STUDIES ON LABELLING OF GRAPHS

Thesis submitted to the University of Calicut for the award of the degree of

DOCTOR OF PHILOSOPHY IN MATHEMATICS

under the Faculty of Sciences

By

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ABSTRACT

STUDIES ON LABELLING OF GRAPHS

Graph labeling is the fascinating and active area of graph theory with widespread applications to combinatorics and arithmetic aspects. It is an assignment of real values or subsets of a set to the vertices or edges or both subject to certain conditions. If the domain is the set of vertices we call it vertex labeling. If the domain is the set of edges we call it edge labeling. If the labels are assigned to both vertices and edges we call it total labeling. Actually, graph labeling problems are not of recent origin, example, the problem of colouring the vertices emerged in connection with Thomas Gutherie's famous Four Color Conjecture, which was solved in 1976 after more than 150 years of waiting.

In the first part of the thesis, the study was done by connecting the two topics namely graph labeling and graph convexity. In the second part a new vertex labeling, LH labeling is introduced and studied it in some class of graphs.

We made an attempt to study geodesic and monophonic convexity in a graph G with respect to a labeling function $\mathscr{L}$ defined on it. Defined $\mathscr{L}_g$ convexity space, $g-$ convex label, $\mathscr{L}_m$ convexity space, $m-$ convex label and strong $m-$ convex label. A new class of graphs, geodesically elegant graphs are introduced and studied it in some class of graphs. Also, geodetic and edge geodetic number in labeled graphs are studied. Defined $\mathscr{L}-$ geodetic number and $\mathscr{L}-$ edge geodetic number of graphs. The concept of geodetic label, edge geodetic label, strong geodetic label and strong edge geodetic label are defined and established it in some family of graphs.

Defined the LH labeling of graphs and the LH labeling of Petersen graph, Hyper cube Q_3, Grotzch graph and the Heawood graph are established. The complete graph $K_n, n \geq 4$, the wheel graph $W_{1,5}$ and the complete bipartite graph $K_{3,3}$ are non LH.

For a non LH graph G, we defined the LH completion $\Omega^*(G)$ and LH completion number Λ_G of G. An upper bound for the size of an LH graph is obtained. Demonstrated the LH labeling of a perfect binary tree T_n and spider graph $S_3(m)$. LH completion number of the complete graph K_n, $n \leq 10$ is obtained. Studied LH labeling in some family of graphs such as path related graphs, cycle related graphs, star related graphs, splitting graphs, line graphs and theta graphs.

Key Words

1. Graph Labeling
2. Graph Convexity Space
3. Geodesically Elegant Graphs
4. LH Labeling
5. $\mathscr{L}-$geodetic Number
6. $\mathscr{L}-$Edge Geodetic Number

This thesis is heartily dedicated to my father K Moidheen Kutty who departed us for heaven before the completion of this work, my mother Nazeera M who stood with me all times, my teachers and friends.

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"Indeed, those who patiently persevere will truly receive a reward without measure'
(QS. AZ Zumar : 10).

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FARISA M

Contents

CHAPTER 1

Introduction

Discrete Mathematics is a branch of Mathematical sciences which deals with the systematic study of discrete structures and has numerous applications in our day - to day life. One of the major category in the subject of discrete mathematics is the theory of graphs, which has applications in various fields like operation research, biology, information theory, architecture, chemistry, anthropology, economics, psychology, computer science, clustering analysis to name a few. Also, the concept of graph theory can be used in medical science to study the structures of RNA and DNA. In the present situation, network plays an important role in many fields including society, internet, transportation etc and any network related problem can be modeled as one of the graph problems and hence solved. Thus any problem of real life situations can be modeled through graphs.

Graph theory emerged from the famous Koningsberg bridge problem by the Swiss Mathematician Leonhard Paul Euler in 1736. It is now one of the fastest growing research field. Some of the potential field of research are topological graph theory, domination in graphs, graph labeling, algebraic graph theory and fuzzy graph theory. Our interest is on Graph labeling, the flourishing, fascinating and emerging area of graph theory with widespread applications to arithmetic and combinational aspects. This chapter is a collection of some basic definitions, literature reveiw of the research topic and an overview of the remaining chapters. The definitions and theorems on graphs are useful for the subsequent chapters.

1.1 Preliminaries

This section provides basic definition and terminology required for the advancement of the topic. For all other terminology and notations we refer to Douglas B West [10], Buckley and Harary [20], Ignacio M.Pelayo [66].

Definition 1.1.1. By a *graph* we mean an ordered pair $G = (V, E)$, where $V = V(G)$ is a finite nonempty set of objects called *vertices* and $E = E(G)$ is a set of unordered pairs of distinct vertices, i.e., two-element subsets of V called *edges*. Each edge $\{u, v\}$ of G is usually denoted by uv or vu. If $e = uv$ is an edge of G, then e is said to *join* u and v. If uv is an edge of G, then u and v are *adjacent vertices*. Two adjacent vertices are referred to as *neighbours* of each other. If uv and vw are distinct edges in G, then uv and vw are adjacent edges. The vertex u and the edge uv are said to be *incident* with each other.

Definition 1.1.2. The number of vertices in a graph G (ie, the cardinality $|V|$ of V) is the *order* of G and the number of edges (ie, the cardinality $|E|$ of E) is the *size* of G. A graph of order 1 is called a *trivial graph*. A *nontrivial graph* therefore has two or more vertices. A graph of size 0 is called an *empty graph*.

Definition 1.1.3. The *open neighborhood* $N(v)$ of a vertex $v \in V(G)$ is the set of neighbors of v in G, i.e., $N(u) = \{v \in V(G) : uv \in E(G)\}$. The set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v.

Definition 1.1.4. The *degree of a vertex* v in a graph G is defined to be the number of edges incident with v and is denoted by deg_v. A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end vertex* or a *pendant vertex* or a *leaf*.

Definition 1.1.5. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, in which case we write $H \subseteq G$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. An *induced subgraph* H of a graph G is any subgraph satisfying the following property: for every pair u, v of vertices of H, if they are adjacent in G, then they are also adjacent in H. If H is an induced subgraph of a graph G and $S = V(H)$, then we say that H is the subgraph induced by S in G and we write $H = G[S]$. If H is a subgraph of a graph G and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a *proper subgraph* of G.

Definition 1.1.6. Two or more edges that join the same pair of distinct vertices are called *parallel edges.* An edge joining a vertex to itself is called a *loop.* A graph which has neither loops nor parallel edges is called a *simple graph.*

Definition 1.1.7. A graph G is *regular* if the vertices of G have the same degree and is *regular of degree* r if this degree is r. Such graphs are also called r- *regular.* In particular, a 3- regular graph is called a *cubic graph.*

Definition 1.1.8. The *complete graph* K_n is the graph of order n in which any two distinct vertices are adjacent.

Definition 1.1.9. A graph G is *bipartite* if $V(G)$ can be partitioned into two sets U and W (called *partite sets*) so that every edge of G joins a vertex of U and a vertex of W. If G contains every edge joining U and W, then G is a *complete bipartite graph.* In this case, if U and W have m and n vertices, we write $G = K_{m,n}$. Obviously $K_{m,n}$ has mn edges and $m + n$ vertices. The complete bipartite graph $K_{1,t}$ is called a *star.*

Definition 1.1.10. A *walk* in a graph G is an alternating sequence of vertices and edges $v_0, e_0, v_1, e_2, v_2, ..., v_{n-1}, e_n, v_n$ such that every $e_i = v_{i-1}v_i$ is an edge of G, $1 \leq i \leq n$. It is important to mention that the vertices need not be distinct and the same holds for the edges. The walk connects v_0 and v_n is called a $v_0 - v_n$ walk. This walk has *length* n, the number of occurrences of edges in it. A walk in a graph G is a *trail* if all its edges are distinct and a *path* if all its vertices (and thus necessarily all its edges) are distinct. A path on n vertices is denoted by P_n. The walk is closed if $v_0 = v_n$ and is open otherwise.

Definition 1.1.11. A closed walk is a *cycle* provided its n vertices are distinct and $n \geq 3$. A cycle of even length is an *even cycle*, a cycle of odd length is an *odd cycle*. A cycle on n vertices is denoted by C_n. A cycle of length n is an *n-cycle*. A 3-cycle is referred to as a *triangle*.

Definition 1.1.12. Two vertices u and v in a graph G are *connected* if G contains a $u - v$ path. A graph G is *connected* if it has a $u - v$ path whenever $u, v \in G$. A graph G that is not connected is a *disconnected* graph.

Definition 1.1.13. A shortest $u - v$ path is called a $u - v$ *geodesic*. The *distance* $d_G(u, v)$ *(or* $d(u, v)$*)* between the vertices u and v is defined as the length of a $u - v$ geodesic.

Definition 1.1.14. Let G_1 and G_2 be two graphs with disjoint vertex sets.

1. The *union* $G = G_1 \cup G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.

2. The *join* $G = G_1 + G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

Definition 1.1.15. [70]

1. The *cartesian product* of G and H is a graph, denoted as $G \Box H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Thus, $V(G \Box H) = \{(g, h)/g \in V(G) \text{ and } h \in V(H)\}$,
$E(G \Box H) = \{(g, h)(g', h') | g = g', hh' \in E(H), \text{ or } gg' \in E(G), h = h'\}$.

2. The *direct product* of G and H is the graph, denoted as $G \times H$, whose vertex set is $V(G) \times V(H)$, and for which vertices (g, h) and (g', h') are adjacent precisely if $gg' \in E(G)$ and $hh' \in E(H)$. Thus, $V(G \times H) = \{(g, h)/g \in V(G)$ and $h \in V(H)\}$, $E(G \times H) = \{(g, h)(g', h')/gg' \in E(G)$ and $hh' \in E(H)\}$. Other names for the direct product that have appeared in the literature are tensor product, Kronecker product, cardinal product, relational product, cross product, conjunction, weak direct product, or categorical product.

3. The *strong product* of G and H is the graph denoted as $G \boxtimes H$, and defined by $V(G \boxtimes H) = \{(g, h)/g \in V(G)$ and $h \in V(H)\}$, $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$. Note that $G \square H$ and $G \times H$ are subgraphs of $G \boxtimes H$.

4. The *lexico graphic product* of graphs G and H is the graph $G[H]$ with vertex set $V(G) \times V(H)$, vertices (g_1, h_1) and (g_2, h_2) are adjacent if either $g_1 g_2 \in E(G)$ or $g_1 = g_2$ and $h_1 h_2 \in E(H)$.

Definition 1.1.16. The *corona* $G_1 \circ G_2$ was defined by Frucht and Harary as the graph G obtained by taking one copy of G_1 of order p_1 and p_1 copies of G_2, and then joining the i'th vertex of G_1 to every vertex in the i'th copy of G_2.

Definition 1.1.17. An *acyclic graph* has no cycles. A *tree* is a connected acyclic graph.

Definition 1.1.18. [40, 35] A *perfect binary tree* is a binary tree in which every parent has two children and all leaves are at the same depth. For any non negative integer n, a perfect binary tree of height n denoted by T_n. A perfect binary tree of n levels has exactly $2^n - 1$ vertices and all its internal vertices must have two children .

Definition 1.1.19. *[64]*A spider graph or spider is a tree with at most one vertex of degree greater than 2 and this vertex is called the **branch vertex** and is denoted by v_0. A **leg** of a spider graph is a path from the branch vertex to a leaf of the tree. Let $S_n(m_1, m_2, m_3, ..., m_k), n \geq k$, denote a spider of n legs such that its legs has length one except for k legs of lengths m_1, m_2, ..., m_k, where $m_i \geq 2$ for all $i = 1, 2, ..., k$. Let $S_n(m)$ denote a spider graph of n legs and each leg has length m.

Definition 1.1.20. The $n-$*cube* Q_n is defined recursively by $Q_1 = K_2$ and $Q_n = K_2 \square Q_{n-1}$. Thus Q_n has 2^n vertices which may be labeled $a_1 a_2 \ldots a_n$, where each a_i is either 0 or 1. Two vertices of Q_n are adjacent if their binary sequences differ in exactly one place.

Definition 1.1.21. [39] The *Crown* $(C_n \circ K_1)$ is obtained by joining a pendant edge to each vertex of C_n.

Definition 1.1.22. [80, 46] The *triangular snake* T_n is obtained from the path P_n by replacing each edge of the path by a triangle C_3.

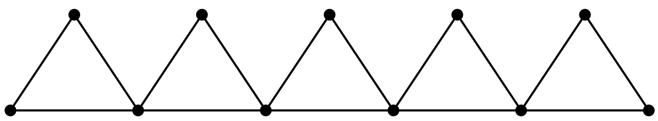


Figure 1.1: T_6

Similarly, A *quadrilateral snake* G_n is obtained from a path $u_1, u_2, ..., u_{n+1}$ by replacing every edge of a path by a cycle C_4, and is denoted by G_n.

Definition 1.1.23. [80] The corona graph $P_n \circ K_1$ is called a *Comb*.

Definition 1.1.24. [2] The *wheel graph* $W_{1,n}$ is the join of the graphs C_n and K_1. i.e. $W_{1,n} = C_n + K_1$. Here vertices corresponding to C_n are called *rim vertices* and C_n is called *rim* of $W_{1,n}$ while the vertex corresponds to K_1 is called *apex vertex*.

Definition 1.1.25. [42] The *Helm* $H_n, n \geq 3$ is the graph obtained from a wheel W_n by attaching a pendant edge at each rim vertex.

Definition 1.1.26. [42] The *Flower graph* Fl_n is obtained from the helm graph by joining each pendant vertices to the central vertex.

Definition 1.1.27. [81] The friendship graph ${F_3}^n$ is the union of n triangles with a common vertex. It has $2n + 1$ vertices and $3n$ edges.

Definition 1.1.28. [30] The *generalized friendship graph* $f_{q,p}$ is a collection of p-cycles (all of order q) meeting at a common vertex. It is also called Dutch windmill graph in literature.

Definition 1.1.29. [81] The *Windmill graph* $Wd(k, n)$ is an undirected graph obtained by taking k copies of the complete graph K_n with a vertex in common.

Definition 1.1.30. [42] *Bistar* is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ and is denoted by $B_{m,n}$.

Definition 1.1.31. [44] A *Twig* $TW(n), n \geq 3$, is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

Definition 1.1.32. [12]A $Y-$ *tree* is a graph obtained from a path by appending an edge to a vertex of a path adjacent to an end point and it is denoted by Y_n, where n is the number of vertices in a tree.

Definition 1.1.33. [9] A *sparkler*, denoted as ${P_m}^{+n}$, is a graph obtained from the path P_m and appending n edges to n end point. This is a special case of a caterpillar. We refer to the *hub* of ${P_m}^{+n}$, sparkler, as the vertex of degree $n + 1$.

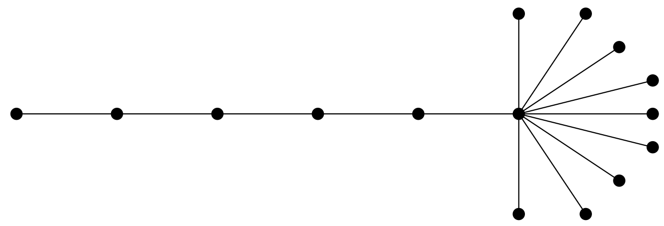


Figure 1.2: $(P_6)^{+9}$

Definition 1.1.34. [80] Let S_m be a star with central vertex v_0 and pendant vertices $v_1, v_2, ..., v_m$ and let $[P_n; S_m]$ be the graph obtained from n copies of S_m with vertices $v_{0_j}, v_{1_j}, v_{2_j}, ..., v_{m_j} (1 \leq j \leq n)$ and joining v_{0_j} and $v_{0_{j+1}}$ by means of an edge, $1 \leq j \leq n-1$.

Definition 1.1.35. [55] The *$H-$ graph* of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_n$ by joining the vertices $v_{(n+1)/2}$ and $u_{(n+1)/2}$ by means of an edge if n is odd and the vertices $u_{(n/2)+1}$ and $v_{n/2}$ if n is even.

Definition 1.1.36. [76, 28] The *generalized theta graph* denoted by $\theta(l_1, l_2, \ldots, l_k)$ consists $k \geq 3$ pairwise internally disjoint paths of length $l_1, l_2, \ldots, l_k$ that share a pair of common endpoints u and w. In this thesis, we consider the generalized theta graph $\theta(3, 2, 3)$ as theta graph and it is denoted by T_α. It has two non- adjacent vertices of degree 3 and all other vertices of degree 2.

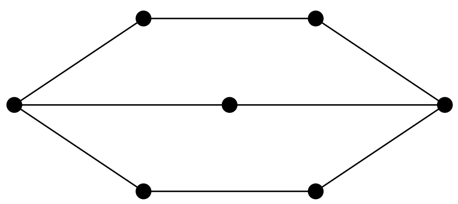


Figure 1.3: $\theta(3, 2, 3) = T_\alpha$

Definition 1.1.37. The *Petersen graph* is the simple graph whose vertices are the $2-$element subset of a $5-$ element set and whose edges are the pairs of dijoint $2-$

element subsets. Peterson graph is a graph with 10 vertices and 15 edges.

Definition 1.1.38. [34] A *weighted graph* $G = (V, E, w)$ is one in which every edge is assigned a non negative number $w(e)$ called the weight of e. An ordinary graph is a weighted graph with unit weight assigned for all edges.

Definition 1.1.39. A *chord* of a path P in a graph G is any edge joining a pair of nonadjacent vertices of P.

Definition 1.1.40. [78] For a graph G, *the splitting graph* S' of G is obtained by adding to each vertex v a new vertex v' so that v' is adjacent to every vertex that is adjacent to v in G, that is $N(v) = N(v')$.

Definition 1.1.41. [68] Let $G = (V, E)$ be a graph with $V = S_1 \cup S_2 \cup ... \cup S_t \cup T$ where each S_i is a set of vertices having atleast two vertices and having the same degree and $T = V / \cup_i S_i$. The *degree splitting graph of* G denoted by $DS(G)$ is obtained from G by adding $w_1, w_2, ..., w_t$ and joining v_i to each vertex of $S_i (1 \leq i \leq t)$.

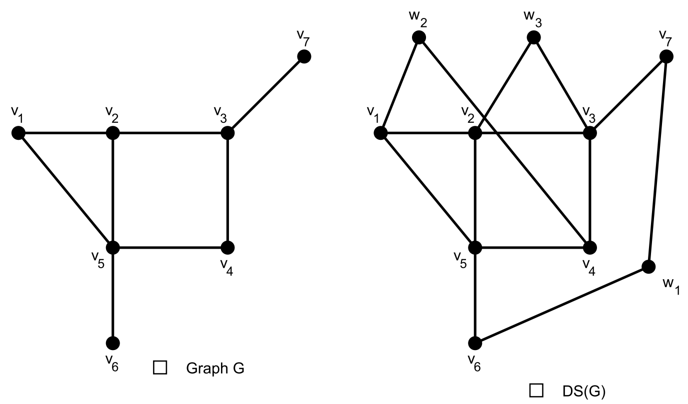


Figure 1.4: *$DS(G)$ of a graph G*

Definition 1.1.42. A *convexity* $\mathcal{C}$ on a nonempty set V is a collection of subsets of V such that:

- $\emptyset, V \in \mathcal{C}$.

- Arbitrary intersection of convex sets are convex.
- Every nested union of convex sets is convex .

A *convexity space* is an ordered pair $(V, \mathcal{C})$, where V is a nonemptyset and $\mathcal{C}$ is a convexity on V. The members of $\mathcal{C}$ are called *convex sets.* Union of nested family of sets is the nested union.

Definition 1.1.43. A *graph convexity space* is a an ordered pair $(G, \mathcal{C})$ formed by a connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that$(V, \mathcal{C})$ is a convexity space and satisfying the condition that every member of $\mathcal{C}$ induces a connected subgraph of G.

Definition 1.1.44. [76] A *vertex switching of a graph* G is a graph G_v obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.

Definition 1.1.45. A *vertex labeling* of a graph G is a mapping f from the vertices G to a set of elements, often integers. Each edge xy has a label that depends on adjacent vertices x and y and their labels $f(x)$ and $f(y)$.

By a graph $G = (V, E)$, we mean a finite, connected graph without loops and parallel edges.

1.2 Back ground of the study

A *graph labeling*[**69**, **60**] is an assignment of real values or subsets of a set to the vertices or edges or both subject to certain conditions. If the domain is the set of vertices we call it vertex labeling. If the domain is the set of edges we call it edge labeling. If the labels are assigned to both vertices and edges we call it total labeling. Usually in graph labeling problems, we label the vertices and then correspondingly get the labels on the edges. Actually, graph labelling problems are not of recent origin, example, the problem of colouring the vertices emerged in connection with

Thomas Gutherie's famous Four Color Theorem, which was solved in 1976 after more than 150 years of waiting. [65].

Labeled graphs are useful mathematical model for a variety of applications. It is used in Coding theory problems, Social networking, design of good Radar location codes and in electrical circuit theory. Also, it is applied to design Communication Network addressing systems. X- ray crystallography is the primary method for characterizing the atomic structure of new materials. Labeled graphs are used to find out the ambiguities in X- ray crystallographic analysis [29].

Most of the graph labeling, trace their origin to that one introduced by Alexander Rosa in 1967. Rosa introduced a function f from the set of vertices in a graph G with q edges to a set of integers $\{0, 1, 2, ..., q\}$ so that each edge xy is assigned the label $|f(x)–f(y)|$, the resulting edge labels are distinct. Rosa studied this labeling as a tool for attacking the conjecture of Ringel and he called this labeling as β -valuation. S.W.Golomb called these type of labeling as graceful labeling in 1972 and this is now the popular term [39].

Harmonious graphs naturally emerged in the study by Ron Graham and Neil Sloane in 1980. They defined a graph G with q edges to be harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge xy is assigned the label $f(x) + f(y)(mod q)$, the resulting edge labels are distinct [39].

Relatively prime numbers play a key role in both algebraic and analytic number theory. Roger Entringer used the concept of primes in graph labeling and introduced prime labeling of graphs. In 1980, Entrinjer postulated that every tree is prime. The notion of prime labeling was further studied by Tout, Dabboucy, and Howalla in 1982. In prime graph labeling, vertices are labeled with distinct positive integers less than or equal to the number of vertices in the graph such that labels of adjacent vertices are relatively prime.

Due to the Ringel-Kotzig tree-conjecture, All trees are graceful, many new ideas are added in the development of graph labelling. In 1987, Cahit introduced the concept of cordial labeling as a weaker version of graceful and harmonious labeling. A binary vertex labeling $f : V(G) \to \{0, 1\}$ of a graph G is called cordial labeling if for each edge $xy \in G$, the induced edge function $f^* : E(G) \to \{0, 1\}$ is defined as $f^*(xy) = |f(x) - f(y)|$ and it satisfies the condition $|e_f(0) - e_f(1)| \leq 1$ and $|v_f(0) - v_f(1)| \leq 1$. Here $v_f(i)$ and $e_f(i)$ of G are respectively the number of vertices and edges having label i, $i \in \{0, 1\}$ [79]. Cahit was inspired by the failure conditions for the existence of all trees are harmonious or graceful.

With various applications, variations of these labelings and other labelings have been introduced over time like permutation labeling, combination labeling, fibonacci labeling to mention a few. Joseph A Gallian, in his brilliant dynamic survey, has collected everything on graph labeling, which can be acessed freely from the website of ' The electronic journal of combinatorics'. It provides resource materials and most recent developments in the field of labeling of graphs for the new scholars. Note that it does not address the another major labeling problem, called the band - width problem. A detailed survey of this can be seen in [8].

Numerous connections exist between labeling and other branches of mathematics, including algebra, combinatorics, and number theory. In the opinion of L W Beineke and S. M. Hegde graph labeling serves as a boundary between number theory and the structure of graph. They introduced the concept of strongly multiplicative graphs in 2001. Also known as productive graphs, thus fitting in with the names harmonious and graceful. A graph with p vertices is said to be strongly multiplicative if the vertices can be labeled $1, 2, ..., p$ so that the values on the edges, obtained as the products if the labels of their end vertices, are all distinct [2].

Beineke and Hegde proved that every tree has a strongly multiplicative labeling in which an arbitrary vertex is labeled with 1. Also, every spanning subgraph of a strongly multiplicative graph is strongly multiplicative and every graph is an induced subgraph of a strongly multiplicative graph. They obtained an upper bound for the size of a strongly multiplicative graph with n vertices [2]. Later

Chandrashekar Adiga, H. N. Ramaswamy and D.D. Somashekara obtained a sharp bound for the maximum number of edges in a strongly multiplicative graph [7]. Since then, so many authors studied and contributed to the concept of strongly multiplicative graphs including K. K. Kanani, T M Chhaya M, Muthusamy, Joice Punitha, A. Josephine Lissie, K.C. Raajasekar and J Baskar Babujee [42, 60, 41, 59] .

Basically, convexity is a branch of Geometry. It plays an important role in other branch of Mathematics sucha as algebra, analysis and topology and other branch of sciences. There is a vast amount of literature on convexity theory from different perspective. The axiomatic approach to convexity developed in the works of Levi, Jamison, Vande Vel and Sierksma [62]. An elegant survey on convex structures has been done by Vande Vel. A convexity space is an ordered pair $(V, \mathcal{C})$, where V is a nonempty set and $\mathcal{C}$ is a family of subsets of V called convex sets, that satisfies

- V and ϕ are in $\mathcal{C}$.
- arbitrary intersection of convex sets are convex
- every nested union of convex sets is convex.

A graph convexity space is a an ordered pair $(G, \mathcal{C})$ formed by a connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space and satisfying the condition that every member of $\mathcal{C}$ induces a connected subgraph of G . Thus classical convexity can be extended to graphs in a natural way [66]. Convexity in graphs is discussed in the book by Buckley and Harary [20]. For unweighted and weighted graphs different types of convexities and other related parameters are introduced and studied by many authors including Chepoi, Dutchet, Bandelt, Jamison, Juhani Nieminen, Ignacio M Pelayo, Changat, Vijayakumar, Parvathy, Sunil Mathew and Jill K Mathew and details are available in the literature [19, 6, 67, 62, 63, 33, 34]. There are many types of convexity geodesic or metric convexity, monophonic convexity or the minimal path convexity, triangle path convexity, P_3 convexity and so on and most prominent among them are geodesic (which arises when we consider shortest paths) and monophonic convexity (when we consider chordless paths).

The distance $d_G(u, v)$ between vertices u and v is defined as the length of a $u - v$ geodesic. The geodesic closed interval $I[u, v]$ is the set of all vertices in all $u - v$ geodesic including u and v. For $W \subseteq V$, the union of all geodesic closed interval $I[u, v]$ over all pairs $u, v \in W$ is called the geodetic closure of W and is denoted by $I[W]$. Any subset W of V is called geodesic convex if $I[W] = W$. That is, the vertices in the shortest path connecting any two vertices in W is also in W [6]. A $u - v$ path P is called a monophonic path if it is a chordless path. It is also known as induced path or minimal path in literature. The monophonic closed interval $J[u, v]$ is the set of vertices of all induced paths linking u and v. For $W \subseteq V$, the monophonic closure J[W] of W is the union of the intervals $J[u, v]$ over all pairs u and v of W. In addition, if $W = J[W]$ then W is said to be monophonicaly convex or simply m- convex.

The geodetic number of a graph was introduced by Frank Harary, Emmanuel Loukakis and Constantine Tsouros in 1986 and is published in [17]. It was further studied in [21, 22]. A set of vertices of G is called a geodetic set in G if $I[W] = V$ and a geodetic set of minimum cardinality is a minimum geodetic set. The cardinality of a minimum geodetic set in G is called the geodetic number $g(G)$. If G is a non trivial connected graph of order n, then $2 \leq g(G) \leq n$ [21, 22]. The geodetic number of a graph can find applications in location theory and convexity theory [17]. Ping Zhang and Gary Chartrand studied geodetic number of an oriented graph [24].

For a non empty subset S of $V, T[S] = \{e \in E$, e is in some geodesic connecting two vertices of $S\}$. A set S of vertices of V is defined to be an edge geodetic set in G if $T[S] = E$. The cardinality of a minimum edge geodetic set in G is the edge geodetic number $g_e(G)$ [58]. The edge geodetic number of graphs was studied by Mustafa Atici, A P Santhakumaran and J John and can be seen in [73, 58]. The geodetic and edge geodetic number of some class of graphs are seen in [17, 21, 73, 58, 81]. For any connected graph G, $g(G) \leq g_e(G)$. There have been many versions of geodetic and edge geodetic number of a graph in literature like double geodetic number, split geodetic number and so on.

1.3 Organization of the Thesis.

This thesis contains six chapters. The chapterwise description of the report is given below.

In order to make the thesis as self contained as possible, we begin with an introductory chapter arranged in 3 sections. This chapter includes some basic definitions and results needed for the upcoming chapters, developments in the study of graph labeling and summarize the contents covered in each chapter.

'Graph Convexities and Graph Labeling' are the two major research areas of Graph theory. Different kinds of convexities and different types of graph labeling can be seen in literature. Studies connecting these two major topics is not found in literature. This motivated us to define the concept 'convexity in labeled graphs'. Chapter 2 deals with the study of geodesic and monophonic convexities in labeled graphs. In the second section, we introduce the concept of $\mathcal{L}_g$ convexity space and geodesic convex label or $g-$convex label. Geodesically elegant graphs are discussed in the next section. We give a necessary condition for a graph G to be geodesically elegant. The fourth section discuss monophonic convex label or $m-$ convex label and $\mathcal{L}_m$ convexity space.

In chapter 3, we studied geodetic and edge geodetic numbers in labeled graphs. We defined $\mathcal{L}-$geodetic label, strong $\mathcal{L}-$ geodetic label, $\mathcal{L}-$edge geodetic label and strong $\mathcal{L}-$ edge geodetic label and studied these concepts in Petrsen graph, cycle C_n, Mesh $P_r \Box P_s$, Wheel graph $W_{1,n}$, Friendship graph ${F_3}^n$ and the Windmill graph $Wd(k, n)$.

In chapter 4, a new type of vertex labeling, 'LH Labeling' is introduced. A graph G with n vertices is said to have an LH labeling if there exists a bijective function$f : V$ to $\{1, 2, 3, ..., n\}$ such that the induced map $f^* : E \rightarrow N$, the set of natural numbers defined by $f^*(uv) = \frac{LCM(f(u),f(v))}{HCF(f(u),f(v))}$ is injective. A graph that admits an LH labeling is called an LH graph. If the labels of each pair of adjacent vertices of a given graph G are relatively prime, then the LH labeling coincides with the strong multiplicative labeling. All prime graphs are LH. Most remarkable result in this chapter is an upper bound for the size of an LH graph. We proved that for any non LH graph G there exist an LH graph $\Omega^*(G)$, called the LH completion of G. Defined LH completion number and LH completion number of the complete graph K_n up to $n = 10$ are obtained. LH labeling of a perfect binary tree and spider graph $S_3(m)$ are demonstrated.

In chapter 5, LH labeling of some class of graphs are discussed. In the second section LH labeling of path related graphs like comb $P_n \circ K_1$, triangular snake T_n, quadrilateral snake G_n, twig graph $TW(n), n \geq 3$, sparkler graph $(P_m)^+n$, $H-$ graph and the graph $[P_n : S_2]$ are discussed. Third section discusses the LH labeling of cycle related graphs namely cycle C_n, wheel graph $W_{1,n}$, Crown $C_n \circ K_1$, Helm graph H_n, flower graph Fl_n and the friendship graph F_3^n. Next section deals with the LH labeling of bistar graph $B_{m,n}$. In the fifth section, LH labeling of splitting graphs namely splitting graph of a path $S'(P_n)$, comb $S'(P_n \circ K_1)$ and star $S'(K_{1,n})$ are discussed. LH labeling of line graph a comb $L(P_n \circ K_1)$ is established in section 6. Last section deals with the LH labeling of a Theta graph T_α, splitting graph of a theta graph $S(T_\alpha)$, degree splitting graph $DS(T_\alpha)$ and the graph obtained by switching any vertex of T_α.

Chapter VI briefly sums up the overall work carried out and some directions for application and further research.

CHAPTER 2

Convexity in Labeled Graphs

2.1 Introduction

In the study of convexity in graphs, two types of convexity have played a prominent role - geodesic convexity (also called metric convexity) which arises when we consider shortest paths and the monophonic convexity (also called the minimal path convexity) when we consider chordless paths [6]. The concepts, main ideas and the results related to geodesic convexity in graphs can be seen in [66]. In this chapter we made an attempt to study geodesic convexity and monophonic convexity in a graph G with respect to a labeling function $\mathcal{L}$ defined on the vertex set of G.
If a non negative integer $\mathcal{L}(v)$ (may or may not be distinct) is assigned to each vertex v of G, then the vertices of G are said to be labeled (numbered). In this thesis, we consider only distinct vertex labels. G is itself a labeled graph if each edge $e = uv$ is given the value $\mathcal{L}(uv) = \mathcal{L}(u) * \mathcal{L}(v)$ where $*$ is any mathematical operation like addition, multiplication, modulo addition or absolute difference. Here we take $*$ as absolute difference of the vertex labels.

Definition 2.1.1. A shortest $u - v$ path is called a $u - v$ geodesic. The distance $d_G(u, v)$ (or $d(u, v)$ between vertices u and v is defined as the length of a $u - v$

geodesic.

The distance function $d_G : G \times G \to N$ associated to a connected graph G satisfies, for every $u, v, w \in V$, the following properties:

- $d_G(u, v) \geq 0$, equality holding iff $u = v$.
- $d_G(u, v) = d_G(v, u)$.
- $d_G u, v) \leq d_G(u, w) + d_G(w, v)$.

Definition 2.1.2. [6] The geodesic closed interval $I[u, v]$ is the set of all vertices in all $u - v$ geodesic including u and v. For $W \subseteq V$, the union of all geodesic closed interval $I[u, v]$ over all pairs $u, v \in W$ is called the geodetic closure of W and is denoted by $I[W]$. Any subset W of V is called geodesic convex if $I[W] = W$.

Definition 2.1.3. [67] A chord of a path $u_0, u_1, u_2, ..., u_n$ is an edge $u_i u_j$ with $j \geq i + 2(0 \leq i, j \leq n)$. A $u - v$ path P is called a monophonic path or an induced path if it is a chordless path. The monophonic closed interval $J[u, v]$ is the set of vertices of all induced paths linking u and v. For $W \subseteq V$, the monophonic closure J[W] of W is the union of the intervals $J[u, v]$ over all pairs u and v of W. In addition, if $W = J[W]$ then W is said to be monophonicaly convex or simply m-convex.

2.2 Geodesic Convexity in Labeled Graphs

This section explores the study of geodesic convexity in a graph G with respect to a labeling function $\mathcal{L}$ defined on the vertex set of G. The concept of geodesic convex label and $\mathcal{L}_g$ convexity space are introduced.

The distance between two vertices in a labeled graph is defined in the same way, as in weighted graphs. Let $G(V, E)$ be an undirected, connected graph without loops and multiple edges. A bijective function $\mathcal{L} : V \to \{1, 2, 3, .., |V|\}$ be a vertex labeling of G and it induces a function $\mathcal{L}^* : E \to \{1, 2, 3, .., |V|\}$ defined by

$\mathcal{L}^*(uv) = |\mathcal{L}(u) - \mathcal{L}(v)|$. We use $\Gamma_{\mathcal{L}}$ to denote a labeled graph, $\Gamma_{\mathcal{L}} = (G, \mathcal{L})$.

Definition 2.2.1. For any $u-v$ path P in $\Gamma_{\mathcal{L}}$, the path sum denoted by $\mathcal{L}(P)$ is defined as the sum of the edge labels present in the path. That is $\mathcal{L}(P) = \sum_{e \in P} \mathcal{L}^*(e)$.

Definition 2.2.2. For any two vertices u and v in V, the distance between u and v denoted by $d_{\mathcal{L}}(u, v)$ is defined as $d_{\mathcal{L}}(u, v) = min_P\{\mathcal{L}(P)$ where P is a $u-v$ path in $\Gamma_{\mathcal{L}}\}$.

Let $G(V, E, w)$ be a connected weighted graph and u, v be any two vertices of G. Then the geodesic distance between u and v is defined and denoted by $d(u, v) = min_P \sum_{e \in P} w(e)$ where P is a $u-v$ path in G and $w(e)$ is the weight associated with the edge e.

A labeled graph can be treated as a weighted graph, we define the distance between any two vertices in $\Gamma_{\mathcal{L}}$, by replacing $w(e)$ by $\mathcal{L}^*(e)$, $\mathcal{L}^*(e)$ is the label associated with the edge e.

Clearly the distance function $d_{\mathcal{L}}(u, v) : \Gamma_{\mathcal{L}} \times \Gamma_{\mathcal{L}} \to N$ associated to a labeled graph satisfies all the conditions of a metric. Hence for every labeled graph $\Gamma_{\mathcal{L}}$, $d_{\mathcal{L}}$ is a metric on V and the label induces a convexity $\mathcal{C}_{\mathcal{L}}$ on V such that the vertices in any shortest path between each pair of vertices in a set $S \subset V$ is contained in it.

Definition 2.2.3. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. A shortest $u-v$ path in $\Gamma_{\mathcal{L}}$ is called $(u, v)\mathcal{L}$-geodesic. For any two vertices u and v of $\Gamma_{\mathcal{L}}$ the $\mathcal{L}-$ geodesic closed interval $I_{\mathcal{L}}[u, v]$ is defined as
$I_{\mathcal{L}}[u, v] = \{w \in V : d_{\mathcal{L}}(u, v) = d_{\mathcal{L}}(u, w) + d_{\mathcal{L}}(w, v)\}$.

Definition 2.2.4. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. Let $S \subset V$. The union of all $\mathcal{L}-$ geodesic closed intervals $I_{\mathcal{L}}[u, v]$ over all pairs $u, v \in S$ is called a $\mathcal{L}-$ geodesic closure of S and is denoted by $I_{\mathcal{L}}[S]$. That is for every $u, v \in S$, the vertices on an $u-v$ $\mathcal{L}$- geodesic belongs to S. If $I_{\mathcal{L}}[S] = S$, we say that S is $\mathcal{L}_g$ convex.

Equivalently, a set S is $\mathcal{L}_g$ convex if for every pair of $u, v \in S$ the interval $I_{\mathcal{L}}[u, v] \subseteq S$.

For example consider C_4 with different vertex labeling as given in Figure 2.1.

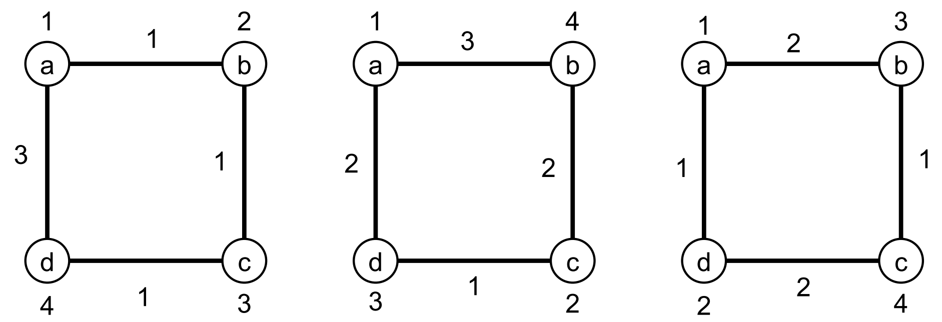


Figure 2.1: *3 different labelings of C_4 : $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$*

Geodesic convexity of the cycle C_4 is given by
$\mathcal{C} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{a, b, c, d\}\}$.

$\mathcal{L}_g$ convex sets of $\Gamma_{\mathcal{L}_1}$, $\Gamma_{\mathcal{L}_2}$ and $\Gamma_{\mathcal{L}_3}$ are given in the following table.

Table 2.1: *Geodesic convex sets with respect to the vertex labeling.*

$\mathcal{C}_{\mathcal{L}_1}$	$\mathcal{C}_{\mathcal{L}_2}$	$\mathcal{C}_{\mathcal{L}_3}$
$\emptyset, \{a\}, \{b\}$, $\{c\}, \{d\}, \{a, b\}$, $\{b, c\}, \{c, d\}$, $\{a, b, c\}, \{b, c, d\}$ and $\{a, b, c, d\}$	$\emptyset, \{a\}, \{b\}$, $\{c\}, \{d\}, \{a, b\}$, $\{b, c\}, \{c, d\}$, $\{a, d\}, \{a, d, c\}$, $\{b, c, d\}$ and $\{a, b, c, d\}$	$\emptyset, \{a\}, \{b\}$, $\{c\}, \{d\}, \{a, b\}$, $\{b, c\}, \{c, d\}$, $\{a, d\}$ and $\{a, b, c, d\}$.

Comparing $\mathcal{C}_{\mathcal{L}_1}$, $\mathcal{C}_{\mathcal{L}_2}$ and $\mathcal{C}_{\mathcal{L}_3}$ with $\mathcal{C}$ we conclude the following:

In any graph, the empty set, the whole vertex set, every one point sets and every two point sets consisting of adjacent vertices are members of $\mathcal{C}$. Clearly, the empty set, the whole vertex set and every one point sets are in $\mathcal{C}_{\mathcal{L}}$; but every two

point subsets consisting of adjacent vertices need not be in $\mathcal{C}_{\mathcal{L}}$.
Number of elements in $\mathcal{C}_{\mathcal{L}}$ may exceed (subceed) the number of elements in $\mathcal{C}$. In some cases these are equal.

Definition 2.2.5. An $\mathcal{L}_g$ convexity space is an ordered pair $(\Gamma_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}})$ where , $\Gamma_{\mathcal{L}}$ is a labeled graph and $\mathcal{C}_{\mathcal{L}}$ is the convexity induced by the label $\mathcal{L}$.

It is interesting to find a label in which the convex sets induced by it coincides with the geodesic convex sets. Based on this concept g-convex label is defined in the next section.

2.3 Geodesically Elegant graphs

In this section, a new class of graphs, geodesically elegant graphs are introduced and studied it in some classes of graphs.

Definition 2.3.1. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The label $\mathcal{L}$ is called a geodesic convex label or simply g-convex label if the convexity $\mathcal{C}_{\mathcal{L}}$ induced by the label $\mathcal{L}$ coincides with the geodesic convexity $\mathcal{C}$ on V. In other words, the $\mathcal{L}_g$ convex sets of $\mathcal{C}_{\mathcal{L}}$ are the same as the g- convex sets of $\mathcal{C}$ in G. A graph G is **geodesically elegant** if there exist a $g-$convex label for G.

Remark 2.3.1. In a tree each pair of vertices are connected by a unique path. Therefore, $\mathcal{C}_{\mathcal{L}} = \mathcal{C}$ for any vertex labeling. Hence a tree T is always geodesically elegant.

In the following proposition we characterize the necessary condition for the existence of geodesic convex label in a graph G.

Proposition 2.3.2. All geodesically elegant graphs are triangle free.

Proof. On the contrary suppose that G contains a triangle C_3 or K_3. Let us label the vertices of C_3 using the numbers a, b, and c with $a < b < c$. Then

$d_{\mathcal{L}}(v_1, v_3) = d_{\mathcal{L}}(v_1, v_2) + d_{\mathcal{L}}(v_2, v_3)$. Hence $v_2 \in I[\{v_1, v_3\}]$. Thus the two point subset $\{v_1,\, v_3\}$ is not convex.

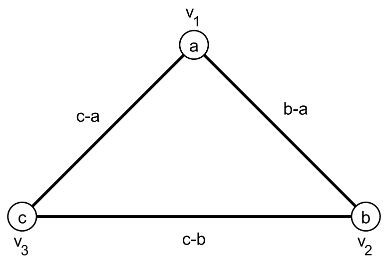


Figure 2.2

Hence $\mathcal{C}_{\mathcal{L}} \neq \mathcal{C}$, we conclude that if the graph G is geodesically elegant, then G is triangle free. □

2.3.1 Some Family of Geodesically Elegant graphs.

Now we check, which of the following graphs or graph families are geodesically elegant.

Theorem 2.3.3. The cycle C_n for all $n > 3$ is geodesically elegant.

Proof. To prove the existence of a g-convex label, find a vertex label $\mathcal{L}$ such that the convexity induced by the label coincides with the geodesic convexity in $C_n, n > 3$.
Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n.
Define $\mathcal{L} : V \to \{1, 2, 3, ..., n\}$ by

Case 1 : n is even

$$\mathcal{L}(v_i) = 2i - 1, \quad 1 \leq i \leq \frac{n}{2}$$
$$\mathcal{L}(v_i) = n, \quad i = \frac{n}{2} + 1$$
$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) - 2, \quad i = \frac{n}{2} + 2 \leq i \leq n.$$

Case 2 : n is odd

$$\mathcal{L}(v_i) = 2i - 1, \quad 1 \leq i \leq \frac{n+1}{2}$$

$$\mathcal{L}(v_i) = 2, \quad i = [\frac{n}{2}] + 2$$

$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) + 2, \quad [\frac{n}{2}] + 3 \leq i \leq n.$$

The label $\mathcal{L}$ satisfies the conditions of a g-convex label, C_n, for all $n > 3$ is geodesically elegant.

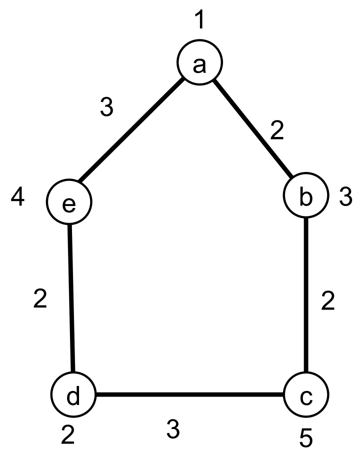


Figure 2.3: *Geodesic convex labeling of* C_5

□

Theorem 2.3.4. The crown graph $C_n \circ K_1$ for all $n > 3$ is geodesically elegant.

Proof. Let $\{v_i, i = 1 \text{ to } n\}$ be the vertices of the cycle C_n and $\{u_i, i = 1 \text{ to } n\}$ be the pendant vertices.
To prove the existence of a g-convex label, it is enough to find a vertex label $\mathcal{L}$ such that the convexity induced by the label coincides with the geodesic convexity in $C_n \circ K_1$.
Define $\mathcal{L} : V \to \{1, 2, 3, ..., 2n\}$ by

Case 1 : n is even

$$\mathcal{L}(v_i) = 2i - 1, \quad 1 \leq i \leq \frac{n}{2}$$

$$\mathcal{L}(v_i) = n, \quad i = \frac{n}{2} + 1$$
$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) - 2, \quad \frac{n}{2} + 2 \leq i \leq n.$$
$$\mathcal{L}(u_1) = n + 1$$
$$\mathcal{L}(u_i) = \mathcal{L}(u_{i-1}) + 2, \quad 2 \leq i \leq \frac{n}{2}$$
$$\mathcal{L}(u_{\frac{n}{2}} + 1) = 2n$$
$$\mathcal{L}(u_i) = \mathcal{L}(u_{i-1}) - 2, \quad \frac{n}{2} + 2 \leq i \leq n.$$

vspace3mm Case 2 : n is odd

$$\mathcal{L}(v_i) = 2i - 1, \quad 1 \leq i \leq \frac{n+1}{2}$$
$$\mathcal{L}(v_i) = 2, \quad i = [\frac{n}{2}] + 2$$
$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) + 2, \quad [\frac{n}{2}] + 3 \leq i \leq n$$
$$\mathcal{L}(u_1) = n + 1$$
$$\mathcal{L}(u_i) = \mathcal{L}(u_{i-1}) + 2, \; 2 \leq i \leq \frac{n+1}{2}$$
$$\mathcal{L}(u_{\frac{n+1}{2}} + 1) = n + 2$$
$$\mathcal{L}(u_i) = \mathcal{L}(u_{i-1}) + 2, \quad \frac{n+1}{2} + 2 \leq i \leq n.$$

The label $\mathcal{L}$ satisfies the conditions of a geodesic convex label and hence $C_n \circ K_1$ for all $n > 3$ is geodesically elegant.

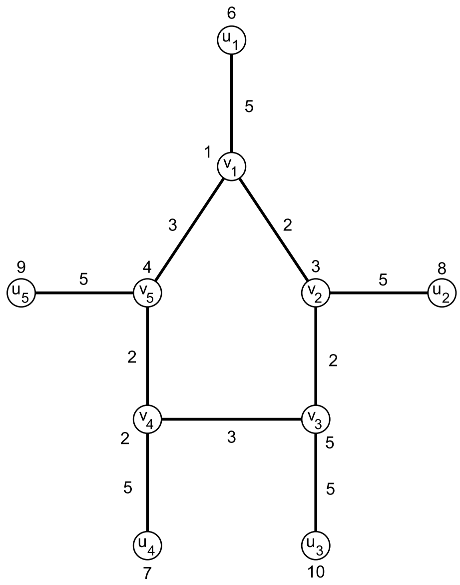


Figure 2.4: *Geodesic convex labelings of* $C_5 \circ K_1$

□

Theorem 2.3.5. The hypercube graph Q_n is geodesically elegant.

Proof. For $n = 1$ and 2, using the Remark 2.3.1 and Theorem 2.3.3, the result is true.
Suppose $n \geq 3$.
Let $\{u_i, 1 \leq i \leq 2^n\}$ be the vertices of Q_n. Each u_i is labeled by the binary $n-$ tuples $(x_1, x_2, \ldots, x_n)$ (that is $x_i = 0$ or 1, $1 \leq i \leq n$). In Q_n, two vertices u_i and u_j are adjacent if their corresponding $n-$ tuples differ in exactly one position.
Then $|V| = 2^n$ and $|E| = n2^{(n-1)}$.
Define $\mathcal{L} : V \to \{1, 2, 3, \ldots, 2^n\}$ as

$$\mathcal{L}(u_i) = \mathcal{L}(x_1, x_2, \ldots, x_n) = 1 + x_1 + 2x_2 + 2^2x_3 + \cdots + 2^{(n-1)}x_n.$$

The label $\mathcal{L}$ satisfies the conditions of a geodesic convex label and hence Q_n is geodesically elegant.

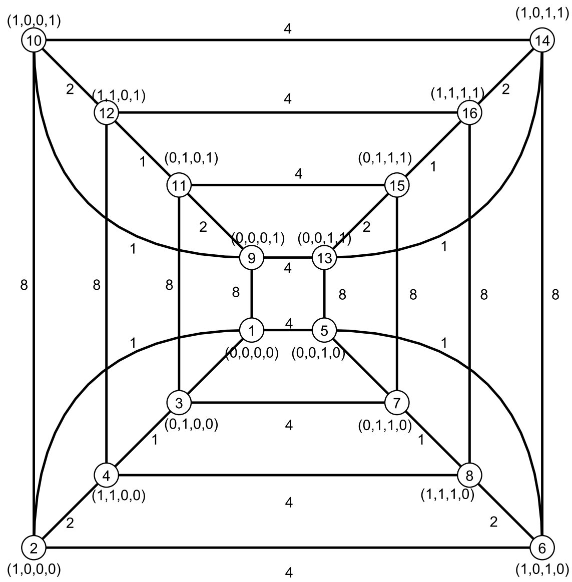


Figure 2.5: *Geodesic convex labeling of* Q_4

□

Theorem 2.3.6. The generalized friendship graph $f_{4,n}$ is geodesically elegant.

Proof. The generalized friendship graph $f_{4,n}$ is a collection of n cycles of order 4 meeting at a common vertex v_0.
Let $V(f_{4,n}) = \{v_0, v_1, v_2, ...v_{3n}\}$. Then $|V(f_{4,n})| = 3n + 1$
in $f_{4,n}$.

Ordinary labeling of $f_{4,n}$ is shown in Figure 2.6

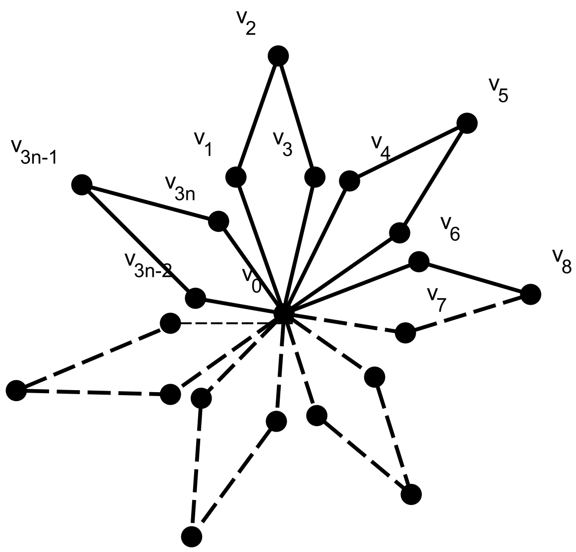


Figure 2.6: *Ordinary labelings of* $f_{4,n}$

Define $\mathcal{L} : V(f_{4,n}) \to \{1, 2, 3, ..., 3n + 1\}$ by

$$\mathcal{L}(v_0) = 3n + 1$$

$$\mathcal{L}(v_{3i-2}) = \mathcal{L}(v_0) - i, \quad 1 \leq i \leq n$$

$$\mathcal{L}(v_{3n}) = \mathcal{L}(v_{3n-2}) - 1$$

$$\mathcal{L}(v_{3i}) = \mathcal{L}(v_{3n}) - i, \quad 1 \leq i \leq n - 1$$

$$\mathcal{L}(v_{3n-4}) = \mathcal{L}(v_{3(n-1)}) - 1$$

$$\mathcal{L}(v_{3n-1}) = \mathcal{L}(v_{3n-4}) - 1$$

$$\mathcal{L}(v_{3i-1)} = \mathcal{L}(v_{3n-1}) - i, \quad 1 \leq i \leq n - 2.$$

The label $\mathcal{L}$ satisfies the conditions of a g-convex label and hence $f_{4,n}$ is geodesically elegant.

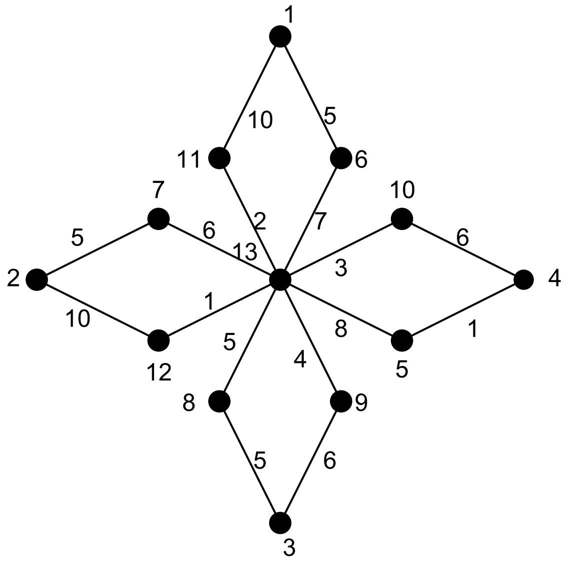


Figure 2.7: *Geodesic convex labelings of* $f_{4,4}$

□

Theorem 2.3.7. A Theta graph T_α is geodesically elegant.

Proof. If $v_0, v_1, v_2, v_3, v_4, v_5$ and v_6 are the vertices of a Theta graph T_α with centre v_0. We define the vertex labeling
$\mathcal{L} : V(T_\alpha) \to \{1, 2, 3, 4, 5, 6, 7\}$ as in Figure 2.8.

$\mathcal{C}_{\mathcal{L}}(T_\alpha) = \mathcal{C}(T_\alpha) = \{\emptyset, \{v_0\}, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}$
$\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_1\}, \{v_1, v_0\}, \{v_0, v_4\},$
$\{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \{v_4, v_3, v_5\}, \{v_4, v_5, v_6\}, \{v_1, v_5, v_6\}, \{v_2, v_1, v_0\}, \{v_1, v_0, v_4\}$
$, \{v_4, v_3, v_0\}, \{v_4, v_5, v_0\}, \{v_6, v_1, v_0\},$
$\{v_0, v_1, v_2, v_3, v_4\}, \{v_0, v_1, v_4, v_5, v_6\}$
$\{v_0, v_1, v_2, v_3, v_4, v_5, v_6\}\}.$

Since $\mathcal{C}_{\mathcal{L}}(T_\alpha) = \mathcal{C}(T_\alpha)$, the graph T_α is geodesically elegant.

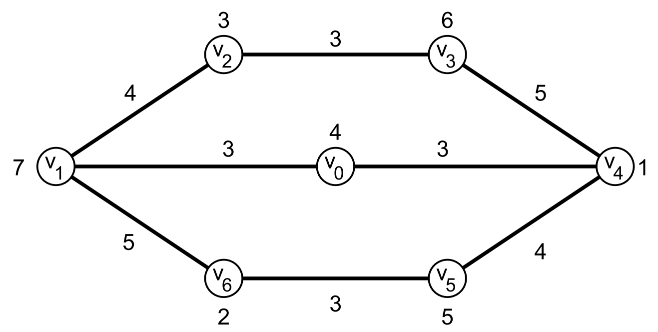


Figure 2.8: *Geodesic convex labelings of T_α*

□

Theorem 2.3.8. The graph G_v obtained by switching of any vertex v in a graph G where the non-neighbours of v contains atleast one edge is not geodesically elegant.

Proof. The vertex switching of any vertex v in a graph G where the non-neighbours of v contains atleast one edge produces a triangle, by Proposition 2.3.2 geodesic convex label does not exist in the graph G_v.

Corollary 2.3.9. *The graph obtained by switching of any vertex in a Theta graph T_α is not geodesically elegant.*

Figure 2.9: *The switching of a vertex v_1 and v_5 in T_α*

□

Remark 2.3.2. *The graph G_v obtained by switching of any vertex v in G such that $G - N[v]$ is an independent set, may or may not be geodesically elegant.*

Take $G = C_4$, geodesically elegant. Then the graph obtained by switching of a vertex a in G is geodesically elegant.

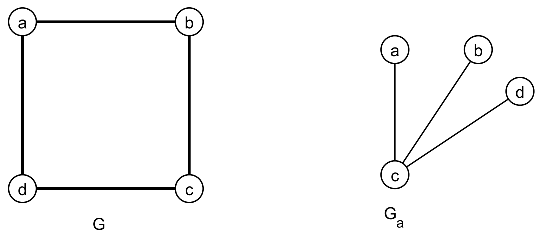


Figure 2.10: *The switching of the vertex a in $G = C_4$, $G_v = K_{1,3}$*

Take $G = K_{3,3}$, not geodesically elegant. Here, the graph obtained by switching of a vertex a in G is not geodesically elegant.

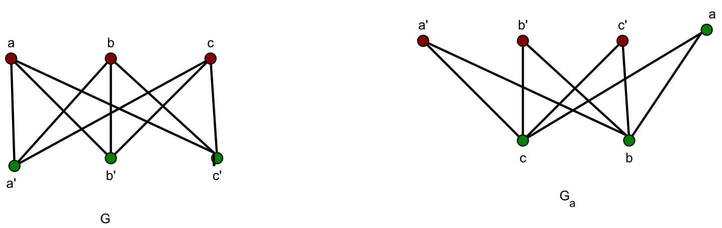


Figure 2.11: *The switching of the vertex a in $G = K_{3,3}$, $G_v = K_{2,4}$*

Theorem 2.3.10. Geodesic convex label does not exist in the complete bipartite graph $K_{m,n}$ except for $m = 1$ or $n = 1$ or $m = n = 2$.

Proof. Case 1: $m = 1$ or $n = 1$

$K_{m,n}$ is a tree, by remark 2.3.1 it is geodesically elegant.

Case 2: $m, n = 2$

The $g-$ convex label of $K_{2,2}$ is shown in the Figure 2.12.

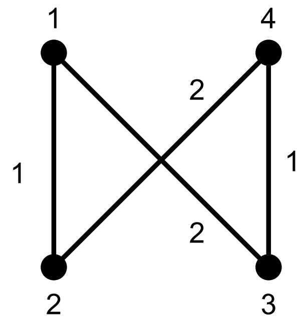


Figure 2.12: $K_{2,2}$

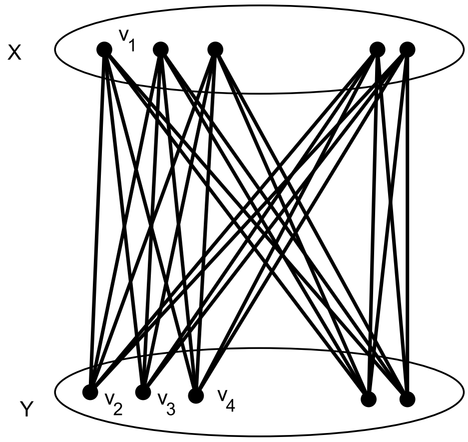
Case 3: $m \geq 2$, $n \geq 2$, $\quad m+n \geq 5$

Let $\{v_i, 1 \leq i \leq m+n\}$ be the vertices of $K_{m,n}$. Then $|V| = m+n$.
Suppose X and Y be the partition of V with $|X| = m$ and $|Y| = n$. $g-$ convex sets of $K_{m,n}$ are $\emptyset$, singleton sets, two point sets consisting of adjacent vertices and V.

Define $\mathcal{L} : V \rightarrow \{1, 2, 3, ..., m+n\}$ as $\mathcal{L}(v_i) = i, \quad 1 \leq i \leq m+n$.

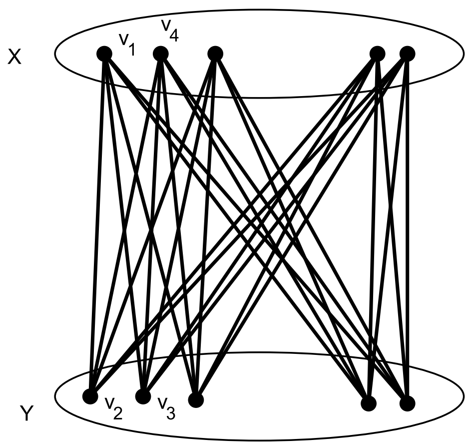
If $\mathcal{L}$ is a geodesic convex label, then the $\mathcal{L}_g$ convex sets are the same as $g-$ convex sets.
Let $v_1 \in X$.
Let $r = minimum\{t : v_t \in X, t \geq 2\}$.

Subcase 1 : suppose $r > 4$.
Then v_2, v_3 and $v_4 \in Y$. So $v_2 - v_1 - v_3$ is the only $\mathcal{L}$-geodesic connecting v_2 and v_3. For all other $v_i \in X$, $v_2 - v_i - v_3$ i s a path of length greater than 3. Thus the set $\{v_1, v_2, v_3\}$ is an $\mathcal{L}_g$ convex set.



Subcase 2 : suppose $r = 4$.

Then v_2 and $v_3 \in Y$ and $v_4 \in X$.
The $\mathcal{L}$- geodesic connecting v_2 and v_3 are $v_2 - v_1 - v_3$ and $v_2 - v_4 - v_3$. Also, $\mathcal{L}$- geodesic connecting v_1 and v_4 are $v_1 - v_2 - v_4$ and $v_1 - v_3 - v_4$. Thus the set $\{v_1, v_2, v_3, v_4\}$ is an $\mathcal{L}_g$ convex set.



Subcase 3 : (i) suppose $r = 3$
Then $v_3 \in X$ and $v_2 \in Y$. As in sub case 1, $\{v_1, v_2, v_3\}$ is an $\mathcal{L}_g$ convex set.

(ii) Suppose $r = 2$.
Then v_1 and v_2 are in X and let $s = minimum\{i, v_i \in Y\}$. So $v_1 - v_s - v_2$ is the

only $\mathcal{L}$-geodesic connecting v_1 and v_2. Thus $\{v_1, v_2, v_s\}$ is an $\mathcal{L}_g$ convex set. Hence, geodesic convex label does not exist in the complete bipartite graph $K_{m,n}$, except for $m = 1$ or $n = 1$ or $m = n = 2$.

□

We have proved that geodesically elegant graphs are triangle free, so we can't find the geodesic convex label in K_n for $n \geq 3$. In that case we can find the number of $\mathcal{L}_g$ convex sets.

Theorem 2.3.11. The number of $\mathcal{L}_g$ convex sets of K_n for $n \geq 3$ with respect to any label $\mathcal{L}$ is $\frac{n^2+n+2}{2}$.

Proof. Let $G \cong K_n$
. $V = V(G) = \{v_1, v_2, v_3, ..., v_n\}$.
Let $\mathcal{L} : V \to \{1, 2, 3, .., n\}$ be a labeling. Assume without loss of generality that $\mathcal{L}(v_i) = i$, for $i = 1$ to n.
Claim : For every $i = 1, 2, ..., n$, $j = 0, 1, 2, .., n - i$,
the set $C = \{v_i, v_{i+1}, ..., v_{i+j}\}$ is $\mathcal{L}_g$ convex.
For any $i \leq k \leq l \leq i + j$
$I_{\mathcal{L}}[v_k, v_l] = \{v_s : d_{\mathcal{L}}(v_k, v_l) = d(v_k, v_s) + d(v_s, v_l)\} = \{v_s : k \leq s \leq l\} \subset C$.
Hence C is convex.
on the other hand let W be an $\mathcal{L}_g$ convex subset of V.
Let $i = Minimum\{k : v_k \in W\}$
and $j = Maximum\{k : v_k \in W\}$. Then for any s such that $i < s < j$ we have $d_{\mathcal{L}}(v_i, v_s) = s - i$, $d_{\mathcal{L}}(v_s, v_j) = j - s$, and

$$\begin{aligned} d_{\mathcal{L}}(v_i, v_j) &= j - i = (j - s) + (s - i) \\ &= d_{\mathcal{L}}(v_s, v_j) + d_{\mathcal{L}}(v_i, v_s). \end{aligned}$$

Therefore $v_s \in W$ for every S such that $i \leq s \leq j$.
Hence the convex sets are precisely $\emptyset$ and those sets of the form $\{v_i, v_{i+1}, ..., v_{i+j}\}$.
Let m_i denote the number of convex sets with i vertices for $i = 0, 1, 2, .., n$.
Then $m_0 = 1$, $m_1 = n$
, $m_2 = n - 1$,....$m_n = 1$. Hence

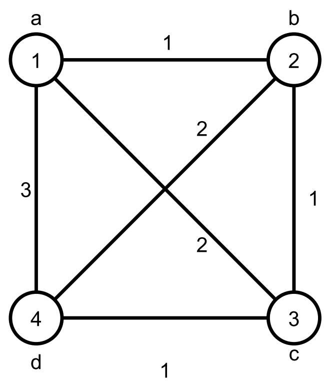
$$|\mathcal{C}_{\mathcal{L}}| = 1 + n + n - 1 + ... + 1.$$

$$= 1 + \frac{n(n+1)}{2} = \frac{n^2+n+2}{2}$$

□

Illustration:

Consider K_4



$\mathcal{L}_g$ convex sets of K_4 are $\emptyset$, one point sets, $\{a, b\}, \{b, c\}$, $\{c, d\}, \{b, c, d\}, \{a, b, c\}, \{a, b, c, d\}$
That is there are 11 $\mathcal{L}_g$convex sets.
when $n = 4$, $\frac{n^2+n+2}{2} = \frac{22}{2} = 11$.

2.3.2 $g-$ convex label in Graph Products and Join

In this section, we are discussing about the geodesically elegant graphs in some graph products and graph operations.

Remark 2.3.3. Strong product of any two graphs G and H, both having atleast one edge is not geodesically elegant. By the definition of strong product, $G \boxtimes H$ contains a subgraph K_3. Using Proposition 2.3.2, $G \boxtimes H$ is not geodesically elegant.

Remark 2.3.4. Lexicographic product of any two graphs G and H, both having atleast one edge is not geodesically elegant. By the definition of lexico graphic product, $G[H]$ contains a subgraph K_3. Using Proposition 2.3.2, $G[H]$ is not geodesically elegant.

Example 2.3.1. The join of two geodesically elegant graphs need not be geodesically elegant. For, Consider $G = C_4$ and $H = P_2$ are two geodesically elegant graphs. Then $G+H$ contains a subgraph K_3, by Proposition 2.3.2, $G+H$ is not geodesically elegant.

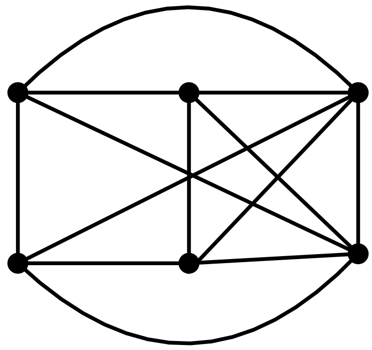


Figure 2.13: $G + H$

Example 2.3.2. The corona product of two geodesically elegant graphs may not be geodesically elegant. For, example, $G = C_4$ and $H = P_2$ are two geodesically elegant graphs. Then $G \circ H$ contains the subgraph K_3, by Proposition 2.3.2, $G \circ H$ is not geodesically elegant.

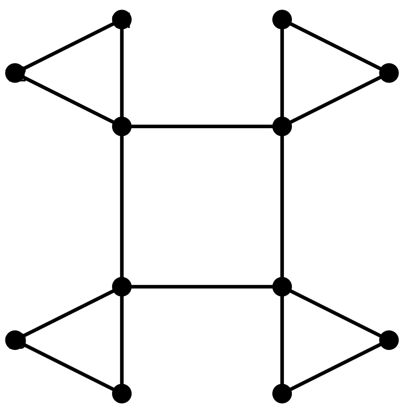


Figure 2.14: $G \circ H$

Theorem 2.3.12. The graph $C_m \square P_n$ is geodesically elegant.

Proof. To prove the existence of a g-convex label, find a vertex label $\mathcal{L}$ such that

the convexity induced by the label coincides with the geodesic convexity in $C_m \Box P_n$. Let $V = \{u_{ij}, 1 \leq i \leq m, 1 \leq j \leq n\}$ and
$E = \{u_{ij}u_{(i+1)j}, 1 \leq i \leq (m-1), 1 \leq j \leq n\} \cup \{u_{ij}u_{i(j+1)}, 1 \leq j \leq (n-1), 1 \leq i \leq m\} \cup \{u_{mj}u_{1j}, 1 \leq j \leq n\}$.

Then $|V| = mn$.

Ordinary labeling of $C_m \Box P_n$ is shown in figure 2.15

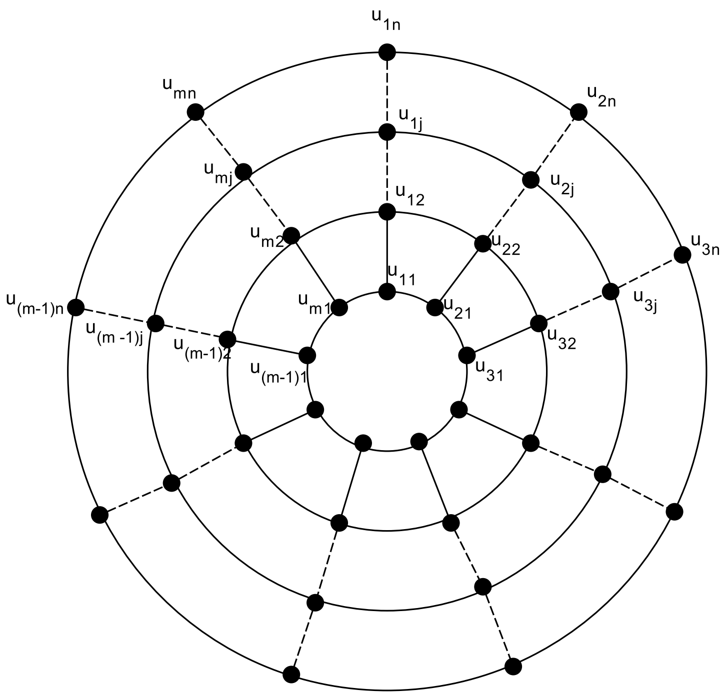


Figure 2.15: *Ordinary labeling of* $C_m \Box P_n$

Define $\mathcal{L} : V \rightarrow \{1, 2, 3, ..., mn\}$ by

Case 1 : m is even

$$\mathcal{L}(u_{i1}) = 2i - 1, \quad 1 \leq i \leq \frac{m}{2}$$
$$\mathcal{L}(u_{i1}) = m, \quad i = \frac{m}{2} + 1$$
$$\mathcal{L}(u_{i1}) = \mathcal{L}(u_{(i-1)1}) - 2, \quad i = \frac{m}{2} + 2 \leq i \leq m.$$
$$\mathcal{L}(u_{ij}) = \mathcal{L}(u_{i(j-1)}) + m, 1 \leq i \leq m \quad and \quad 2 \leq j \leq n.$$

Case 2 : m is odd

$$\mathcal{L}(u_{i1}) = 2i-1, \quad 1 \leq i \leq \frac{m+1}{2}$$
$$\mathcal{L}(u_{i1}) = 2, \quad i = [\frac{m}{2}]+2$$
$$\mathcal{L}(u_{ij}) = \mathcal{L}(u_{(i-1)1})+2, \quad i = [\frac{m}{2}+3] \leq i \leq m.$$
$$\mathcal{L}(u_{ij}) = \mathcal{L}(u_{i(j-1)})+m, 1 \leq i \leq m \text{ and } 2 \leq j \leq n.$$

The label $\mathcal{L}$ satisfies the conditions of a geodesic convex label and hence $C_m \square P_n$ for all $m > 3$ is geodesically elegant.

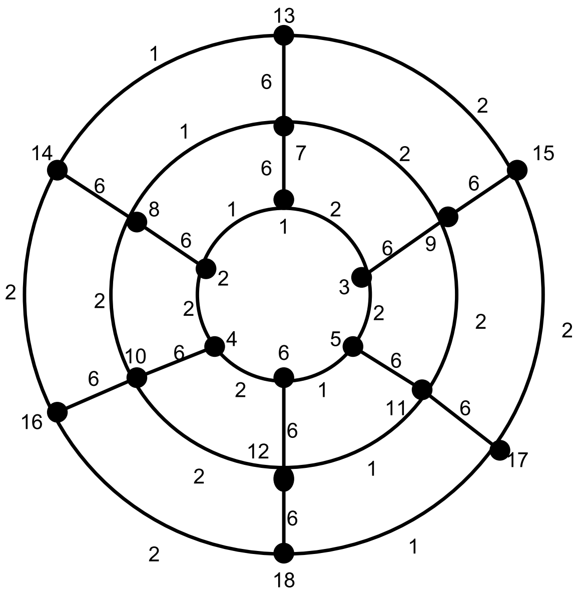


Figure 2.16: *Geodesic convex labelings of* $C_6 \square P_3$

□

Theorem 2.3.13. The square mesh graph $P_r \square P_r$ is geodesically elegant.

Proof. Let $V = a_{i,j} : 1 \leq i \leq r, 1 \leq j \leq r\}$ and

$E = \{a_{(i-1),j}a_{i,j} : 2 \leq i \leq r, 1 \leq j \leq r\} \cup \{a_{i,j}a_{i,(j-1)} : 1 \leq i \leq r, 2 \leq j \leq r\}$
The $g-$ convex sets of $P_r \square P_r$ are of the form $\mathcal{C} = \{A \times B/A$ and B are the $g-$ convex sets of $P_r\}$.
To prove the existence of a g-convex label, find a vertex label $\mathcal{L}$ such that the convexity induced by the label coincides with the geodesic convexity in $P_r \square P_r$.

Define $\mathcal{L} : V \rightarrow \{1, 2, 3, ..., r^2\}$ by

$$\mathcal{L}(a_{i,1}) = i, \quad i = 1 \ \ to \ \ r$$
$$\mathcal{L}(a_{i,j}) = \mathcal{L}(a_{i,(j-1)}) + r, \ \ i = 1 \ \ to \ \ r \ \ and \ \ j = 2 \ \ to \ \ r$$

The label $\mathcal{L}$ satisfies the conditions of a geodesic convex label and hence $P_r \square P_r$ is geodesically elegant.

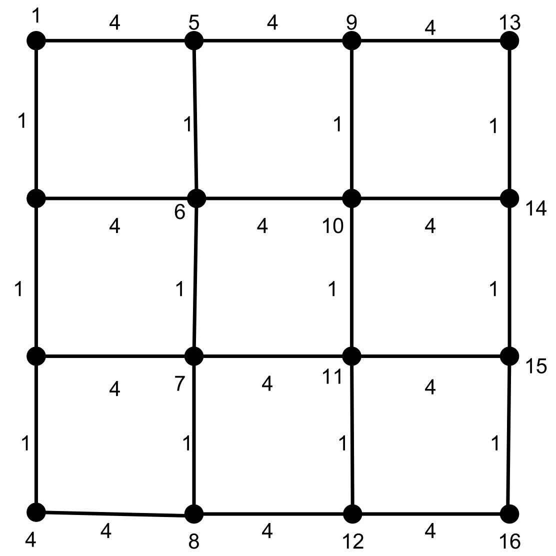


Figure 2.17: *Geodesic convex labelings of* $P_4 \square P_4$

□

Theorem 2.3.14. The ladder graph $L_n = P_n \square P_2$ is geodesically elegant.

Proof. Let $u_1, u_2, \ldots, u_{(n-1)}, u_n$ and $u'_1, u'_2, \ldots, u'_{(n-1)}, u'_n$ be the vertices of the ladder graph $L_n = P_n \square P_2$. Then
$E = \{u_iu_{i+1}, 1 \leq i \leq (n-1)\} \cup \{u'_iu'_{i+1}, 1 \leq i \leq (n-1)\} \cup \{u_iu'_i, 1 \leq i \leq n\}$.
$|V| = 2n$ and $|E| = 3n - 2$.

Define $\mathcal{L} : V \to \{1, 2, 3, ..., 2n\}$ by

$$\mathcal{L}(u_i) = 2i - 1,\ 1 \leq i \leq n$$
$$\mathcal{L}(u_i') = 2i,\ 1 \leq i \leq n$$

The function $\mathcal{L}$ satisfies all the conditions of a $g-$ convex label. Hence the ladder graph $L_n = P_n \Box P_2$ is geodesically elegant.

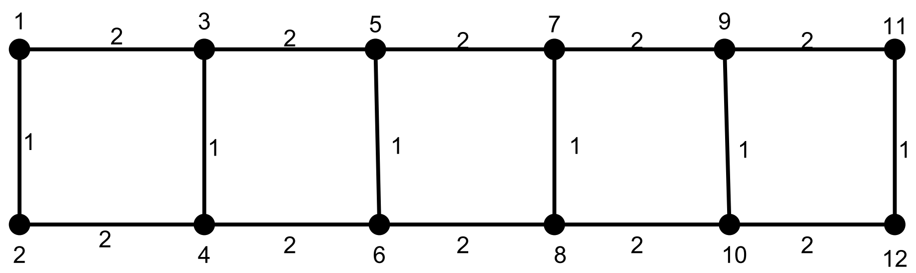


Figure 2.18: *Geodesic convex labelings of L_6*

□

2.4 Monophonic Convexity in Labeled Graphs

Monophonic convex sets in graphs were discussed for the first time by Farber and Jamison [34]. As we studied geodesic convexity in labeled graphs, we focused on to the next convexity - monophonic convexity. Like geodesic convex label, we define monophonic convex label for a graph G.

A labeled graph can be treated as a weighted graph and weighted monophonic convexity can be seen in the literature in [34]. Jill K Mathew and Sunil Mathew [34, 33] defined chord of a path in weighted graph using the concept strength of a path, which seems to be not apt for our discussion. So we have redefined the definition of a chord to use in labeled graphs. The following definitions constitute a background for the monophonic convexity in labeled graphs.

Definition 2.4.1. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. Let $P = v_0e_1v_1e_2v_2.....e_nv_n$ be a $v_0 - v_n$ path. A chord of P is an edge $e = (v_i, v_j), j \geq i + 2$ (that is it joins two non-consecutive vertices of the path) such that $\sum_{e_i \epsilon P} \mathcal{L}(e_i) \geq \sum_{e_i \epsilon P'} \mathcal{L}(e_i)$ where $\mathcal{L}(e_i)$ denotes the label of an edge e_i and $P' = v_0 e_1 v_1 ... v_i e v_j .... e_n v_n$. If strict inequality occurs, it is called a strong chord.

Definition 2.4.2. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. A $u - v$ path P in $\Gamma_{\mathcal{L}}$ is called a $\mathcal{L}$-monophonic $u - v$ path ($\mathcal{L}$- induced $u - v$ path) if it has no chords and it is called a strong $\mathcal{L}$-monophonic $u - v$ path if it has no strong chords.

Definition 2.4.3. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The $\mathcal{L}$-monophonic closed interval $J_{\mathcal{L}}[u, v]$ is the set of all vertices in all $\mathcal{L}$-monophonic $u - v$ path including u and v. The strong $\mathcal{L}$-monophonic closed interval $J'_{\mathcal{L}}[u, v]$ is the set of all vertices in all strong $\mathcal{L}$-monophonic $u - v$ path including u and v.

Definition 2.4.4. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The union of all $\mathcal{L}$-monophonic closed intervals $J_{\mathcal{L}}[u, v]$ over all pairs $u, v \in S$ is called $\mathcal{L}$-monophonic closure of S and is denoted by $J_{\mathcal{L}}[S]$. A subset S of V is called $\mathcal{L}$-monophonic (or simply $\mathcal{L}_m$ convex) if $J_{\mathcal{L}}[S] = S$. That is for every $x, y \in S$, the vertices on an $x - y$ $\mathcal{L}$-monophonic path belongs to S. The union of all strong $\mathcal{L}$-monophonic closed intervals $J'_{\mathcal{L}}[u, v]$ over all pairs $u, v \in S$ is called strong $\mathcal{L}$-monophonic closure of S and is denoted by $J'_{\mathcal{L}}[S]$. A subset S of V is called strong $\mathcal{L}$-monophonic (or simply strong $\mathcal{L}_m$convex) if $J'_{\mathcal{L}}[S] = S$. That is for every $x, y \in S$, the vertices on an $x - y$ strong $\mathcal{L}$-monophonic path belongs to S.

Consider four different labelings of the cycle C_4

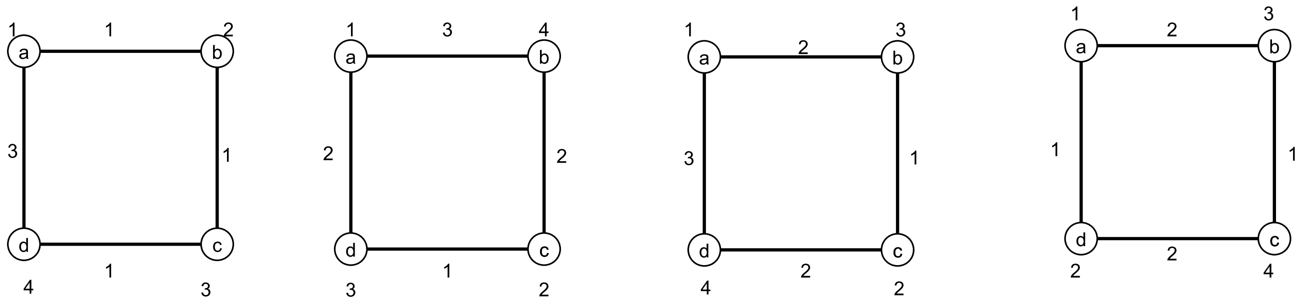


Figure 2.19: *4 different labelings of* C_4 : $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$

The $m-$ convex sets of C_4 are c singleton subsets, $\{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}$ and $\{a,b,c,d\}$.

$\mathcal{L}_m$ convex sets with respect to the vertex labelings $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ are $\emptyset$, $\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}$ and $\{a,b,c,d\}$ In all four cases the function $\mathcal{L}$ preserves the m- convex sets , that is the m - convex sets of G and $\Gamma_{\mathcal{L}}$ are same. In all the cases, except $\mathcal{L}_1$, are strong $\mathcal{L}_m$ convex sets.
Consider K_4.
The m-convex sets of K_4 are $\emptyset$, singleton subsets, $\{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, \{a,c\}, \{b,d\}$ and $\{a,b,c,d\}$.
The $\mathcal{L}_m$ convex sets are $\emptyset$, singleton subsets, $\{a,b\}, \{b,c\}, \{c,d\}, \{a,d\}, \{a,c\}, \{b,d\}$ and $\{a,b,c,d\}$ and strong $\mathcal{L}_m$ convex sets are $\emptyset$, singleton subsets, $\{a,b\}, \{b,c\}, \{c,d\}$, $\{b,c,d\}, \{a,b,c\}$ and $\{a,b,c,d\}$.

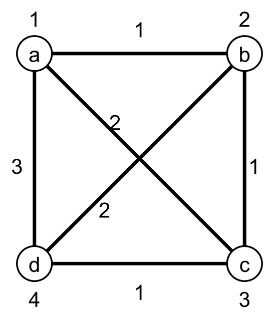


Figure 2.20: *Labeling of* K_4

The empty set, the whole vertex set and every one point sets are convex with respect to any vertex labeling $\mathcal{L}$.
In weighted graph, any edge is a weighted monophonic path between its end vertices [34], in labeled graph, any edge is a $\mathcal{L}$ - monophonic path between its end vertices, not true for strong $\mathcal{L}_m$ convex.
Thus we have the following definitions.

Definition 2.4.5. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The function $\mathcal{L}$ is called a monophonic label or simply m-convex label if the convexity $\mathcal{C}_{\mathcal{L}}$ induced by the function $\mathcal{L}$ is the same as the monophonic convexity $\mathcal{C}$ in V. That is, the m-convex sets of $\mathcal{C}$ of G are the same as $\mathcal{L}_m$ convex sets of $\Gamma_{\mathcal{L}}$. A graph G is monophonically elegant if there exist $m-$ convex label for G. The function $\mathcal{L}$ is called a strong monophonic label (strong m- convex label) if the strong $\mathcal{L}_m$ convex sets are the same as the m- convex sets of G.

Definition 2.4.6. An $\mathcal{L}_m$ convexity space is an ordered pair $(\Gamma_{\mathcal{L}}, \mathcal{C}_{\mathcal{L}})$ where, $\Gamma_{\mathcal{L}}$ is a labeled graph and $\mathcal{C}_{\mathcal{L}}$ is the convexity induced by the label $\mathcal{L}$

Remark 2.4.1. In a tree each pair of vertices are connected by a unique path. Therefore, the $\mathcal{L}_m$ convex sets with respect to any labeling function is always same as the m-convex sets of V. Hence m-convex label exists in a tree.

Remark 2.4.2. In K_4, The $\mathcal{L}_m$ convex sets are the same as the m- convex sets of V, monophonic convex label exist in K_4. But convexity with respect to strong monophonic convex label does not exist in K_4.

In the following Proposition, we characterize the necessary condition for the existence of strong monophonic convex label in a graph $\Gamma_{\mathcal{L}}$.

Proposition 2.4.7. If a strong monophonic convex label exists in a graph G then G is triangle free.

Proof. Suppose that G contains a triangle. Let us label the vertices of using the numbers a, b, and c with $a < b < c$ as in Figure2.2. Then $d_{\mathcal{L}}(v_1, v_3) = d_{\mathcal{L}}(v_1, v_2) +$

$d_{\mathcal{L}}(v_2, v_3)$. Hence $v_2 \in J'_{\mathcal{L}}[\{v_1, v_3\}]$. Thus the two point subset $\{v_1, v_3\}$ is not convex. Hence we conclude that strong monophonic convex label exists in G if G is triangle free.

□

Example 2.4.1. Strong monophonic convex label exists in the Petersen graph. The labeling defined in Figure 2.21 satisfies the conditions of a strong monophonic convex label.

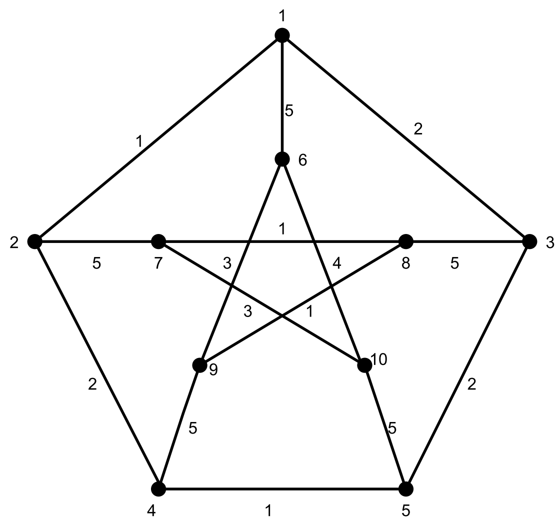


Figure 2.21: *strong monophonic labeling of the petersengraph*

Theorem 2.4.8. *Strong monophonic convex label exist in the cycle C_n for $n > 3$.*

Proof. To prove the existence of a strong monophonic convex label, we have to find a vertex labeling function $\mathcal{L}$ such that the convexity induced by the labeling function coincides with the $m-$ convexity in $C_n, n > 3$. Let $v_1, v_2, v_3, ..., v_n$ be the vertices of the cycle C_n. Define $\mathcal{L} : V \to \{1, 2, 3, ..., n\}$ by
Case 1: if n is even

$$\mathcal{L}(v_i) = 2i - 1, 1 \leq i \leq \tfrac{n}{2}$$
$$\mathcal{L}(v_i) = n, \quad i = \tfrac{n}{2} + 1$$
$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) - 2, i = \tfrac{n}{2} + 2 \leq i \leq n.$$

Case 2 : if n is odd

$$\mathcal{L}(v_i) = 2i - 1, 1 \leq i \leq [\tfrac{n}{2}]$$
$$\mathcal{L}(v_i) = n, \quad i = [\tfrac{n}{2}] + 1$$
$$\mathcal{L}(v_i) = n - 1, i = \tfrac{n+3}{2}$$
$$\mathcal{L}(v_i) = \mathcal{L}(v_{i-1}) - 2, i = \tfrac{n+5}{2} + 2 \leq i \leq n.$$

The function defined above satisfies the condition for a strong monophonic convex label. □

2.5 Conclusion

We made an attempt to study the concept of geodesic convexity and monophonic convexity in labeled graphs. A new class of graphs namely, geodesically elegant graphs are introduced. Studied some family of geodesically elegant graphs. We have'nt obtained the general result - the cartesian product of two geodesically elegant graphs are geodesically elegant. It is an open problem. To investigate similar results with other labeling function and other convexities in the literature is a future area of research.

Problem 2.5.1. *Is the Petersen graph geodesically elegant?*

CHAPTER 3

Geodetic Number and Edge Geodetic Number in Labeled Graphs

3.1 Introduction

In this chapter we discuss the geodetic and edge geodetic number in labeled graphs. For a non empty subset S of $V, I[S] = \{u \in V,\ u$ is in some geodesic connecting two vertices of $S\}$. The set S is convex if $I[S] = S$. A set of vertices of V is called a geodetic set in G if $I[S] = V$. The cardinality of a minimum geodetic set in G is called the geodetic number $g(G)$. The geodetic number of a graph was introduced by Frank Harary, Emmanuel Loukakis and Constantine Tsouros[17]. Further studied by Gary Chartrand, Ping Zhang and it is proved that If G is a non trivial connected graph of order n, then $2 \leq g(G) \leq n$ [21, 22]. For a non empty subset S of $V, T[S] = \{e \in E,\ e$ is in some geodesic connecting two vertices of $S\}$. A set S of vertices of V is defined to be an edge geodetic set in G if $T[S] = E$. The cardinality of a minimum edge geodetic set in G is the edge geodetic number $g_e(G)$ [58]. The edge geodetic number of graphs was studied in [73, 58]. For any connected graph G, $g(G) \leq g_e(G)$.

For any non- trivial tree T, $g(T) = g_e(T)$. The geodetic number of a disconnected

graph is the sum of the geodetic number of its components and the edge geodetic number of a disconnected graph is the sum of the edge geodetic number of its components. For any graph G of order n, $2 \leq g_e(G) \leq n$. The geodetic and edge geodetic number of some classes of graphs are seen in [17, 21, 73, 58] and some of them are given below.
$g(P_n) = 2$.

$$g(C_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Every tree T with k end vertices has $g(T) = k$. The complete graph K_n has $g(K_n) = n$ $g_e(K_n) = n$. For the path graph P_n, $g_e(P_n) = 2$. Every mesh $M = M_{r,s} = P_r \square P_s$ has $g(M) = 2$.

The Petersen graph has geodetic number 4. For any non- trivial tree T, $g(T) = g_e(T)$. The Wheel graph $W_{1,n}$ has

$$g(W_{1,n}) = \begin{cases} 4 & \text{if } n = 3 \\ \lceil n/2 \rceil & n \geq 4 \end{cases}$$

For any friendship graph $F_3^n, n \geq 2, g_e(F_3^n) = 2n$. For the Windmill graph $Wd(k,n)$, $g_e(Wd(k,n)) = k(n-1)$.

3.2 The $\mathcal{L}$-geodetic number and $\mathcal{L}$-edge geodetic number.

In this section we investigate the geodetic and edge geodetic number in labeled graphs. The concept of $\mathcal{L}$- geodetic number and $\mathcal{L}$ - edge geodetic number is introduced.
In the previous chapter we have defined an $\mathcal{L}$- geodesic set. Here, we define an $\mathcal{L}$-geodetic set.

Definition 3.2.1. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. Let $S \subset V$. The $\mathcal{L}$- geodesic closure $I_{\mathcal{L}}[S]$ of a set $S \subset V$ is defined as $I_{\mathcal{L}}[S] = \{u \in V, u$ is in some $\mathcal{L}$- geodesic connecting two vertices of $S\}$. A set S of vertices is defined to be a $\mathcal{L}$-geodetic set if $I_{\mathcal{L}}[S] = V$. The $\mathcal{L}$- geodetic number $g_{\mathcal{L}}(\Gamma_{\mathcal{L}})$ is the cardinality of a smallest $\mathcal{L}$- geodetic set. Simply, we can denote $g_{\mathcal{L}}(\Gamma_{\mathcal{L}})$ as $g_{\mathcal{L}}(G)$.

Definition 3.2.2. For any two vertices u and v in $\Gamma_{\mathcal{L}}$, we define $T_{\mathcal{L}}[u, v]$ to be the set of all edges lying on uv $\mathcal{L}$-geodesic. For a nonempty subset S of V, $T_{\mathcal{L}}[S] = \{T_{\mathcal{L}}[u, v], u, v \in S\}$. .

Definition 3.2.3. In a labeled graph $\Gamma_{\mathcal{L}}$, a set S of vertices is defined to be an $\mathcal{L}$-edge geodetic if $T_{\mathcal{L}}[S] = E$. The $\mathcal{L}$ - edge geodetic number denoted by $g'_{\mathcal{L}}(\Gamma_{\mathcal{L}})$ is the cardinality of a smallest $\mathcal{L}$- edge geodetic set. It is also denoted by $g'_{\mathcal{L}}(G)$.

For a connected graph G with n vertices, we have $2 \leq g_{\mathcal{L}}(G) \leq n$ and $2 \leq g'_{\mathcal{L}}(G) \leq n$.

Consider a graph G as shown in Figure 3.1.

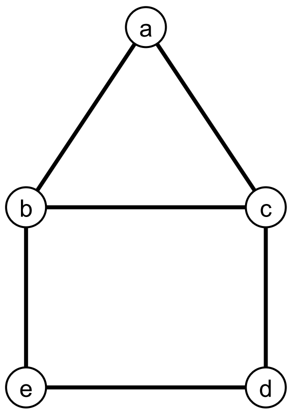


Figure 3.1

The set $S = \{a, c, e\}$ is a minimum geodetic set and a minimum edge geodetic set of G, so $g(G) = g_e(G) = 3$.

Consider different vertex labelings, denoted by $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ of G.

In $\Gamma_{\mathcal{L}_1}$, the set $S = \{a, b\}$ is a minimum $\mathcal{L}$- geodetic set and a minimum $\mathcal{L}$- edge geodetic set. So $g_{\mathcal{L}_1}(G) = g'_{\mathcal{L}_1}(G) = 2$.

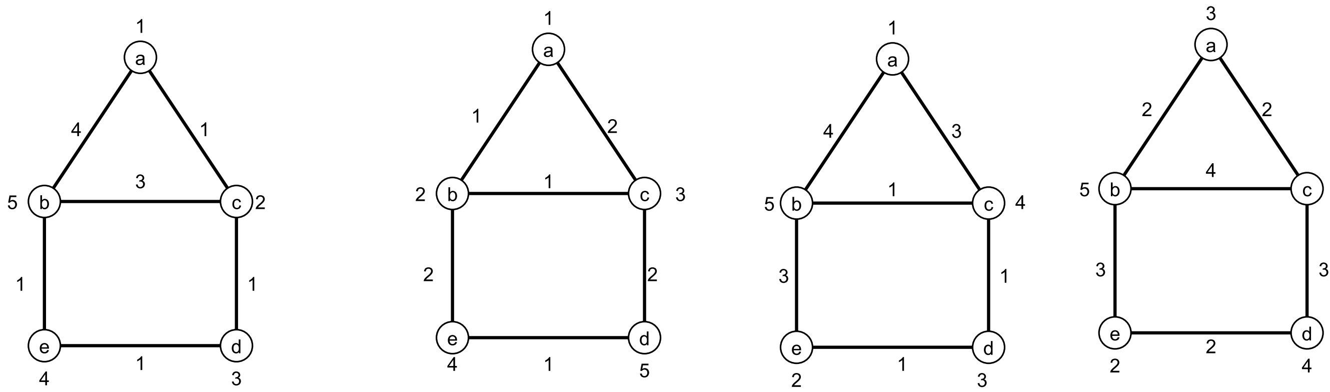


Figure 3.2: *Different labelings of a grah G : $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$*

In $\Gamma_{\mathcal{L}_2}$, $I_{\mathcal{L}_2}[S] = \{a, b\}$ and $T_{\mathcal{L}_2}[S] = \{a, b\}$ for $S = \{a, b\}$.

Take $S = \{a, d\}$, then S is is a minimum $\mathcal{L}$- geodetic set and a minimum $\mathcal{L}$-edge geodetic set. So $g_{\mathcal{L}_2}(G) = g'_{\mathcal{L}_2}(G) = 2$.

In $\Gamma_{\mathcal{L}_3}$, any two element subset of V is not a minimum $\mathcal{L}$- geodetic set and a minimum $\mathcal{L}$- edge geodetic set. So $g_{\mathcal{L}_3}(G) \geq 3$ and $g'_{\mathcal{L}_3}(G) \geq 3$.

Consider $S = \{a, b, e\}$, then $I_{\mathcal{L}_3}[S] = \{a, b, c, d, e\}$ and $T_{\mathcal{L}_3}[S] = \{ab, bc, cd, de, eb, ac\}$, $g_{\mathcal{L}_3}(G) = g'_{\mathcal{L}_3}(G) = 3$.

In $\Gamma_{\mathcal{L}_4}$, the set $S = \{a, e, d\}$ is a minimum $\mathcal{L}$ - geodetic set, but not a minimum $\mathcal{L}$- edge geodetic set. $S = \{b, c, d\}, I_{\mathcal{L}_4}[S] = V$ and $T_{\mathcal{L}_4}[S] = E$. So $g_{\mathcal{L}_4}(G) = g'_{\mathcal{L}_4}(G) = 3$.
For a given $\Gamma_{\mathcal{L}}$, $g_{\mathcal{L}}(G)$ and $g'_{\mathcal{L}}(G)$ may vary with respect to the label $\mathcal{L}$. Also an $\mathcal{L}$-geodetic set may not be an $\mathcal{L}$-edge geodetic set. Note that $g(G)$ need not be equal to $g_{\mathcal{L}}(G)$ and $g_e(G)$ need not be equal to $g'_{\mathcal{L}}(G)$.

Remark 3.2.1. In any vertex labeling $\mathcal{L}$ of the path P_n, the end vertices form a minimum $\mathcal{L}$- geodetic set and $\mathcal{L}$-edge geodetic set and so $g_{\mathcal{L}}(P_n) = g'_{\mathcal{L}}(P_n) = 2$.

Remark 3.2.2. For a tree T, each pair of vertices are connected by a unique path, therefore if T has k end nodes, then $g_{\mathcal{L}}(T) = g'_{\mathcal{L}}(T) = k$ with respect to any label $\mathcal{L}$. In particular for the star graph $K_{1,n}, g_{\mathcal{L}}(T) = g'_{\mathcal{L}}(T) = n$.

Theorem 3.2.4. $g_{\mathcal{L}}(K_n) = g'_{\mathcal{L}}(K_n) = 2$.

Proof. Label the vertices of K_n using the numbers $1, 2, 3, ..., n$. Since all the vertices are adjacent in K_n, the vertices with labels 1 and n is a minimum $\mathcal{L}$-geodetic and minimum $\mathcal{L}$-edge geodetic set, hence $g_{\mathcal{L}}(K_n) = g'_{\mathcal{L}}(K_n) = 2$.

□

For any labeled graph $\Gamma_{\mathcal{L}}$, $g_{\mathcal{L}}(G) \leq g'_{\mathcal{L}}(G)$. Sometimes strict inequality may occur, which is illustrated in F igure 3.3. Here, $\{3, 6\}$ is a minimum $\mathcal{L}$-geodetic set and$\{3, 6, 1, 8\}$ minimum $\mathcal{L}$-edge geodetic set.

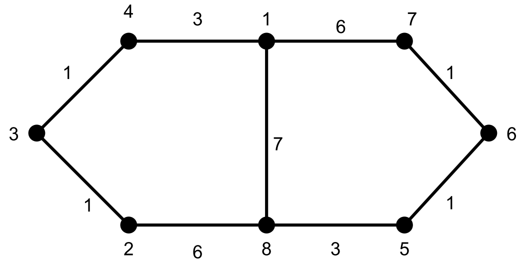


Figure 3.3

In Figure 3.2,
$g_{\mathcal{L}_1}(G) \leq g(G)$ and $g'_{\mathcal{L}_1}(G) \leq g_e(G)$;
$g_{L_2}(G) \leq g(G)$ and $g'_{\mathcal{L}_2}(G) \leq g_e(G)$;
$g_{L_3}(G) = g(G)$ and $g'_{\mathcal{L}_3}(G) = g_e(G)$.
In some cases, for a given label $\mathcal{L}$ we get $g_{\mathcal{L}}(G) > g(G)$ and $g'_{\mathcal{L}}(G) > g_e(G)$, as illustrated in Figure 3.4.

Consider C_4 with a label $\mathcal{L}$ as shown in Figure 3.4.
There does not exist any two element subset S of V with $I_{\mathcal{L}}[S] = V$ and $T_{\mathcal{L}}[S] = E$.
Then $g'_{\mathcal{L}}(C_4) > 2$ and $g_{\mathcal{L}}(C_4) > 2$.
If $S = \{a, b, c\}$, then $I_{\mathcal{L}}[S] = V$ and $T_{\mathcal{L}}[S] = E$. So $g'_{\mathcal{L}}(C_4) = g_{\mathcal{L}}(C_4) = 3$.
Thus $g'_{\mathcal{L}}(C_4) > g_e(C_4)$ and $g_{\mathcal{L}}(C_4) > g(C_4)$.

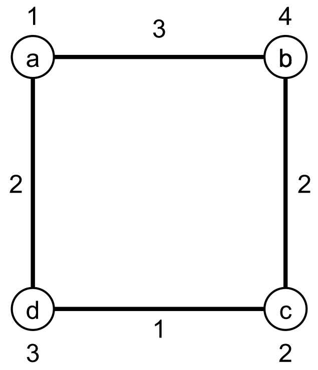


Figure 3.4: *A vertex labeling of C_4*

3.3 Geodetic Label and Edge Geodetic Label

In this section we discuss geodetic and edge geodetic labels and investigate in which classes of graphs these labels exists.

Definition 3.3.1. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The label $\mathcal{L}$ is said to be a geodetic label if the geodetic number $g(G)$ and the $\mathcal{L}$ geodetic number $g_{\mathcal{L}}(\Gamma_{\mathcal{L}})$ are the same. That is $g_{\mathcal{L}}(G). = g(G)$.
The label $\mathcal{L}$ is said to be an edge geodetic label if the edge geodetic number $g_e(G)$ and the $\mathcal{L}$- edge geodetic number$g'_{\mathcal{L}}(\Gamma_{\mathcal{L}})$ are the same. That is $g'_{\mathcal{L}}(G) = g_e(G)$.

Definition 3.3.2. Let $\Gamma_{\mathcal{L}}$ be a labeled graph. The label $\mathcal{L}$ is said to be a strong geodetic label if $g_{\mathcal{L}}(G) < g(G)$ and a weak geodetic label if $g_{\mathcal{L}}(G) > g(G)$.
The label $\mathcal{L}$ is said to be a strong edge geodetic label if $g'_{\mathcal{L}}(G) < g_e(G)$ and a weak geodetic label if $g'_{\mathcal{L}}(G) > g_e(G)$.

Remark 3.3.1. In any vertex labeling of K_n, $g_{\mathcal{L}}(K_n) = g'_{\mathcal{L}}(K_n) = 2$, strong geodetic label and strong edge geodetic label exist in $K_n, n > 2$.

In any vertex labeling of a tree T, $g_{\mathcal{L}}(T) = g(T)$ and $g'_{\mathcal{L}}(T) = g_e(T)$. So geodetic label and edge geodetic label exist in every tree.

Theorem 3.3.3. Geodetic label and edge geodetic label exist for every even cycle. Strong geodetic label and Strong edge geodetic label exist for every odd cycle.

Proof. To prove the existence, we have to find a vertex label $\mathcal{L}$ such that $g_{\mathcal{L}}(C_n) \leq g(C_n)$ and $g'_{\mathcal{L}}(C_n) \leq g_e(C_n)$. Let $v_1, v_2, ..., v_n$ be the vertices of the cycle C_n. Define $\mathcal{L} : V \to \{1, 2, 3, ..., n\}$ by

$\mathcal{L}(v_i) = i$, $i = 1$ to n.

The vertices labeled with 1 and n that is $\{v_1, v_n\}$ is a $\mathcal{L}$-geodetic set and an $\mathcal{L}$-edge geodetic set, that is $g_{\mathcal{L}}(C_n) = g'_{\mathcal{L}}(C_n) = 2$. If n is even, $g(C_n) = g_{\mathcal{L}}(C_n)$ and $g_e(C_n) = g'_{\mathcal{L}}(C_n)$.
If n is odd, $g(C_n) = 3 > g_{\mathcal{L}}(C_n)$ and $g_e(C_n) = 3 > g'_{\mathcal{L}}(C_n)$.

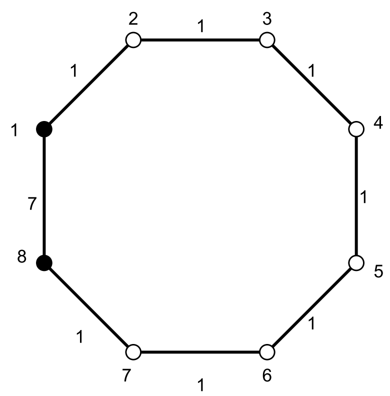


Figure 3.5: *The coloured vertices is the $g_{\mathcal{L}}$ set and $g'_{\mathcal{L}}$ set of C_8*

□

Theorem 3.3.4. Strong geodetic label exist in the Petersen graph.

Proof. To prove the existence, find a vertex label $\mathcal{L}$ such that $g_{\mathcal{L}}(G) < 4$. Label the vertices $\{v_i, i = 1$ to $10\}$ using the set $\{1, 2, 3, ..., 10\}$ as shown in Figure 3.6. The set $\{v_1, v_8, v_9\}$ is an $\mathcal{L}$-geodetic set and $g(G) = 4 > g_{\mathcal{L}}(G) = 3$.

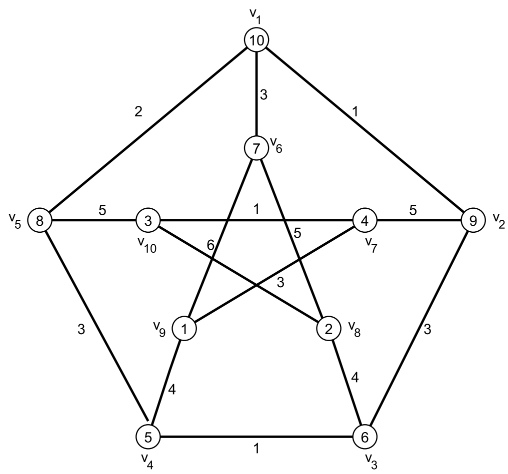


Figure 3.6: *vertex labeling of the Petersen graph*

□

Theorem 3.3.5. Strong geodetic label exists in the Wheel graph $W_{1,n}$ except for $n = 4$. Geodetic label exist in the graph $W_{1,4}$.

Proof. Let $v_0, v_1, v_2, ..., v_n$ be the vertices of the wheel graph $W_{1,n}$ with centre at v_0.

Define $\mathcal{L} : V \to \{1, 2, 3, ..., n+1\}$ by

$$\mathcal{L}(v_i) = i,\ 1 \leq i \leq n$$

$$\mathcal{L}(v_0) = n + 1.$$

The set $\{v_0, v_1\}$ is an $\mathcal{L}$- geodetic set.

When $n \neq 4$, $g(W_{1,n}) > g_{\mathcal{L}} = 2$

When $n = 4, g(W_{1,n}) = g_{\mathcal{L}} = 2.$

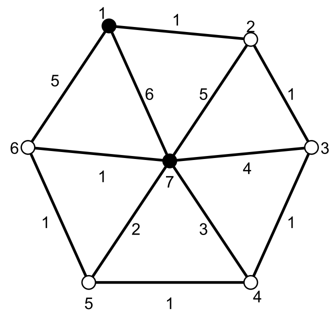


Figure 3.7: *The coloured vertices is the $g_{\mathcal{L}}$ set of $W_{1,6}$*

□

Theorem 3.3.6. *Geodetic label exist in the mesh* $M = M_{r,s} = P_r \square P_s$

Proof. Let $V = a_{i,j} : 1 \leq i \leq r, 1 \leq j \leq s\}$ and
$E = \{a_{(i-1),j}a_{i,j} : 2 \leq i \leq r, 1 \leq j \leq s\} \cup \{a_{i,j}a_{i,(j-1)} : 1 \leq i \leq r, 2 \leq j \leq s\}$
Define $\mathcal{L} : V \to \{1, 2, 3, ..., rs\}$ by

$$\mathcal{L}(a_{1,j}) = j, \quad j = 1 \quad to \quad s$$
$$\mathcal{L}(a_{i,j}) = \mathcal{L}(a_{(i-1),s}) + j, \ \ j = 1 \quad to \quad s \quad and \quad i = 2 \quad to \quad r$$

The set $\{a_{1,1}, a_{r,s}\}$ is an $\mathcal{L}$- geodetic set, so $g_{\mathcal{L}}(P_r \square P_s) \leq 2$. Clearly, $g_{\mathcal{L}}(P_r \square P_s) \geq 2$. Hence, $g_{\mathcal{L}}(P_r \square P_s) = 2$.

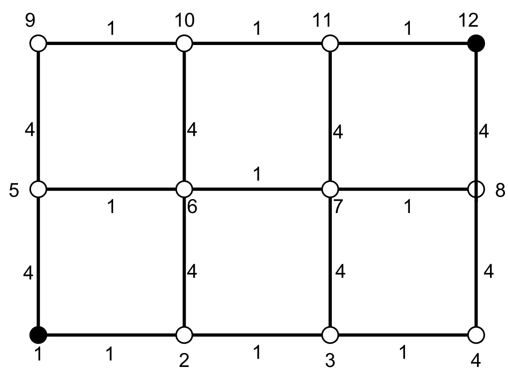


Figure 3.8: *The coloured vertices is a* $g_{\mathcal{L}}$ *set of* $P_3 \square P_4$

□

Theorem 3.3.7. Strong edge geodetic label exist in the friendship graph ${F_3}^n$.

Proof. The theorem is trivially true if $n = 1$.
For $n \geq 2$, let $G = {F_3}^n$ with $V = \{v_0, v_1, v_2, ..., v_n, ..., v_{2n}\}$ such that $|V| = 2n + 1$ and $|E| = 3n$ and $\{v_0\}$ be a common vertex.
Define a labeling $\mathcal{L} : V \to \{1, 2, 3, ..., 2n + 1\}$ as

$$\mathcal{L}(v_0) = n + 1.$$
$$\mathcal{L}(v_i) = i \text{ for } i = 1 \text{ to } n.$$
$$\mathcal{L}(v_i) = i + 1 \text{ for } i = n + 1 \text{ to } 2n.$$

Case 1: when n is even

Let $S = \{v_1, v_3, v_5, ..., v_{n-1}, v_{n+2}, v_{n+4}, ..., v_{2n-2}, v_{2n}\}$ be an $\mathcal{L}$- edge geodetic set which contains all the edges of G. Therefore, $g'_{\mathcal{L}}(G) = n$.

Case 2 : when n is odd

Let $S' = \{v_1, v_3, v_5, ..., v_n, v_{n+1}, v_{n+3}, v_{n+5}, ..., v_{2n}\}$ be an $\mathcal{L}$- edge geodetic set which contains all the edges of G. Therefore, $g'_{\mathcal{L}}(G) = n + 1$.

□

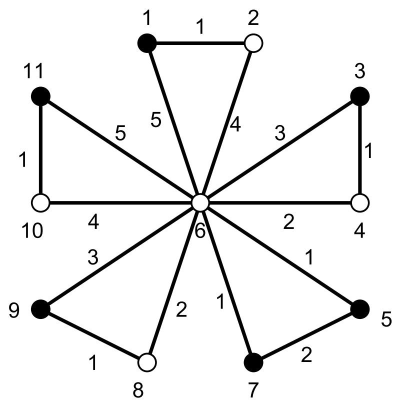


Figure 3.9: *The coloured vertices is the $g'_{\mathcal{L}}$ set of ${F_3}^5$*

Theorem 3.3.8. Strong edge geodetic label exist in the Windmill graph $Wd(k, n)$.

Proof. Let $G = Wd(k, n)$ with $V = \{v, v_1, v_2, v_3, \ldots\ldots, v_{k(n-1)}$ and v be the central vertex. then $|V| = k(n-1) + 1$ and $E = \frac{kn(n-1)}{2}$.

Define $\mathcal{L} : V \to \{1, 2, 3, \ldots, k(n-1) + 1\}$ as

$$\mathcal{L}(v) = 1$$
$$\mathcal{L}(v_i) = i + 1.$$

Then the set $\{v_{n-1}, v_{2(n-1)}, v_{3(n-1)}, \ldots, v_{k(n-1)}\}$is an $\mathcal{L}-$ edge geodetic set which contains all the edges of G.

Thus $g_{\mathcal{L}}(e) = k$, which is $< g_e(G)$. Hence the result.

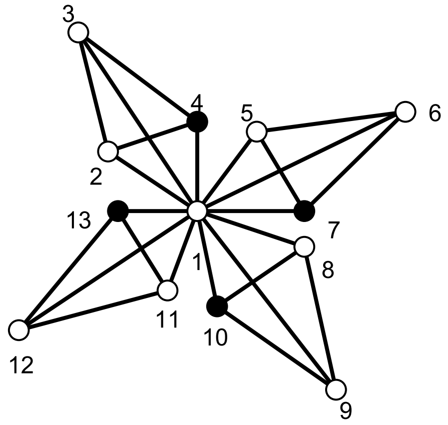


Figure 3.10: *The coloured vertices is the* $g'_{\mathfrak{L}}$ *set of* $Wd(k, n)$

□

3.4 Conclusion

In this article, the authors made an attempt to study the concept of geodetic and edge geodetic number in labeled graphs and defined geodetic label, edge geodetic label, strong geodetic label and strong edge geodetic label. Also studied the existence of these labels in some classes of graphs. To investigate similar results with other labeling function in the literature is a future area of research.

CHAPTER 4

LH Labeling Of Graphs

4.1 Introduction

Each graph can be labeled in infinitely many ways. Prime labeling and strong multiplicative labeling are two important concepts in graph labeling that have the same flavour as harmonious and graceful labelings. In prime labeling the vertices of a graph G is labeled with distinct elements from the set $\{1, 2, 3, \ldots, |V|\}$, so that every edge $e = (u, v)$ the greatest common factor of their labels $gcd(f(u, f(v)) = 1$. A graph that admits a prime labeling is called a prime graph. In strong multiplicative labeling we have to label the vertices with distinct elements from the set $\{1, 2, 3, \ldots, |V|\}$, so that the resulting edge labels are the product of corresponding vertex labels and all are different. Recent update of these labelings can be seen in [39]. Motivated from the research works of these labelings mentioned in literature, a new type of vertex labeling called *LH labeling* of graphs is introduced. Here, the elementary class concepts LCM (least common multiple) and HCF (highest common factor) are used in the definition. This chapter explores the results on LH labeling of graphs.

4.2 LH Labeling of Graphs

The concept of LH labeling of graphs, Examples and its properties are discussed in this section.

Definition 4.2.1. A graph G with n vertices is said to have an LH labeling if there exists a bijective function$f : V \to \{1, 2, 3, ..., n\}$ such that the induced map $f^* : E \to N$, the set of natural numbers defined by $f^*(uv) = \frac{LCM(f(u),f(v))}{HCF(f(u),f(v))}$ is injective (where LCM and HCF denotes the least common multiple and highest common factor respectively). A graph that admits an LH labeling is called an LH graph.

By labeling the vertex v_i by i, we observe that the path P_n and odd cycles are LH graphs. If we label the apex vertex of the star $K_{1,n}$ by 1 and the pendant vertices by using the remaining numbers serially from 2 to $n+1$, then the star graph $K_{1,n}$ becomes an LH graph.
The Y- tree Y_n is an LH graph. Since the vertex labeling using the numbers n, $n-1$, $n-2$,........, 3, 2 and 1 satisfies the conditions of an LH graph.

Observation 4.2.2. For any LH graph G with n vertices, $2 \leq f^*(e) \leq n^2 - n$, where $f^*(e)$ denotes the label of the edge e.

In prime labeling, the vertices are labeled with relatively prime numbers. So a prime graph is always an LH graph. But the converse is not true.

Example 4.2.1. The hyper cube Q_3 and the Petersen graph are LH graphs.

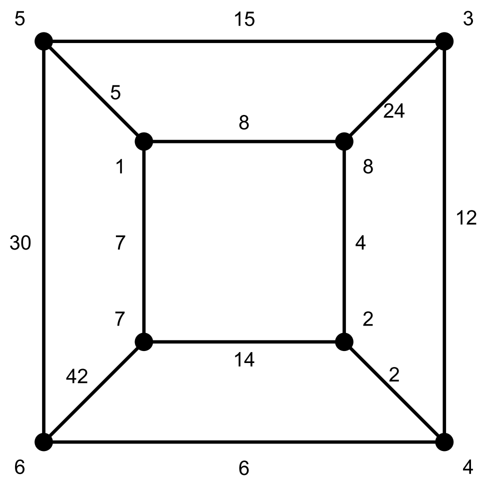


Figure 4.1: *LH labeling of Q_3*

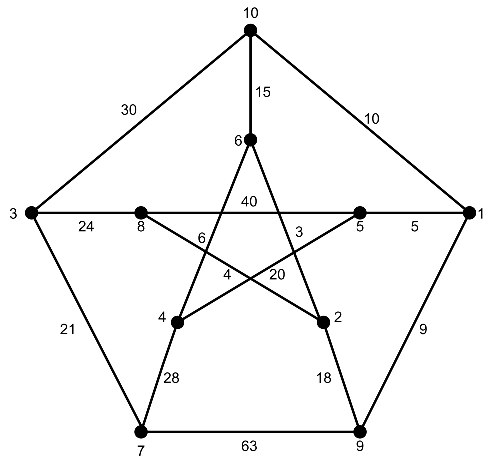


Figure 4.2: *LH labelling of the Petersen graph*

Example 4.2.2. The **Heawood graph**[27, 20], famous in graph theory literature, is an undirected 6-cage graph having 14 vertices and 21 edges. It is named after **Percy John Heawood. It is an LH graph.**

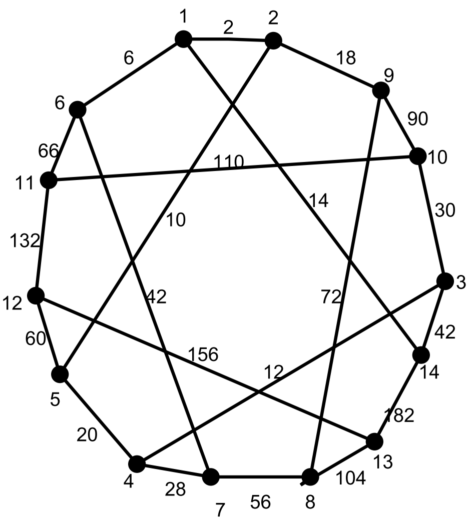


Figure 4.3: *LH labelling of Heawood graph*

Example 4.2.3. [26] The *Grotzsch graph* G_Z is a triangle – free graph with 11 vertices and 20 edges. It contains a star $K_{1,5}$ in which each pendant vertex of $K_{1,5}$ is connected with two rim vertices of the cycle C_5 whose vertex set , $V = \{v_1, v_2, \ldots, v_5, v_1', v_2', \ldots, v_5', v\}$. Grotzsch graph is an LH graph.

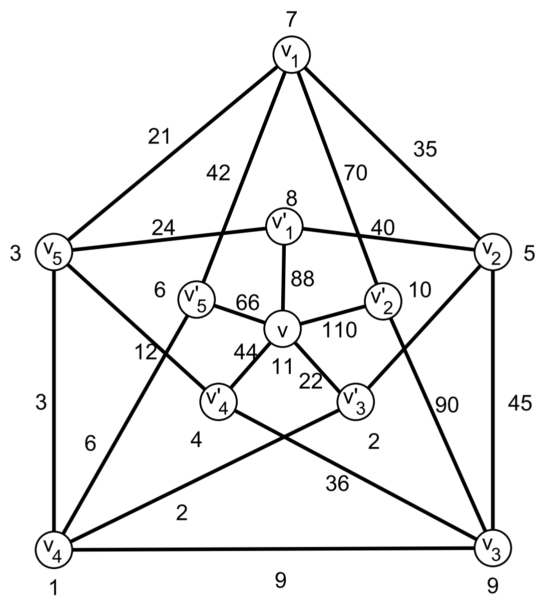


Figure 4.4: *LH labelling of grotzch graph G_Z*

L.W. Beineke and S.M. Hegde have introduced strong multiplicative graphs in [2]. The next remark shows that strong multiplicative labeling is a special case of LH labeling.

Remark 4.2.1. If the labels of each pair of adjacent vertices of a given graph G are relatively prime, then the LH labeling coincides with the strong multiplicative labeling.

The complete graphs K_2 and K_3 are LH graphs.

Theorem 4.2.3. The complete graph K_4 is not an LH graph.

Label the vertices of K_4 using the numbers 1,2,3 and 4. Since all the vertices are adjacent, the edge label 2 is obtained two times. Hence K_4 is not an LH graph. So we conclude that K_n is not an LH gaph for $n \geq 4$.

Theorem 4.2.4. K_n is not an LH graph for $n \geq 4$.

Note that $K_4 - e$ is an LH graph.

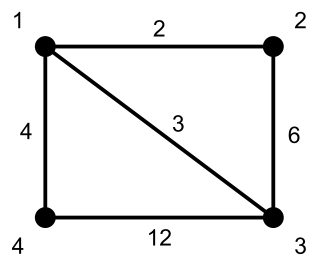


Figure 4.5: *LH labeling of $K_4 - e$*

Theorem 4.2.5. The complete bipartite graph $K_{2,s}$ is an LH graph.

Proof. Let $V = V_1 \cup V_2$ be the partition of the vertex set. $|V| = (s + 2)$.
Define $f : V \to \{1, 2, 3, ..., (s + 2)\}$ as follows.
Let p be the highest prime in the set $\{1, 2, 3, ..., (s+2)\}$. Label the vertices of V_1 with 1 and p. The vertices in V_2 are labeled using the numbers $\{1, 2, 3, ..., (s + 2)\}\backslash\{1, p\}$. Induced edge labels are 1, 2, 3,...., $p - 1$, $p + 1$,..., $s + 2$, p, $2p$,.....,$p(p - 1)$, $p(p + 1)$,....$p(s + 2)$. By Betrand's postulate [76], there exist a prime p' such that $p < p' < 2p$. Since p is the highest prime in the set $\{1, 2, 3, ..., (s + 2)\}$, we conclude that $s + 2 < 2p$. Therefore the edge labels obtained are all different. Hence the result. □

Theorem 4.2.6. $K_{3,3}$ is a non LH graph.

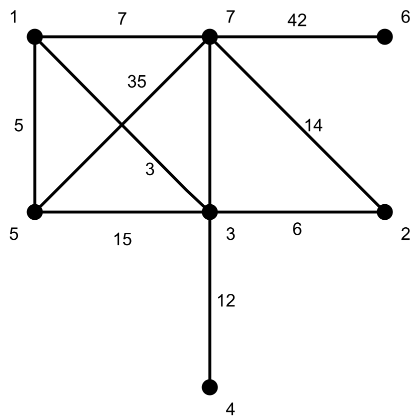
Proof. Let $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{v_1, v_2, v_3\}$ be the partition of V. Label the vertices of V_1 and V_2 using the set $\{1, 2, 3, 4, 5, 6\}$. Note that the pairs $(1, 2), (2, 4)$ and $(3, 6)$ produce the same edge label 2. Also, the pairs $(2, 3)$, $(1, 6)$ and $(4, 6)$ produce the same edge label 6. We consider the following cases.
Case 1: The vertices with labels 1 and 2 are not adjacent.
Suppose the vertices with labels 1 and 2 are in V_1. Then $f(u_1) = 1$ and $f(u_2) = 2$. Then the vertex u_3 can be labeled with 3 or 4 or 5 or 6.
If $f(u_3) = 3$, the vertices of V_2 are labeled with 4, 5 and 6. Then the pairs $(2, 4)$ and $(3, 6)$ produce the same edge label.
If $f(u_3) = 4$ or 5, then the pairs $(2, 3)$ and $(1, 6)$ produce the same edge label.
If $f(u_3) = 6$, then the pairs $(2, 3)$ and $(4, 6)$ produce the same edge label.

Case 2 **:** The vertices with labels 1 and 2 are adjacent.
Suppose $f(u_1) = 1$ and $f(v_1) = 2$. Since $(1, 2)$ and $(2, 4)$ produce the same edge label, vertex with label 4 must be in V_2, say $f(v_2) = 4$. Then v_3 can be labeled with 3 or 5 or 6. The vertices with label 3 and 6 must be in the same vertex set, so v_3 is assigned the label 5. Then the pairs $(2, 3)$ and $(4, 6)$ produce the same edge label. In all the cases the edge labels produced are not distinct and hence K_{33} cannot be an LH graph. □

In the next result, an observation about the subgraphs of an LH graph is given.

Theorem 4.2.7. (i). Every spanning subgrah of an LH graph is an LH graph.
(ii). Every induced subgraph of an LH graph need not be LH.

Proof. The first part of the Theorem follows directly from the definition of an LH graph.
We give an illustration for the second part of the Theorem. K_4 is a subgraph of the given LH graph given below, by Theorem 4.2.3, which is not LH.



□

4.2.1 Size of an LH graph

Here, we consider the question of finding the maximum number of edges in an LH graph with a given number of vertices.

To find the maximum number of edges in an LH grah with n vertices, label the vertices of the complete graph K_n with integers $1, 2, ..., n$ and then successively delete edges whose label is duplicated on another edge.
Let $f : V \to \{1, 2, ...n\}$.
Consider the set $W = \{(k, l)/k < l, k, l \in \{1, 2, ...n\}\}$.
Let e and e' are any two elements in E. Then $e = (k, l)$ and $e' = (k', l')$
If $f^*(e) = f^*(e')$ then

$$\frac{kl}{[gcd(k,l)]^2} = \frac{k'l'}{[gcd(k',l')]^2}$$

Define a relation $\sim$ on the set W such that $(k_1, l_1) \sim (k_2, l_2)$ if

$$\frac{k_1 l_1}{[gcd(k_1,l_1)]^2} = \frac{k_2 l_2}{[gcd(k_2,l_2)]^2}$$

Clearly this relation is reflexive, symmetric and transitive. Hence it is an equivalence relation and it partitions the set W into equivalence classes. Label the edges of G using an element from each class. Thus the number of equivalence classes represents the maximum possible size of an LH graph.

Let μ_n denote the distinct number of ratios $\frac{lcm(a,b)}{hcf(a,b)}$, for $1 \leq a < b \leq n$. That is μ_n is the number of equivalence classes possible with a given n. Thus if the size of the graph is greater than μ_n, then the graph is not an LH graph. But, if the size is less than or equal to μ_n the graph may or may not be LH. The table below shows the value of μ_n for $n \leq 15$.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15
μ_n	1	3	5	9	10	16	20	26	28	38	41	53	57	62

The size of $K_{3,3}$ is 9 which is less than μ_6, still it is not an LH graph by Theorem 4.2.6.
The following theorem shown that $\mu_6 =$ the size of $W_{1,5}$, still the graph is not LH.

Theorem 4.2.8. The wheel $W_{1,5}$ is not an LH graph.

Proof. Let v_1, v_2, v_3, v_4, v_5 and v_6 be the vertices of $W_{1,5}$. We can assume that each vertex v_i is labeled with i. The possible equivalence classes are $[(1,2),(2,4),(3,6)]$, $[(1,3),(2,6)]$, $[(1,6),(2,3),(4,6)]$, $[(1,4)]$, $[(3,4)]$, $[(1,5)]$, $[(2,5]$, $[(3,5)]$, $[(4,5)]$ and $[(5,6)]$. Since the size of $W_{1,5}$ is 10, exactly one pair from each class is required to label the edges.
The equivalence classes $[(1,5)], [(2,5], [(3,5)], [(4,5)]$ and $[(5,6)]$ has only one element, each pair is required to label the edges. Thus v_5 is the central vertex, since the central vertex is of degree 5. Also the equivalence class $[(1,4)]$ and $[(3,4)]$ have only one element and is required to label the edges. Thus v_1, v_3 and v_4 are the rim vertices forming the path $v_1 - v_4 - v_3$. Now we have two cases,

Case 1: v_1 is adjacent to v_2.
If v_1 is adjacent to v_2, then v_3 is adjacent to v_6, producing the edge label 2 two times.
Case 2: v_1 is not adjacent to v_2.
Then v_1 is adjacent to v_6 and v_3 is adjacent to v_2. So edge label 6 is produced two times.
Thus in all cases the edge labels obtained are not distinct. Hence, there will not be an LH labeling possible in $W_{1,5}$. □

4.3 LH completion of a graph G

In the previous section, examples of non LH graphs such as K_4, $K_{3,3}$ and $W_{1,5}$ are discussed. It is interesting to find an LH graph in which the given non LH graph as its subgraph. In this section, the LH completion of a non LH graph is studied.

Theorem 4.3.1. Given a non LH graph G, there exist an LH graph $\Omega^\star(G)$ such that G is an induced subgraph of $\Omega^\star(G)$.

Proof. Let G be a non LH graph on n vertices. Let q be the number so that the set $\{1, 2, 3, ..., n+q\}$ contains first $n-1$ primes. Form $\Omega^\star(G)$ by attaching q pendant edges to any one vertex, say v of G. Let p be the highest prime in the set $\{1, 2, 3, ...n+q\}$. Label v by p. The remaining vertices excluding the pendant vertices are labeled using the $n-2$ primes in the set $\{1,2,3,....n+q\}\backslash\{p\}$ and 1. Pendant vertices of $\Omega^{\star}(G)$ are labeled using the remaining numbers in the set $\{1,2,3,....n+q\}$. Clearly, the induced edge labels are distinct and hence the graph $\Omega^{\star}(G)$ constructed is an LH graph. □

The LH graph $\Omega^{\star}(G)$ constructed in the above Theorem is called the LH completion of G. That is,

Definition 4.3.2. For any non LH graph G there exist an LH graph $\Omega^{\star}(G)$ such that G is an induced subgraph of $\Omega^{\star}(G)$, known as the LH completion of G.

Note that LH completion of G is not unique. One of the LH completion of K_4 is discussed in theorem 4.2.7.

Definition 4.3.3. Let $\Omega^{\star}(G)$ be an LH completion of a non LH graph G. LH completion number of G, denoted by Λ_G is defined as $\Lambda_G = minimum\{|V(\Omega^{\star}(G))| - |V(G)|\}$.

One of the LH completion of $W_{1,5}$ is given below. $\Lambda_{W_{1,5}} = 1$

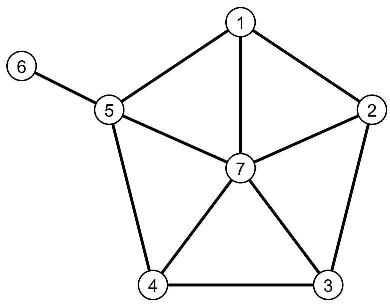


Figure 4.6: $\Omega^{\star}(W_{1,5})$

One of the LH completion of $K_{3,3}$ is shown in Figure 4.7. $\Lambda_{K_{3,3}} = 1$

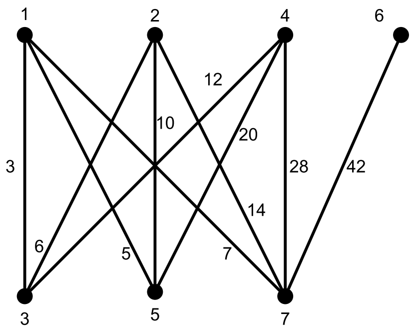


Figure 4.7: $\Omega^{\star}(K_{3,3})$

LH completion number of the complete graph K_n, $n \leq 10$ is given in the following table.

Table 4.1: *LH completion number of* K_n, $4 \leq n \leq 10$

n .	Number of edges.	μ_n	Λ_{K_n}
4	6	5	1
5	10	9	1
6	15	10	2
7	21	16	4
8	28	20	5
9	36	26	8
10	45	28	9

It is a challenging task to find the LH completion number of G when the number of vertices are large. We found a bound for q discussed in the Theorem 4.3.1 using the following number theory result.

Theorem 4.3.4. [83] For $n \geq 1$ the nth prime p_n satisfies the inequalities
$\frac{1}{6}nlogn < p_n < 12(nlogn + nlog\frac{12}{e})$.

By using the above Theorem, we have
$\frac{1}{6}(n-1)log(n-1) < n+q < 12((n-1)log(n-1)+(n-1)log\frac{12}{e})$.
where q is as discussed in Theorem 4.3.1

4.4 Trees

Trees are building blocks of a data structure in computer science and involved in many network applications. Among the different types of trees, binary trees are most frequently used. In this section, LH labeling of a perfect binary tree and spider graph $S_{3(m)}$ are analyzed.

Theorem 4.4.1. Spider graph $S_3(m)$ is an LH graph.

Proof. Let $\{v_0, v_1, v_2, ..., v_{3m}\}$ be the vertices and $\{e_1, e_2, e_3, ..., e_{3m}\}$ be the edges of $S_3(m)$.
$|V| = 3m + 1$.
Define $f : V \to \{1, 2, 3, ..., 3m + 1\}$ as
Case 1: m is even.

$$f(v_0) = 3m + 1$$
$$f(v_i) = i,\ 1 \le i \le 3m.$$

The induced edge labels are the product of consecutive integers from 1 to $3m$ and $(3m + 1)$, $(3m + 1)(m + 1)$, $(3m + 1)(2m + 1)$. Clearly, all are different.
Case 2: m is odd.

$$f(v_0) = 1$$
$$f(v_i) = i + 1,\ 1 \le i \le 2m.$$
$$f(v_{2m+1}) = 3m + 1$$
$$f(v_i) = f(v_{i-1}) - 1,\ 2m + 2 \le i \le 3m.$$

The induced edge labels are 1×2, 2×3,....., $m(m + 1)$, $(m + 2)$, $(m + 2)(m + 3)$, $(m+3)(m+4)$,...., $2m(2m+1)$, $(2m+2)(2m+3)$, $(2m+3)(2m+4)$,......, $3m(3m+1)$ and $(3m + 1)$. All edge labels induced are even except $(m + 2)$. To prove the

distinctness of edge labels, we have to consider only one case, that is $(3m+1) = i(i+1)$, $i \in \{1, 2, ..., 3m\}$. $\implies 3m = i^2 + i - 1$
$\implies m = \frac{i^2+i-1}{3}$, a contradiction.

In all the cases, the edge labels induced are distinct and hence the spider $S_3(m)$ is an LH graph.

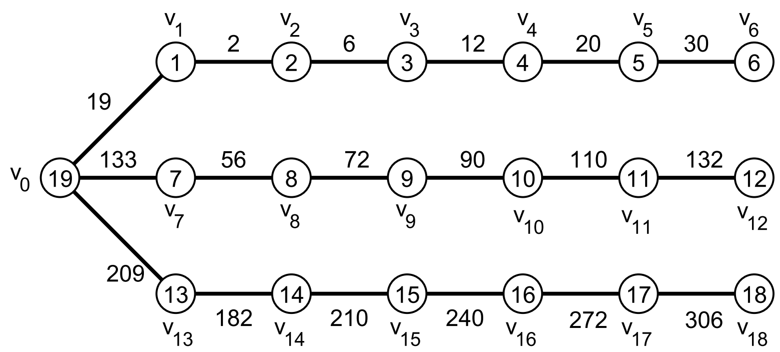


Figure 4.8: *LH labeling of $S_3(6)$*

□

Theorem 4.4.2. A perfect binary tree T_n is an LH graph.

Proof. Consider a perfect binary tree T_n. Note that T_n has n levels namely, 1, 2, 3,..., n and level k, $1 \le k \le n$ contains 2^{k-1} vertices and $|V| = 2^n - 1$.
Let a_{pq} be the q^{th} vertex from the left of th p^{th} level from the top in a perfect binary tree of n vertices. Define $f : V \to \{1, 2, 3, ..., 2^n - 1\}$ as

$$f(a_{11}) = 2^n - 1$$

Case 1: p is odd

$$f(a_{pq}) = 2^p(2^{n-p} - 1) + q$$

Case 2: p is even

$$f(a_{pq}) = 2^n - 2^{p-1} - q + 1$$

In view of the above labeling, the vertices of T_n can be numbered as shown below.

Level .	Labels.
T_1	$2^n - 1$
T_2	$2(2^{n-1} - 1)$, $2(2^{n-1} - 1) - 1$
T_3	$2^2(2^{n-2} - 1)$, $2^2(2^{n-2} - 1) - 1$, $2^2(2^{n-2} - 1) - 2$, $2^2(2^{n-2} - 1) - 3$
....	....
T_k	2^{k-1}, $(2^{n-k+1} - 1)$, $2^{k-1}(2^{n-k+1} - 1) - 1$, $2^{k-1}(2^{n-k+1} - 1) - 2$ ...... $2^{k-1}(2^{n-k+1} - 1) - 2^{k-1} - 1)$
.....	....
T_n	1, 2, 3, ... 2^{k-1}

Each edge of T_n is a path connecting a parent and their children. Each parent has one odd numbered children and one even numbered children. An even numbered parent in the level T_i and even numbered child in the level T_{i+1} has a common factor in powers of 2. Then the edge labels obtained is the product of two non adjacent odd numbered labels or one odd numbered and one even numbered labels. If any two adjacent vertex label has a common factor say $n \in N$, the edge labels produced is the product of two pendant vertices. Thus the induced edge labels are distinct. Hence the theorem.

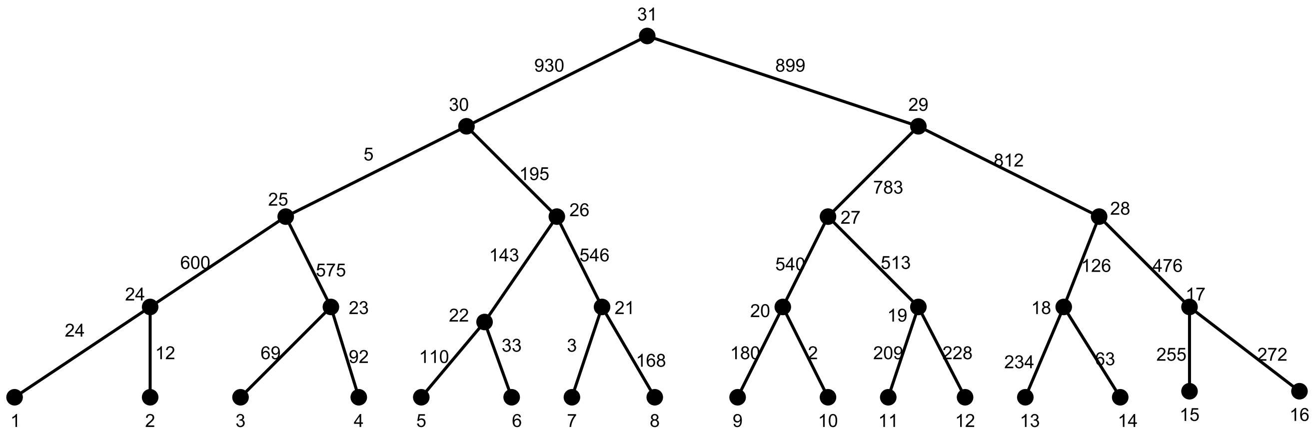


Figure 4.9: *LH labeling of* T_5

□

4.5 Conclusion

In this chapter, a new type of vertex labeling is introduced and some of the properties are analyzed. Discussed the LH completion number of a non LH graph. Also, the LH labeling of a perfect binary tree and spider graph $S_3(m)$ are demonstrated. We conclude this chapter by putting some problems for future research.

Problem 4.5.1. *Find the LH completion number of The complete graph K_n.?*

Problem 4.5.2. *Is the spider graph $S_n(m)$ an LH graph?*

Problem 4.5.3. *Find the LH labeling of the tree derived networks like X- tree?.*

CHAPTER 5

Some Families of LH Graphs

5.1 Introduction

Finding classes of graphs which admits a specific type of labeling is mostly seen in literature. In this chapter we discussed the LH labeling of some of the path related graphs, cycle related graphs, splitting graphs, bistar graph, a theta graph and line graphs.

5.2 LH Labeling of Path Related Graphs

In this section, the LH labeling of comb graph $P_n \circ K_1$, triangular snake T_n, quadrilateral snake G_n, Twig graph $TW(n), n \geq 3$, $[P_n : S_2]$, Sparkler graph $(P_m)^{+n}$ and $H-$ graph are discussed.

Theorem 5.2.1. The comb graph $P_n \circ K_1$ is an LH graph.

Proof. Consider a comb graph $P_n \circ K_1$ with vertex set $\{v_t, v'_t, 1 \leq t \leq n\}$ where $v'_t, t = 1$ to n are the pendant vertices. $|V| = 2n$.
$E = E' \cup E''$ where $E' = \{v_t v_{t+1}, 1 \leq t \leq (n-1)\}$ and
$E'' = \{v_t v'_t, 1 \leq t \leq n\}$

Define the vertex labeling f from V to $\{1, 2, 3, ..., 2n\}$ by

$$f(v_t) = 2t - 1, \ \ 1 \leq t \leq n$$
$$f(v_t') = 2t, \ \ 1 \leq t \leq n$$

To prove the graph $P_n \circ K_1$ is LH, we have to show that the edge labels induced are all distinct. That is the elements within the sets E' and E'' are distinct and to show that, no label is common to both E' and E''.

Let $f^* : E \to N$ be the induced edge function.
Claim : All the edge labels are distinct.
Note that the edge labels in the set E' are of the form $(2t-1)(2t+1), 1 \leq t \leq (n-1)$ and that of E'' are of the form $2t(2t-1), 1 \leq t \leq n$. Clearly the elements in E' are odd numbers and in E'' are even numbers. So no element is common to both E' and E''. Thus the edge labels are all different.
Hence the comb graph $P_n \circ K_1$ is an LH graph.

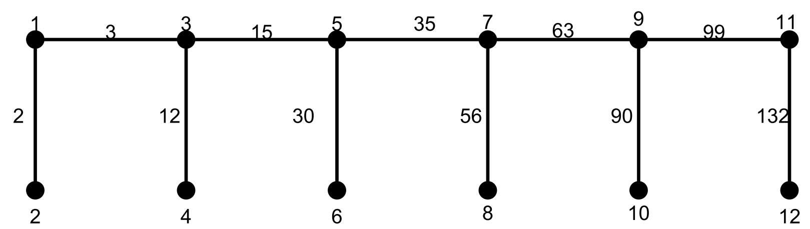


Figure 5.1: *LH labeling of* $P_6 \circ K_1$

□

Theorem 5.2.2. The triangular snake T_n is an LH graph.

Proof. Let T_n denote the triangular snake, which can be obtained from a path $u_1, u_2, ..., u_n, u_{n+1}$, by joining u_i, u_{i+1} to a new vertex $v_j, j = 1$ to n.
$V = \{u_t, i = 1 \text{ to } n + 1, v_t, t = 1 \text{ to } n\}$
$|V| = 2n + 1$.
$E = E' \cup E'' \cup E'''$ where $E' = \{u_t u_{t+1}, 1 \leq t \leq n\}$, $E'' = \{u_t v_t, 1 \leq t \leq n\}$ and $E''' = u_{t+1} v_t, 1 \leq t \leq n\}$.

Define $f : V \to \{1, 2, 3, ..., 2n+1\}$ by

$$f(u_t) = 2t - 1, \quad 1 \le t \le n+1$$
$$f(v_t) = 2t, \qquad 1 \le t \le n$$

Let $f^* : E \to N$ be the induced edge function.

Claim : All the edge labels induced by the function f are distinct.

The edge labels in the set E' are of the form $(2t-1)(2t+1), 1 \le t \le n$, product of two consecutive odd numbers. The edge labels in E'' are of the form $2t(2t-1), 1 \le t \le n$ and that of E''' are $2t(2t+1), 1 \le t \le n$. Clearly, the edge labels within E'' and E''' are even numbers. Thus to prove the claim, it is enough to show that, no label is common to both the sets E'' and E'''.
Consider $e_i \in E''$ and $e_j \in E'''$.
Then $f^*(e_i) = 2i(2i-1)$ and $f^*(e_j) = 2j(2j+1)$.
Suppose $f^*(e_i) = f^*(e_j)$.

$$\begin{aligned} 2i(2i-1) &= 2j(2j+1) \\ \implies 2i^2 - i &= 2j^2 + j \\ \implies 2(i^2 - j^2) &= i + j \\ \implies i - j &= \frac{1}{2}, acontradiction. \end{aligned}$$

Thus no edge label is common to both E'' and E'''.
Hence the labeling function defined above satisfies the conditions of an LH labeling and the graph under consideration is an LH graph.

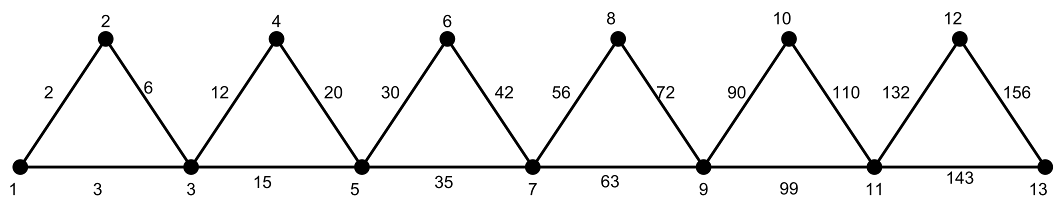


Figure 5.2: *LH labeling of T_6*

□

Theorem 5.2.3. A quadrilateral snake G_n is an LH graph.

Proof. Consider a quadrilateral snake G_n with vertex set $\{v_i, i = 1 \text{ to } n+1, u_j$ and $w_j, j = 1 \text{ to } n\}$ which is obtained from the path $v_1, v_2, ..., v_{n+1}$ by joining v_i, v_{i+1} to new vertices u_i and w_i respectively and joining u_i and $w_i, 1 \leq i \leq n$.
$|V| = 3n + 1$.
$E = E' \cup E \cup E''' \in E^{iv}$ where $E' = \{v_t v_{t+1} t = 1 \text{ to } n\}$, $E'' = \{u_t w_t, t = 1 \text{ to } n\}$, $E''' = \{u_i v_i, i = 1 \text{ to } n\}$ and $E^{iv} = \{v_{i+1} w_i, i = 1 \text{ to } n\}$.

Define $f : V \to \{1, 2, 3, ..., 3n + 1\}$ by

$$f(v_m) = 3m - 2,\ 1 \leq m \leq n + 1$$
$$f(u_i) = 3m - 1,\ 1 \leq i \leq n$$
$$f(w_m) = 3m,\ 1 \leq i \leq n.$$

Let $f^* : E \to N$ be the induced edge function.
Claim : All the edge labels are distinct.

To prove the claim, we have to show that the edge labels in each set E', E'', E''', E^{iv} and the labels in each pair $E'$and E'', E' and E''', E' and E^{iv}, E'' and E''', E'' and E^{iv} and E''' and E^{iv} are distinct.

Case 1: Let e_i and $e_j \in E'$, $i \neq j$
Then $f^*(e_i) = f^*(v_i v_{i+1}) = (3i - 2)(3i + 1)$ and
$f^*(e_j) = f^*(v_j v_{j+1}) = (3j - 2)(3j + 1)$.
Suppose $f^*(e_i) = f^*(e_j)$.
Then

$$(3i - 2)(3i + 1) = (3j - 2)(3j + 1).$$
$$9i^2 - 3i - 2 = 9j^2 - 3j - 2$$

$$which \implies i + j = 1/3, acontradiction.$$

Therefore, the edge labels within E' are distinct.

Case 2: Let e_i and $e_j \in E''$, $i \neq j$
Then $f^*(e_i) = f^*(u_iw_i) = (3i-1)3i$ and $f^*(e_j) = f^*(u_jw_j) = (3j-1)3j$. If $i \neq j \implies f^*(e_i) \neq f^*(e_j)$.
Similarly, it can be easily proved that the edge labels within E''' and E^{iv} are distinct.

Case 3: Let $e_i \in E'$ and $e_j \in E''$, $i \neq j$ Then $f^*(e_i) = f^*(v_iv_{i+1}) = (3i-2)(3i+1)$ and
$f^*(e_j) = f^*(u_jw_j) = (3j-1)3j$.
Suppose $f^*(e_i) = f^*(e_j)$.
Then

$$\begin{aligned}
(3i-2)(3i+1) &= (3j-1)3j. \\
ie, 9i^2 - 3i - 2 &= 9j^2 - 3j. \\
ie, 9i^2 - 3i - 2 - 9j^2 + 3j &= 0. \\
Therefore, i = \frac{3 \pm \sqrt{9 - 36(-2 - 9j^2 + 3j)}}{18}&. \\
i = \frac{1 \pm \sqrt{9 + 36j^2 - 12j}}{6}&, acontradiction,
\end{aligned}$$

since i and j are positive integers.
Therefore, there is no edge label common to E' and E''.

Case 4: Let $e_i \in E''$ and $e_j \in E^{iv}$, $i \neq j$. Then
$f^*(e_i) = f^*(u_iw_i) = (3i-1)3i$
$f^*(e_j) = f^*(v_{i+1}w_j) = (3j+1)3j$.
Suppose $f^*(e_i) = f^*(e_j)$.

$$\begin{aligned}
&\implies (3i-1)3i = (3j+1)3j. \\
&\implies 3i^2 - i = 3j^2 + j.
\end{aligned}$$

$$\implies 3(i^2 - j^2) = i + j.$$
$$\implies i - j = \frac{1}{3}, acontradiction.$$

Therefore, the edge labels within E'' and E^{iv} are distinct.
Similarly we can prove that the edge labels in the pairs E' and E''', E' and E^{iv}, E'' and E''' and E''' and E^{iv} are all different.
Hence the graph G_n is an LH graph.

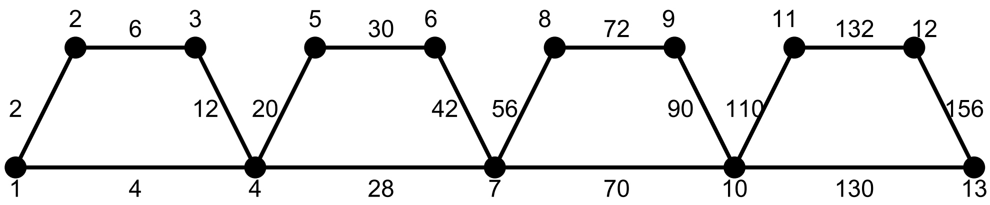


Figure 5.3: *LH labeling of* G_4

□

Theorem 5.2.4. A Twig graph $TW(n), n \geq 3$ is an LH graph.

Proof. Consider a twig graph $TW(n)$, which is obtained from a path $u_1, u_2, ..., u_{n-1}, u_n$ by attaching exactly two pendent edges v_i and w_i to each internal vertex of the path.

$V = \{u_i, i = 1 \text{ to } n \text{ and } v_i, w_i, i = 1\text{to } n-2\}$.
$|V| = 3n - 4$.
$E = E' \cup E'' \cup E'''$ where $E' = \{u_i u_{i+1}, i = 1 \text{ to } n\}$, $E'' = \{u_{j+1} v_j, j = 1 \text{ to } n-2\}$ and $E''' = \{u_{j+1} w_j, j = 1 \text{ to } n-2\}$

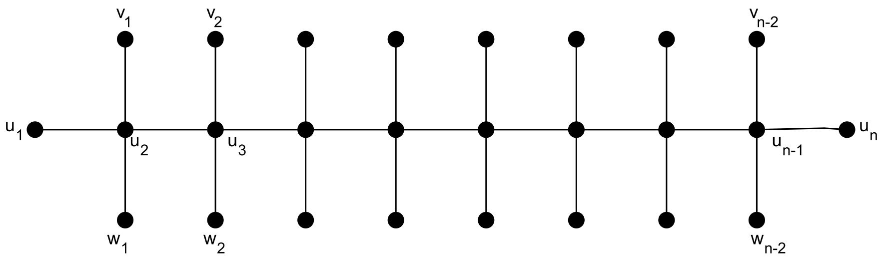


Figure 5.4: *Ordinary labeling of* $TW(n)$

We define the function $f : V \to \{1, 2, 3, 4..., 3n - 4\}$ by

$$f(u_m) = 3m - 2,\ 1 \leq m \leq n - 1.$$
$$f(v_m) = 3m - 1,\ 1 \leq m \leq n - 2.$$
$$f(w_m) = 3m,\ 1 \leq m \leq n - 2.$$
$$f(u_n) = 3n - 4.$$

Next we have to show that the edge labels are distinct.
Let $f^* : E \to N$ be the induced edge function. To prove the edge labels are distinct, it is enough to show that the edge labels within in the set E', E'', E''', and the labels in each pair E' and E'', E' and E''' E'' and E''' are distinct.
Claim : All the edge labels are distinct.

Case 1: Let e_i and $e_j \in E'$, $i \neq j$
Then $f^*(e_i) = f^*(v_i v_{i+1}) = (3i - 2)(3i + 1)$ and
$f^*(e_j) = f^*(v_j v_{j+1}) = (3j - 2)(3j + 1)$.
Suppose $f^*(e_i) = f^*(e_j)$.
Then

$$(3i - 2)(3i + 1) = (3j - 2)(3j + 1).$$
$$9i^2 - 3i - 2 = 9j^2 - 3j - 2\, which \implies i + j = 1/3,\ a\, contradiction.$$

Therefore, the edge labels within E' are distinct.

Case 2: Let e_i and $e_j \in E$", $i \neq j$.
Then $f^*(e_i) = f^*(u_{i+1} v_i) = (3i + 1)(3i - 1)$
and $f^*(e_j) = f^*(u_{j+1} v_j) = (3j + 1)(3j - 1)$.
If $i \neq j \implies f^*(e_i) \neq f^*(e_j)$.

Case 3: Let $e_i \in E'$ and $e_j \in E''$, $i \neq j$
Then $f^*(e_i) = f^*(u_i u_{i+1}) = (3i - 2)(3i + 1)$ and
$f^*(e_j) = f^*(u_{j+1} v_j) = (3j + 1)(3j - 1)$.
Suppose $f^*(e_i) = f^*(e_j)$.

Then

$$(3i-2)(3i+1) = (3j-1)(3j+1).$$
$$ie, 9i^2 - 3i - 2 = 9j^2 - 1.$$
$$ie, 9i^2 - 3i - 9j^2 - 1 = 0.$$
$$Therefore, i = \frac{3 \pm \sqrt{9 + 36(9j^2+1)}}{18}.$$
$$i = \frac{1 \pm \sqrt{5 + 36j^2}}{6},\ a\ contradiction,$$

since i and j are positive integers.

Case 4: Let $e_i \in E''$ and $e_j \in E'''$, $i \neq j$
Then $f^*(e_i) = f^*(u_{i+1}v_i) = \frac{(3i-1)(3i+1)}{4}$ and
$f^*(e_j) = f^*(u_{j+1}w_j) = (3j+1)3j$.
Suppose $f^*(e_i) = f^*(e_j)$. Then

$$\frac{(3i-1)(3i+1)}{4} = (3j+1)3j.$$
$$\implies 9i^2 - 1 = 4(9j^2 + 3j)$$
$$\implies `36j^2 + 12j - 9i^2 + 1 = 0.$$
$$j = \frac{-12 \pm \sqrt{144-144(1-9i^2}}{72}\quad j = \frac{-1 \pm 3i}{6}, \text{ a contradiction.}$$

Similar proof holds for the other cases. Hence the twig graph $TW(n), n \geq 3$ is an LH graph.

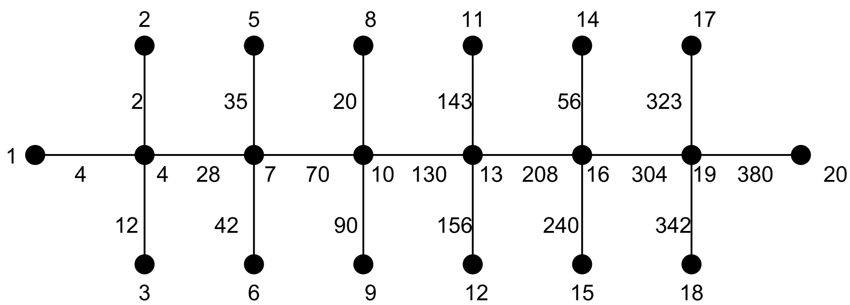


Figure 5.5: *LH labeling of* $TW(8)$

□

Theorem 5.2.5. The graph $[P_n : S_2]$is an LH graph.

Proof. Consider the graph $[P_n : S_2]$. Let the vertex set of the path P_n is given by $v_i, i = 1$ to n and $u_i, w_i, 1 \leq i \leq n$ be the vertices which are made adjacent with v_i.
$V = \{u_i, v_i, w_i, 1 \leq i \leq n\}$
$|V| = 3n$.
$E = E' \cup E'' \cup E'''$ where, $E' = \{v_i v_{i+1}, i = 1$ to $n - 1\}$, $E'' = \{v_i w_i, 1 \leq i \leq n\}$ and $E''' = \{v_i u_i, 1 \leq i \leq n\}$.

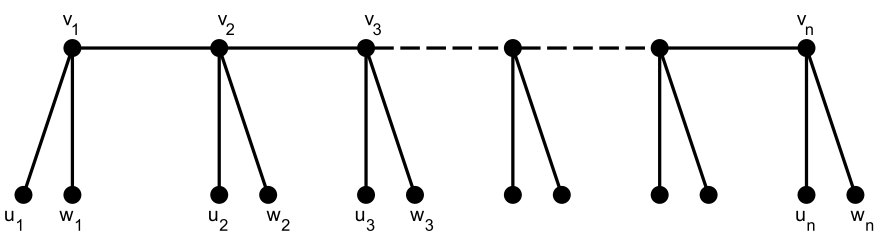


Figure 5.6: *Ordinary labeling of* $[P_n : S_2]$

We define $f : V \to \{1, 2, 3, ..., 3n\}$ by

$$f(v_m) = 3m - 2,\ 1 \leq m \leq n$$
$$f(u_m) = 3m - 1,\ 1 \leq m \leq n$$
$$f(w_m) = 3m, \quad 1 \leq m \leq n.$$

Let $f^* : E \to N$ be the induced edge function.
Claim : All the edge labels are distinct.

To prove the edge labels are distinct, it is enough to show that the edge labels within in the set E', E", E''', and the labels in each pair E' and E'', E' and E''' & $E''$and E''' are distinct.
The edge labels within the sets E' are of the form $(3m - 2)(3m + 1), 1 \leq m \leq n$ and that of E''' are $(3m - 1)(3m - 2), 1 \leq m \leq n$. In E'' the edge labels are

$$3m(3m - 2), \text{ if } m \text{ is odd}$$
$$\text{and } \frac{3m(3m-2)}{4}, \text{ if } m \text{ is even}, 1 \leq m \leq n$$

Clearly, all the labels within the sets E', E'' and E''' are different.

Now we consider two cases.

Case 1: Let $e_i \in E'$ and $e_j \in E''$, $i \neq j$

Then $f^*(e_i) = f^*(v_i v_{i+1}) = (3i-2)(3i+1)$ and

$f^*(e_j) = f^*(v_j w_j) = \frac{(3j-2)3j}{4}$ if j is even, otherwise $(3j-2)3j$.

Suppose $f^*(e_i) = f^*(e_j)$, j is even.. Then

$$(3i-2)(3i+1) = \frac{(3j-2)3j}{4}.$$

$$\implies 4(9i^2 - 3i - 2) = (9j^2 - 6j)$$

$$\implies \text{‘}36i^2 - 12i - 9j^2 + 6j - 8 = 0.$$

$$i = \frac{12 \pm \sqrt{144 - 144(-8 - 9j^2 + 6j}}{72}$$

$$i = \frac{1 \pm \sqrt{9 + 9j^2 - 6j}}{6}, \textit{ a contradiction.}$$

Similar result holds for the case when j is odd.

Case 2: $e_i \in E''$ and $e_j \in E'''$, $i \neq j$

If i is odd, $f^*(e_i) = (3i-2)3i$ and $f^*(e_j) = (3j-1)(3j-2)$

Suppose $f^*(e_i) = f^*(e_j)$. Then

$$(3i-2)3i = (3j-2)(3j-1).$$

$$\implies 9i^2 - 6i = (9j^2 - 9j + 2)$$

$$\implies \text{‘}9i^2 - 6i - 9j^2 + 9j - 2 = 0.$$

$$i = \frac{6 \pm \sqrt{36 - 36(9j - 9j^2 - 2}}{18}$$

$$i = \frac{1 \pm \sqrt{(3 + 9j^2 - 9j)}}{3}, \textit{ a contradiction.}$$

Similar result holds for the case when i is even.

Similarly we can prove that the edge labels in the pair E' and E''' are distinct. Thus the induced edge labels are different and hence $[P_n : S_2]$ is an LH graph.

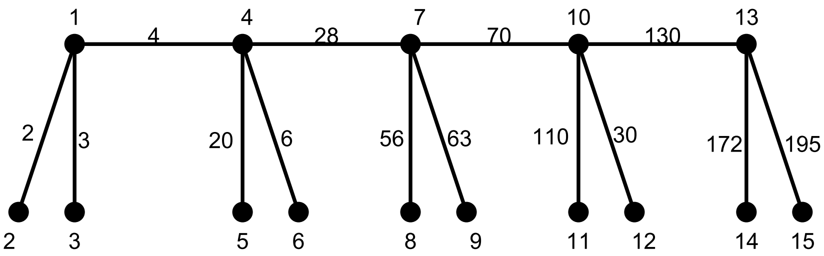


Figure 5.7: *LH labeling of* $[P_5 : S_2]$

□

Theorem 5.2.6. The Sparkler graph $(P_m)^{+n}$ is an LH graph.

Proof. Let $\{v_i, i = 1 \text{ to } m\}$ be the vertices of the path P_m and $\{u_i, i = 1 \text{ to } n\}$ be the vertices joined to the vertex v_m to form the sparkler graph $(P_m)^{+n}$.
$|V| = m + n$.
$E = E' \cup E''$ where, $E' = \{v_i v_{i+1}, 1 \leq i \leq m-1\}$ and $E'' = \{v_m u_i, 1 \leq i \leq n\}$.

Define $f : V \to \{1, 2, 3, ..., m + n\}$ as follows

$$f(v_m) = 1$$
$$f(u_i) = i + 1,\ 1 \leq i \leq n.$$
$$f(v_1) = m + n$$
$$f(v_i) = f(v_{i-1}) - 1,\ 2 \leq i \leq m - 1.$$

The induced edge function $f^* : E \to \{1, 2, 3....., m + n\}$ is injective, since the edge labels within the set E'' are $2, 3, 4, 5, ..., (n + 1)$ and that of E' are $(m + n)(m + n - 1), (m + n - 1)(m + n - 2), ..., (n + 2)$, and are distinct.

Hence $(P_m)^{+n}$ is an LH graph. □

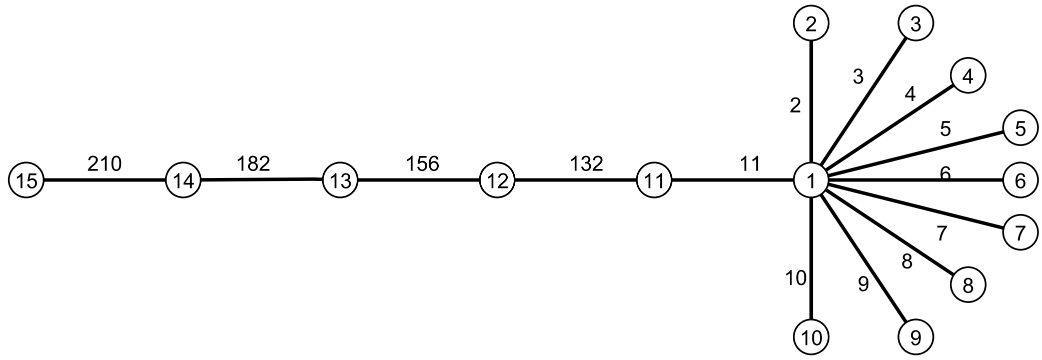


Figure 5.8: *LH labeling of* $(P_6)^{+9}$

Theorem 5.2.7. The $H-$ graph is an LH graph.

Proof. The vertex and the edge set of H-graph are given by $V = \{u_i, v_i/1 \leq i \leq n\}$ and $E = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \cup \{u_{(n+1)/2}v_{(n+1)/2}$ if n is odd (or) $u_{(n/2)+1}v_{n/2}$ if n is even $\}$. Then $|V| = 2n$ and $|E| = 2n - 1$.

Define $f : V \to \{1, 2, 3, ..., 2n\}$ as follows

$$f(v_i) = 2i - 1,\ 1 \leq i \leq n$$
$$f(u_1) = 2n$$
$$f(u_i) = f(u_{i-1}) - 2,\ 2 \leq i \leq n$$

The induced edge function $f^* : E \to \{1, 2, 3....., 2n\}$ is injective, Since the edge labels induced by the edge set $\{u_iu_{i+1}, 1 \leq i \leq n-1\}$ are odd numbers and the edge set $\{v_iv_{i+1}, 1 \leq i \leq n\}$ are even numbers. It is clear that the edge labels induced are distinct.

Hence the $H-$ graph is an LH graph. □

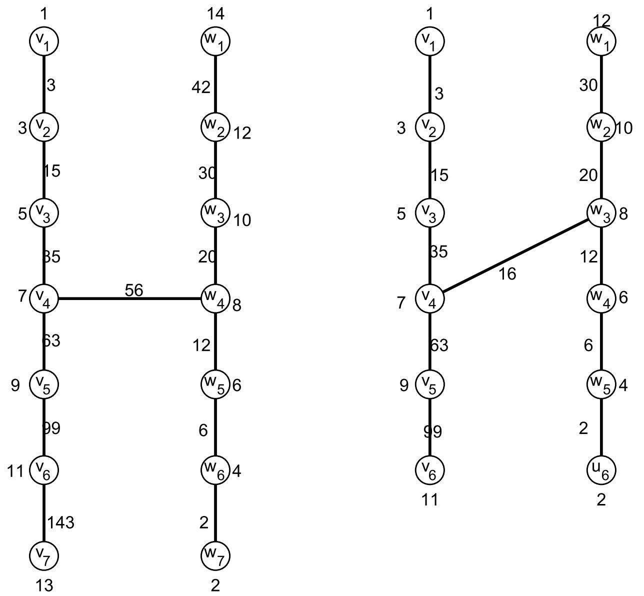


Figure 5.9: *LH labeling of H_7 and H_6*

5.3 LH Labeling of Cycle Related Graphs

In this section LH labeling of cycle C_n, friendship graph F_3^n, crown graph $C_n \circ K_1$, helm H_n and the flower graph Fl_n are discussed.

Theorem 5.3.1. The cycle C_n is an LH graph.

Proof. Let v_0, v_1, v_2,.......,v_n be the vertices of C_n. Let $f : V \to \{1, 2, 3, ..., n\}$ be the vertex labeling function.
Suppose n cannot be expressed as the product of two consecutive natural numbers, ie, $n \neq l(l+1), l \in N$.
Define $f(v_i) = i$, clearly the induced edge labels are all different.
Suppose $n = l(l+1)$. There are 2 cases:

Case 1 : l is even, $l > 2$.
Define the vertex labeling as

$$f(v_1) = l$$

$$f(v_2) = l - 1$$

$$f(v_3) = 1$$
$$f(v_j) = f(v_{j-1}) + 1 \quad 4 \leq j \leq l$$
$$f(v_i) = i, \ Otherwise.$$

The induced edge labels are $l(l-1)$, $(l-1)$, 1×2, 2×3, ...,$(l-2)(l+1)$,......, $(l+1)$, and are all different.

Case 2 : l is odd.

$$f(v_1) = l + 1$$
$$f(v_j) = f(v_{j-1}) - 1 \quad 2 \leq j \leq l + 1$$
$$f(v_i) = i, \ \ Otherwise.$$

The edge lab els produced are $(l+1)l$, $l(l-1)$,.....,1.$(l+2)$,....,l, which are all different.

When $l = 2$, ie, $n = 6$, the labeling defined in Figure 5.10 is an LH labeling. Thus C_6 is LH.
Hence C_n is an LH graph.

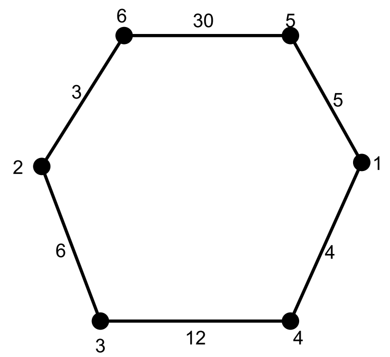


Figure 5.10: *LH labeling of* C_6

□

Theorem 5.3.2. The Friendship graph F_3^n is an LH graph.

Proof. Let $V = \{u_i, i = 0 \text{ to } 2n\}$ be the vertex set of F_3^n with u_0 be the center vertex.
$E = E' \cup E''$ where $E' = \{u_0 u_i, 1 \leq i \leq n\}$ and $E'' = \{u_i u_i + 1, i = 1, 3, 5...., 2n-1\}$.
$|V| = 2n + 1$ and $|E| = 3n$.

Define $f : V \to \{1, 2, 3, ..., 2n + 1\}$ as

$f(u_0) = p$, where p is the highest prime in the set $\{1, 2, 3, ..., 2n + 1\}$.
Now, label the remaining vertices $v_1, v_2, ..., v_{2n}$ consecutively from the set $\{1, 2, 3, ..., 2n + 1\}/\{p\}$.
The induced edge function is $f^* : E \to \{1, 2, 3....., 2n + 1\}$.
Claim : All the edge labels are distinct.

The edge labels within the set E' are p, $2p$, $3p$, $\ldots$, $(p-1)p$, $(p+1)p$, $\ldots (2n+1)p$.
The edge labels within the set E'' are respectively of the following form:
If $2n + 1$ is prime, then 1×2, 3×4, 5×6,....$(2n - 1)2n$; Otherwise, 1×2, 3×4, 5×6,....$(2n + 1)2n$. clearly, the edge labels produced are all different. Hence, the friendship graph F_3^n is an LH graph.

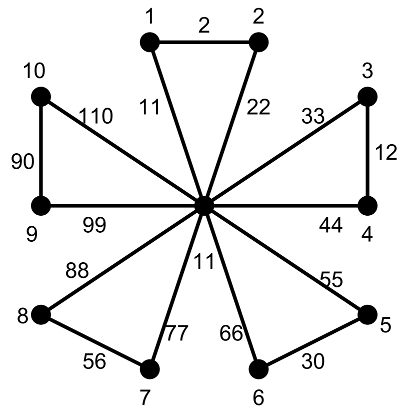


Figure 5.11: *LH labeling of F_3^5*

□

Theorem 5.3.3. The Wheel graph $W_{1,n}$ is an LH graph if $(n + 1)$ is prime or $n + 1$ is the product of two consecutive natural numbers.

Proof. Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of the wheel $W_{1,n}$ with centre at v_0. Let $(n+1) = p$. Then $|V| = p$ and $E = E' \cup E''$ where $E' = \{v_0v_t, 1 \leq t \leq n\}$ and $E'' = \{v_1v_n, v_tv_{t+1}, 1 \leq t \leq n-1\}$.
Define $f : V \to \{1, 2, 3, ..., p\}$.
If n cannot be expressed as the product of two consecutive natural numbers, ie, $n \neq m(m+1), m \in N$. Define $f(v_0) = p$ and $f(v_i) = i$ for $1 \leq i \leq n$. Clearly the induced edge labels induced are all different.
Suppose $n = m(m+1), m \in N$. We consider 4 cases.

Case 1: m is even and $m > 2$
Define the vertex labeling as

$$\begin{aligned} f(v_0) &= p \\ f(v_1) &= m \\ f(v_2) &= (m-1) \\ f(v_3) &= 1 \\ f(v_j) &= f(v_{j-1}) + 1, \;\; 4 \leq j \leq m \\ f(v_i) &= i, \;\; otherwise. \end{aligned}$$

The edge labels induced are p, $2p$, ...,np, $m(m-1)$, $(m-1)$, 1×2, 2×3, ..., $(m-2)(m+1)$, $(m+1)(m+2)$, ..., $(m+1)$. Clearly all the labels are distinct.

Case 2: m is odd.
Define the vertex labeling as

$$\begin{aligned} f(v_0) &= p \\ f(v_1) &= (m+1) \\ f(v_j) &= f(v_{j-1}) - 1, \; 2 \leq j \leq (m+1) \\ f(v_i) &= i, \;\; otherwise. \end{aligned}$$

The edge labels induced are p, $2p$, ...,np, $m(m+1)$, $m(m-1)$, ..., 2×1, $(m+2)$, $(m+2)(m+3)$, ..., m. Clearly all the labels are distinct.

Case 3: $m = 2$ ie, $p = 7$.

The LH Labeling of $W_{1,6}$ is shown in the figure given below.

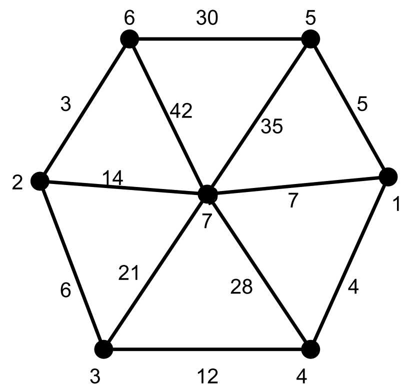


Figure 5.12: *LH labeling of* $W_{1,6}$

Case 4: Suppose $n+1$ is the product of two consecutive natural numbers, ie, $n+1 = m(m+1), m \in N$.

Let p be the highest prime in the set $\{1, 2, 3, \ldots, (n+1)\}$

Define the vertex labeling as

$$f(v_0) = p$$
$$f(v_1) = 1$$
$$f(v_2) = m(m+1) - 2$$
$$f(v_3) = 2$$
$$f(v_4) = m(m+1)$$
$$f(v_5) = 3$$
$$f(v_j) = f(v_{j-1}) + 1, \;\; 6 \leq j \leq n$$

The induced edge labels are p, $2p$, $3p$, $\ldots$, $p(p-1)$, $p(p+1)$, $\ldots$, $(n+1)p$, $m(m+1)-2$, $\frac{m(m+1)-2}{2}$, $\frac{m(m+1)}{2}$, $\frac{m(m+1)}{3}$ or $3m(m+1)$, 3×4, 4×5, $\ldots$, $[m(m+1)-4][m(m+1)-3]$.

All the edge labels induced are distinct.

Thus in all cases the graph under consideration is an LH graph.

□

Theorem 5.3.4. The crown graph $C_n \circ K_1$ for all $n \geq 3$ is an LH graph.

Proof. Let $\{u_i, i = 1 \text{ to } n\}$ be the vertices of the cycle C_n and $\{v_i, i = 1 \text{ to } n\}$ be the pendant vertices. Then
$E = \{u_i u_{i+1}, 1 \leq i \leq (n-1)\} \cup \{u_i v_i, 1 \leq i \leq n\} \cup \{u_1 u_n\}$.
$|E| = |V| = 2n$
Define $f : V \to \{1, 2, 3, \ldots, 2n\}$ as

Case 1: $(2n-1)$ cannot be expressed as the product of two consecutive odd numbers.

$$f(u_i) = 2i - 1, \ \ 1 \leq i \leq n$$
$$f(v_i) = 2i, \ \ 1 \leq i \leq n$$

The induced edge labels are $(2i-1)(2i+1)$, $1 \leq i \leq (n-1)$, $2i(2i-1)$, $1 \leq i \leq n$ and $(2n-1)$.
Clearly all are different.

Case 2: $(2n-1)$ is the product of two consecutive odd numbers, ie,
$(2n-1) = m(m+2)$, m is odd.

Subcase i: $m > 3$
Define the vertex labeling as

$$f(u_1) = m$$
$$f(u_2) = (2n-1) = m(m+2)$$
$$f(u_3) = (m+2)$$
$$f(u_i) = f(u_{i-1} + 2, \ \ 4 \leq i \leq (n-2)$$
$$f(v_i) = f(u_i) + 1, \ \ 1 \leq i \leq n$$

The remaining vertices are numbered consecutively using the set $\{1, 3, 5, \ldots, (m-2)\}$. The induced edge labels are $(m+2)$, m, 1×3, 3×5, $\ldots$, $(m-4)(m-2)$, $(m+2)(m+4)$, $\ldots$, $(2n-3)(2n-5)$ and $2i(2i-1), 1 \le i \le n$.
Clearly, all the labels induced are distinct.

Subcase ii: $m = 3$, or $(2n-1) = 15$.
LH labeling of $C_8 \circ K_1$ is shown in the 5.13

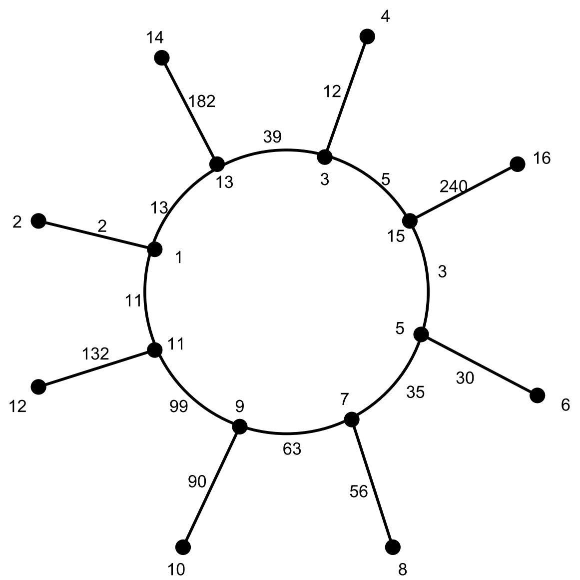


Figure 5.13: *LH labeling of* $C_8 \circ K_1$

Hence the crown graph $C_n \circ K_1$ is an LH graph.

□

Theorem 5.3.5. The flower graph Fl_n is an LH graph.

Proof. Let u_0 be the central vertex, $u_1, u_2, \ldots\ldots, u_n$ be the rim vertices and $u'_1, u'_2, \ldots\ldots, u'_n$ be the vertices of Fl_n as shown in figure 5.13.
$V = \{u_0, u_t, u'_t, 1 \le t \le n\}$.
$E = E' \cup E'' \cup E'''$ where $E' = \{u_0u_t, u_0u'_t, 1 \le t \le n\}$, $E'' = \{u_tu'_t, 1 \le t \le n\}$ and $E''' = \{u_1u_n, u_tu_{t+1}, 1 \le t \le n-1\}$.
$|V| = 2n + 1$ and $|E| = 4n$.

Define the vertex labeling $f : V \to \{1, 2, 3 \ldots., 2n + 1\}$ as
Case 1: $(2n + 1)$ is prime.

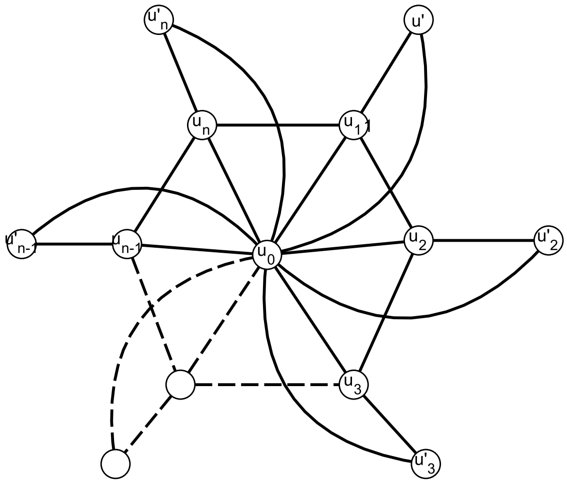


Figure 5.14: *Ordinary labeling of Fl_n*

$$f(u_0) = (2n + 1)$$
$$f(u_t) = (2t - 1)$$
$$f(u'_t) = 2t$$

The edge labels induced are $(2t - 1)(2n + 1)$, $(2n + 1)2t$, $(2t - 1)2t$, $1 \leq t \leq n$, $(2n - 1)$ and $(2t - 1)(2t + 1), 1 \leq t \leq n - 1$. Clearly all the labels are different.

Case 2: $(2n + 1)$ is not a prime number.
Let p be the highest prime in the set $\{1, 2, 3, ..., 2n + 1\}$. We have two sub cases.

Sub case (i) : $(2n + 1)$ cannot be expressed as the product of two consecutive odd numbers.
Define the vertex labeling as

$$f(u_0) = p$$
$$f(u_t) = (2t - 1), \ \ 1 \leq t \leq \frac{(p - 1)}{2}$$
$$f(u_t) = (2t + 1), \ \ \frac{(p + 1)}{2} \leq t \leq n$$
$$f(u'_t) = f(u_t) + 1, \ \ 1 \leq t \leq n.$$

The edge labels induced within E' are p, $2p$ $3p$, $\ldots$, $p(p-2)$, $p(p-1)$, $p(p+2)$, $\ldots$, $p(2n+1)$.
The edge labels within E'' are 1×2, 2×3, $\ldots$, $(p-2)(p-1)$, $(p+2)(p+1)$, $\ldots$, $2n(2n+1)$.
The edge labels within E''' are 1×3, 3×5, $\ldots$, $(2n-1)(2n+1)$, $(2n-1)$.
Clearly all are different.

Sub case (ii) : $(2n+1)$ is the product of two consecutive odd numbers. ie, $(2n+1)=m(m+2)$, m is odd.

When $m=3$, ie,$(2n+1)=15$. LH labeling of Fl_7 is shown in the figure 5.15

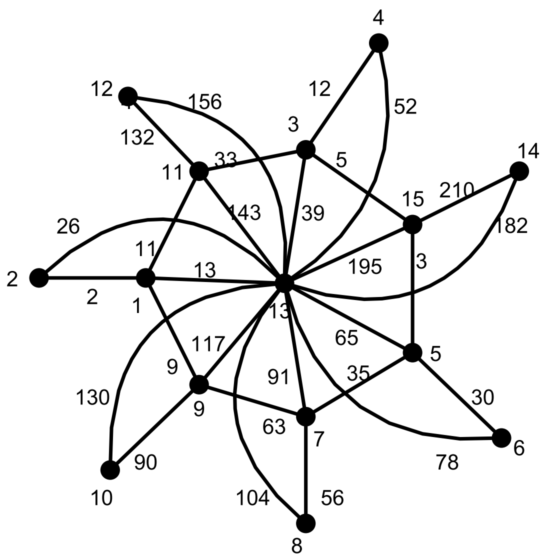


Figure 5.15: *LH labeling of Fl_7*

When $m>3$.
Define the vertex labeling as

$$f(u_0)=p$$
$$f(u_1)=m$$
$$f(u_1')=(m+1)$$
$$f(u_2)=(2n+1)=m(m+2)$$

$$f(u_2') = 2n$$
$$f(u_3) = (m+2)$$
$$f(u_t) = f(u_{t-1}) + 2, \quad 1 \le t \le \frac{(p-1)}{2}$$
$$f(u_t') = f(u_t) + 1, \quad 3 \le t \le n.$$

The remaining vertices $u_{\frac{p+1}{2}}$, $u_{\frac{p+3}{2}}$, ..., u_n are numbered consecutively using the set $\{1, 3, 5, \ldots, (m-2)\}$.
The edge labels induced within E' are p, $2p$ $3p$, ..., $p(p-2)$, $p(p-1)$, $p(p+2)$, ..., $p(2n+1)$. The edge labels within E'' are 1×2, 2×3, ..., $(p-2)(p-1)$, $(p+2)(p+1)$, ..., $2n(2n+1)$. The edge labels within E''' are $(m+2)$, m, $(m+2)(m+4)$, ..., $(2n-1)$, 1×3, 3×5, ..., $m(m-2)$. Thus all the labels induced are distinct.
The labeling pattern defined above satisfies the vertex conditions and edge conditions of an LH graph. Hence the graph Fl_n is an LH graph.

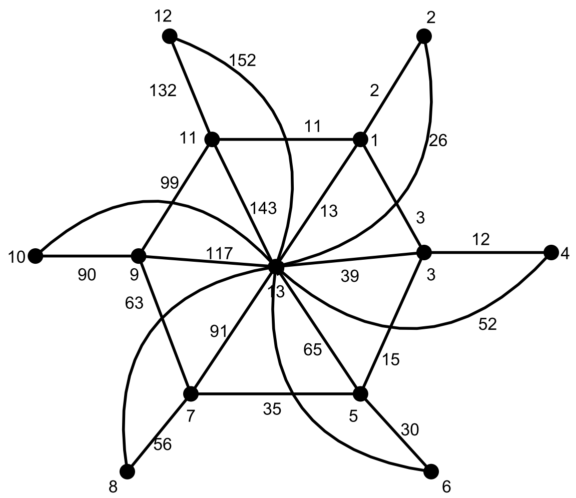


Figure 5.16: *LH labeling of Fl_6*

□

Corollary 5.3.6. The helm $H_n, n \ge 3$ is an LH graph

Proof. H_n is a spanning subgraph of Fl_n and the result follows from the theorems 4.2.8 and 5.3.4.

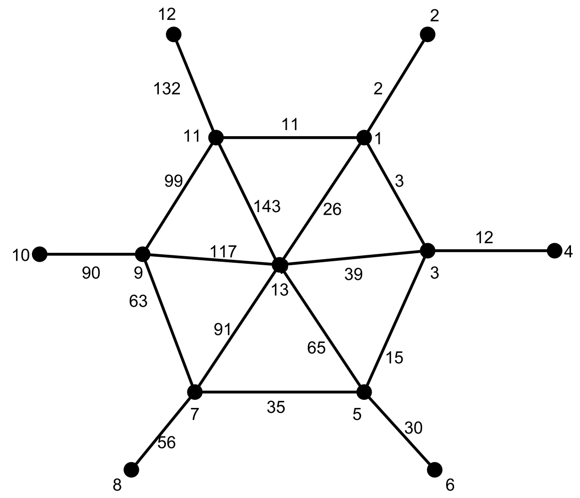


Figure 5.17: *LH labeling of H_6*

□

5.4 Star related Graphs

LH labeling of bistar graph is demonstrated in this section.

Theorem 5.4.1. The bistar graph $B_{m,n}$ is an LH graph.

Proof. First number the m pendant vertices by $u_1, u_2, ..., u_m$ and n pendant vertices by $v_1, v_2, v_3...., v_n$. The vertex adjacent to u_i is u and vertex adjacent to v_i is v.
$V = \{u, v\} \cup \{u_i, i = 1 \text{ to } m\} \cup \{v_j, j = 1 \text{ to } n\}$.
$|V| = m + n + 2$.
$E = E' \cup E'' \cup E'''$ where $E' = \{uu_i, i = 1 \text{ to } m\}$, $E'' = \{vv_j, j = 1 \text{ to } n\}$ and $E''' = \{uv\}$.
Vertex labeling f from V to the set $\{1, 2, 3, ..., m + n + 2\}$ is defined as follows:
We have 3 cases.

Case 1 : When $m = n$
$|V| = 2n + 2$

$$f(u) = 2n + 1$$

$$f(u_i) = 2i + 2, \ \ 1 \leq i \leq n$$
$$f(v) = 2n - 1$$
$$f(v_i) = 2i - 1, \ \ 1 \leq i \leq (n - 1)$$
$$f(v_n) = 2$$

The induced edge labels within the set E' are of the form $(2n + 1)(2i + 2)$, $i = 1$ to n and within the set E'' are $2(2n - 1)$, $(2n - 1)(2i - 1)$, $i = 1$ to $(n - 1)$. In E''' only one label $(2n + 1)(2n - 1)$. Clearly, all labels are different.

Case 2 : When $m = 1$
$V = \{u, u_1, v, v_i, i = 1 \text{ to } n\}$
$|V| = n + 3$

$$f(u) = n + 2$$
$$f(u_1) = n + 3$$
$$f(v) = 1$$
$$f(v_i) = i + 1, \ \ 1 \leq i \leq n.$$

The induced edge labels within the set E' is $(n + 2)(n + 3)$ and within the set E'' are $(i + 1)$, $i = 1$ to n. In E''' only one label $(n + 2)$. Clearly, all labels are different.

Case 3 : When $m < n$
i) n odd and m even or n even and m odd

$$f(u) = m + n + 2$$
$$f(u_1) = m + n + 1$$
$$f(u_i) = f(u_{i-1}) - 2, \ \ 2 \leq i \leq m$$
$$f(v) = m + n$$

The remaining $v_i, i = 1$ to n can be numbered using the digits $1, 3, 5, 7, ..., m + n - 2, 2, 4, 6.., f(u_m) - 2$.

The induced edge labels within the set E' are of the form $(m+n+2)(m+n+1)$,

$(m+n+2)(m+n-1)$,.... $(m+n+2)(n-m+3)$ and within the set E'' are $1(m+n)$, $3(m+n)$, $5(m+n)$, ...,$(m+n-2)(m+n)$, $2(m+n)$, $(n-m+1)(m+n)$. In E''', the edge label is $(m+n+2)(m+n)$. Clearly all labels are different.

ii) n even and m even or n odd and n odd

$$f(u) = m+n+1$$
$$f(u_1) = m+n+2$$
$$f(u_i) = f(u_{i-1}) - 2, \ \ 2 \le i \le m$$
$$f(v) = m+n-1$$

The remaining $v_i, i = 1$ to n can be numbered using the digits $1, 3, 5, 7, ..., m+n-3, 2, 4, 6.., f(u_m) - 2$.
The induced edge labels within the set E' are of the form $(m+n+2)(m+n+1)$, $(m+n+1)(m+n)$,.... $(m+n+1)(n-m+4)$ and within the set E'' are $1(m+n-1)$, $3(m+n-1)$, $5(m+n-1)$, ...,$(m+n-1)(m+n-3)$, $2(m+n-1)$, $4(m+n-1)$, ..., $(n-m+2)(m+n-1)$. In E''', the edge label is $(m+n+1)(m+n-1)$. Clearly all labels are different.

Hence $B_{m,n}$ is an LH graph.

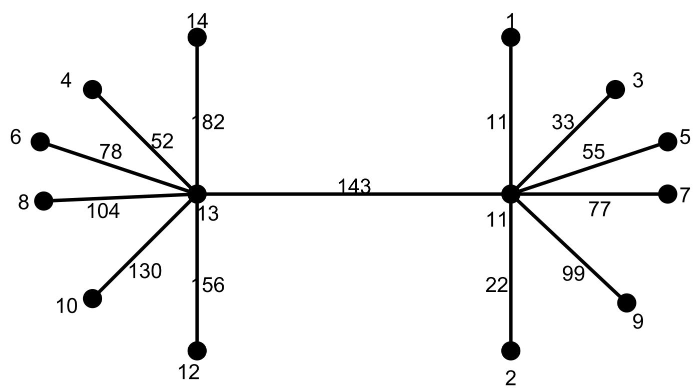


Figure 5.18: *LH labeling of* $B_{6,6}$

□

5.5 Splitting graphs

The concept of splitting graphs are originated to yield new large graph from the given graph. Splitting graph was introduced by Sampath Kumar and Walikar [68]. In this section LH labeling of splitting graph of a path, comb and star are discussed.

Theorem 5.5.1. Splitting graph of a path $S'(P_n)$ is an LH graph.

Proof. Let $\{v_t, 1 \leq t \leq n\}$ be the vertices of a path P_n and Let $\{u_t, 1 \leq t \leq n\}$ be the new set of vertices added to P_n to obtain the splitting graph of P_n, denoted by $S'(P_n)$. Then $V = \{v_t, u_t, 1 \leq t \leq n\}$
and $E = E' \cup E'' \cup E'''$ where $E' = \{v_t v_{t+1}, 1 \leq t \leq n-1\}$, $E'' = \{v_{t+1} u_t, 1 \leq t \leq n-1\}$ and $E''' = \{v_t u_{t+1}, 1 \leq t \leq n-1\}$.
$|V| = 2n$.

Define $f : V \to \{1, 2, ...2n\}$ by

$$f(v_1) = 1$$
$$f(v_2) = 2n - 1$$

$$f(v_i) = \begin{Bmatrix} f(v_{i-2)} + 2 & if \;\; i \;\; is \;\; odd \\ f(v_{i-2)} - 2 & if \;\; i \;\; is \;\; even. \end{Bmatrix}$$
$$f(u_1) = 2n$$
$$f(u_2) = 2$$
$$f(u_i) = \begin{Bmatrix} f(u_{i-2)} - 2 & if \;\; i \;\; is \;\; odd \\ f(u_{i-2)} + 2 & if \;\; i \;\; is \;\; even. \end{Bmatrix}$$

The induced edge function is $f^* : E \to N$.

Claim : All the edge labels are distinct.

Note that the edge labels with in the set E' are odd. To prove the claim, it is enough to show that the edge labels within E'' and E''' are distinct. The edge labels within the set E'' are of the form 2×3, 4×5, 6×7,......,$(2n-1)2n$ and that in E''' are 1×2, 3×4, 5×6,......, $(2n-1)(2n-2)$. Therefore, no edge label is common to the sets E'' and E'''. Thus all the edge labels induced are distinct. Hence the proof.

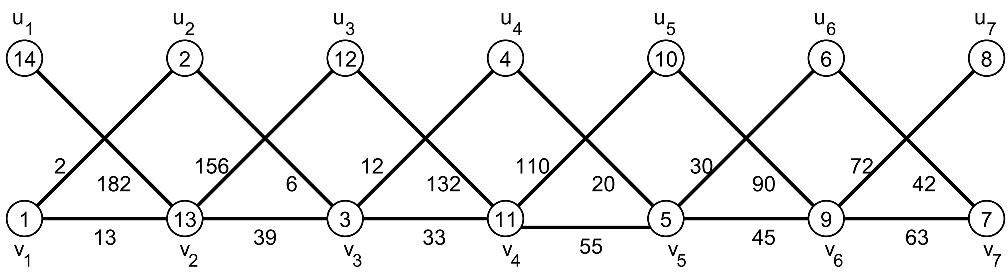


Figure 5.19: *LH labeling of $S'(P_7)$*

□

Theorem 5.5.2. The Splitting graph of of a comb $S'(P_n \circ K_1)$ is an LH graph.

Proof. Let $\{v_i, 1 \leq i \leq n\}$ and $\{v_i', 1 \leq i \leq n\}$ be the vertices of comb in which $\{v_i', 1 \leq i \leq n\}$ are the pendant vertices.
Let $\{u_i, 1 \leq i \leq n\}$ and $\{u_i', 1 \leq i \leq n\}$ be the newly added vertices to obtain the splitting graph.
Then $|V| = 4n$.
$E = E' \cup E'' \cup E''' \cup E^{iv} \cup E^{v} \cup E^{vi}$ where $E' = \{v_iv_{i+1}, 1 \leq i \leq (n-1)\}$, $E'' = \{v_iv_i', 1 \leq i \leq n\}$, $E''' = \{u_iv_i', 1 \leq i \leq n\}$, $E^{iv} = \{v_iu_i', 1 \leq i \leq n\}$, $E^{v} = \{v_iu_{i+1}, 1 \leq i \leq (n-1)\}$ and $E^{vi} = \{u_iv_{i+1}, 1 \leq i \leq (n-1)\}$.

Define $f : V \to \{1, 2, 3, ..., 4n\}$ as follows:

$$f(v_1) = 4n - 1$$
$$f(v_i) = f(v_{i-1}) - 2, \ \ 2 \leq i \leq n$$
$$f(v_1') = 4n$$
$$f(v_i') = f(v_{i-1}') - 2, \ \ 2 \leq i \leq n$$
$$f(u_i) = 2i - 1, \ \ 1 \leq i \leq n$$
$$f(u_i') = f(v_n') - 2i, \ \ 1 \leq i \leq n$$

The induced edge function is $f^* : E \to N$.

Claim : All the edge labels are distinct.

Table 5.1: *Induced edge labels .*

Edge set	Edge labels
E'	$(4n-1)(4n-3)$, $(4n-3)(4n-5)$,......., $(2n+3)(2n+1)$
E''	$(4n-1)4n$, $(4n-3)(4n-2)$, ...,$(2n+1)(2n+2)$
E'''	$4n$, $3(4n-2)$, $5(4n-4)$ ....,$(2n-1)(2n+2)$
E^{iv}	$(4n-1)2n$, $(4n-3)(2n-2)$, $(4n-5)(2n-4)$,......,$(2n+1)2$
E^{v}	$3(4n-1)$, $5(4n-3)$, $7(4n-5)$,....., $(2n-1)(2n+3)$
E^{vi}	$(4n-3)$, $3(4n-5)$, $5(4n-7)$,........,$(2n-3)(2n+1)$.

In E''', if two vertex label have a common factor then the edge label is an even number between 1 and n.
Similarly, In E''', if two vertex label have a common factor then the edge label is an odd number between 1 and n.
In view of the above defined labeling pattern f satisfies the conditions of an LH labeling. Hence the proof.

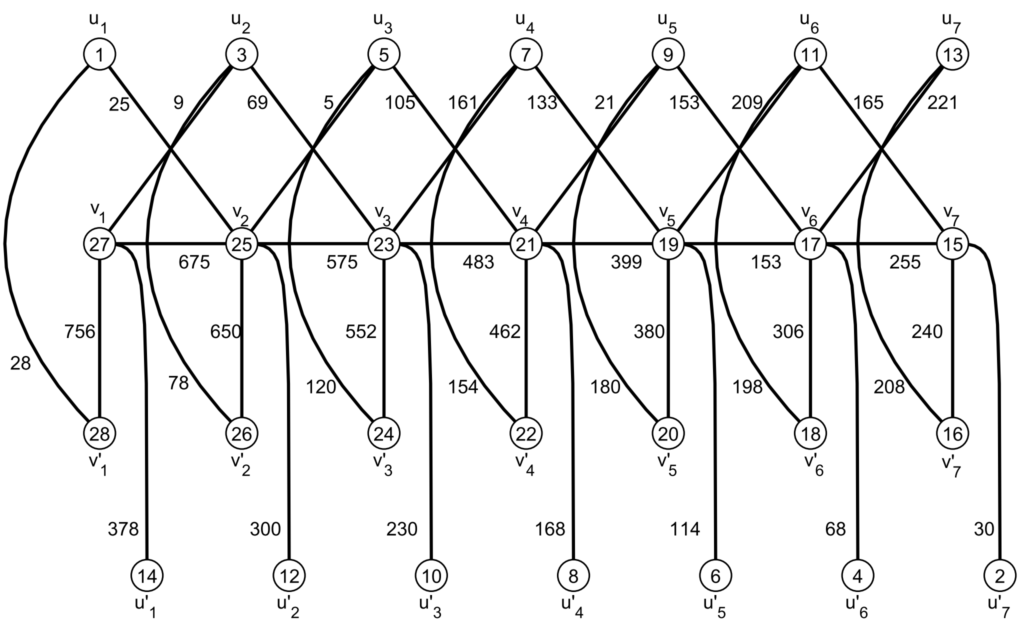


Figure 5.20: *LH labeling of* $(P_7 \circ K_1)$

□

Theorem 5.5.3. The splitting graph of a star $S'(K_{1,n})$ is an LH graph.

Proof. Let $v, v_i, 1 \leq i \leq n$ are the vertices of the star $K_{1,n}$ with central vertex v. Let $u, u_i, 1 \leq i \leq n$ are the newly added vertices to obtain the splitting graph $S'(K_{1,n})$. Then $V = \{u, v, v_i, u_i, 1 \leq i \leq n\}$. $E = E' \cup E'' \cup E'''$ where $E' = \{u_i v, 1 \leq i \leq n\}$, $E'' = \{vv_i, 1 \leq i \leq n\}$ and $E''' = \{uv_i, 1 \leq i \leq n\}$.
$|V| = 2n + 2$

Define $f : V \rightarrow \{1, 2, 3, ..., 2n + 2\}$ as follows:

$f(v) = p$, p is the highest prime in the set $\{1, 2, 3, ..., 2n + 2\}$

$$f(v_i) = (2i + 1), \ \ 1 \leq i \leq (n - 1)$$
$$f(v_n) = (2n + 2)$$
$$f(u_i) = 2i, \ \ 1 \leq i \leq n$$
$$f(u) = 1.$$

The induced edge function is $f^* : E \rightarrow N$.
Claim : All the edge labels are distinct.

Table 5.2: *Induced edge labels*

Edge set.	Edge labels.
E'	$2p, 4p, 6p, ..., 2np$
E''	$3p, 5p, 7p, ...(2n + 1)p, (2n + 2)p$
E'''	$3, 5, 7, ..., (2n + 1), (2n + 2)$

.

‘ It is clear that the induced edge labels are distinct. Hence the splitting graph of a star $S'(K_{1,5})$ is an LH graph.

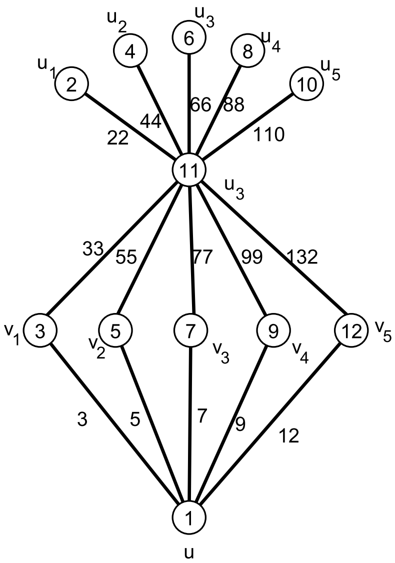


Figure 5.21: *LH labeling of* $S'(K_{1,5})$

□

5.6 Line Graphs

Line graphs are first introduced by Harary and Norman. Let G be a graph, then the line graph of G is denoted by $L(G)$ and it is a graph whose vertex set is in $1-1$ correspondence with the edge set of G and two vertices of $L(G)$ are joined by an edge if and only if the corresponding edges of G are adjacent in G [48]. In this section LH labeling of line graph of comb is derived .

Theorem 5.6.1. The line graph of a comb $L(P_n \circ K_1)$ is an LH graph.

Proof. Consider a comb graph $P_n \circ K_1$ with vertex set $\{v_t, v'_t, 1 \leq t \leq n\}$ where $v'_t, t = 1$ to n are the pendant vertices.
$E = \{v_i v'_i, 1 \leq i \leq n\} \cup \{v_i v_{i+1}, 1 \leq i \leq (n-1)\}$ and
$|E| = 2n - 1$.
Let $p_i = v_i v'_i, 1 \leq i \leq n$ and $q_i = v_i v_{i+1}, 1 \leq i \leq (n-1)$ are the vertices of $L(P_n \circ K_1)$. $|V| = 2n - 1$. The edge set of $L(P_n \circ K_1)$ is $E = E' \cup E'' \cup E'''$, where $E' = \{q_i q_{i+1}, 1 \leq i \leq (n-2)\}$, $E'' = \{q_i p_i, 1 \leq i \leq (n-1)\}$ and

$E''' = \{q_i p_{i+1}, 1 \leq i \leq (n-1)\}$.

Define $f : V \to \{1, 2, 3, ..., (2n-1)\}$ as follows:

Case 1: n is even.

$$f(q_i) = (2i-1),\ 1 \leq i \leq (n-1)$$
$$f(p_1) = (2n-2)$$
$$f(p_i) = 2i-2,\ 2 \leq i \leq n-1$$
$$f(p_n) = (2n-1).$$

Case 2: n is odd.

$$f(q_i) = (2i-1),\ 1 \leq i \leq (n-1)$$
$$f(p_1) = (2n-1)$$
$$f(p_i) = 2i-2,\ 2 \leq i \leq n.$$

The induced edge function is $f^* : E \to N$.

Claim : All the edge labels are distinct.

For n even, the edge labels are given in the following table.

Table 5.3: *Induced edge labels*

Edge set.	Edge labels.
E'	1×3, 3×5, 5×7,.....,$(2n-5)(2n-3)$
E''	$1(2n-2)$, 2×3, 4×5, ..., $(2n-4)(2n-3)$
E'''	1×2, 3×4, 5×6,....., $(2n-5)(2n-4)$

.

Clearly the edge labels are distinct. Similar proof holds for the case when n is odd. Hence the proof.

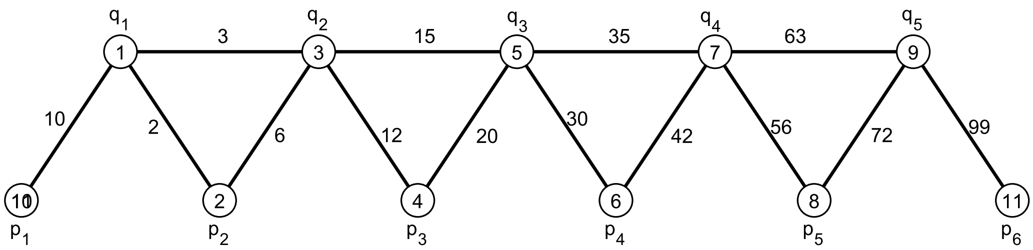


Figure 5.22: *LH labeling of* $L(P_6 \circ K_1)$

□

5.7 A Theta graph

In this section, the LH labeling of a theta graph T_α, splitting graph of T_α, degree splitting graph of T_α are discussed. Also studied the LH labeling of the graph obtained by the switching of a vertex in T_α.

Theorem 5.7.1. A Theta graph T_α is an LH graph.

Proof. Let $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ are the vertices of a Theta graph T_α with centre v_0 and $|V| = 7$.
We define the vertex labeling $f : V \to \{1, 2, 3, 4, 5, 6, 7\}$ as follows:
$f(v_0) = 6, f(v_1) = 7, f(v_2) = 1, f(v_3) = 2, f(v_4) = 5, f(v_5) = 4$ and $f(v_6) = 3$ In view of the labeling defined above, it is clear that the induced edge labels are distinct. Hence the proof. □

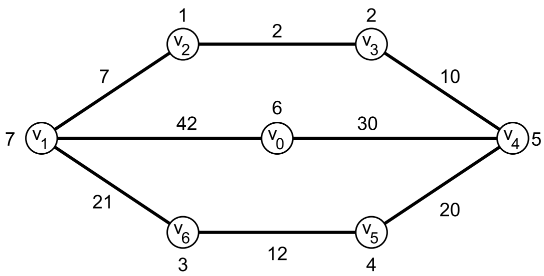


Figure 5.23: *LH labeling of* T_α

Theorem 5.7.2. The splitting graph of a Theta graph $S'(T_\alpha)$ is an LH graph.

Proof. Let $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ are the vertices of a Theta graph T_α with centre v_0 and $u_0, u_1, u_2, u_3, u_4, u_5, u_6$ be the newly added vertices corresponding to $v_i, 0 \leq i \leq 6$ to obtain the splitting graph of a Theta graph. $|V| = 14$.
We define the vertex labeling $f : V \to \{1, 2, 3, 4, ..., 14\}$ as follows:
$f(v_0) = 11, f(v_1) = 7, f(v_2) = 1, f(v_3) = 3, f(v_4) = 13, f(v_5) = 9, f(v_6) = 5$
$f(u_0) = 6, f(u_1) = 12, f(u_2) = 2, f(u_3) = 4, f(u_4) = 14, f(u_5) = 8, f(u_6) = 10.$
It is clear that the induced edge labels are distinct. Hence the splitting graph of a Theta graph $S'(T_\alpha)$ is an LH graph. □

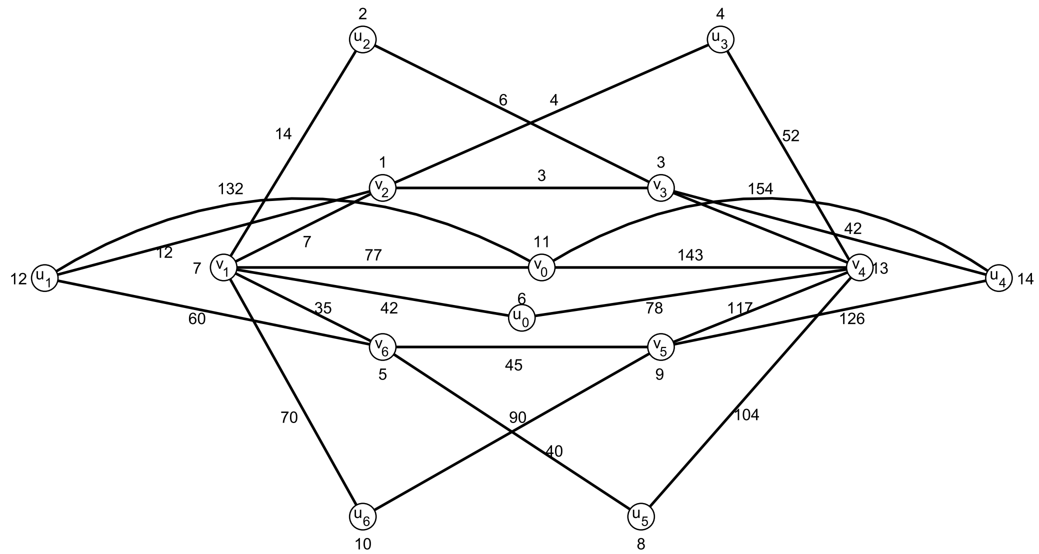
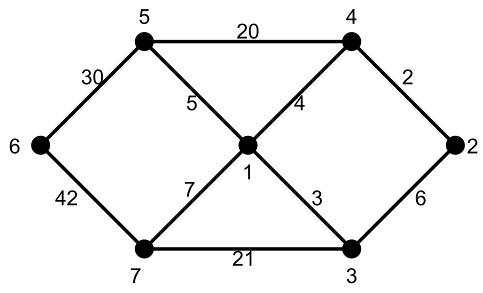


Figure 5.24: *LH labeling of $S'(T_\alpha)$*

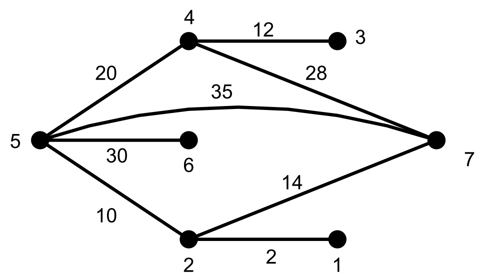
Theorem 5.7.3. The switching of any vertex in a Theta graph is an LH graph.

Proof. Let $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ are the vertices of a Theta graph T_α with centre v_0 and $|V| = 7$.
Let G_v be the graph obtained from T_α after switching the vertex $v_i, 0 \leq i \leq 6$. Clearly $|V(G_v)| = 7$
We define the vertex labeling $f : |V(G_s)| \to \{1, 2, 3...., 7\}$ as follows :
Case 1 : switching of the center vertex v_0
$f(v_0) = 1, f(v_1) = 6, f(v_2) = 5, f(v_3) = 4, f(v_4) = 7, f(v_5) = 3$ and $f(v_6) = 2$



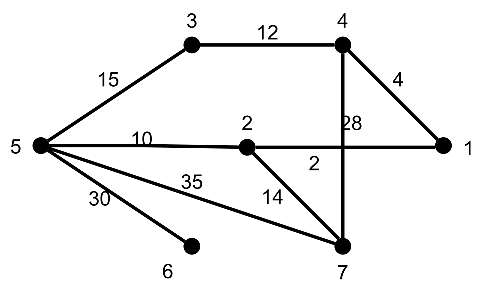
Case 2 : switching of any vertex of degree 3

In T_α, only two vertices are of degree 3. The vertex in which switching is done is labeled with 5, centre vertex with 6 and the vertices adjacent to 5 with 7,2 and 4. The vertex adjacent to 4 is labeled with 3 and the vertex which is adjacent to 2 with 1.



Case 3 : switching of any vertex of degree 2 other than the centre

The vertex in which switching is done is labeled with 7, centre vertex with 2 and the pendant vertex with 6. Label the adjacent vertices of centre with 5 and 1. The vertex adjacent to 5 is labeled with 3 and the vertex adjacent to 4 is labeled with 1.



□

Theorem 5.7.4. The degree splitting graph of a Theta graph $DS(T_\alpha)$ is an LH graph.

Proof. Let $v_0, v_1, v_2, v_3, v_4, v_5, v_6$ are the vertices of the Theta graph T_α with centre v_0.

It has two vertices of degree 3 and all other vertices are of degree 2. Therefore, $V = S_1 \cup S_2$ where $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_0, v_2, v_3, v_5, v_6\}$. Add two vertices w_1 and w_2 corresponding to the sets S_1 and S_2 and make w_1 adjacent to v_1 and v_4 and w_2 adjacent to v_2, v_3, v_5, v_6 and v_0, to obtain the degree splitting graph $DS(T_\alpha)$.

$|V| = 9$

We define the vertex labeling $f : V \to \{1, 2, 3, 4, ..., 9\}$ as follows: $f(v_0) = 6, f(v_2) = 1, f(v_3) = 2, f(v_4) = 5, f(v_5) = 8, f(v_6) = 3f(w_1) = 9, f(w_2) = 7$ The vertex labeling defined above satisfies the conditions of an LH graph. therefore, the degree splitting graph of a Theta graph $DS(T_\alpha)$ is an LH graph. □

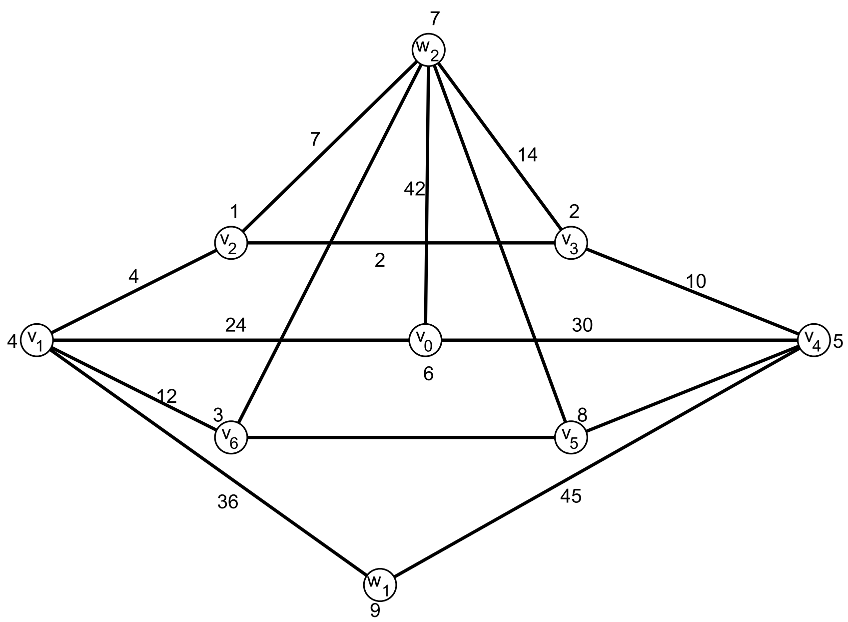


Figure 5.25: *LH labeling of $DS(T_\alpha)$*

5.8 Conclusion

In this chapter, LH labeling of some well-known and familiar families of graphs are investigated. Exploring the LH labeling of other families of graphs like middle graphs, shadow graphs, total graphs etc is a future area of research.

Problem 5.8.1. *Is it true that the cartesian product of two LH graphs is an LH graph?*

Problem 5.8.2. *Find the LH labeling of arbitrary super subdivision of paths and cycles?*

CHAPTER 6

Conclusion and Further Scope of Research

In the first section, we give a summary of the thesis and the second section includes application and proposals for further study.

6.1 Conclusion

We begin with an introductory chapter which includes preliminaries and literature survey of the topic.

In the first part of the report, the study was done by connecting the two topics namely graph labeling and graph convexity. In the second part a new vertex labeling, LH labeling is introduced and studied it in some classes of graphs.

In chapter II, we made an attempt to study geodesic and monophonic convexity in a graph G with respect to a labeling function $\mathcal{L}$ defined on it. Defined L_g convexity space, $g-$ convex label, $\mathcal{L}_m$ convexity space and $m-$ convex label. A new class of graphs, geodesically elegant graphs are introduced. Some of the results of this chapter are published in Malaya Journal of Mathematik [14]. The main results in this chapter are

1. All geodesically elegant graphs are triangle free.
2. The cycle C_n for all $n > 3$ is geodesically elegant.
3. The crown graph $C_n \circ K_1$ for all $n > 3$ is geodesically elegant.
4. The generalized friendship graph $f_{4,n}$ is geodesically elegant.
5. The hypercube graph Q_n is geodesically elegant.
6. The graph $C_m \square P_n$ is geodesically elegant.
7. The square mesh graph $P_r \square P_r$ is geodesically elegant.
8. The ladder graph $L_n = P_n \square P_2$ is geodesically elegant.
9. A Theta graph T_α is geodesically elegant.
10. The graph obtained by switching of any vertex in a Theta graph T_α is not geodesically elegant.
11. Geodesic convex label does not exist in the complete bipartite graph $K_{m,n}$ except for $m = 1$ or $n = 1$ or $m = n = 2$.
12. A tree T is always geodesically elegant.
13. The number of $\mathcal{L}_g$ convex sets of K_n for $n \geq 3$ with respect to any label $\mathcal{L}$ is $\frac{n^2+n+2}{2}$.
14. Strong product of any two graphs G and H, both having atleast one edge is not geodesically elegant.
15. Lexicographic product of any two graphs G and H, both having atleast one edge is not geodesically elegant.
16. The corona product of two geodesically elegant graphs may not be geodesically elegant
17. The join of two geodesically elegant graphs need not be geodesically elegant.
18. m-convex label exists in a tree.

19. If a strong monophonic convex label exists in a graph G then G is triangle free.

20. Strong monophonic convex label exist in the cycle C_n for $n > 3$.

21. Strong monophonic convex label exists in the Petersen graph.

Chapter III deals with the study of geodetic and edge geodetic number in labeled graphs. Defined $\mathcal{L}-$ geodetic number and $\mathcal{L}-$ edge geodetic number of graphs. The concept of geodetic label, edge geodetic label, strong geodetic label and strong edge geodetic label are studied. The results of this chapter are communicated to AIP Conference Prooceedings. The main results are

22. $g_{\mathcal{L}}(G) \leq g'_{\mathcal{L}}(G)$.

23. Geodetic label and edge geodetic label exist in every tree.

24. Strong geodetic label and strong edge geodetic label exist in $K_n, n > 2$.

25. Geodetic label and edge geodetic label exist for every even cycle. Strong geodetic label and Strong edge geodetic label exist for every odd cycle.

26. Strong geodetic label exist in the Petersen graph.

27. Strong geodetic label exists in the Wheel graph $W_{1,n}$ except for $n = 4$. Geodetic label exist in the graph $W_{1,4}$.

28. Geodetic label exist in the mesh $M = M_{r,s} = P_r \times P_s$.

29. Strong edge geodetic label exist in the friendship graph ${F_3}^n$.

30. Strong edge geodetic label exist in the Windmill graph $Wd(k, n)$.

Chapter IV deals with the LH labeling of graphs. For a non LH graph G, we defined the LH completion $\Omega^*(G)$ and LH completion number Λ_G of G. An upper bound for the size of an LH graph is obtained. The results of this chapter are published in Advances in Mathematics: Scientific Journal[15]. Main results of this chapter are

31. Petesrsen graph, Grotzch graph, Heawood graph and hyper cube Q_3 are LH.

32. For any LH graph G with n vertices, $2 \leq f^*(e) \leq n^2 - n$, where $f^*(e)$ denotes the label of the edge e.

33. All prime graphs are LH.

34. If the labels of each pair of adjacent vertices of a given graph G are relatively prime, then the LH labeling coincides with the strong multiplicative labeling.

35. The complete graph $K_n, n \geq 4$, the wheel graph $W_{1,5}$ and the complete bipartite graph $K_{3,3}$ are non LH.

36. The complete bipartite graph $K_{2,s}$ is an LH graph.

37. Every spanning subgrah of an LH graph is an LH graph.

38. Every induced subgraph of an LH graph need not be LH.

39. Given a non LH graph G, there exist an LH graph $\Omega^\star(G)$ such that G is an induced subgraph of $\Omega^\star(G)$, called the LH completion of G.

40. Let $\Omega^\star(G)$ be an LH completion of a non LH graph G. LH completion number of G, denoted by Λ_G is defined as $\Lambda_G = minimum\{|V(\Omega^\star)| - |V(G)|\}$.

41. $\Lambda_{K_{3,3}} = 1$ and $\Lambda_{W_{1,5}} = 1$.

42. LH completion number of the complete graph K_n, $n \leq 10$ is given in the following table.

Table 6.1: *LH completion number of K_n, $4 \leq n \leq 10$*

n .	Number of edges.	μ_n	Λ_{K_n}
4	6	5	1
5	10	9	1
6	15	10	2
7	21	16	4
8	28	20	5
9	36	26	8
10	45	28	9

43. A perfect binary tree T_n and spider graph $S_3(m)$ are LH graphs.

Finally, we discussed the LH labeling of some class of graphs in chapter V. The results of this chapter can be seen in [13, 15]. The following classes of graphs are proved to be LH.

44. Path related graphs : comb graph $P_n \circ K_1$, triangular snake T_n, quadrilateral snake G_n, Twig graph $TW(n), n \geq 3$, $[P_n : S_2]$, sparkler graph $(P_m)^{+n}$ and the $H-$ graph.
45. Cycle related graphs : cycle C_n, Crown $C_n \circ K_1$, Friendship graph F_n, Wheel graph $W_{1,n}$ if $(n+1)$ is prime or $n+1$ is the product of two consecutive natural numbers, flower graph Fl_n and helm $H_n, n \geq 3$.
46. Bistar graph $B_{m,n}$.
47. Splitting graph of a path $S'(P_n)$, star $S'(K_{1,n})$ and comb $S'(P_n \circ K_1)$.
48. Line Graph of comb $P_n \circ K_1$.
49. A Theta graph T_α.
50. The splitting graph of a Theta graph $S'(T_\alpha)$.
51. The degree splitting graph of a Theta graph $DS(T_\alpha)$.
52. Graph obtained by switching of any vertex in a Theta graph T_α.

6.2 Application and Proposal for Further Study

Any real life situation can be studied and solved using a mathematical model. A weighted graph is used to model all the practical problematic situations like rail networks, communication networks, electrial power systems, road networks etc. Usually, the weight in a weighted graph is single. But in a real life situation, a single weight is not enough to describe the information totally.

For example: in a road network, we always prefer to travel in the shortest path between two places. Sometimes these shortest path includes bumpy roads and toll roads with huge amount. There are narrow roads and bridges in the network in which huge vehicles can't pass through. Sometimes, the shortest path includes these two which is to be avoided. There is a restriction for heavy vehicles in some routes in the network. So small vehicles can pass through that route without any traffic congestion. Also the network consist of roads with chaotic traffic which takes hours to cross the road. To calculate the shortest distance, we have to consider all these factors. So to model a road network in the real world multi-dimension weighted graphs are needed. A multi-dimension graph was introduced and studied by Shuo Jiang, Zhiyong Feng, Xiawoang Zhang, Xinwang and Guozheng Rao in [75].

Multi-dimension weighted graph is an extension of weighted graph. Multi dimension weighted graph is, not single weight on every edge. It can be represented as $G = (V, E, w_1, w_2, w_3, ..., w_n)$ [75]. A Multi dimension weighted graph can be transformed to a single dimension weigted graph, so that the properties which are used in the single dimension weighted graphs can also be applied in multi-dimension weighted graph. Jill K mathew and Sunil mathew defined weighted geodesic convexity in weighted graphs in [33]. If we transform the multidimensional weighted graph into a weighted graph, then it should coincide with the weighted geodesic convexity defined by them. But we can see that these studies on weighted graph only focus on one dimension weight and the weighted distance between the vertices u and v in G is defined and denoted by $d_w(u, v) = min\{l(P) * S(P)/P$ is a $u - v$ path$\}$ where $l(P)$ represents the length and $S(P)$ is the strength of the path. The length of a path is the number of edges present in the path and strength is the minimum of the edge weights in it. While discussing the connectivity concept, the definition of strength of a path is apt. But in the context of discussing shortest distance in a transportation network, $S(P)$ defined to be the minimum edge weight is not enough and the distance function does not satisfy the metric property. Also in a road network usually we avoid weak strength roads and prefer to travel in shortest and strong routes. We can study geodesic convexity in multi-dimension weighted graphs as in the case of labeled graphs. Since we are able to find the least common multiple and highest common factor of a group of numbers, we can apply the concept of LH labeling in these graphs also.

We know that the study carried out in this report is not complete. The following are some of the questions in our mind for future research.

Problem 6.2.1. *Find the geodetic iteration number $gin(G)$ in labeled graphs for different labelings.*

Problem 6.2.2. *Does there exist two graphs G and H such that one of them is not geodesically elegant but thier cartesian product is not geodesically elegant.*

Problem 6.2.3. *Find $\mathfrak{L}-$ geodetic number of power of cycles?.*

Problem 6.2.4. *Is all trees LH?.*

Problem 6.2.5. *Study convexity determined by LH labeling..*

Problem 6.2.6. *If G is LH, is it true that the Myscielskian $M(G)$ is LH or not?*

Problem 6.2.7. *Find the (G, D) number of graphs with respect to the labeling function $\mathfrak{L}$*

Problem 6.2.8. *Find the LH labeling of power of cycles.*

Publications in Journals and Presentations

Publications:

1. Farisa M and Parvathy K S, "LH LABELING OF GRAPHS," *Advances in Mathematics: Scientific Journal,* 10 (2021), no.4, 2167-2179.

2. Farisa M and Parvathy K S, "Geodesic convexity in labeled graphs," *Malaya journal of Matematik,* Vol.9, No.1, 735-740, 2021.

3. Farisa M and Parvathy K S, "Lh Labeling of Some Graphs," *International Journal of Recent Technology and Engineering,* ISSN: 2277 – 3878, Volume- 7, Issue- 682, April 2019.

4. Farisa M and Dr. Parvathy K S, *Geodetic and Edge Geodetic Number in Labeled Graphs*, *AIP Conference Proceedings (Accepted for publication).*

Presentations:

1. Farisa M, *Geodetic and Edge Geodetic Number in Labeled Graphs*,2nd International Conference on Computational Sciences - Modelling, Computing and Soft Computing , conducted online by the Department of Mathematics of Manipal Institute of Technology, Manipal, India, held on 28 - 30, March 2022.

2. Farisa M, *Geodesic Convexity in Labeled Graphs*, in the interntional conference on Emerging Trends in Graph Theory (ICETGT – 2019), Bangalore, India, held on 27 – 28 February 2019.

3. Farisa M, *LH Labeling of Some Graphs*, in the international conference on Pure and Applied Mathematics (ICPAM – 2018), SCSMV, Enathur, Kanchipuram, India, held on 17 – 19, December 2018.

4. Farisa M, *LH Labeling of Graphs*, in the international conference on Discrete Mathematics and its Application to Network Science (ICDMANS- 2018) at BITS, Pilani Goa, India, held on 07 – 10, July 2018.

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