sum_{i=1}^{m}\beta_i\phi_i, (m \leq n)$ where ϕ_i's are orthogonal pure states such that $\sum_{i=1}^{m}\beta_i = 1$, a complex number λ and a unitary u in $\mathcal{A}$ so that

$$\phi((z_0-\lambda)^*(z_0-\lambda)) = 0 = \phi(u(z_0-\lambda)(z_0-\lambda)^*u^*); \tag{4.2}$$

for some non-zero vector $z_0 \in span\,(x_1, x_2..., x_{n-1})$. In the case where $m < n$, we further have

$$\phi(u\mathcal{V}) = 0. \tag{4.3}$$

Hence, when $m < n$, equations 4.2 and 4.3 together will imply that $\tilde{1}_{\mathcal{A}}$ and $\tilde{u}^*$ are orthogonal eigenvectors of $\tilde{\pi}_i(z_0)$ and $\tilde{\pi}_i(z_0^*)$ corresponding to eigenvalues λ and $\overline{\lambda}$ respectively. Here, orthogonality violates condition (ii).

In the second case ($m = n$), there exist n orthogonal pure states $\phi_i, i = 1, 2, .., n$ satisfying equation 4.2.

Case(a): At least two of the ϕ_i's are equivalent, say ϕ_1 and ϕ_2. Then the corresponding G.N.S representations π_1 and π_2 are equivalent.

We have for unit vectors $\xi_i \in \mathcal{H}_i$, $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle, a \in \mathcal{A}, i = 1, 2.$

Let $\pi_1(a) = w_1^*\pi_2(a)w_1$ where $w_1 : \mathcal{H}_1 \to \mathcal{H}_2$ is unitary.

Therefore,

$$\begin{aligned}
\phi_2(a) &= \langle \pi_2(a)\xi_2, \xi_2 \rangle = \langle w_1\pi_1(a)w_1^*\xi_2, \xi_2 \rangle \\
&= \langle \pi_1(a)w_1^*\xi_2, w_1^*\xi_2 \rangle = \left\langle \pi_1(a)\tilde{\xi_1}, w_1^*\tilde{\xi_1} \right\rangle.
\end{aligned}$$

where $\tilde{\xi_1} = w_1^*\xi_2$.

Claim : ξ_1 and $\widetilde{\xi_1}$ are independent.

If possible, let $\widetilde{\xi_1} = \beta\xi_1$ where β is a scalar such that $|\beta| = 1$. This implies $\phi_2(a) = \langle \pi_1(a)\beta\xi_1, \beta\xi_1 \rangle = \langle \pi_1(a)\xi_1, \xi_1 \rangle = \phi_1(a)$ which is not possible. Now it will follow that λ and $\overline{\lambda}$ are eigenvalues of $\pi_1(z_0)$ and $\pi_1(z_0^*)$ with multiplicity 2. Consequently the condition on the multiplicities of the eigenvalues in (ii) will be violated.

Case(b): None of the ϕ_i's are equivalent. In this case it is easy to see that we get n

inequivalent irreducible representations which will again violate condition (ii). □

The following theorem establishes that the representation mentioned in Proposition 4.2.1 is indeed a boundary representation for the subspace, provided it generates the C^*-algebra.

Theorem 4.3.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $x_0 \in \mathcal{A}$ such that $G = span\,(1_{\mathcal{A}}, x_0)$ is a two-dimensional Čebyšev subspace of $\mathcal{A}$, with $\mathcal{A} = C^*(G)$. Given $\lambda \in \mathbb{C}$, let π_0 be an irreducible representation of $\mathcal{A}$ on $\mathcal{H}_{\pi_0}$ such that $\pi_{0|G}$ is pure with $\pi_0(x_0)(u_0) = \lambda u_0$ and $\pi_0(x_0^*)(v_0) = \overline{\lambda} v_0$ for some unit vectors $u_0, v_0 \in \mathcal{H}_{\pi_0}$. Also assume that every pure $\Psi \in \left\{\Phi \in CP\,(\mathcal{A}, \mathcal{H}_{\pi_0}) : \Phi_{|G} = \pi_{0|G}\right\}$ satisfies the condition $\|\Psi(u)(\xi_0)\| = \|\xi_0\|$ for every unitary $u \in \mathcal{A}$ and some $\xi_0 \in \mathcal{H}_{\pi_0}$. Then π_0 is a boundary representation for G.*

Proof. Let $K = \left\{\Phi \in CP\,(\mathcal{A}, \mathcal{H}_{\pi_0}) : \Phi_{|G} = \pi_{0|G}\right\}$. Then K is a compact convex set with respect to the BW-topology.

By Krein-Milman theorem, there exists an extreme element Φ_0 of K. Since Φ_0 is linearly extreme and $\Phi_{0|G}$ is pure, Φ_0 is pure ([23], Proposition 2.2 and Crollory 2.3). Let $(V, \mathcal{H}_{\pi_0'}, \pi_0')$ be the minimal Stinespring triple corresponding to Φ_0 where π_0' is an irreducible representation. Then $\Phi_0(.) = V^*\pi_0'(.)V$.

We now claim that V is unitary. Since Φ_0 is unital, $\Phi_0(1_{\mathcal{A}}) = V^*\pi_0'(1_{\mathcal{A}})V = V^*V = I$, so V is isometric and it suffices to show that $[V\mathcal{H}_{\pi_0}] = \mathcal{H}_{\pi_0'}$. But $[V\mathcal{H}_{\pi_0}]$ is cyclic for $\pi_0'(\mathcal{A})$. We prove that the self-adjoint family of operators $\pi_0'(\mathcal{A})$ leaves $[V\mathcal{H}_{\pi_0}]$ invariant.

Choose a unitary element u in $\mathcal{A}$ and some $\xi_0 \in \mathcal{H}_{\pi_0}$. Then we have

$$\|\pi_0'(u)V\xi_0 - V\Phi_0(u)\xi_0\|^2 = \|\pi_0'(u)V\xi_0\|^2 - 2Re\,\langle V^*\pi_0'(u)V\xi_0, \Phi_0(u)\xi_0\rangle + \|V\Phi_0(u)\xi_0\|^2$$

$$= \langle \pi_0'(u)V\xi_0, \pi_0'(u)V\xi_0\rangle - 2Re\,\langle \Phi_0(u)\xi_0, \Phi_0(u)\xi_0\rangle + \|\Phi_0(u)\xi_0\|^2$$

$$= \|\xi_0\|^2 - \|\Phi_0(u)\xi_0\|^2 = \|\xi_0\|^2 - \|\xi_0\|^2 = 0.$$

Thus, $\pi_0'(u)V\xi_0 = V\Phi_0(u)\xi_0 \in [V\mathcal{H}_{\pi_0}]$. But π_0' being irreducible, $V(\xi_0)$ is cyclic for it and this implies that $\pi_0'(u)$ leaves $[V\mathcal{H}_{\pi_0}]$ invariant. Since $\mathcal{A}$ is the norm closed span of its unitary elements, $\pi_0'(A)$ leaves $[V\mathcal{H}_{\pi_0}]$ invariant. Therefore V is unitary. Since $x_0 \in G$, $\Phi_0(x_0) = \pi_0(x_0)$ and therefore $\Phi_0(x_0)(u_0) = \pi_0(x_0)(u_0) = \lambda u_0$. Thus,

$$\begin{aligned} V^*\pi_0'(x_0)V(u_0) &= \lambda u_0 \quad \text{and} \\ V^*\pi_0'(x_0^*)V(v_0) &= \overline{\lambda} v_0. \end{aligned}$$

Now let $u \in \mathcal{H}_{\pi_0'}$. Then we have,

$$\begin{aligned} \langle V^*\pi_0'(x_0)V(u_0), u\rangle &= \lambda\,\langle u_0, u\rangle \\ &= \lambda\,\langle V^*Vu_0, u\rangle \\ &= \langle \lambda V u_0, Vu\rangle\,. \end{aligned}$$

Therefore,

$$\langle \pi_0'(x_0)Vu_0, Vu\rangle = \langle \lambda Vu_0, Vu\rangle$$

which implies that $\pi_0'(x_0)Vu_0 = \lambda Vu_0$, since $u \in \mathcal{H}_{\pi_0}$ is arbitrary and $V\mathcal{H}_{\pi_0} = \mathcal{H}_{\pi_0'}$. Similarly, $\pi_0'(x_0^*)Vv_0 = \overline{\lambda}Vv_0$. Thus λ and $\overline{\lambda}$ are eigenvalues of $\pi_0'(x_0)$ and $\pi_0'(x_0^*)$ respectively. Then by proposition 4.2.1, $\pi_0 \sim \pi_0'$. Therefore, $\pi_0' = U^*\pi_0 U$ for some unitary $U : H_{\pi_0'} \mapsto H_{\pi_0}$. Hence $\Phi_0 = V^*\pi_0'V = V^*U^*\pi_0 UV = V_1^*\pi_0 V_1$ where $V_1 = UV$. Thus, $\Phi_0(g) = V_1^*\pi_0(g)V_1$ for every $g \in G$. Since $\Phi_{0|G} = \pi_{0|G}$, we have $\pi_0(g) = V_1^*\pi_0(g)V_1$ for every $g \in G$.

$$\text{i.e.,} \quad T = V_1^* T V_1 \ \text{ for every } \ T \in \pi_0(G).$$

Now, note that since U and V are unitaries, V_1 is a unitary.

For $T_1, T_2 \in \pi_0(G)$; $T_i = V_1^* T_i V_1; i = 1, 2$. Hence $T_1 T_2 = V_1^* T_1 V_1 V_1^* T_2 V_1 = V_1^* T_1 T_2 V_1$.

Also, clearly $T^* = V_1^* T^* V_1$ for all $T \in \pi_0(G)$.

Thus, $T = V_1^* T V_1$ for all $T \in C^*(\pi_0(G)) = \pi_0(C^*(G)) = \pi_0(\mathcal{A})$.

Hence $\pi_0(a) = V^*\pi_0'(a)V$ for all $a \in \mathcal{A}$ and hence $\pi_0(a) = \Phi_0(a)$ for all $a \in \mathcal{A}$. Thus $\pi_0 = \Phi_0$, which proves that π_0 is a boundary representation for G. □

Remark 4.3.1. *Theorem 4.3.1 can be applied wherever the irreducible representations of the C^*-algebra under consideration are completely known. One such special case of interest is $C(X) \otimes M_N$ where X is a compact Hausdorff space and M_N is the set of all $N \times N$ matrices over $\mathbb{C}$ which is nothing but the C^*-algebra of all M_N-valued continuous* *functions on X. Let $\mathcal{A}$ be the C^*-algebra $C(X) \otimes M_N$ with identity $1_X \otimes I_N$ where 1_X and I_N be the constant function 1 on X and the $N \times N$ identity matrix respectively. Let $G = span(f_0 \otimes a_0, f_1 \otimes a_1, ..., f_{n-1} \otimes a_{n-1})$ be an n dimensional subspace of $\mathcal{A}$ where $f_k \in C(X)$ and $a_k \in M_N$ for $k = 1, 2, ...n-1$, and $f_0 \otimes a_0 = 1_X \otimes I_N$. By Theorem 4.3.1, G is a Čebyšev subspace of $\mathcal{A}$ if and only if the spanning set satisfies the noncommutative Haar condition. In the following results the conditions (a) and (b) in the noncommutative Haar condition are made more explicit in comparison with the classical case by proving equivalent conditions for (a) and (b) for $C(X) \otimes M_N$.*

Proposition 4.3.2. *Let $\omega_0 \in span\{f_j \otimes a_j; j = 0, 1, ..., n-1\}$. Then there exist at most $n-1$ irreducible representations $\pi_k, k = 1, 2, ..., n-1$ such that 0 is an eigenvalue of $\pi_k(\omega_0)$ and $\pi_k(\omega_0^*)$ if and only if given n distinct points $x_1, x_2, ...x_n$ in X and non-zero vectors $\xi_1, \xi_2 ... \xi_n$ in $\mathbb{C}^N$, the $Nn \times n$ matrix $(f_j(x_k)a_j(\xi_k))$ is of rank n.*

Proof. Assume that the condition (a) of the non-commutative Haar condition holds. If possible, let there be n distinct points $x_1, x_2, ...x_n$ in X and non-zero vectors $\xi_1, \xi_2, ...\xi_n$ in $\mathbb{C}^N$ such that $(f_j(x_k)a_j(\xi_k))$ is of rank less than or equal to n-1. Then there will exist n scalars $\lambda_0, \lambda_1, ..., \lambda_{n-1}$ not all zero such that

$$\begin{pmatrix} f_0(x_1)a_0(\xi_1) & f_1(x_1)a_1(\xi_1) & ...f_{n-1}(x_1)a_{n-1}(\xi_1) \\ f_0(x_2)a_0(\xi_2) & f_1(x_2)a_1(\xi_2) & ...f_{n-1}(x_2)a_{n-1}(\xi_2) \\ . & & \\ . & & \\ . & & \\ f_0(x_n)a_0(\xi_n) & f_1(x_n)a_1(\xi_n) & ...f_{n-1}(x_n)a_{n-1}(\xi_n) \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ . \\ . \\ . \\ \lambda_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix}$$

$\Longleftrightarrow \sum_{j=1}^{n-1} \lambda_j f_j(x_k)a_j(\xi_k) + \lambda_0 f_0(x_k)a_0(\xi_k) = 0, k = 1, 2, ..., n. \Longleftrightarrow \pi_k(\omega_0)(\xi_k) = 0, k = 1, 2, ...n \Longleftrightarrow$ 0 is an eigenvalue of $\pi_k(\omega_0)$ and $\pi_k(\omega_0^*)$ for $k = 1, 2, ...n$ which violates condition (a). Converse follows similarly. □

Proposition 4.3.3. *A vector $\omega_0 = \sum_{j=0}^{n-1} \beta_j(f_j \otimes a_j)$ in G satisfies condition (b) of the non-commutative Haar condition if and only if there exist at most m cyclic vectors $\xi_1, \xi_2, ..., \xi_m$ ($m \leq n-1$) in $\mathbb{C}^N$, distinct points $x_1, x_2, ...x_m$ in X for the identity representation on C^N and unitary matrices $u_1, u_2, ...u_m$ in M_N such that*

$$AB = \bar{\lambda}B \tag{4.4}$$

where

$$A = \begin{pmatrix} \bar{\beta}_1\bar{f}_1(x_1)a_1^* & \bar{\beta}_2\bar{f}_2(x_1)a_2^* & ...\bar{\beta}_{n-1}\bar{f}_{n-1}(x_1)a_{n-1}^* \\ \bar{\beta}_1\bar{f}_1(x_2)a_1^* & \bar{\beta}_2\bar{f}_2(x_2)a_2^* & ...\bar{\beta}_{n-1}\bar{f}_{n-1}(x_2)a_{n-1}^* \\ . & & \\ . & & \\ . & & \\ \bar{\beta}_1\bar{f}_1(x_m)a_1^* & \bar{\beta}_2\bar{f}_2(x_m)a_2^* & ...\bar{\beta}_{n-1}\bar{f}_{n-1}(x_m)a_{n-1}^* \\ . & & \end{pmatrix},$$

$B = diagonal(u_1(\xi_1), ..., u_m(\xi_m))$ *and the diagonal matrix on the right side of (4.4) with non-zero diagonal entries is non-singular. Also the multiplicities* n_i *(respectively* $\overline{n}_i$*) of* λ_0 *(respectively* $\bar{\lambda}_0$ *) satisfy the inequality*

$$\sum_{i=1}^{m} n_i \leq n-1 \left(\textit{respectively} \sum_{i=1}^{m} \overline{n}_i \leq n-1 \right).$$

Proof. Assume that the condition (b) of the non-commutative Haar condition holds. Hence,if there are $m \leq n-1$ irreducible representations $(\pi_i, \mathcal{H}_i, i = 1, 2, ..., m)$ (up to equivalence) and a non-zero vector $z_0 \in span\,(f_1 \otimes a_1, f_2 \otimes a_2 ..., f_{n-1} \otimes a_{n-1})$ such that λ and $\overline{\lambda}$ are eigenvalues of $\pi_i(z_0)$ and $\pi_i(z_0^*)$ respectively ($i = 1, 2, ..., n-1$). Let $z_0 = \sum_{j=1}^{n-1} \beta_j(f_j \otimes a_j)$. If n_i (respectively $\overline{n}_i$) $i = 1, 2, ..., m$ are the multiplicities of λ (respectively $\overline{\lambda}$) in $\mathcal{H}_i$, then $\sum_{i=1}^{m} n_i \leq n-1 \left(respectively \sum_{i=1}^{m} \overline{n}_i \leq n-1 \right)$. Moreover, at least one eigenvector of $\tilde{\pi}_i(z_0^*)$ of the form $\tilde{\pi}_i(u_i)$ for some unitary $u_i \in \mathcal{A}$ which is not in $\mathcal{V}$ corresponding to $\overline{\lambda}$ is not orthogonal to $\tilde{\pi}_i(\mathcal{V})$ where $\left(\tilde{\pi}_i, \tilde{\mathcal{H}}_i\right)$ is the G.N.S representation corresponding to $(\mathcal{A}, \phi_i)$, ϕ_i is the pure state defined by $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i \rangle\,; i = 1, 2, ..., m, a \in \mathcal{A}$, ξ_i is a cyclic vector of $\pi_i(z_0)$ corresponding to λ.

Let $\omega_0 = z_0 - \overline{\lambda} e_{\mathcal{A}}$ which is in $\mathcal{V}$. Now, put $\mathcal{J}_i = \{a \in \mathcal{A} : \phi_i(a^*a) = 0\} = \{a \in \mathcal{A} : \pi_i(a)\xi_i = 0\}$. Then $\widetilde{\mathcal{H}_i} = \overline{(\mathcal{A}/\mathcal{J}_i)}$ and $\tilde{\pi}_i(a)(\tilde{h_i}) = \tilde{\pi}_i(a)(h + \mathcal{J}_i) = ah + \mathcal{J}_i$, where $\tilde{h_i} = h + \mathcal{J}_i \in \mathcal{A}/\mathcal{J}_i$.

Now, $\tilde{\pi}_i(z_0^*)\tilde{\pi}_i(u_i) = \overline{\lambda}\tilde{\pi}_i(u_i)$ in $\widetilde{\mathcal{H}_i}$, if and only if

$$\sum_{j=1}^{m-1} \bar{\beta}_j \pi_i(\bar{f}_j \otimes a_j^*)(\pi_i)(u_i)(\xi_i) = \bar{\lambda}\pi_i(u_i)\xi_i, i = 1, 2, ..., m. \tag{4.5}$$

As discussed in Remark 4.3.1, we can write $\pi_i(\bar{f}_j \otimes a_j^*) = \pi_i^{(1)}(\bar{f}_j)\pi_i^{(2)}(a_j^*)$ where $\pi_i^{(1)}$ and $\pi_i^{(2)}$ are irreducible representations. But the irreducible representations acting on $C(X)$ are the evaluation functionals. Also the only irreducible representation on $M_N(\mathbb{C})$ is the identity matrix. Therefore $\pi_i^{(1)}(\bar{f}_j) = \bar{f}_j(x_i)$ for some $x_i \in X$. Again, $\pi_i^{(2)}(a_j^*) = a_j^*$. This implies that there exists m distinct points $x_1, x_2, ..., x_m$ in X satisfying the matrix equation in the proposition. □

Remark 4.3.2. *The above two propositions bring clarity to the obscure nature in the definition of noncommutative Haar condition. Note that condition (b) comes from mainly non-commutativity while condition (a) is shared by commutative case. When $N = 1$, the $Nn \times n$ matrix $(f_j(x_k)a_j(\xi_k))$ of rank n becomes a non-singular matrix of order n. Thus* *in this case, condition (a) becomes the classical Haar condition. Further, when N=1, condition (b) holds trivially because each of the diagonal elements* $u_i\xi_i$ *on the right hand side of equation 4.4 becomes a product of two non-zero scalars.*

Chapter 5

Separating and Quasi hyperrigid Operator Systems

Hyperrigidity of operator systems introduced by Arveson attracted a lot of attention. Weaker analogues of hyperrigidity also emerged and proved to be worth studying. For example, M.N.N Namboodiri [32] introduced the notion of weakly hyperrigid sets in the setting of W*-algebras and obtained a Arveson-type characterization theorem. Quasi hyperrigidity introduced in [28] is another case in point. Arveson [4] characterised boundary representations for subalgebras of C^*-algebras in terms of finite representations and separating subalgebras. In this chapter, we connect these two notions with quasi hyperrigidity.

5.1 Separating subalgebras of C^*-algebras

First we will define representation of a subalgebra of a C^*-algebra.

Definition 5.1.1. *[4] Let A be a subalgebra of a C^*-algebra B such that A contains the identity of B. A representation of A is a homomorphism ϕ of A into the algebra of operators on some Hilbert space such that*

(i) $\phi(e) = I.$

(ii) $\|\phi(a)\| \leq \|a\|$ *for every* $a \in A.$

It can be verified that when $A = B$, ϕ will become the usual representation of a C^*-algebra. Now let us recall the definition of a semi-invariant subspace. A closed subspace M of a Hilbert space H is called semi-invariant for a sublalgebra A of $B(H)$ (A containing identity) if the map $\Psi(T) = P_M T|_M$ is multiplicative on A for every $T \in A$. Sarason [42], who introduced the concept gave the following characterization: If M is semi-invariant for A, then $M_0 := [AM] \ominus M$ is A-invariant. From this we can deduce that when A is self-adjoint, M will be a reducing subspace. Now, if ϕ is a representation of A on a Hilbert space H and if M is a semi-invariant subspace for $\phi(A)$ we define a new representation ϕ_0 of A on M by $\phi_0(a) = P_M\phi(a)|_M, a \in A$. We call ϕ_0 a *subrepresentation* of ϕ. We say that two representations ϕ_1 and ϕ_2 of A are *equivalent*, if there exists a unitary operator U between their Hilbert spaces such that $U\phi_1(a) = \phi_2(a)U$ for all $a \in A$. Now we define separating subalgebra of a C^*-algebra.

Definition 5.1.2. *Let A be a subalgebra of a C^*-algebra B. Let ω be an irreducible representation of B. We say that A separates ω if whenever π is any irreducible representation of B such that $\omega|A$ is equivalent to a subrepresentation of $\pi|A$, we have π equivalent to ω. A is called a separating subalgebra if it separates every irreducible representation of B.*

5.2 Quasi Hyperrigidity and Separating Operator Systems

The notion of quasi hyperrigidity for an operator system was first introduced in [28]. We give below the definition of a quasi hyperrigid operator system and give an example to show that the notion is weaker than the notion of hyperrigidity.

Definition 5.2.1. *[28] An operator system S is said to be quasi hyperrigid if for every irreducible representation π of $C^*(S)$ and for every isometry $V : H_\pi \to H_\pi$ such that $V^*\pi(s)V = \pi(s)$ for all s in S, then $V^*\pi(a)V = \pi(a)$ for all a in $C^*(S)$.*

Example 5.2.1. *Let $M_n(\mathbb{C})$ denote the set of all $n \times n$ matrices over $\mathbb{C}$, where $n \geq 3$. Let $M \in M_n(\mathbb{C})$ be given by*

$$M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots\ldots & a_{3n} \\ . & . & . & \ldots\ldots & . \\ . & . & . & \ldots\ldots & . \\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots & a_{nn} \end{bmatrix}$$

be arbitrary. Now define Φ on $M_n(\mathbb{C})$ as $\Phi(M) = N$, where

$$N = \begin{bmatrix} a_{11} & a_{12} & 0 & \ldots\ldots & 0 \\ a_{21} & a_{22} & 0 & \ldots\ldots & 0 \\ 0 & 0 & a_{22} & \ldots\ldots & 0 \\ 0 & 0 & 0 & a_{22} & 0 \\ . & . & . & \ldots\ldots & . \\ 0 & 0 & 0 & \ldots\ldots & a_{22} \end{bmatrix}.$$

Observe that Φ is a unital completely positive map. Let T be the $n \times n$ matrix in $M_n(\mathbb{C}$ where $a_{21} = 1$ and all other entries equal to 0. If $S = span\{I, T, T^*\}$ and $A = C^*(S)$, then $\Phi(s) = s$ for all s in S, but $\Phi(TT^*) \neq TT^*$. i.e, S is not a hyperrigid set. However, if V is any isometry such that $V^*V = I$, then $VV^* = I$, since A is finite dimensional. Thus S is quasi hyperrigid, but fails to be a hyperrigid set.

In fact, numerous examples of quasi hyperrigid systems are given in [28].

Now we give the definition of a finite representation.

Definition 5.2.2. *[4] Let A be a subalgebra of a C^*-algebra and let ϕ be a representation of A. Then ϕ is called finite if it is not equivalent to any proper subrepresentation $\phi_0 \neq \phi$.*

Arveson gave the following characterization for finite representations:

Theorem 5.2.1. *[4] Let A be a subalgebra of a C^*-algebra and let ϕ be a representation of A on H. Then ϕ is finite if and only if for every isometry $V \in B(H)$, the condition $V^*\phi(a)V = \phi(a)$ for all $a \in A$ implies V is unitary.*

. The following result connects the notion of finite representations with quasi hyperrigidity.

Proposition 5.2.1. *Let A be a unital operator algebra in $B(H)$. Consider the operator system $S = A + A^*$and let $\mathcal{B} = C^*(S)$ be the C^*-algebra generated by S. Then every irreducible representation $\pi \in \widehat{\mathcal{B}}$ is a finite representation of A if and only if S is a quasi hyperrigid system.*

Proof. Assume that every irreducible representation $\pi \in \widehat{\mathcal{B}}$ is a finite representation of A. Let $\pi : \mathcal{B} \longrightarrow B(H_\pi)$ be an irreducible representation and let $V : H_\pi \longrightarrow H_\pi$ be an isometry satisfying

$$V^*\pi(s)V = \pi(s) \text{ for all } s \in S.$$

In particular, $V^*\pi(a)V = \pi(a)$ for all $a \in A$.

But by our assumption π is a finite representation of A and therefore V is unitary. Hence we have

$$V^*\pi(s)V = \pi(s) \text{ for all } s \in \mathcal{B}.$$

This will imply that S quasi hyperrigid.

Conversely, assume that S is a quasi hyperrigid system. Let $\pi : \mathcal{B} \longrightarrow B(H_\pi)$ be an irreducible representation and let $V : H_\pi \longrightarrow H_\pi$ be an isometry satisfying

$$V^*\pi(a)V = \pi(a) \text{ for all } a \in A.$$

$$V^*\pi(s)V = \pi(s) \text{ for all } s \in S = A + A^*.$$

Since S is quasi hyperrigid, we have

$$V^*\pi(s)V = \pi(s) \text{ for all } s \in \mathcal{B}.$$

V is an isometry and from the above equation it is clear that the range of V reduces the irreducible C^*-algebra $\pi(\mathcal{B})$. Therefore V is unitary and hence π is a finite representation of A. □

Now we define a separating operator system in line with the definition of a separating subalgebra defined by Arveson. The necessary and sufficient condition due to Arveson for a subalgebra to separate an irreducible representation is as follows. A subalgebra A of a C^*-algebra $\mathcal{B}$ separates an irreducible representation ω of $\mathcal{B}$ on a Hilbert space H if and only if the following condition is satisfied: for every irreducible representation π of $\mathcal{B}$ on K and every isometry $V \in L(H, K), V^*\pi(a)V = \omega(a)$ for all $a \in A$ implies that π and ω are equivalent representations of $\mathcal{B}$. Replacing subalgebra with a mere linear subspace we give the definition of a separating operator system as follows.

Definition 5.2.3. *Let S be an operator system and $\mathcal{A} = C^*(S)$-the C^*- algebra generated by S. Let $\pi : \mathcal{A} \longrightarrow B(H)$ be an irreducible representation. We say that S separates π if for every irreducible representation $\rho : \mathcal{A} \longrightarrow B(K)$ and every isometry $V : H \longrightarrow K$, $V^*\rho(s)V = \pi(s),$ for all $s \in S$ implies that π and ρ are unitarily equivalent representations of $\mathcal{A}$. We say that S is a separating operator system of A if S separates every irreducible representation of $\mathcal{A}$.*

In the classical case, a set S in $C(X)$ (where X is a compact Hausdorff) is said to separate points of X if for each pair of points $x_1, x_2 \in X$ where $x_1 \neq x_2$, there exists $g \in S$ such that $g(x_1) \neq g(x_2)$. As irreducible representations of $C(X)$ correspond to points of X, our notion of separating operator system will coincide with the subspace which separates points in the classical sense. Further, when S is a Korovkin set in $C(X)$, it separates points of X([9], page 163). In the same way, in non commutative setting, if S is a separable operator system and $\mathcal{A} = C^*(S)$, then by ([7], Theorem 2.1) it follows that every separable hyperrigid operator system is separating.

Theorem 5.2.2. *Let S be an operator system and $\mathcal{A} = C^*(S)$. Then every irreducible representation of $\mathcal{A}$ is a boundary representation of $\mathcal{A}$ for S if and only if the following conditions are satisfied:*

(i) *S is quasi hyperrigid.*

(ii) *every irreducible representation of $\mathcal{A}$ restricted to S is pure.*

(iii) *S is a separating operator system.*

Proof. First we will assume that every irreducible representation for S is a boundary representation. Consider an irreducible representation $\rho : \mathcal{A} \longrightarrow B(H_\rho)$. Let $V : H_\rho \longrightarrow H_\rho$ be an isometry such that $V^*\rho(s)V = \rho(s)$ for every $s \in S$. Then $V^*\rho(.)V$ is a completely positive map on $\mathcal{A}$ which agrees with ρ on S. But ρ is a boundary representation of $\mathcal{A}$ for S. This implies that $V^*\rho(a)V = \rho(a)$ for all $a \in \mathcal{A}$. Therefore S is quasi hyperrigid. Let $\rho_{|_S} = \phi_1 + \phi_2$ for some $\phi_i \in CP(S, B(H_\rho)), i = 1, 2$. Then by ([4],Theorem 1.2.3), there exists $\xi_i \in CP(\mathcal{A}, B(H_\rho))$ such that $\xi_{i|_S} = \phi_i, i = 1, 2$. Then $\xi_1+\xi_2$ is a completely positive extension of $\rho_{|_S}$. But ρ is a boundary representation and hence $\rho = \xi_1+\xi_2$. Since ρ is an irreducible representation, ρ is pure ([4],Theorem 1.4.3). Therefore there exists $t_i \geq 0, i = 1, 2$ such that $\xi_i = t_i\rho, i = 1, 2$. This implies that $\phi_i = t_i\rho_{|_S}, i = 1, 2$ and therefore $\rho_{|_S}$ is pure.

Now we will prove that S is a separating operator system by showing that S separates ρ. Let $\pi : \mathcal{A} \longrightarrow B(H_\pi)$ be any other irreducible representation and let $V : H_\rho \longrightarrow H_\pi$ be an isometry satisfying the condition $V^*\pi(s)V = \rho(s)$, for all $s \in S$. We know that $V^*\pi(.)V$ is a completely positive extension of $\rho_{|_S}$ and since ρ is a boundary representation of $\mathcal{A}$ for S, we have $V^*\pi(a)V = \rho(a)$, for all $a \in \mathcal{A}$. But then VH_ρ is a reducing subspace for $\pi(\mathcal{A})$. Since $\pi(\mathcal{A})$ is irreducible, we must have $VH_\rho = H_\pi$ and this gives that V is unitary. Therefore ρ and π are unitarily equivalent. Since ρ is arbitrary, we get that S is a separating operator system.

Conversely, assume that conditions (i), (ii) and (iii) are satisfied. Let π be an irreducible representation of $\mathcal{A}$ on a Hilbert space H_π. In order to prove that π is a boundary representation of $\mathcal{A}$ for S. Consider $K = \{\xi \in CP(\mathcal{A}, B(H_\pi)) : \psi_{|S} = \pi_{|S}\}$. We will show that $K = \{\pi\}$. With respect to BW-topology, K is a compact convex subset of $CP(\mathcal{A}, B(H_\pi))$ ([4], page 146). Clearly K is non-empty. By Krein-Milman theorem, K is the closed convex hull of its extreme points. Let $\phi \in K$ is an extreme point. We will show that $\phi = \pi$.

We first claim that ϕ is a pure element of $CP(\mathcal{A}, B(H_\pi))$. Choose non-zero elements ϕ_1 and ϕ_2 of $CP(\mathcal{A}, B(H_\pi))$ such that $\phi(a) = \phi_1(a) + \phi_2(a), a \in \mathcal{A}$. Then π and ϕ are bounded linear maps of $\mathcal{A}$ agreeing on S. But by condition (2) of the theorem there exist scalars $t_i \geq 0, i = 1, 2$ such that $\phi_i(s) = t_i\pi(s)$, for every $s \in S$. If we take $t_i = 0$, and since $e \in S$, we get $\phi_i(e) = 0$. Hence $\phi_i = 0, i = 1, 2$ which is not possible because of our selection of ϕ_i. This gives that $t_i > 0, i = 1, 2$. Since $e \in S, \pi(e) = 1 = t_1\pi(e) + t_2\pi(e)$ we get $t_1 + t_2 = 1$. Now put $\psi_i = t_i^{-1}\phi_i$. Then $\psi_i \in K, i = 1, 2$. Therefore we get $\phi = t_1\psi_1 + t_2\psi_2$. But by our assumption, ϕ is an extreme point of K, $\phi = \psi_1 = \psi_2$. Then $\phi_i = t_i\phi, i = 1, 2$. This proves that ϕ is pure.

By ([4], Theorem 1.4.3), there exists an irreducible representation $\rho : \mathcal{A} \longrightarrow B(H_\rho)$ and a bounded operator $V : H_\pi \longrightarrow H_\rho$ such that $\phi(a) = V^*\rho(a)V$ for all $a \in \mathcal{A}$. Then

$\pi(s) = V^*\rho(s)V$ for all $s \in S$. Putting $s = e$ we get $V^*V = I$ and hence V is an isometry. Because of our assumption that S is a separating operator system, we get that $\pi \sim \rho$. Therefore, there exists a unitary operator $U : H_\rho \longrightarrow H_\pi$ such that $\rho = U^{-1}\pi U$. Hence we can write $\pi(s) = (UV)^*\pi(s)(UV)$ for all $s \in S$. But UV is an isometry. By our assumption (i) S is quasi hyperrigid and this implies that UV is unitary which in turn gives $V = U^{-1}UV$ is unitary. Therefore, we can write $\pi(s) = V^{-1}\rho(s)V, s \in S$. Then $V^{-1}\rho(.)V$ is a representation of $\mathcal{A}$ which agrees with π on S. This gives that $\pi(a) = V^{-1}\rho(a)V, a \in \mathcal{A}$. Therefore we have $\pi = \phi$ and the proof is complete. □

The following example illustrates the above theorem.

Example 5.2.2. *Let $A = span\,(I, S, S^*, SS^*)$, where S is the unilateral right shift in $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and I the identity operator. Let $B = C^*(A)$ be the C^*-algebra generated by A. It is a well known fact that $K(\mathcal{H}) \subseteq B$ where $K(\mathcal{H})$ denotes the set of compact operators on $\mathcal{H}$. Again we have $B/K(\mathcal{H}) \cong C(\mathbb{T})$ is commutative, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and the spectrum $\hat{B}$ of B can be identified with $\{Id\} \cup \mathbb{T}$. Since S is an isometry, A is hyperrigid ([7], Theorem 3.3) this in turn will imply that all the irreducible representations of B are boundary representations for A. Clearly A is quasi hyperrigid. Also A is separating operator system.*

Further, $Id_{|A}$ is pure and the irreducible representations corresponding to $\mathbb{T}$ are one dimensional and their restrictions to A are also pure.

Chapter 6

Conclusion

In this thesis, we introduced the noncommutative Haar condition (Definition 4.3.1) and proved that it is equivalent to Haar condition in the classical case (Proposition 4.3.1). We proved in Theorem 4.3.1 that for a unital C^*-algebra, if we consider a linearly independent subset of n elements which includes the identity of the C^*-algebra, the n-dimensional subspace spanned by the set being a Čebyšev subspace is equivalent to the set satisfying noncommutative Haar condition. We also proved in Theorem 4.3.2 that for a unital C^*-algebra which is generated by a two-dimensional Čebyšev subspace which contains the identity of the C^*-algebra, an irreducible representation of it under certain conditions will become a boundary representation.

Two propositions (Proposition 4.3.2 and Proposition 4.3.3) are also given to give clarity to the two conditions in the definition of noncommutative Haar condition.

We give the definition of a separating operator system. We prove in Theorem 5.2.2 that if S is an operator system generating a C^*-algebra, every irreducible representation is a boundary representation for S exactly when S is quasi hyperrigid separating operator system such that restriction of every irreducible representation to S is pure.

Now we will mention some problems for further research.

Arveson's hyperrigid conjecture [7] is not proven in its generality till this date. It has been proven for cases where the C^*-algebra generated by a separable operator system has countable spectrum [7], the C^*-algebra generated by an operator system is a Type-1 C^*-algebra [24] and for function systems in the commutative case [14] .

Classification of non-commuting self-adjoint operators $a_1, a_2, ..., a_n$ satisfying non-commutative Haar condition needs further study. A special case of interest shall be non-commutative Toeplitz operators with continuous, periodic, real symbols. Analyzing non-commutative Haar condition for tensor products of more general C^*-algebras can also be undertaken. The connection between n-dimensional Čebyšev subspaces ($n > 2$) of C^*-algebras and the associated boundary representations is to be investigated.

Bibliography

[1] J. Agler, *An abstract approach to model theory*, in Surveys of Some Recent Results in Operator Theory, vol. II, Longman Sci. Tech., Harlow 1988, 1-23.

[2] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, Appendix A by Michael Pannenberg and Appendix B by Ferdinand Beckhoff, De Gruyter Studies in Mathematics **17**, Walter de Gruyter & Co., Berlin, 1994.

[3] F. Altomare, *Korovkin-type theorems and approximation by positive linear operators*, Surv. Approx. Theory **5** (2010), 92-164.

[4] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141-224.

[5] W. B. Arveson, *An invitation to C^*-algebras*, Springer-Verlag, New York, 1976.

[6] W. B. Arveson, *The noncommutative Choquet boundary*, J. Amer. Math. Soc. **21** (2008), no. 4, 1065-1084.

[7] W. B. Arveson, *The noncommutative Choquet boundary II: hyperrigidity*, Israel J. Math. **184** (2011), 349-385.

[8] W. B. Arveson, *An approximation theorem for function algebras*,preprint, University of Texas at Austin, 1970, SC 1.2, 7.3.c.

[9] H. Berens and G. G. Lorentz, *Geometric theory of Korovkin sets*, J. Approx. Theory **15** (1975), no. 3, 161-189.

[10] E. Bishop and K. de Leeuw, *The representations of linear functionals by measures on sets of extreme points*, Ann. Inst. Fourier (Grenoble) **9** (1959), 305-331.

[11] D. Blecher, Z. J. Ruan and A. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), 188-201.

[12] M. D. Choi and E. Effros, *Injectivity and operator spaces*, J. Funct. Anal. **24** (1977), 156-209.

[13] K. R. Davidson and M. Kennedy, *The Choquet boundary of an operator system*, Duke Math. J. **164** (2015), 2989-3004

[14] K. R. Davidson and M. Kennedy, *Choquet order and hyperrigidity for function systems*, (2016), arXiv:1608.02334v1 (to appear).

[15] M. Dritschel and S. McCullough, *Boundary representations for families of representations of operator algebras and spaces*, J. Operator Theory **53** (2005), no. 1, 159-167.

[16] N. Efimov and S. B. Stechkin *Some properties of Čebyšev sets*, Dokl.Akad.Aauk.SSSR(1958),17-19.

[17] A. L. Garkavi, *Compact admitting Čebyšev systems of measures* Matem. Sbornik **74**,(1967)641-656.

[18] J. G. Glimm and R. V. Kadison, *Unitary operators on C^*-algebras*, Pacific Jour. Math.10 (1960) 547-556.

[19] A. Haar, *Minkowskische geometrie und die annaherung an stetige funktionen*, Math.Ann **8** (1918), 294-311.

[20] M. Hamana, *Injective envelopes of operator systems*, Publ. Res. Inst. Math. Sci. **15** (1979), 773-785.

[21] Y. Ikebe, *A characterization of Haar subspace in C(a,b)*,Proc. Japan Acad.,44, (1968), 219-220.

[22] R. V. Kadison, *The trace in finite operator algebras*, Proc. Amer. Math. Soc. **12** (1961), 973-977.

[23] C. Kleski, *Boundary representations and pure completely positive maps*, J. Operator Theory **71** (2014), no. 1, 101-118.

[24] C. Klesky, *Korovkin-type properties for completely positive maps*, Illinois J. Math. **58** (2014), no. 4, 1107-1116.

[25] P. P. Korovkin, *Linear Operators and Approximation Theory*, Hindustan publishing corp., Delhi, 1960.

[26] D. A. Legg, B. E. Scranton and J.D. Ward *Chebyshev subspaces in the space of compact operators*, J.of Approximation Theory **15** (1975) 326-334.

[27] B. V. Limaye and M. N. N. Namboodiri, *Korovkin-type approximation on C^*-algebras*, J. Approx. Theory **34** (1982), no. 3, 237-246.

[28] M. N. N. Namboodiri, Pramod S, Shankar P and Vijayarajan A K, *Quasi hyper-rigidity and weak peak points for noncommutative operator systems*, Proc. Indian. Acad.Sci.Math.Sci **128** (2018), no. 5, Art.66.

[29] P. Muhly and B. Solel, *Hilbert modules over operator algebras*, Memoirs of the Amer. Math. Soc. **117** #559 (1995).

[30] P. Muhly and B. Solel, *An algebraic characterization of boundary representations*, Nonselfadjoint operator algebras, operator theory, and related topics, 189-196, Oper. Theory Adv. Appl. **104**, Birkhauser, Basel, 1998.

[31] M. N. N. Namboodiri, *Developments in noncommutative Korovkin theorems*, RIMS Kokyuroku Series **737** (2011), 91-104.

[32] M. N. N. Namboodiri, *Geometric theory of weak Korovkin sets*, Oper. Matrices **6** (2012), no. 2, 271-278.

[33] T. W. Palmer, *Characterizations of W^*-homomorphisms and expectations*, Proc. Amer. Math. Soc. **46** (1974), 265-272.

[34] V. Paulsen, *Completely bounbed maps and operator algebras*, Cambridge University Press, New York, 2003.

[35] G. K. Pedersen, *Cebysev subspaces of C^*-algebras*, Math.Scand. **45** (1979); 147-156.

[36] R. R. Phelps, *Lectures on Choquet's theorem*, Second edition, Lecture Notes in Mathematics 1757, Springer-Verlag, Berlin, 2001.

[37] W. M. Priestley, *A noncommutative Korovkin theorem*, J. Approximation Theory **16** (1976), no. 3, 251-260.

[38] A. G. Robertson, *Best approximation in von Neumann algebras*, Math.Proc.Cambridge.Phil.Soc. **81**(1977), 233-236.

[39] A. G. Robertson, *A Korovkin theorem for Schwarz maps on C^*-algebras*, Math. Z. **156** (1977), no. 2, 205-207.

[40] A. G. Roberston and D. Yost, *Chebyshev subspaces of operator algebras*, J.Lond.Math.Soc.(2)19(1979), 523-531.

[41] S. Sakai, *C^*-Algebras and W^*-algebras*, Springer-Verlag, New York, 1971.

[42] D. Sarason, *On spectral sets having connected complement*, Acta Sci. Math. (Szeged) **26** (1965), 289-299.

[43] Y. A. Saskin, *Korovkin systems in spaces of continuous functions*, Amer. Math. Soc. Transl. **54** (1966), no. 2, 125-144.

[44] I. Singer, *Best approximation in normed linear spaces by elements of linear subspaces*, Springer-Verlag, 1970.

[45] I. Singer, *Best approximation in normed linear spaces*, Constructive aspects of functional analysis, G.Geymonat (Ed.), CIME summer schools, Erice Italy, Springer, 1970; 683-792.

[46] Karl George Steffens, *History of Approximation Theory, from Euler to Benstein*, Birkhause, 2006.

[47] W. F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6**, (1955), 211–216.

[48] V. S. Sunder, *Functional analysis - Spectral theory*, Birkhauser Verlag, Basel, 1997.

[49] M. Takesaki, *On the cross-norm of the direct product of C^*-algebras*, Tôhoku Math. J. **16** (1964), no. 2, 111-122.

[50] S. Takahasi, *Korovkin's theorems for C^*-algebras*, J. Approx. Theory **27** (1979), no. 3, 197-202.

[51] M. Uchiyama, *Korovkin-type theorems for Schwarz maps and operator monotone functions in C^*-algebras*, Math. Z. **230** (1999), no. 4, 785–797.

[52] D. E. Wulbert, *Convergence of operators and Korovkin's theorem*. J. Approximation Theory **1** (1968), 381-390.

Publications

1. M. N. N. Namboodiri, S. Pramod, and A. K. Vijayarajan, *Finite Dimensional Čebyšev Subspaces of C^*-algebras* J. of Ramanujan. Math. Soc.**29**, No.1 (2014), 63-74.

2. M. N. N. Namboodiri, S. Pramod, and A. K. Vijayarajan, *Čebyšev Subspaces of C^*-algebras-a Survey* Operator Theory: Advances and Applications, Springer International Publishing, **Vol.247**(2015), 101-121.

3. S. Pramod, Shankar P and A. K. Vijayarajan, *Separating and quasi hyperrigid operator systems in C^* -algebras* Tbilisi Mathematical J., **10** (4)2017, 55-61.