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NEWFORMS OF SIEGEL MAASS SPACE OF DEGREE TWO AND SAITO-KUROKAWA LIFTS

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by

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CERTIFICATE

I hereby certify that the thesis entitled "**NEWFORMS OF SIEGEL MAASS SPACE OF DEGREE TWO AND SAITO-KUROKAWA LIFTS**" is a bona fide work carried out by **Mr. Sreejith M. M.** under my guidance with Dr. A. K. Vijayarajan, Professor, KSoM, as Co-Guide, for the award of Degree of Ph.D. in Mathematics of the University of Calicut and that this work has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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DECLARATION

I hereby declare that the thesis, entitled "**NEWFORMS OF SIEGEL MAASS SPACE OF DEGREE TWO AND SAITO-KUROKAWA LIFTS**" is based on the original work done by me under the supervision of Dr. M Manickam, Adjunct Professor, IISER Bhopal with Dr. A. K. Vijayarajan, Professor, KSoM, as Co-Guide, and it has not been included in any other thesis submitted previously for the award of any degree either to this University or to any other University or Institution.

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Dedicated to my parents

Rema A. P. and Mani M. V.

ABSTRACT

This thesis deals with the following main themes: plus subspaces associated with modular forms of half-integer weight, a generalisation of the Saito-Kurokawa lift which connects elliptic modular forms and Siegel modular forms, and kernel functions functions for L-values of half-integral weight Hecke eigenforms. Specifically, we focus on the following problems:

1. To develop the theory of newforms on the space of cusp forms of weight $k-1/2$ for $\Gamma_0(M)$ and character $\chi_0 = \left(\frac{4\chi(-1)}{\cdot}\right)\chi$ where $k \geq 2,\ M \geq 1$ are integers with $32|M$ and χ modulo M, χ^2 modulo $M/2$ are primitive Dirichlet characters. We denote this space by $S_{k-1/2}(M,\chi_0)$.

2. With the same assumptions on k, M, χ as above, we next develop the theory of newforms for the space of Jacobi forms $J_{k,1}^{cusp}(M,\chi)$ and the for the Maass spezialschar $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ when k is even. Then, we obtain the required isomorphisms between the spaces of newforms regarding the Saito-Kurokawa correspondence.

3. Final problem in this thesis is to obtain a non-cusp form of half-integral weight $k+1/2$ ($k \geq 2$ even) for $\Gamma_0(4)$ in the Kohnen plus space whose Petersson scalar product against a cuspidal Hecke eigenform g is equal to a constant multiple (the constant is explicit) the L value $L(g, k-1/2)$, and then derive certain arithmetical information related with the special values of L-function.

പ്രബന്ധ സംഗ്രഹം

മിശ്രസംഖ്യാ പ്രതലത്തിന്റെ മുകൾ പകുതിയിൽ നിർവചിച്ചിരിക്കുന്നതും, ചില സവിശേഷ ഗണിത സ്വഭാവങ്ങൾ ഉള്ളതുമായ വിശ്ലേഷകഫലനങ്ങളെയാണ് മോഡുലാർ ഫോമുകൾ എന്ന് വിളിക്കുന്നത്. ഇവയുടെ മൂല്യം മിശ്ര സംഖ്യകൾ ആയിരിക്കും. ഗണിതശാസ്ത്രത്തിലെ പല മേഖലകളിലും മോഡുലാർ ഫോമുകൾ കാണപ്പെടുന്നു, കൂടാതെ വൈവിധ്യമാർന്ന ഗണിതശാസ്ത്ര വെല്ലുവിളികൾ പരിഹരിക്കുന്നതിലും അവ ഒരു പ്രധാന പങ്ക് വഹിക്കുന്നു. ഉദാഹരണത്തിന്, നൽകിയിരിക്കുന്ന സ്വാഭാവിക സംഖ്യയെ രണ്ട് സ്ക്വയറുകളുടെ ആകെത്തുകയായി പ്രതിനിധീകരിക്കുന്നതിനുള്ള വഴികളുടെ എണ്ണം ഒരു പ്രത്യേക തരം മോഡുലാർ ഫോമിന്റെ ഫൊറിയർ ഗുണകങ്ങൾ നൽകുന്നു. 2022-ലെ സംയുക്ത ഫീൽഡ്സ് മെഡൽ ജേതാവായ മറീന വിയാസോവ്സ്കയുടെ E-8 ലാറ്റിസുകളെ പറ്റിയുള്ള പഠനത്തിൽ മോഡുലാർ ഫോമുകൾ ഒരു സുപ്രധാന പങ്കു വഹിച്ചിട്ടുണ്ട്.

അതിനാൽ, മോഡുലാർ ഫോമുകളെയും അനുബന്ധ വസ്തുക്കളെയും കുറിച്ച് കൂടുതൽ മനസ്സിലാക്കേണ്ടത് പ്രധാനമാണ്. ഈ പ്രബന്ധത്തിൽ, ഇനിപ്പറയുന്ന പ്രധാന വിഷയങ്ങൾ കൈകാര്യം ചെയ്യുന്നു: അർദ്ധ-പൂർണ്ണസംഖ്യാ ഭാരമുള്ള മോഡുലാർ ഫോമുകളുടെ ന്യൂഫോം സിദ്ധാന്തം, എലിപ്റ്റിക് മോഡുലാർ ഫോമുകളെയും സീഗൽ മോഡുലാർ ഫോമുകളെയും ബന്ധിപ്പിക്കുന്ന സൈറ്റോ-കുറോകാവ ലിഫ്റ്റിന്റെ സാമാന്യവൽക്കരണം, കൂടാതെ അർദ്ധ-പൂർണ്ണസംഖ്യാ ഭാരമുള്ള ഹെക്കേ ഐഗൻഫോമുകളുടെ L-മൂല്യങ്ങളുമായി ബന്ധപ്പെട്ട കെർണൽ ഫലനങ്ങൾ.

Contents

Chapter 1

Introduction

We focus on the following main problems in this thesis:

1. To develop the theory of newforms on the space of cusp forms of weight $k-1/2$ for $\Gamma_0(M)$ and character $\chi_0 = \left(\frac{4\chi(-1)}{\cdot}\right)\chi$ where $k \geq 2,\ M \geq 1$ are integers with $32|M$ and χ modulo M, χ^2 modulo $M/2$ are primitive Dirichlet characters. We denote this space by $S_{k-1/2}(M, \chi_0)$.

2. With the same assumptions on k, M, χ as above, we next develop the theory of newforms for the space of Jacobi forms $J_{k,1}^{cusp}(M, \chi)$ and the for the Maass spezialschar $\mathcal{S}_k^*(\Gamma_0^2(M), \chi)$ when k is even. Then, we obtain the required isomorphisms between the spaces of newforms regarding the Saito-Kurokawa correspondence.

3. Final problem in this thesis is to obtain a non-cusp form of half-integral weight $k+1/2$ ($k \geq 2$ even) for $\Gamma_0(4)$ in the Kohnen plus space whose Petersson scalar

product against a cuspidal Hecke eigenform g is equal to a constant multiple(the constant is explicit) the L value $L(g, k-1/2)$, and then derive certain arithmetical information related with the special values of L-function.

Let us explain each of the above in detail. Before that, we give a brief summary of the existing works in this direction.

1.1 Survey of literature

Saito-Kurokawa lift for level 1:

Let f be a modular form of weight $2k-2$ ($k \geq 3$) for $\mathrm{SL}_2(\mathbb{Z})$, and let f be a Hecke eigenform. Then after normalizing f by letting $a_f(1) = 1$, we get that the eigenvalues are the Fourier coefficients $a_f(n)$. Using certain explicit eigenvalues through numerical computations, N. Kurokawa [16], and H. Saito independently observed the existence of certain eigenvectors F in the space of Siegel modular form of weight k and degree 2 for the full group $\mathrm{Sp}_4(\mathbb{Z})$ having a relation with the set of eigenvalues. This correspondence connecting f and F is known as the Saito-Kurokawa correspondence. In this connection, Maass introduced a subspace in Siegel modular form of degree two, known as Maass spezialchar. The Maass spezialchar is the space of all Siegel modular forms F as described above. The Fourier coefficients of F are indexed by a set of binary quadratic forms $T = \left(\begin{smallmatrix} n & r/2 \\ r/2 & m \end{smallmatrix}\right)$ where n, m are positive integers and $r \in \mathbb{Z}$ such that $r^2 \leq 4mn$. The Hecke operators on Siegel modular forms introduced by Andrianov preserve the Maass space and moreover, the forms F and f connected under the Saito-Kurokawa correspondence. But, the determination of F uniquely up to scalar for each of the normalized Hecke eigenform f of weight $2k-2$ for $\Gamma_0(N)$ with character χ is an unknown problem. it is proved that the answer is affirmative when $N=1$ by D. Zagier [32], and then by M. Manickam, B. Ramakrishnan and T. C. Vasudevan [25] when N is odd and square-free. However in the general case where N is arbitrary and χ is arbitrary, the Saito-Kurokawa correspondence has been studied in [8], [7]. We explain this connection when $N=1$.

Let $2|k$, $k \geq 4$ be an integer. Let $S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ denote the space of cusp forms of weight $2k-2$ for $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{M}_k(\mathrm{Sp}_4(\mathbb{Z}))$ denote the space of Siegel modular forms of weight k, degree two for $\mathrm{Sp}_4(\mathbb{Z})$. H. Maass introduced and studied a canonical subspace inside $\mathcal{M}_k(\mathrm{Sp}_4(\mathbb{Z}))$, called the Maass 'Spezialschar' $\mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$ which satisfy a certain condition on the Fourier coefficients: let $F \in \mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$, and $A(n,r,l)$ denote the Fourier coefficients of F. Then, we have

$$A(n,r,l) = \sum_{d|gcd(n,r,l)} d^{k-1} A\left(\frac{nl}{d^2}, \frac{r}{d}, 1\right).$$

The above relations are called Maass relations.

Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ be a Hecke eigenform. Then the works of Andrianov, Maass and Zagier proved the existence of a unique (up to a constant) Siegel Hecke eigenform $F \in \mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$ corresponding to f via the Saito-Kurokawa correspondence given by

$$L(F,s) = \zeta(s-k+2)\zeta(s-k+1)L(f,s), \tag{1.1}$$

where $\zeta(s)$ is the Ramanujan zeta function and $L(F,s)$ and $L(f,s)$ are the L-functions associated with F and f, respectively.

The above connection relating Hecke eigenforms in $M_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$ was first realized by D. Zagier through a set of isomorphisms. Let $M_{k-1/2}(4)$ denote the space of modular forms of weight $k-1/2$ for $\Gamma_0(4)$. W. Kohnen [11] identified a special subspace $M^+_{k-1/2}(4)$, called the plus space in $M_{k-1/2}(4)$ and proved that the spaces $M_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $M^+_{k-1/2}(4)$ are Hecke equivariantly isomorphic. Let $2|k$ and $k \geq 2$. Let $J_{k,1}$ denote the space of Jacobi forms of weight k and index 1 for the full Jacobi group. The Eichler-Zagier map which sends a Jacobi form to a half-integral weight modular form in the plus space, gives an isomorphism between the spaces $J_{k,1}$ and $M^+_{k-1/2}(4)$. Now, let V_m for $m \geq 1$ denote the index shifting operator which sends forms in $J_{k,1}$ to $J_{k,m}$. If $m = 0$, the operator V_0 is the one defined by Eichler and Zagier in [6] (page 43). The Maass lift ϑ, defined on $\phi \in J_{k,1}$ by

$$\phi|\vartheta := \sum_{m \geq 0} \phi|V_m(\tau, z) e^{2\pi m \tau'}$$

acts as an isomorphism between $J_{k,1}$ and $\mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$. We refer to the following diagram from [6]:

$$\begin{array}{c}\text{Maass `Spezialschar'} \subset \mathcal{M}_k(\mathrm{Sp}_4(\mathbb{Z})) \\ \big\downarrow \wr \\ \text{Jacobi forms of weight } k \text{ and index } 1 \\ \big\downarrow \wr \\ \text{Kohnen's `plus' space } \subset M_{k-1/2}(4) \\ \big\downarrow \wr \\ M_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))\end{array}$$

The composition of the maps as explained above gives an isomorphism between the spaces $M_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{M}_k^*(\mathrm{Sp}_4(\mathbb{Z}))$ (when k is even). This is called the Saito-Kurokawa lift (for level 1), which preserves the Hecke eigenforms connected via equation (1.1) in the respective spaces. We refer to [6] for a comprehensive theory on the topic.

Extension of Saito-Kurokawa lift:

It is natural to expect certain theory of Saito-Kurokawa correspondence when one replaces $\mathrm{SL}_2(\mathbb{Z})$ by its congruence subgroups. Specifically, let f be a Hecke eigenform of weight $2k-2$ for the congruence subgroup $\Gamma_0(N)$. Does there exists a corresponding Siegel Hecke eigenform F of weight k for the congruence subgroup $\Gamma_0^2(N)$ such that an equation of the type 1.1 holds true? Also, corresponding to a normalized newform $f \in M_{2k-2}(N)$, how many linearly independent forms F are there in $\mathcal{M}_k(\Gamma_0^2(N))$? These questions were answered for the space $S_{2k-2}(N)$ where N is an odd and square-free integer in [25], and later generalised to arbitrary odd level in [22]. Let us explain this in detail.

The Saito-Kurokawa isomorphism in the case of level 1 was realised through a series of isomorphisms. To set up such an isomorpshim for higher levels, one requires the theory of newforms in the respective spaces. In the case of modular forms of integral weight, the newform theory was developed by Atkin-Lehner in [3] by identifying a certain subspace inside $S_{2k-2}(N)$ called the space of newforms, denoted by $S^{new}_{2k-2}(N)$. They also proved that the multiplicity one theorem holds true on $S^{new}_{2k-2}(N)$. Later, Kohnen set up a parallel theory of newforms inside the plus space $S^{+}_{k-1/2}(4N)$ where N is odd and square-free [12]. He proved that the spaces $S^{new}_{2k-2}(N)$ and $S^{+,new}_{k-1/2}(4N)$ are Hecke equivariantly isomorphic with each other. Now, let $J^{cusp}_{k,1}(N)$ denote the space of Jacobi cusp forms of weight k, index 1 and level N. Also, let $\mathcal{S}^{*}_{k}(\Gamma^2_0(N))$ denote the space of Maass spezialschar inside the space of Siegel cusp forms of weight k, genus two for the congruence subgroup $\Gamma^2_0(N)$. Using the theory of newforms in $S^{+,new}_{k-1/2}(N)$, the authors in [25] set up a parallel theory of newforms inside $J^{cusp}_{k,1}(N)$ and $\mathcal{S}^{*}_{k}(\Gamma^2_0(N))$. They also proved that the spaces $S^{+,new}_{k-1/2}(N)$ and $J^{cusp,new}_{k,1}(N)$ are isomorphic with each other under a generalised version of the Eichler-Zagier map, and the spaces $J^{cusp,new}_{k,1}(N)$ and $\mathcal{S}^{*,new}_{k}(\Gamma^2_0(N))$ are isomorphic with each other under the Maass lift. Combining the above three isomorphisms, we get the generalized version of the Saito-Kurokawa isomorphism for odd and square-free level. Specifically, we have :

Theorem 1.1.1 (Theorem 8, [25]). *Let N be an odd and square-free integer. Then, there is a one-one correspondence between the spaces $S^{new}_{2k-2}(N)$ and $\mathcal{S}^{*,new}_{k}(\Gamma^2_0(N))$. For a normalized Hecke eigenform $f \in S^{new}_{2k-2}(N)$ and a Hecke eigenform $F \in \mathcal{S}^{*,new}_{k}(\Gamma^2_0(N))$,* *the correspondence is given by*

$$Z_F^*(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s)$$

where $Z_F(s)$ *is the Andrianov zeta function corresponding to* F *(as in [1]) and*

$$Z_F^*(s) = \prod_{p|N}(1-p^{k-1-s})^{-1}(1-p^{k-2-s})^{-1}Z_F(s).$$

Later, such a correspondence was obtained for arbitrary level N with trivial character in [22].

Question: What happens if one replaces the congruence subgroup $\Gamma_0(N)$ with $\Gamma_1(N)$?

1.2 Statement and results

As a natural extension to the above problem mentioned in the previous subsection, we try to derive similar results in the case of modular forms for the congruence subgroup $\Gamma_1(N)$ for an integer $N \geq 1$. Note that we have the following direct sum decomposition:

$$S_{2k-2}(\Gamma_1(N)) = \bigoplus_{\psi} S_{2k-2}(N,\psi), \tag{1.2}$$

where $S_{2k-2}(\Gamma_1(N))$ denote the space of cusp forms for the subgroup $\Gamma_1(N)$ and the direct sum varies over all Dirichlet characters modulo N.

Hence, it is enough to study the problem on each of the individual spaces $S_{2k-2}(N,\psi)$, where ψ is a Dirichlet character modulo N.

1.2.1 Newforms and Saito-Kurokawa lifts

The aim is to derive the Saito-Kurokawa correspondence on newforms inside the Maass subspace $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ under the assumptions that $32|M$ and the involved characters are primitive, and prove that a newform $f\in S_{2k-2}(M/2,\chi^2)$ is lifted into two linearly independent Maass forms under the Saito-Kurokawa correspondence. For this, we first set up the theory of newforms on the spaces $S^+_{k-1/2}(M,\chi_0)$ and $S^+_{k-1/2}(4M,\chi_0)$. Then, using the Eichler-Zagier map $\mathcal{Z}_1$ and the Maass lift $\iota_{M,\chi}$, we develop the theory of newforms on the spaces $J^{cusp}_{k,1}(M,\chi)$ and $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ respectively. Finally as a consequence we get the required Saito-Kurokawa isomorphism.

More precisely, we make the following assumptions. Let $k\geq 2$, $M=2^{\alpha-2}N$ ($2\nmid N$, $\alpha>6$) be integers. Let χ modulo M be a Dirichlet character with $\epsilon=\chi(-1)$ such that $\chi_0=\left(\frac{4\epsilon}{\cdot}\right)\chi$ is an even character modulo M. Let $cond(\chi)=M$ and $cond(\chi^2)=M/2$. Let $S_{k-1/2}(M,\chi_0)$ denote the space of cusp forms of weight $k-1/2$ for $\Gamma_0(M)$ and character χ_0. We denote the Petersson scalar product by

$$\langle f,g\rangle=\frac{1}{i_M}\int_{\Gamma_0(M)\backslash\mathbb{H}} f(\tau)\overline{g(\tau)}y^{k-1/2}\frac{dxdy}{y^2},$$

where $\tau=x+iy$, $y>0$, i_M denotes the index of $\Gamma_0(M)$ in $SL_2(\mathbb{Z})$ and $f,g\in S_{k-1/2}(M,\chi_0)$. Let $S^+_{k-1/2}(M,\chi_0)$ denote the Kohnen plus space which consists of the cusp forms in $S_{k-1/2}(M, \chi_0)$ whose n-th Fourier coefficients vanish unless $\epsilon(-1)^{k-1}n \equiv 0, 1 \pmod 4$.

We state the following dimension equality (see Lemma 3.2.7): If $k \geq 2$, $cond(\chi) = M$ and $cond(\chi^2) = M/2$, then

$$\dim S_{k-1/2}(M, \chi_0) = \dim S_{2k-2}(M/2, \chi^2). \tag{1.3}$$

If $D \equiv 1(4)$ is a fundamental discriminant with $\epsilon(-1)^{k-1}D > 0$, we consider the D-th Shimura-Kohnen lift S_D which is the same as the D-th Shimura lift, given by

$$g|S_D = \sum_{n\geq 1} \left(\sum_{d|n} \chi(d) \left(\frac{D}{d}\right) d^{k-2} a_g \left(\frac{|D|n^2}{d^2}\right)\right) e^{2\pi i n z},$$

where $g \in S_{k-1/2}(M, \chi_0)$. In order to set up the Shimura lifts S_D on $S^+_{k-1/2}(M, \chi_0)$, we consider the image of m-th Poincaré series in the plus space $S^+_{k-1/2}(M, \chi_0)$ under S_D and derive its explicit image as a period function in $S_{2k-2}(M/2, \chi^2)$. By varying the integers $m \geq 1$ with $\epsilon(-1)^{k-1}m \equiv 0, 1 \pmod 4$, we get that all the Poincaré series $P^+_{k-1/2, M, \chi_0; m}$ span the space $S^+_{k-1/2}(M, \chi_0)$. Hence, the Shimura-Kohnen lifts S_D maps $S^+_{k-1/2}(M, \chi_0)$ into $S_{2k-2}(M/2, \chi^2)$ and the adjoint Shintani lifts S^*_D maps $S_{2k-2}(M/2, \chi^2)$ into $S^+_{k-1/2}(M, \chi_0)$. If $f \in S_{2k-2}(M/2, \chi^2)$ is a normalised Hecke eigenform, it is known that there exists a fundamental discriminant D ($\epsilon(-1)^{k-1}D > 0$, $(D, M) = 1$) such that the special value $L(f, \overline{\chi}\left(\frac{D}{\cdot}\right), k-1) \neq 0$ (see the remark after Theorem 1.1 in Chapter 6, [28]). Therefore, using the condition $cond(\chi) = M$ and following the computations as in ([15],p. 137), we derive that the $|D|$-th Fourier coefficient of $f|S_D^*$ is non-zero. Since this is valid for each of the normalised Hecke eigenform $f \in S_{2k-2}(M/2, \chi^2)$ and by using the dimension equality as given by the equation (1.3), we observe that the space $S^+_{k-1/2}(M, \chi_0)$ coincides with the full space $S_{k-1/2}(M, \chi_0)$. We state the theory of newforms for the space $S^+_{k-1/2}(M, \chi_0)$:

Theorem 1.2.1. *Let $k \geq 2$, $32|M$, $cond(\chi) = M$ and $cond(\chi^2) = M/2$. Then,*

$$S^+_{k-1/2}(M, \chi_0) = S_{k-1/2}(M, \chi_0).$$

There exists a finite linear combination of Shimura lifts ψ_K which defines an isomorphism from $S^+_{k-1/2}(M, \chi_0)$ into $S_{2k-2}(M/2, \chi^2)$. In particular, we have the strong multiplicity one theorem on $S^+_{k-1/2}(M, \chi_0)$.

Next, we consider the spaces $S^+_{k-1/2}(4M, \chi_0)$ and $S_{2k-2}(M, \chi^2)$. The condition $cond(\chi^2) = M/2$ along with the dimension formula and the theory of newforms for $S_{2k-2}(M, \chi^2)$ gives $S^{new}_{2k-2}(M, \chi^2) = \{0\}$. To get the same for the plus space $S^+_{k-1/2}(4M, \chi_0)$, we derive that the Shimura lifts S_D ($D \equiv 0, 1 \pmod 4$, $\epsilon(-1)^{k-1}D > 0$, $(D, M) = 1$) map $S^+_{k-1/2}(4M, \chi_0)$ into $S_{2k-2}(M, \chi^2)$. Then we find the relation between the spaces $S^+_{k-1/2}(M, \chi_0)$ and $S^+_{k-1/2}(4M, \chi_0)$. We state the following main theorem for modular forms of half-integral weight (see §3.2.2):

Theorem 1.2.2.

$$S^{+,new}_{k-1/2}(4M, \chi_0) = \{0\} \text{ and } S^{new}_{2k-2}(M, \chi^2) = \{0\},$$

$$S^+_{k-1/2}(4M,\chi_0) = S^+_{k-1/2}(M,\chi_0) \bigoplus S^+_{k-1/2}(M,\chi_0)|B_4,$$

$$S_{2k-2}(M,\chi^2) = S_{2k-2}(M/2,\chi^2) \bigoplus S_{2k-2}(M/2,\chi^2)\big|B_2.$$

Moreover, the isomorphism ψ_K maps $S^+_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$.

In order to get the Maass lift and theory of newforms for the Maass space, we let $k \geq 2$, $\epsilon = (-1)^k$ and we first develop the theory of newforms for the space of Jacobi cusp forms $J^{cusp}_{k,1}(M,\chi)$. This can be done by deriving the Eichler-Zagier canonical map (see preliminaries for definition)

$$\mathcal{Z}_1 : J^{cusp}_{k,1}(M,\chi) \longrightarrow S^+_{k-1/2}(4M,\chi_0).$$

This is an isomorphism preserving the Hecke eigenforms and the scalar product structures (see Proposition 4.2.2 and Lemma 4.2.3). Using the Theorem 1.2.2 as above and the existence of Eichler-Zagier canonical isomorphism we get the following. There exists a non-zero cusp form $\phi \in J^{cusp}_{k,1}(M,\chi)$ such that $\phi|B_J(4)$ belongs to $J^{cusp}_{k,1}(M,\chi)$ and $\phi|\mathcal{Z}_1 \in S^+_{k-1/2}(M,\chi_0)$. We call the inverse image of the space $S^+_{k+1/2}(M,\chi_0)$ as the space of newforms in $J^{cusp}_{k,1}(M,\chi)$. Let $P_{k,1,M,\chi_0;D,r}$ denote the (D,r)-th Poincarè series in $J^{cusp}_{k,1}(M,\chi)$; let $U_J(4)$ and $B_J(4)$ be operators on $J^{cusp}_{k,1}(M,\chi)$ (see preliminaries for definitions). Let $J^{cusp;new}_{k,1}(M,\chi)$ denote the linear span over complex numbers of the set $\{P_{k,1,M,\chi;D,r}|U_J(4) : D\}$, where D varies over all the discriminants with $4\big|D$ and $D/4 \equiv 0, 1 \pmod 4$. We call the elements of an orthogonal basis consisting of simultaneous eigenforms under all the Hecke operators $T_J(n)$ in $J^{cusp;new}_{k,1}(M,\chi)$ as Jacobi newforms of weight k, index 1, level M and character χ. We have (see, the

Lemma 4.2.7):

$$J_{k,1}^{cusp;new}(M,\chi)|\mathcal{Z}_1 = S_{k-1/2}^{+}(M,\chi_0).$$

We now state the following main theorem for the theory of newforms of Jacobi cusp forms:

Theorem 1.2.3.

$$J_{k,1}^{cusp}(M,\chi) = J_{k,1}^{cusp;new}(M,\chi) \bigoplus J_{k,1}^{cusp;new}(M,\chi)|B_J(4).$$

The space $J_{k,1}^{cusp;new}(M,\chi)$ is isomorphic to the space $S_{2k-2}(M/2,\chi^2)$ under a certain linear combination of Shimura lifts. Hence, The multiplicity one result holds good on $J_{k,1}^{cusp;new}(M,\chi)$.

Let $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ denote the space of degree two Siegel cusp forms of level M and character χ, where χ is primitive Dirichlet character modulo M as above. Let $\phi \in J_{k,1}^{cusp}(M,\chi)$. Let $\iota_{M,\chi}$ denote the Maass embedding (as in [8]) defined by

$$\phi|\iota_{M,\chi}(\tau,z,w) = \sum_{m=1}^{\infty}(\phi|_{k,1}V_{m,\chi})(\tau,z)e^{2\pi imw},$$

where $V_{m,\chi}$ is the index shifting operator on $J_{k,1}^{cusp}(M,\chi)$ (defined in §4.3). Then by Theorem 3.2 of [8], the map $\iota_{M,\chi}$ is an embedding from $J_{k,1}^{cusp}(M,\chi)$ to $\mathcal{S}_k(\Gamma_0^2(M),\chi)$. Denote the image of $J_{k,1}^{cusp}(M,\chi)$ under this embedding by $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$. We define the space of Maass newforms by

$$\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi) := J_{k,1}^{cusp;new}(M,\chi)|\iota_{M,\chi}.$$

Finally, combining the above results we get the relevant Saito-Kurokawa lifting for level M and a primitive character χ modulo M. For a comprehensive theory on Saito-Kurokawa correspondence we refer to [6, 8]. Let $f \in S_{2k-2}(M/2,\chi^2)$ be a normalised newform. Let $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ be the associated unique (up to a scalar) newform. The corresponding Andrianov zeta function $Z_F(s)$ (see §4.3.2) has an Euler product expansion

$$Z_F(s) = \prod_{p|M}(1-\mu_p p^{-s})^{-1} \prod_{p\nmid M} Q_p(p^{-s})^{-1},$$

where

$$\begin{aligned} Q_p(p^{-s}) = 1 - \gamma_p p^{-s} + (p\omega_p + (p^2+1)\chi(p^2)p^{2k-5})p^{-2s} \\ - \omega_p\chi(p^2)p^{2k-3-3s} + \chi(p^4)p^{4k-6-4s}. \end{aligned}$$

Now, using the isomorphisms ψ_k, $\mathcal{Z}_1$, $\iota_{M,\chi}$, and the Theorems 1.2.1, 1.2.2, 1.2.3, we derive the following:

Theorem 1.2.4.

$$\mathcal{S}_k^*(\Gamma_0^2(M),\chi) = \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi) \bigoplus \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)|B_S(4).$$

The multiplicity one theorem is valid on $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$*. Also,* $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ *is in one to one correspondence with* $S_{2k-2}(M/2,\chi^2)$ *under the Saito-Kurokawa isomorphism. A given normalised Hecke eigenform* $f \in S_{2k-2}(M/2,\chi^2)$ *is lifted into two equivalent Hecke eigenforms* $F, F|B_S(4)$*, where* $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ *is the newform satisfying*

$$Z_F(s) = L(s-k+1,\chi)L(s-k+2,\chi)L(f,s).$$

1.2.2 Kernel function for L-values of half-integral weight Hecke eigenforms

Let D be a positive fundamental discriminant and $k \geq 2$ be an even integer. W. Kohnen and D. Zagier [14] considered a modular form $\theta_D G_{k,4D}|Tr_4^{4D}Pr_+$ of weight $k+1/2$ for $\Gamma_0(4)$ in the Kohnen plus space. They proved that its image under D^{th} Shimura-Kohnen lift equals a known constant times the square of an Eisenstein series of weight $2k$, level 1 (see proposition 3 of [14]). We consider a similar modular form $\Theta_D E_{k,4D,(\frac{D}{\cdot})}|Tr_4^{4D}Pr_+$ of weight $k+1/2$ ($k \geq 2$, even) for $\Gamma_0(4)$ in the Kohnen plus space. We first sum up its cusp parts over all positive fundamental discriminants D and get a cusp form of weight $k+1/2$ for $\Gamma_0(4)$ in the plus space, and characterize the resulting cusp form (see theorem 5.2.1 in section 3). Finally, we characterize the above modular form $\theta_D E_{k,4D,(\frac{D}{\cdot})}|Tr_4^{4D}Pr_+$ for each positive fundamental discriminant D, and get the following algebraic information.

The results of G. Shimura and Y. Manin ([26], [31]) show that the values of L-functions associated with a normalized newform of integral weight belongs to a number field. More precisely, if $f \in S_{2k}^{\text{new}}(N)$ and $L(f,s) = \sum_{n\geq 1} a_f(n)n^{-s}$ $(Re(s) >> 1)$ is the associated L-series, and if $\mathbb{Q}_f$ denotes the number field generated by all the eigenvalues of f over $\mathbb{Q}$, then they proved the existence of a real number ω such that

$$\frac{L(f,n)}{\pi^n \omega} \in \mathbb{Q}_f, \quad \text{where } 1 \leq n \leq 2k-1.$$

Let $2|k$ and consider a normalized Hecke eigenform f of weight $2k$, level 1 and the associated unique non-zero Hecke eigenform g of weight $k+1/2$ for $\Gamma_0(4)$. Both of them are cusp forms with $g|\mathcal{S}_D = a_g(D)F$ for all positive fundamental discriminants D, where $\mathcal{S}_D$ denotes the D^{th} Shimura-Kohnen lift (see [11]). We prove that

$$\frac{a_g(D)L(f,2k-1)}{\pi^{k-1}\langle g,g\rangle L(D,k)} \in \overline{\mathbb{Q}}.$$

To get this result, we use the fact that the Fourier coefficients of $\theta_D E_{k,4D,(\frac{D}{\cdot})}|Tr_4^{4D}Pr_+$ are rational numbers.

1.3 Organisation of the thesis

This thesis is organised in the following way. In Chapter 2, we gather the necessary definitions and preliminary results which will be used throughout the thesis. In Chapter

3, we consider both plus spaces of the same weight $k - 1/2$ and character χ_0 for the groups $\Gamma_0(M), \Gamma_0(4M)$, respectively. We derive the Shimura-Kohnen lifts on each of the spaces and develop the respective theory of newforms for both the spaces. Next, in Chapter 4 we derive the Eichler-Zagier canonical map and get the theory of newforms for the space of Jacobi forms. Using this, we develop the theory of newforms for the Maass space, and obtain the necessary Saito-Kurokawa isomorphism. Chapter 5 deals with a certain kernel function associated with the L-values of a half-integral weight Hecke eigenform. Finally, Chapter 6 concludes the thesis where we also mention few problems for future work.

Chapter 2

Preliminaries

This chapter gives the necessary definitions and properties regarding modular forms of integral and half integral weight, Jacobi forms and Siegel modular forms of genus two. We start with few notations.

Let (a, b) denote the gcd of given integers a, b. For complex numbers x and y with $y \neq 0$, define $e_y(x) := e^{2\pi ix/y}$. For a real number r, let $\lfloor r \rfloor$ denote the greatest integer less than or equal to r. For integers a and n, let $\left(\frac{a}{n}\right)$ denote the Kronecker symbol.

Let $M_{2\times 2}(\mathbb{R})$ denote the collection of all 2×2 matrices with real entries. For a matrix A, let $det(A)$ denote the determinant of A. We have the following subgroups of $M_{2\times 2}(\mathbb{R})$:

- $\mathrm{GL}_2(\mathbb{R}) := \big\{A \in M_{2\times 2}(\mathbb{R}) : det(A) \neq 0\big\}$.
- $\mathrm{GL}_2^+(\mathbb{R}) := \big\{A \in M_{2\times 2}(\mathbb{R}) : det(A) > 0\big\}$.

- $\mathrm{SL}_2(\mathbb{R}) := \big\{A \in \mathrm{GL}_2(\mathbb{R}) : det(A) = 1\big\}$.

Let $\mathrm{GL}_2(\mathbb{Z})$ denote the collection of invertible 2×2 matrices with integer entries. Let $\mathrm{GL}_2^+(\mathbb{Z})(= \mathrm{SL}_2(\mathbb{Z}))$ denote the subgroup of $\mathrm{GL}_2(\mathbb{Z})$ with determinant 1.

For an integer N, let $\Gamma_0(N)$ denote the congruence subgroup (of *level* N) of $\mathrm{SL}_2(\mathbb{Z})$, defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

Note that when $N = 1$, $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. For integers M, N with $M|N$ we denote $[\Gamma_0(M) : \Gamma_0(N)]$ to be the index of $\Gamma_0(N)$ in $\Gamma_0(M)$. Also, let

$$i_M := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(M)].$$

Let $\mathbb{H}$ denote the upper half plane. i.e.,

$$\mathbb{H} := \{z = x + iy : z \in \mathbb{C} \text{ and } y > 0\}.$$

The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau) := \frac{a\tau + b}{c\tau + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $\tau \in \mathbb{H}$.

Dirichlet character

Let $N \geq 1$ be an integer. A complex valued arithmetical function $\chi : \mathbb{Z} \longrightarrow \mathbb{C}$ is called a Dirichlet character χ modulo N if it satisfies the following properties:

i) periodic with period N:

$$\chi(n+N) = \chi(n) \text{ for all } n \in \mathbb{Z},$$

ii) completely multiplicative:

$$\chi(mn) = \chi(m)\chi(n) \text{ for all } m, n \in \mathbb{Z},$$

iii) and

$$\chi(n) = 0 \text{ if } (n, N) > 1.$$

2.1 Modular forms of integral weight

Stroke operator:

Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a holomorphic function. Let $k \geq 1$ be an integer and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$. Then, the stroke operator '$\big|_k$' is defined by

$$f\big|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(z) := (ad-bc)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

When there is no ambiguity, we usually omit the subscript k.

Definition 2.1.1. *(Modular form of integral weight)*: *Let $k \geq 1$, $N \geq 1$ be integers and χ be a Dirichlet character modulo N. A holomorphic function $f : \mathbb{H} \longrightarrow \mathbb{C}$ is called a modular form of weight k for $\Gamma_0(N)$ with character χ modulo N if it satisfies*

i) automorphic property:

$$(f\big|_k A)(z) = \chi(d) f(z)$$

for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

ii) and it is holomorphic at all the cusps of $\Gamma_0(N)$.

- The vector space of modular forms of weight k for $\Gamma_0(N)$ with character χ is denoted by $M_k(N, \chi)$.
- A form $f \in M_k(N, \chi)$ is called a cusp form if if vanishes at all its cusps. The vector space of all cusp forms in $M_k(N, \chi)$ is denoted by $S_k(N, \chi)$.

When χ is the trivial character, the space $M_k(N, \chi)$ (resp. $S_k(N, \chi)$) is denoted by $M_k(N)$ (resp. $S_k(N)$). Moreover, when $N = 1$, the space $M_k(1)$ (resp. $S_k(1)$) is denoted by M_k (resp. S_k).

Definition 2.1.2. *(Petersson inner product)*: *For $f_1, f_2 \in M_k(N, \chi)$ with at least one of them in $S_k(N, \chi)$, we define the Petersson inner product of f_1 and f_2 by*

$$\langle f_1, f_2 \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathbb{H}} f_1(z)\overline{f_2(z)} y^{k-2} \,\mathrm{d}x \,\mathrm{d}y.$$

Definition 2.1.3. *(Hecke operators)*: *For an integer $m \geq 1$, define the m-th Hecke operator T_m on $M_k(N,\chi)$ by :*

$$T_m(f) := m^{k/2-1} \sum_{\substack{ad=m \\ (a,N)=1 \\ a>0}} \chi(a) \sum_{b \,(\mathrm{mod}\ d)} f\Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Certain shift operators:

Let $f = \sum_{n\geq 0} a_f(n)e^{2\pi i n\tau}$ be a formal power series where $a_f(n)$ denotes the n-th coefficient of f. Let $m \geq 1$ be an integer. The operator B_m is defined on formal series by

$$B_m : \sum_{n\geq 0} a_g(n)e^{2\pi i n\tau} \longrightarrow \sum_{n\geq 0} a_g(n)e^{2\pi i n m\tau},$$

and the operator U_m is defined on formal series by

$$U_m : \sum_{n\geq 0} a_g(n)e^{2\pi i n\tau} \longrightarrow \sum_{n\geq 0} a_g(mn)e^{2\pi i n\tau}.$$

2.2 Modular forms of half-integral weight

Let G denote the collection of all ordered pairs $(A, \phi(\tau))$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic function on $\mathbb{H}$ such that

$$\phi^2(\tau) = t\frac{c\tau + d}{\sqrt{\det(A)}}, \quad t \in \{\pm 1\}.$$

G forms a group under the group law:

$$(A, \phi(\tau))(B, \psi(\tau)) := (AB, \phi(B\tau)\psi(\tau)).$$

For congruence subgroups $\Gamma_0(4M)$ we take the embedding $\Gamma_0(4M) \hookrightarrow G$

$$\Gamma_0^*(4M) = \{(\alpha, j(\alpha,\tau)) : \alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4M), j(\alpha,\tau) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-\frac{1}{2}}(c\tau+d)^{\frac{1}{2}}\}.$$

Stroke operator:

For a complex valued function g defined on the upper half plane $\mathbb{H}$ and $(A, \phi(\tau)) \in G$, we define the stroke operator by

$$g\big|_{k+\frac{1}{2}}(A, \phi(\tau))(\tau) := \phi(\tau)^{-2k-1} g(A\tau).$$

Definition 2.2.1. *(Modular form of half-integral weight)*: *Let $k \geq 1$, $N \geq 1$ be integers and χ be a Dirichlet character modulo $4N$. A holomorphic function $g : \mathbb{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k + 1/2$ for $\Gamma_0(4N)$ with character χ modulo $4N$ if it satisfies*

i) automorphic property:

$$(g\big|_{k+1/2} A^*)(z) = \chi(d) g(z),$$

for all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$, *ii) and it is holomorphic at all the cusps of* $\Gamma_0(4N)$.

- The vector space of modular forms of weight $k+1/2$ for $\Gamma_0(4N)$ with character χ is denoted by $M_{k+1/2}(4N,\chi)$.
- A form $f \in M_{k+1/2}(4N,\chi)$ is called a cusp form if if vanishes at all its cusps. The vector space of all cusp forms in $M_{k+1/2}(4N,\chi)$ is denoted by $S_{k+1/2}(4N,\chi)$.

When χ is the trivial character, the space $M_{k+1/2}(4N,\chi)$ (resp. $S_{k+1/2}(4N,\chi)$) is denoted by $M_{k+1/2}(4N)$ (resp. $S_{k+1/2}(4N)$).

Definition 2.2.2. *(Petersson inner product)*: *For* $g_1, g_2 \in S_{k+1/2}(4N,\chi)$*, we define the Petersson inner product of* g_1 *and* g_2 *by*

$$\langle g_1, g_2\rangle = \frac{1}{[\Gamma_0(N):\Gamma_0(4N)]}\int_{\Gamma_0(4N)\backslash\mathbb{H}} g_1(z)\overline{g_2(z)}y^{k-3/2}\,\mathrm{d}x\,\mathrm{d}y.$$

Definition 2.2.3. *(Hecke operators)*: *Let* $g = \sum_{n\geq 0} a_f(n)e^{2\pi n\tau} \in M_{k+1/2}(4N,\chi)$. *Let* $p \nmid 4N$ *be a prime. Then, the* p*-th Hecke operator* T_{p^2} *is given by its action on the Fourier coefficients in the following way: let*

$$g|T_{p^2} = \sum_{n\geq 0} b(n)e^{2\pi n\tau}.$$

Then,

$$b(n) = a_g(p^2n) + \chi(p)\left(\frac{(-1)^k n}{p}\right)p^{k-1}a_g(n) + \chi(p^2)p^{2k-1}a_g(n/p^2),$$

where $a_g(n/p^2) = 0$ *if* $p^2 \not| n$.

2.3 Jacobi forms

Let $\mathrm{H}(\mathbb{Q})$ denote the Heisenberg group and $\mathrm{G}^J(\mathbb{Q})$ denote the Jacobi group over $\mathbb{Q}$, defined by

$$\mathrm{G}^J(\mathbb{Q}) := \mathrm{GL}_2^+(\mathbb{Q}) \ltimes \mathrm{H}(\mathbb{Q}).$$

For a congruence subgroup $\Gamma_0(N)$, consider

$$\Gamma_0(N)^J := \left\{(M, (\lambda, \mu), \gamma) \in \mathrm{G}^J(\mathbb{Q}) : M \in \Gamma_0(N),\ \lambda, \mu, \gamma \in \mathbb{Z}\right\}.$$

Let y be a complex number and ℓ be a non-negative integer. We use the following notations:

$$e(y) := e^{2\pi i y} \quad \text{and} \quad e^{\ell}(y) := e(\ell y) = e^{2\pi i \ell y}.$$

Stroke operators:

Let $\phi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ be a holomorphic function. Let $k \geq 1$. For $M \in \mathrm{GL}_2^+(\mathbb{R})$ with $det(M) = g$ and $(\lambda, \mu, \gamma) \in \mathrm{H}(\mathbb{Q})$, let

$$(\phi\big|_{k,\ell}[M])(\tau, z) := (c\tau + d)^{-k} e^{\ell g}\left(-\frac{cz^2}{c\tau + d}\right)\phi\left(\frac{a\tau + b}{c\tau + d}\right),$$

$$(\phi\big|_{\ell}[(\lambda, \mu), \gamma]) := e^{\ell}(\lambda^2\tau + 2\lambda z + \lambda\mu + \gamma)\,\phi(\tau + z + \lambda\tau + \mu).$$

We usually omit the subscripts 'k, m' when there is no ambiguity.

Definition 2.3.1. *(Jacobi form)*: *Let $k, \ell, N \geq 1$ be integers and χ be a Dirichlet character modulo N. A holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \longrightarrow \mathbb{C}$ is said to be a Jacobi form of weight k, index ℓ for $\Gamma_0(N)^J$ with character χ if it satisfies*

i) automorphic properties:

a) for any $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$,

$$\phi\big|_{k,\ell}[M] = \chi(d)\phi,$$

b) and for any $\lambda, \mu \in \mathbb{Z}$,

$$\phi\big|_{\ell}[(\lambda, \mu)] = \phi;$$

ii) for any $M \in \mathrm{SL}_2(\mathbb{Z})$, $\phi\big|_{k,\ell}[M]$ has a Fourier expansion of the form

$$\phi\big|_{k,\ell}[M](\tau, z) = \sum_{\substack{n,r \in \mathbb{Z}, \\ r^2 \leq 4n\ell}} c_\phi(n, r) e(n\tau + rz).$$

- The vector space of Jacobi forms of weight k, index ℓ for $\Gamma_0(N)^J$ with character χ modulo N is denoted by $J_{k,\ell}(N, \chi)$.
- A form $\phi \in J_{k,\ell}(N, \chi)$ is called a Jacobi cusp form if $c_\phi(n, r) = 0$ unless $r^2 < 4n\ell$. The vector space of all cusp forms in $J_{k,\ell}(N, \chi)$ is denoted by $J_{k,\ell}^{cusp}(N, \chi)$.

Note: Let $n, r, n', r' \in \mathbb{Z}$ with $r^2 < 4n\ell, r'^2 < 4n'\ell$. Then we have $c_\phi(n, r) = c_\phi(n', r')$ if $r'^2 - 4n'\ell = r^2 - 4n\ell$ and $r' \equiv r \pmod{2\ell}$. Thus, we write the Fourier expansion of ϕ as

$$\phi(\tau, z) = \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod{4n\ell}}} c_\phi(D,r) e\left(\frac{r^2-D}{4\ell}\tau + rz\right).$$

Definition 2.3.2. *(Petersson inner product)*: *For $\phi_1, \phi_2 \in J_{k,1}^{cusp}(N,\chi)$, we define the Petersson inner product of ϕ_1 and ϕ_2 by*

$$\langle \phi_1, \phi_2 \rangle = \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0^J \backslash \mathbb{H}\times\mathbb{C}} \phi_1(\tau, z), \overline{\phi_2(\tau, z)} v^{k-3} e^{(-4\pi y^2)/v}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}v,$$

where $\tau = u + iv, v > 0$, $z = x + iy$ and $\Gamma_0(N)^J$ is the Jacobi group of level N.

Definition 2.3.3. *(Hecke operators)*: *Let*

$$\phi(\tau, z) == \sum_{\substack{n,r\in\mathbb{Z},\\ r^2\leq 4n\ell}} c_\phi(n,r) e(n\tau + rz) \in J_{k,1}(N,\chi).$$

We define the Hecke operators $T_J(p)$ ($p \nmid N$) and $U_J(p)$ ($p|N$) by their action on the Fourier coefficients of ϕ in the following way:

$$c_{\phi|T_J(p)} = c_\phi(p^2D, pr) + p^{k-2}\chi(p)\left(\frac{D}{p}\right) c_\phi(D,r) + p^{2k-3}\chi(p)^2 c_\phi(D/p^2, D/p)$$

and

$$c_{\phi|U_J(p)} = c_\phi(p^2D, pr).$$

For a comprehensive theory of Jacobi forms, we refer to [6, 8].

2.4 Siegel modular forms

For a matrix $Z \in \mathrm{M}_{2\times 2}(\mathbb{C})$, let Z^t denote the transpose of Z. If Z is positive definite, we denote it by $Z > 0$. A matrix T is called a half integral matrix if the diagonal entries of T are integers and off-diagonal entries are half-integers. We have the following definitions:

- Siegel upper half space (of genus 2) :

$$\mathbb{H}_2 := \{Z \in \mathrm{M}_{2\times 2}(\mathbb{C}) : Z = Z^t, \mathrm{Im}(Z) > 0\}.$$

- Siegel modular group :

$$\mathrm{Sp}_4(\mathbb{Z}) := \left\{ M \in \mathrm{GL}_4(\mathbb{Z}) : M^t J M = M, J = \begin{pmatrix} 0_2 & I_2 \\ -I_2 & 0_2 \end{pmatrix} \right\}.$$

- Congruence subgroup

$$\Gamma_0^2(N) := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : C \equiv 0 \,(\mathrm{mod}\ N) \right\}.$$

Definition 2.4.1. *(Siegel modular form)*: *Let $k, N \geq 1$ be integers, and χ be a Dirichlet character modulo N. A holomorphic function $F : \mathbb{H}_2 \longrightarrow \mathbb{C}$ is called a Siegel modular* *form of weight k for $\Gamma_0^2(N)$ with character χ if it satisfies:*

$$F|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(Z) := \chi(\det(A))(\det(CZ+D))^{-k}F((AZ+B)(CZ+D)^{-1}) = F(Z)$$

for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^2(N)$.

- The vector space of Siegel modular forms of weight k for $\Gamma_0^2(N)$ with character χ is denoted by $\mathcal{M}_k(\Gamma_0^2(N), \chi)$.
- To define a cusp form, we embed an $\mathrm{SL}_2(\mathbb{Z})$ element into $\mathrm{Sp}_4(\mathbb{Z})$ by

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \longrightarrow \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$$

  where $A = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} d & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

- For any $F \in \mathcal{M}_k(\Gamma_0^2(N), \chi)$ and for the above embedding of an $\mathrm{SL}_2(\mathbb{Z})$ element in $\mathrm{Sp}_4(\mathbb{Z})$, we have the following Fourier expansion of the form

$$F|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(Z) = \sum_{\substack{T=T^t \geq 0 \\ T \text{ half integral}}} A_{F;\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)}(T)e^{2\pi i \mathrm{Tr}(TZ)}.$$

  Now, F is a cusp form if and only if $A_{F;\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)}(T) = 0$ for all $T \geq 0$ with $\det(T) = 0$ in the above Fourier expansion.

Hecke operators: Let

$$\mathrm{GSp}_4^+(\mathbb{R}) = \{g \in \mathrm{M}_{2\times 2}(\mathbb{R}) : g^t J g = n(g)J, n(g) > 0\}.$$

Let $g \in \mathrm{GSp}_4^+(\mathbb{Q}) \cap \mathrm{M}_{2\times 2}(\mathbb{Z})$ with $gJg^t = n(g)J$. Consider the following double coset decomposition:

$$T = \Gamma_0^2(N) g \Gamma_0^2(N) = \bigcup_{\nu} \Gamma_0^2(N) \begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix}.$$

We assume that each $\det(A_\nu)$ is co prime to N. Then, for any $F \in \mathcal{M}_k(\Gamma_0^2(N), \chi)$, we consider the action of T by

$$F|_{k,\chi} T = n(g)^{2k-3} \sum_{\nu} \chi(\det(A_\nu)) \det(C_\nu Z + D_\nu)^{-k} F((A_\nu Z + B_\nu)(C_\nu Z + D_\nu)^{-1}).$$

When$p \nmid N$, for any diagonal matrix diag(p^a, p^b, p^c, p^d) with $a + c = b + d$ and $p \nmid N$, let

$$T_S(p^a, p^b, p^c, p^d) = \Gamma_0^2(N) \begin{pmatrix} p^a & 0 & 0 & 0 \\ 0 & p^b & 0 & 0 \\ 0 & 0 & p^c & 0 \\ 0 & 0 & 0 & p^d \end{pmatrix} \Gamma_0^2(N).$$

Let us define the following (Hecke operators):

$$\begin{aligned} T_S(p) &= T_S(1, 1, p, p), \\ T_S(p^2) &= T_S(1, p, p^2, p) + T_S(1, 1, p^2, p^2) + T_S(p, p, p, p), \\ T'_S(p) &= T_S(p)^2 - T_S(p^2). \end{aligned}$$

Note that,

$$T'_S(p) = pT_S(1, p, p^2, p) + p(1 + p + p^2)T_S(p, p, p, p).$$

Moreover, when $p|N$ we have

$$U_S(p) = \Gamma_0^2(N)\text{diag}(1, 1, p, p)\Gamma_0^2(N).$$

For the details we refer to [8].

Chapter 3

Newforms of half-integral weight

In this chapter, we develop the theory of newforms for the spaces $S^{+}_{k-1/2}(M,\chi_0)$ and $S^{+}_{k-1/2}(4M,\chi_0)$, where $k \geq 2, 32|M$ and χ is a primitive character modulo M such that χ^2 is a primitive character modulo $M/2$.

3.1 Preliminaries

Throughout this chapter, we let $k \geq 2$, $N \geq 1$ to be integers such that $32|N$. Let χ be a primitive Dirichlet character modulo M such that χ^2 is a primitive Dirichlet character modulo $M/2$. Let $\chi_0 = \chi\left(\frac{4\,\chi(-1)}{\cdot}\right)$.

Let $M_{k-1/2}(M,\chi_0)$ denote the space of modular forms of weight $k-1/2$ for $\Gamma_0(M)$ with character χ_0. Let $S_{k-1/2}(M,\chi_0)$ denote the space of cusp forms in $M_{k-1/2}(M,\chi_0)$.

For $g \in M_{k-1/2}(M, \chi_0)$, we write its Fourier expansion at the cusp ∞ as

$$g(z) = \sum_{n \geq 0} a_g(n) e^{2\pi i n z}.$$

If $m \geq 1$ is an integer, the operator B_m is defined on formal series by

$$B_m : \sum_{n\geq 0} a_g(n) e^{2\pi i n z} \longrightarrow \sum_{n \geq 0} a_g(n) e^{2\pi i n m z}.$$

Projection operator and plus space:

Let

$$\xi = \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \epsilon^{1/2} e^{\pi i/4} \right) \text{ and } \xi' = \left(\begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, \epsilon^{-1/2} e^{-\pi i/4} \right).$$

Then, formal computations show that both $g|\xi$ and $g|\xi'$ belong to $M_{k-1/2}(M, \chi_0)$ for all $g \in M_{k-1/2}(M, \chi_0)$. Also, if g is a cusp form, $g|\xi$ and $g|\xi'$ are cusp forms. The projection operator is defined on $M_{k-1/2}(M, \chi_0)$ by

$$Pr_+ := \left(\frac{8}{2k-1} \right) \frac{1}{2\sqrt{2}\epsilon} (\xi + \xi') + \frac{1}{2} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right),$$

and we let $g|Pr_+$ for its image, where $g \in M_{k-1/2}(M, \chi_0)$. For this we refer to [21]. Define

$$M^+_{k-1/2}(M, \chi_0) := M_{k-1/2}(M, \chi_0)|Pr_+, \text{ and}$$

$$S^+_{k-1/2}(M, \chi_0) := S_{k-1/2}(M, \chi_0) \cap M^+_{k-1/2}(M, \chi_0).$$

Then, formal computations give

$$g|Pr_+ = \sum_{\substack{n\geq 0 \\ \epsilon(-1)^{k-1}n\equiv 0,1 \pmod 4}} a_g(n)e^{2\pi inz},$$

where $g = \sum_{n\geq 0} a_g(n)e^{2\pi inz}$. For an integer $n \geq 1, (n, M) = 1$, let T_{n^2} denote n-th Hecke operator on $M_{k-1/2}(4M, \chi_0)$. It preserves the space of cusp forms $S_{k-1/2}(4M, \chi_0)$. A non-zero form $g \in S^+_{k-1/2}(4M, \chi_0)$ is called a Hecke eigenform if it is a simultaneous eigenform for all the Hecke operators T_{n^2}, $(n, M) = 1$.

Let $S_{2k-2}(M/2, \chi^2)$ denote the space of cusp forms of weight $2k-2$, level $M/2$ and character χ^2. Let T_n ($(n, M) = 1$) and $U_n(n|M)$ denote the Hecke operators, and W_{p^a} (p a prime, $p^a|M/2$ and $p^{a+1} \nmid M/2$) denote the W operator on $S_{2k-2}(M/2, \chi^2)$ as in [3,17]. By a normalised newform $f \in S_{2k-2}(M/2, \chi^2)$ we mean an eigenform for all the operators $T_n((n, M) = 1)$, $U_n(n|M)$ and W_{p^a} (p a prime, $p^a|M/2$ and $p^{a+1} \nmid M/2$) with $a_f(1) = 1$.

Shimura-Kohnen lift: If $D \equiv 1(4)$ is a fundamental discriminant with $\epsilon(-1)^{k-1}D > 0$, we consider the D-th Shimura-Kohnen lift S_D which is the same as the D-th Shimura lift, given by

$$g|S_D = \sum_{n\geq 1}\left(\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}a_g\left(\frac{|D|n^2}{d^2}\right)\right)e^{2\pi inz},$$

where $g \in S_{k-1/2}(M, \chi_0)$.

3.2 Newforms for the spaces $S^{+}_{k-1/2}(M,\chi_0)$ and $S^{+}_{k-1/2}(4M,\chi_0)$

In this section, we develop the theory of newforms for both the spaces $S^{+}_{k-1/2}(M,\chi_0)$ and $S^{+}_{k-1/2}(4M,\chi_0)$, where $k \geq 2$ and $\chi \pmod{M}$ is primitive.

3.2.1 Theory of newforms of $S^{+}_{k-1/2}(M,\chi_0)$

Poincaré series:

Let $n \geq 1$ be an integer. Let $P_{k-1/2,M,\chi_0;n}$ denote the n-th Poincarè series in $S_{k-1/2}(M,\chi_0)$ ([24], page 238) characterized by

$$< g, P_{k-1/2,M,\chi_0;n} >= i_M^{-1}\frac{\Gamma(k-3/2)}{(4\pi n)^{k-3/2}}a_g(n) \quad \text{for all } g \in S_{k-1/2}(M,\chi_0).$$

Let

$$P^{+}_{k-1/2,M,\chi_0;n} = P_{k-1/2,M,\chi_0;n}\big|Pr_{+}.$$

Then using proposition 2 of [24], we have the Fourier expansion of $P^{+}_{k-1/2,M,\chi_0;n}$ in the following lemma.

Lemma 3.2.1. *Let n be a positive integer such that $\epsilon(-1)^{k-1}n \equiv 0,1 \pmod{4}$. We have*

$$P^{+}_{k-1/2,M,\chi_0;n}(\tau) = \sum_{\substack{m\geq 1\\ \epsilon(-1)^{k-1}m\equiv 0,1 \pmod{4}}} g^{+}_{k-1/2,M,\chi_0;n}(m)e^{2\pi i m\tau}, \tag{3.1}$$

where

$$g^+_{k-1/2,M,\chi_0;n}(m) = \delta_{n,m} + \pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(m/n)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(m,n) J_{k-3/2}\left(\frac{4\pi\sqrt{mn}}{Mc}\right),$$

$\delta_{n,m}$ *is the Kronecker delta,* $J_{k-3/2}(.)$ *is the Bessel function and*

$$H_{Mc,\chi}(m,n) = \frac{1}{Mc} \sum_{\delta \pmod{Mc}^*} \overline{\chi}_0(\delta)\left(\frac{Mc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} e_{Mc}(m\delta + n\delta^{-1})$$

with $\delta^{-1} \in \mathbb{Z}$ *and* $\delta\delta^{-1} \equiv 1 \pmod{Mc}$ *is a Kloosterman type sum.*

We state the following which we will use later.

Lemma 3.2.2. *With notations as above,*

$$g^+_{k-1/2,M,\chi_0;n}(m) = (m/n)^{k-3/2} g^+_{k-1/2,M,\overline{\chi}_0;m}(n). \tag{3.2}$$

A certain period function $F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}$:

We first let $k > 2$ and derive the results. If $k = 2$, we mention appropriate changes later in order to get the results (see the paragraph before lemma 3.2.7). Let $m \geq 1$ be an integer with $\epsilon(-1)^{k-1}m \equiv 0, 1 \pmod 4$. Let D be an odd fundamental discriminant with $\epsilon(-1)^{k-1}D > 0$. Let $Q_{M^2/4,|D|M^2m}$ be the set of all integral binary quadratic forms $Q(x,y) := ax^2 + bxy + cy^2$ having discriminant $b^2 - 4ac = |D|M^2m$ and

$a \equiv 0 \pmod{M^2/4}$. If $Q = Q(x,y) \in Q_{M^2/4,|D|M^2m}$, define the genus character χ_D (see, [13]) by $\chi_D(Q) = \left(\frac{D}{r}\right)$ or 0 according as $(a,b,c,D) = 1$ or not, where Q represents r. Define a period function in $S_{2k-2}(M/2,\chi^2)$ by

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{\substack{Q=[a,b,c],\\ Q\in Q_{M^2/4,|D|M^2m}}} \overline{\chi(c)}\chi_D(Q)Q(z,1)^{-(k-1)}.$$

Similar function has been considered by Kohnen in [13] for trivial character and for odd M and for a generic case where the conductor of χ depends on the even part of the level, similar functions have been constructed in [24] in connection with Shimura correspondence. Note that

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M/2,\chi^2).$$

For $f \in S_{2k-2}(M/2,\chi^2)$, let

$$r_{2k-2,M/2,\chi^2}(f;|D|M^2m,D,\chi) = \sum_{\substack{Q \pmod{\Gamma_0(M/2)},\\ |Q|=|D|M^2m,\\ 4a\equiv 0 \pmod{M^2}}} \chi(c)\chi_D(Q)\int_{C_Q} f(z)d_{Q,k}z$$

where C_Q is the image in $\Gamma_0(M/2)\backslash\mathbb{H}$ of the semicircle $az^2+bRe(z)+c=0$ oriented from $(-b-\sqrt{|D|M^2m})/2a$ to $(-b+\sqrt{|D|M^2m})/2a$, if $a \neq 0$ or of the vertical line $bRe(z)+c=0$ oriented from $-c/b$ to $i\infty$ if $b>0$ and from $i\infty$ to $-c/b$, if $a=0$, and $d_{Q,k}z = (az^2-bz+c)^{k-2}dz$. Then, we have

Proposition 3.2.3 (see: [13], proposition 7 and [24], proposition 4)**.** *For* $f \in S_{2k-2}(M/2,\chi^2)$,

$$\langle f(z), \overline{F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(-\overline{z})}\rangle = \frac{\pi\, 2^{-2k+4}\binom{2k-4}{k-2}}{i_{M/2}(|D|M^2m)^{k-3/2}}\, r_{2k-2,M/2,\chi^2}(f;|D|M^2m,D,\chi).$$

We state the Fourier expansion of $F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}$ in the following:

Proposition 3.2.4 (see: [24], proposition 1)**.**

$$F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{n\geq 1} C_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(n;|D|M^2m)e^{2\pi i n z}, \quad (3.3)$$

where

$$\begin{aligned} &C_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}(n;|D|M^2m) \\ &\qquad = \frac{2(-2\pi)^{k-1}}{M^{k-3/2}(k-2)!}(n^2/m|D|)^{(k-2)/2}\Bigg\{(-1)^{\lfloor k/2\rfloor}\chi(n/\sqrt{m/|D|}) \\ &\qquad\qquad \times R_{\overline{\chi},D}\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta(n/\sqrt{m/|D|})|D|^{-1/2} \\ &\qquad + \pi\sqrt{2}(n^2/m|D|)^{1/4} \\ &\qquad\qquad \times \sum_{\substack{a\geq 1,\\ M^2|4a}} a^{-1/2} S_{a,\overline{\chi}}(|D|M^2m,n) J_{k-3/2}\left(\frac{\pi n\sqrt{|D|M^2m}}{a}\right)\Bigg\}, \end{aligned}$$

$R_{\overline{\chi},D}$ *is the Gauss sum given by*

$$R_{\overline{\chi},D} = (M|D|)^{-1/2}\left(\frac{4\epsilon D}{-1}\right)^{-1/2} \sum_{r\,(\mathrm{mod}\ M|D|)} \overline{\chi}(r)\left(\frac{D}{r}\right) e_{M|D|}(r),$$

$S_{a,\overline{\chi}}(|D|M^2m,n)$ *is the finite exponential sum given by*

$$S_{a,\overline{\chi}}(|D|M^2m,n) = \sum_{\substack{b \,(\mathrm{mod}\ 2a),\\ b^2\equiv|D|M^2m \,(\mathrm{mod}\ 4a)}} \overline{\chi}\left(\frac{b^2-|D|M^2m}{4a}\right)\chi_D\left(a,b,\frac{b^2-|D|M^2m}{4a}\right)e_{2a}(nb),$$

and

$$\delta(x) = \begin{cases} 1 \text{ if } x\in\mathbb{Z}, \\ 0 \text{ otherwise.}\end{cases}$$

Action of Shimura map on Poincarè series

To get

$$S_{D;k-1/2,M,\chi_0} : S^+_{k-1/2}(M,\chi_0) \longrightarrow S_{2k-2}(M/2,\chi^2),$$

we derive the image of Poincaré series under Shimura lifts.

Proposition 3.2.5.

$$P^+_{k-1/2,M,\chi_0;m}|S_{D;k-1/2,M,\chi_0} = \lambda_{k,D,M,\chi}F_{2k-2,M/2,\chi^2;|D|M^2m,D,\overline{\chi}}, \tag{3.4}$$

where

$$\lambda_{k,D,M,\chi} = \left(\frac{2(-2\pi)^{k-1}}{(k-2)!}(M|D|)^{-k+3/2}(-1)^{\lfloor k/2\rfloor}R_{\overline{\chi},D}\right)^{-1}.$$

Proof. Let

$$P^+_{k-1/2,M,\chi_0;m}|S_{D;k-1/2,M,\chi_0} = \sum_{n\geq 1} b(n)e^{2\pi inz}.$$

Using the equations (3.1) and (3.2) we get

$$\begin{aligned}
b(n) = &\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}\\
&\times\left(\delta_{\frac{|D|n^2}{d^2},m}+\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(m/\frac{|D|n^2}{d^2})^{k/2-3/4}\right.\\
&\left.\times\sum_{c\geq1}H_{Mc,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi\sqrt{m\frac{|D|n^2}{d^2}}}{Mc}\right)\right)\\
= &\left(\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{-k+1}(|D|n^2/m)^{k-3/2}\delta_{\frac{|D|n^2}{d^2},m}\right)\\
&+\left(\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\right.\\
&\times\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4}\\
&\left.\times\sum_{c\geq1}H_{Mc,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd}\right)\right)\\
= &\left(\chi(n/\sqrt{m/|D|})\left(\frac{D}{n/\sqrt{m/|D|}}\right)\right.\\
&\left.\times(n/\sqrt{m/|D|})^{-k+1}(|D|n^2/m)^{k-3/2}\delta(n/\sqrt{m/|D|})\right)\\
&+\left(\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\right.\\
&\times\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4}\\
&\left.\times\sum_{c\geq1}H_{Mc,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd}\right)\right).
\end{aligned}\tag{3.5}$$

Substituting (3.5) and (3.3) into left- and right-hand sides of equation (3.4) respectively, it is enough to prove that,

$$\begin{aligned}
&\left(\chi(n/\sqrt{m/|D|})\left(\frac{D}{n/\sqrt{m/|D|}}\right)(n/\sqrt{m/|D|})^{-k+1}(|D|n^2/m)^{k-3/2}\delta(n/\sqrt{m/|D|})\right)\\
&\quad+\Bigg(\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\\
&\qquad\times\sum_{d|n}\chi(d)\left(\frac{D}{d}\right)d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4}\\
&\qquad\times\sum_{c\geq1}H_{Mc,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Mcd}\right)\Bigg)\\
&=\Bigg(\lambda_{k,D,M,\chi}\frac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}\\
&\qquad\times(-1)^{\lfloor k/2\rfloor}R_{\overline{\chi},D}\,\chi(n/\sqrt{m/|D|})\left(\frac{D}{n/\sqrt{m/|D|}}\right)\delta(n/\sqrt{m/|D|})|D|^{-1/2}\Bigg)\\
&\quad+\Bigg(\lambda_{k,D,M,\chi}\frac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}\\
&\qquad\times\pi\sqrt{2}(n^2/m|D|)^{1/4}\sum_{\substack{a\geq1,\\ M^2|4a}}a^{-1/2}S_{a,\overline{\chi}}(|D|M^2m,n)J_{k-3/2}\left(\frac{\pi n\sqrt{|D|M^2m}}{a}\right)\Bigg).
\end{aligned}\tag{3.6}$$

The first terms on both sides of equation (3.6) vanishes if $m/|D|$ is not a square of an integer and if $n\neq\sqrt{m/|D|}$. Suppose this happens and if $n=\sqrt{m/|D|}$, then both first terms are equal in the above equation. Now, we compare the second terms on both sides of (3.6). Substituting $cd=a$ on the left-hand side of (3.6), it is enough to prove that

$$\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i) \times \sum_{d|(a,n)} \chi(d)\left(\frac{D}{d}\right) d^{k-2}(|D|n^2/d^2m)^{k-3/2}(m/\frac{|D|n^2}{d^2})^{k/2-3/4} \times \sum_{a\geq 1} H_{Ma/d,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right)$$
$$= \lambda_{k,D,M,\chi}\frac{2(-2\pi)^{k-1}}{(k-2)!}M^{-k+3/2}(n^2/m|D|)^{(k-2)/2}\pi\sqrt{2}(n^2/m|D|)^{1/4} \times \sum_{a\geq 1}((M/2)^2a)^{-1/2}S_{a(M/2)^2,\overline{\chi}}(M^2|D|m,n)J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right).$$

i.e.,

$$\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)\sum_{d|(a,n)} \chi(d)\left(\frac{D}{d}\right) d^{-1/2}|D|^{k/2-3/4}n^{k-3/2}m^{-k/2+3/4} \times \sum_{a\geq 1} H_{Ma/d,\overline{\chi}}(m,\frac{|D|n^2}{d^2})J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right)$$
$$= ((-1)^{\lfloor\frac{k}{2}\rfloor}R_{\overline{\chi},D})^{-1}|D|^{k/2-3/4}n^{k-3/2}m^{-k/2+3/4}\pi\sqrt{2} \times \sum_{a\geq 1}((M/2)^2a)^{-1/2}S_{a(M/2)^2,\overline{\chi}}(M^2|D|m,n)J_{k-3/2}\left(\frac{4\pi n\sqrt{m|D|}}{Ma}\right),$$

which follows from the proposition stated below and whose proof needs a standard set of arguments. Hence we omit the details; for a proof, we refer to ([13], proposition 5; [24], proposition 3).

Proposition 3.2.6. *For all* $a,m,n\in\mathbb{N}$, *we have*

$$S_{aM^2/4,\overline{\chi}}(M^2|D|m,n) = R_{\overline{\chi},D}\sqrt{aM^2/4}(1-(-1)^{k-1}i) \times \sum_{d|(a,n)}\chi(d)\left(\frac{D}{d}\right)d^{-1/2}H_{Ma/d,\overline{\chi}}(m,\frac{|D|n^2}{d^2}).$$

□

Let $D, \epsilon(-1)^{k-1}D > 0$ be a fundamental discriminant. The adjoint of Shimura map $S_{D;k-1/2,M,\chi_0}$ is given by

$$S^*_{D;2k-2,M/2,\chi^2} : S_{2k-2}(M/2,\chi^2) \longrightarrow S^+_{k-1/2}(M,\chi_0).$$

We now let $k=2$. All the stated results in the above subsection are still valid. We define the required period function when $k=2$, by using the standard 'Hecke trick' as in [13]. We leave the details, since we need to proceed along the same lines of arguments of the quoted paper by Kohnen. We make the following observation. Let $k=2$ and N be an arbitrary odd integer. The proof of Theorem 2 of [13] shows that the Shintani map S^*_D maps the first Poincaré series $P_{2,N;1}\in S_2(N)$ into $|D|$-th Poincaré series $P^+_{3/2,4N;|D|}\in S^+_{3/2}(4N)$. These results are also valid in our case.

We now compare the dimensions of the certain spaces under the stated assumptions. We have

Lemma 3.2.7.

$$\dim S_{k-1/2}(M,\chi_0) = \dim S_{2k-2}(M/2,\chi^2) = \frac{1}{2}\dim S_{2k-2}(M,\chi^2).$$

Proof. Let $k > 2$. Using the notations as in [29], we have

$$\begin{aligned}\dim S_{k-1/2}(M,\chi_0) &= \frac{(k-3/2)2^{\alpha-2}N}{12}.\frac{3}{2}\prod_{p|N}(1+1/p) - \frac{\zeta(k-1/2,2^{\alpha-2}N,\chi_0)}{2}\prod_{p|N}2,\\ &= (2k-3)2^{\alpha-6}N\prod_{p|N}(1+1/p) - 2^{\nu(N)}\\ \dim S_{2k-2}(M/2,\chi^2) &= \frac{(2k-3)2^{\alpha-3}N}{12}.\frac{3}{2}\prod_{p|N}(1+1/p) - \frac{\lambda(r_2,s_2,2)}{2}\prod_{p|N}2\\ &= (2k-3)2^{\alpha-6}N\prod_{p|N}(1+1/p) - 2^{\nu(N)},\end{aligned}$$

where $\nu(N)$= number of distinct primes dividing N. Hence, $\dim S_{k-1/2}(M,\chi_0) = \dim S_{2k-2}(M/2,\chi^2)$. In a similar manner we can verify

$$\dim S_{2k-2}(M/2,\chi^2) = \frac{1}{2}\dim S_{2k-2}(M,\chi^2).$$

This completes the proof of the lemma for the case $k > 2$. The proof for $k = 2$ follows using the same computations combined with the following facts. Since $cond(\chi) = M$ the space $M_{1/2}(M,\chi_0)$ becomes trivial, which follows from the work of Serre-Stark (see the Theorem A in [30]). Similarly, since χ is not the trivial character, we also observe that both the spaces $M_0(M/2,\chi^2)$ and $M_0(M,\chi^2)$ are trivial (see, for example, the Theorem 7.4.1 of [5]). □

We now state the theorem regarding newforms for the space $S^+_{k-1/2}(M,\chi_0)$:

Theorem 3.2.8. *Let $k \geq 2$, $32|M$, cond$(\chi) = M$ and cond$(\chi^2) = M/2$. Then,*

$$S^+_{k-1/2}(M,\chi_0) = S_{k-1/2}(M,\chi_0).$$

There exists a finite linear combination of Shimura lifts ψ_K which defines an isomorphism from $S^+_{k-1/2}(M,\chi_0)$ into $S_{2k-2}(M/2,\chi^2)$. In particular, we have the strong multiplicity one theorem on $S^+_{k-1/2}(M,\chi_0)$.

Proof. (Theorem 3.2.8) Let $d = \dim S_{2k-2}(M/2,\chi^2) = \dim S_{k-1/2}(M,\chi_0)$ and $\{f_1, f_2, \ldots, f_d\}$ be the orthogonal basis of normalised Hecke eigenforms of $S_{2k-2}(M/2,\chi^2)$. For each $i, 1 \leq i \leq d$ select a normalised Hecke eigenform $f(= f_i)$ and then a fundamental discriminant $D(= D_i)$ with $\epsilon(-1)^{k-1}D > 0$, $(D,M) = 1$ and $L(f, \overline{\chi}\left(\frac{D}{\cdot}\right), k-1) \neq 0$. Since $|D|$-th Fourier coefficient of $f|S^*_{D;2k-2,M,\chi}$ is equal to a non-zero constant multiple of $L(f, \overline{\chi}\left(\frac{D}{\cdot}\right), k-1)$, by using the arguments which give the Theorem 4.1 of [15], we derive that the d cusp forms $f_i|S^*_{D_i;2k-2,M,\chi}$ constructed as above forms an orthogonal set in $S^+_{k-1/2}(M,\chi_0)$.

Thus we select an orthogonal set of cusp forms $\{g_i : 1 \leq i \leq d\}$ in $S^+_{k-1/2}(M,\chi_0)$

such that

$$g_i|S_{D;k-1/2,M,\chi_0} = a_{g_i}(|D|)f_i \quad \text{and} \quad \frac{f_i}{< f_i, f_i >}|S^*_{D;2k-2,M/2,\chi^2} = \overline{a_{g_i}(|D|)}\frac{g_i}{< g_i, g_i >}$$

hold good.

Now $S^+_{k-1/2}(M,\chi_0)$ is a subspace of $S_{k-1/2}(M,\chi_0)$ having $d = \dim S_{k-1/2}(M\chi_0)$ linearly independent cusp forms, and hence we conclude that

$$S^+_{k-1/2}(M,\chi_0) = S_{k-1/2}(M,\chi_0).$$

This completes the proof of Theorem 3.2.8. □

3.2.2 Theory of newforms of $S^+_{k-1/2}(4M,\chi_0)$

Let $n \geq 1$ be an integer and let $P_{k-1/2,4M,\chi_0;n}$ denote the n-th Poincarè series in $S_{k-1/2}(4M,\chi_0)$ characterized by

$$< g, P_{k-1/2,4M,\chi_0;n} >= i_{4M}^{-1}\frac{\Gamma(k-3/2)}{(4\pi n)^{k-3/2}}a_g(n) \quad \text{for all } g \in S_{k-1/2}(4M,\chi_0).$$

Let

$$P^+_{k-1/2,4M,\chi_0;n} = P_{k-1/2,4M,\chi_0;n}\big|Pr_+.$$

Let $k > 2$ and $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}$ denote the function defined by

$$F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}(z) = \sum_{\substack{Q=[a,b,c],\\ Q\in \mathcal{Q}_{M^2,|D|M^2m}}} \overline{\chi(c)}\chi_D(Q)Q(z,1)^{-(k-1)}.$$

Note that $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M,\chi^2)$.

If $k = 2$, then we define Poincaré series and the period functions in $S_2(M,\chi^2)$ as defined by Kohnen in [13] and the stated results in this subsection are valid.

We have the following result, when $k \geq 2$ and with other conditions assumed on M, χ and χ^2. The proof follows using similar arguments as in the proof of the Proposition 3.2.5:

Proposition 3.2.9.

$$P^+_{k-1/2,4M,\chi_0;m}|S_{D;k-1/2,4M,\chi_0} = \lambda_{k,D,M,\chi}F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}},$$

where

$$\lambda_{k,D,M,\chi} = \left(\frac{2(-2\pi)^{k-1}}{(k-2)!}(M|D|)^{-k+3/2}(-1)^{\lfloor k/2\rfloor}R_{\overline{\chi},D}\right)^{-1}.$$

Hence, image of $P^+_{k-1/2,4M,\chi_0;m}$ under the map $S_{D;k-1/2,4M,\chi_0}$ is a constant multiple of the period function $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}}$. Since $F_{2k-2,M,\chi^2;|D|M^2m,D,\overline{\chi}} \in S_{2k-2}(M,\chi^2)$, and all the Poincaré series $P^+_{k-1/2,4M,\chi_0;m}$ span the space $S^+_{k-1/2}(4M,\chi_0)$ we conclude that

$$S_{D;k-1/2,4M,\chi_0} : S^+_{k-1/2}(4M,\chi_0) \longrightarrow S_{2k-2}(M,\chi^2).$$

We now observe the following.

Lemma 3.2.10. $S^{new}_{2k-2}(M,\chi^2) = \{0\}$.

Proof. Since $cond(\chi^2) = M/2$, the theory of newforms developed by W. Li in [17] gives

$$S_{2k-2}(M,\chi^2) = (S_{2k-2}(M/2,\chi^2) \oplus S_{2k-2}(M/2,\chi^2)|B_2) \bigoplus S^{new}_{2k-2}(M,\chi^2).$$

Now B_2 is an injective linear map and since $\dim S_{2k-2}(M,\chi^2) = 2\dim S_{2k-2}(M/2,\chi^2)$, we get

$$S^{new}_{2k-2}(M,\chi^2) = \{0\}.$$

□

We define

$$S^{+,old}_{k-1/2}(4M,\chi_0) = S^{+}_{k-1/2}(M,\chi_0) \oplus S^{+}_{k-1/2}(M,\chi_0)|B_4$$

and $S^{+,new}_{k-1/2}(4M,\chi_0)$ as the orthogonal complement of $S^{+,old}_{k-1/2}(4M,\chi_0)$ with respect to the Petersson scalar product. We have

Lemma 3.2.11. $S^{+,new}_{k-1/2}(4M,\chi_0) = \{0\}$.

Proof. We first observe the direct sum decomposition

$$S^+_{k-1/2}(4M,\chi_0) = S^+_{k-1/2}(M,\chi_0) \oplus S^+_{k-1/2}(M,\chi_0)|B_4 \bigoplus S^{+,new}_{k-1/2}(4M,\chi_0)$$

and

$$S_{2k-2}(M,\chi^2) = S_{2k-2}(M/2,\chi^2) \oplus S_{2k-2}(M/2,\chi^2)|B_2.$$

Proposition 3.2.5 and Proposition 3.2.9 give

$$S_D : S^+_{k-1/2}(4M,\chi_0) \longrightarrow S_{2k-2}(M,\chi^2)$$

and

$$S_D : S^+_{k-1/2}(M,\chi_0) \longrightarrow S_{2k-2}(M/2,\chi^2),$$

where we use $S_D = S_{D;k-1/2,M,\chi_0}$ or $S_D = S_{D;k-1/2,4M,\chi_0}$. Let $g \in S^+_{k-1/2}(M,\chi_0)$. Since S_D commutes with B_4 and B_2 as

$$g|B_4 S_D = g|S_D B_2,$$

each of the Shimura map S_D maps

$$S^+_{k-1/2}(M,\chi_0) \oplus S^+_{k-1/2}(M,\chi_0)|B_4$$

into the space

$$S_{2k-2}(M/2,\chi^2) \oplus S_{2k-2}(M/2,\chi^2)|B_2.$$

Since S_D maps $S^+_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$ and the Shimura correspondence preserves eigenclasses with respect to Hecke operators T_{n^2}, $(n,M)=1$, if $g \in S^{+,new}_{k-1/2}(4M,\chi_0)$ then $g|S_D$ belongs to $S^{new}_{2k-2}(M,\chi^2)$, and hence $g|S_D = 0$ for all such fundamental discriminants D. This proves $g=0$, if not there exists a fundamental discriminant $D,(D,M)=1$ with $a_g(|D|)\neq 0$ so that $g|S_D \neq 0$, which is not true. Now the result follows. □

We now state the theorem regarding newforms for the space $S^+_{k-1/2}(4M,\chi_0)$, whose proof is obtained by combining the above results.

Theorem 3.2.12. *Let $k\geq 2$, $32|M$, $cond(\chi)=M$ and $cond(\chi^2)=M/2$. Then,*

$$S^{+,new}_{k-1/2}(4M,\chi_0)=\{0\} \ \textit{and} \quad S^{new}_{2k-2}(M,\chi^2)=\{0\},$$

$$S^+_{k-1/2}(4M,\chi_0)=S^+_{k-1/2}(M,\chi_0)\bigoplus S^+_{k-1/2}(M,\chi_0)|B_4,$$

$$S_{2k-2}(M,\chi^2)=S_{2k-2}(M/2,\chi^2)\bigoplus S_{2k-2}(M/2,\chi^2)\big|B_2.$$

Moreover, the isomorphism ψ_K maps $S^+_{k-1/2}(4M,\chi_0)$ into $S_{2k-2}(M,\chi^2)$.

Chapter 4

Saito-Kurokawa lifts

This chapter aims at deriving the Saito-Kurokawa isomorphism on the space of newforms for Maaß spezialschar inside $\mathcal{S}_k(\Gamma_0^2(M), \chi)$ where $32|M$, χ is a primitive character χ modulo M with $\chi(-1) = (-1)^k$ and χ^2 is primitive modulo $M/2$. For this, we first develop the corresponding theory of newforms for respective spaces of Jacobi forms and then for Maass forms. This is achieved using the theory of newforms for the spaces of cusp forms of half-integral weight in the previous chapter. Finally as a consequence we get the Saito-Kurokawa isomorphism.

4.1 Preliminaries

Let $\ell \geq 1$ be an integer. Let $J_{k,\ell}(M, \chi)$ denote the space of Jacobi forms of weight k, index ℓ and character χ, and its sub space of cusp forms is denoted by $J_{k,\ell}^{cusp}(M, \chi)$. We refer to [6] for the development of the theory of Jacobi forms. A Jacobi form

$\phi \in J_{k,\ell}^{cusp}(M,\chi)$ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n,r\in\mathbb{Z},\\ r^2<4n\ell}} c_\phi(n,r)e(n\tau+rz).$$

Let $n,r,n',r' \in \mathbb{Z}$ with $r^2 < 4n\ell, r'^2 < 4n'\ell$. Then we have $c_\phi(n,r) = c_\phi(n',r')$ if $r'^2 - 4n'\ell = r^2 - 4n\ell$ and $r' \equiv r \pmod{2\ell}$. Thus, we write the Fourier expansion of ϕ as

$$\phi(\tau,z) = \sum_{\substack{D<0,r\in\mathbb{Z},\\ D\equiv r^2 \pmod{4n\ell}}} c_\phi(D,r)e\left(\frac{r^2-D}{4\ell}\tau + rz\right).$$

We note that when the index is 1, $J_{k,1}^{cusp}(M,\chi) = \{0\}$ unless $\chi(-1) = (-1)^k$. So whenever we consider the space $J_{k.1}(M,\chi)$, we let $\chi(-1) = (-1)^k$.

The operators $U_J(4)$ and $B_J(4)$ on $J_{k,1}^{cusp}(M,\chi)$ are defined on formal series by

$$\sum_{\substack{D<0,r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c_\phi(D,r)e\left(\frac{r^2-D}{4}\tau + rz\right)\Bigg|U_J(4) = \sum_{\substack{D<0,r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c_\phi(4D,2r)e\left(\frac{r^2-D}{4}\tau + rz\right)$$

and

$$\sum_{\substack{D<0,r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c_\phi(D,r)e\left(\frac{r^2-D}{4}\tau + rz\right)\Bigg|B_J(4) = \sum_{\substack{D<0,r\in\mathbb{Z},\\ D\equiv r^2 \pmod{16}}} c_\phi\left(\frac{D}{4},\frac{r}{2}\right)e\left(\frac{r^2-D}{4}\tau + rz\right).$$

Eichler-Zagier map:

If $\phi \in J_{k,1}^{cusp}(M,\chi)$, the Eichler-Zagier map $\mathcal{Z}_1$ on ϕ is given by

$$\sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c_\phi(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right) \longmapsto \sum_{\substack{D<0,\\ D\equiv r^2 \pmod 4}} c_\phi(D,r) e(|D|\tau).$$

Poincaré series:

Let $D < 0$ be a discriminant and $r \pmod{2\ell}$ with $r^2 \equiv D \pmod{4\ell}$. Then, we denote the (D,r)-th Poincaré series in $J_{k,\ell}^{cusp}(M,\chi)$ by $P_{k,\ell,M,\chi;D,r}$ and it is characterized by the relation

$$\langle \phi, P_{k,\ell,M,\chi;D,r} \rangle = \alpha_{k,\ell,D,M} C_\phi(D,r)$$

for all $\phi \in J_{k,\ell}^{cusp}(M,\chi)$, where

$$\alpha_{k,\ell,D,M} = \frac{\Gamma(k-3/2)}{\pi^{k-3/2} i_M} \ell^{k-2} |D|^{-k+3/2}.$$

4.2 Theory of newforms of $J_{k,1}^{cusp}(M,\chi)$

4.2.1 Eichler-Zagier map $\mathcal{Z}_1$

Let $k \geq 2$ be an integer. $\epsilon = (-1)^k$. Let $\mathcal{Z}_1$ denote the Eichler-Zagier map defined in the previous section. We prove the following.

$$\mathcal{Z}_1 : J_{k,1}^{cusp}(M,\chi) \longrightarrow S_{k-1/2}^{+}(4M,\chi_0).$$

Let $D<0$ be a discriminant and $r \pmod 2$ with $r^2 \equiv D \pmod 4$. Let $P_{k,1,M,\chi_0;D,r}$ denote the (D,r)-th Poincarè series in $J_{k,1}^{cusp}(M,\chi)$. Let $P^{+}_{k-1/2,4M,\chi_0;|D|}$ be the $|D|$-th Poincaré series in $S^{+}_{k-1/2}(4M,\chi_0)$ as defined in §3.2.2.

Fourier expansion of $P_{k,1,M,\chi_0;D,r}$:

The Fourier expansion of $P_{k,1,M,\chi_0;D,r}$ is given by (which can be obtained using standard arguments):

$$P_{k,1,M,\chi_0;D,r}(\tau,z) = \sum_{\substack{D',r'\in\mathbb{Z},\\ D'<0}} g^{\pm}_{k,M,\chi_0;D,r}(D',r')e\left(\frac{r'^2-D}{4}\tau + r'z\right), \qquad (4.1)$$

where $g^{\pm}_{k,1,M,\chi_0;D,r}(D',r')$ is symmetrized or antisymmetrized with respect to r', i.e.,

$$g^{\pm}_{k,1,M,\chi_0;D,r}(D',r') = g_{k,1,M,\chi_0;D,r}(D',r') + \chi(-1)(-1)^k g_{k,1,M,\chi_0;D,r}(D',-r')$$

with $D' = r'^2 - 4n'$, $D = r^2 - 4n$, and

$$g_{k,1,M,\chi_0;D,r}(D',r') = \delta(D,r;D',r') + \Bigg(\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right)\Bigg),$$

where

$$\delta(D,r;D',r') = \begin{cases} 1 \text{ if } D'=D, r'\equiv r (mod\, 2) \\ 0 \text{ otherwise,} \end{cases}$$

$J_{k-3/2}(.)$ is the Bessel function and

$$H_{c,\chi}(D,r;D',r') = \frac{1}{c^{3/2}} \sum_{\substack{\lambda,\delta \,(\mathrm{mod}\ c),\\ \delta^{-1}\delta\equiv 1 (mod\, c)}} \overline{\chi}(\delta) e_c(\delta^{-1}(\lambda^2+r\lambda+n)+n'\delta-r'\lambda)e_{2c}(-rr').$$

Fourier expansion of $P^{+}_{k-1/2,4M,\chi_0;|D|}$:

The Fourier expansion of $P^{+}_{k-1/2,4M,\chi_0;|D|}$ is given by

$$P^{+}_{k-1/2,4M,\chi_0;|D|}(\tau) = \sum_{\substack{m\geq 1,\\ \epsilon(-1)^{k-1}m\equiv 0,1\,(\mathrm{mod}\ 4)}} g^{+}_{k-1/2,4M,\chi_0;|D|}(m)e^{2\pi i m\tau}, \tag{4.2}$$

where

$$g^{+}_{k-1/2,4M,\chi_0;|D|}(m) = \delta_{|D|,m} + \pi\sqrt{2}(-1)^{\lfloor \frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{4Mc,\chi}(m,|D|)J_{k-3/2}\left(\frac{4\pi\sqrt{m|D|}}{4Mc}\right),$$

$\delta_{n,m}$ is the Kronecker delta, $J_{k-3/2}(.)$ is the Bessel function and

$$H_{4Mc,\chi}(m,n) = \frac{1}{4Mc} \sum_{\substack{\delta\,(\mathrm{mod}\ 4Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\ 4Mc)}} \overline{\chi}_0(\delta)\left(\frac{4Mc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k-1/2} e_{4Mc}(m\delta+n\delta^{-1}).$$

We have the following standard identity:

Lemma 4.2.1. *Gauss sum identity: Let* $4|c$, $(c,d)=1$. *Then*

$$\sum_{\lambda \pmod{c}} e_c(d\lambda^2) = (1+i)\sqrt{c}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}.$$

In the following proposition, we prove that the Eichler-Zagier map sends Jacobi Poincaré series into the plus space Poincaré series.

Proposition 4.2.2.

$$P_{k,1,M,\chi_0;D,r}|\mathcal{Z}_1 = 2P^{+}_{k-1/2,4M,\chi_0;|D|}.$$

Proof. To prove the above equation, we compare the $|D'|$-th coefficients on both sides. We need to show that

$$\sum_{\substack{r' \pmod{2}, \\ D' \equiv r'^2 \pmod{4}}} g^{\pm}_{k,M,\chi_0;D,r}(D',r') = 2g^{+}_{k-1/2,4M,\chi_0;|D|}(|D'|). \tag{4.3}$$

Comparing the first terms on both sides of the above equation, using (4.1) and (4.2) we have

$$\delta(D,r;D',r') + \chi(-1)(-1)^k\delta(D,r;D',-r') = \delta(D,r;D',r') + \delta(D,r;D',r')$$
$$= \begin{cases} 2; D = D' \\ 0; D \neq D' \end{cases}$$

$$= 2\delta_{|D|,|D'|}.$$

Hence, the first terms on both sides of equation (4.3) are equal. Now, we compare the second terms on both sides. Second term in the LHS of equation (4.3) is given by

$$\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right) +$$
$$\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',-r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right)$$
$$= 2.\pi\sqrt{2}i^{-k}(D'/D)^{k/2-3/4} \sum_{c\geq 1} H_{Mc,\chi}(D,r;D',r')J_{k-3/2}\left(\frac{\pi\sqrt{D'D}}{Mc}\right).$$

Second term in the RHS of equation (4.3) is given by

$$2.\pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(|D'|/|D|)^{k/2-3/4}\sum_{c\geq 1} H_{4Mc,\chi}(|D'|,|D|)J_{k-3/2}\left(\frac{4\pi\sqrt{|D'||D|}}{4Mc}\right).$$

Note that $(-1)^{\lfloor k/2\rfloor}(1-(-1)^{k-1}i) = i^{-k}(1+i)$. Hence,

$$i^{-k}H_{Mc,\chi}(D,r;D',r')$$
$$= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\lambda,\delta \pmod{Mc},\\ \delta^{-1}\delta\equiv 1 \pmod{Mc}}} \overline{\chi}(\delta)e_{Mc}(\delta^{-1}(\lambda^2+r\lambda+n)+n'\delta-r'\lambda)e_{2Mc}(-rr').$$

Using $\lambda \mapsto \delta\lambda$ the above equals

$$\begin{aligned}
&\frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\lambda,\delta \,(\mathrm{mod}\, Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\, Mc)}} \overline{\chi}(\delta) e_{Mc}(\delta\lambda^2 + r\lambda + n\delta^{-1} + n'\delta - r'\delta\lambda) e_{2Mc}(-rr') \\
&= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\delta\,(\mathrm{mod}\, Mc)} \overline{\chi}(\delta) \left(\sum_{\lambda\,(\mathrm{mod}\, Mc)} e_{Mc}(\delta\lambda^2 + (r - r'\delta)\lambda) \right) \\
&\qquad\qquad \times e_{Mc}(n\delta^{-1} + n'\delta) e_{2Mc}(-rr') \\
&= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\delta\,(\mathrm{mod}\, Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\, Mc)}} \overline{\chi}(\delta) \left(\sum_{\lambda\,(\mathrm{mod}\, Mc)} e_{Mc}\left(\delta \left(\lambda + \frac{r\delta^{-1} - r'}{2} \right)^2 \right) \right) \\
&\qquad \times e_{4Mc}(-r^2\delta^{-1} + 2rr' - r'^2\delta) e_{4Mc}(4n\delta^{-1} + 4n'\delta) e_{4Mc}(-2rr') \\
&= \frac{i^{-k}}{(Mc)^{3/2}} \sum_{\substack{\delta\,(\mathrm{mod}\, Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\, Mc)}} \overline{\chi}(\delta) \left((1+i) \left(\frac{-4}{\delta} \right)^{-1/2} \left(\frac{Mc}{\delta} \right) \sqrt{Mc} \right) \\
&\qquad\qquad \times e_{4Mc}(|D|\delta^{-1} + |D'|\delta) \quad \text{(using lemma 4.2.1)} \\
&= \frac{i^{-k}(1+i)}{Mc} \sum_{\substack{\delta\,(\mathrm{mod}\, Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\, Mc)}} \overline{\chi}(\delta) \left(\frac{4(-1)^k}{\delta} \right) \left(\frac{-4}{\delta} \right)^{k-1/2} \left(\frac{Mc}{\delta} \right) \\
&\qquad\qquad \times e_{4Mc}(|D|\delta^{-1} + |D'|\delta) \\
&= \frac{i^{-k}(1+i)}{4Mc} \sum_{\substack{\delta\,(\mathrm{mod}\, 4Mc),\\ \delta^{-1}\delta\equiv 1\,(\mathrm{mod}\, 4Mc)}} \overline{\chi}_0(\delta) \left(\frac{-4}{\delta} \right)^{k-1/2} \left(\frac{4Mc}{\delta} \right) \\
&\qquad\qquad \times e_{4Mc}(|D|\delta^{-1} + |D'|\delta) \\
&= (-1)^{\lfloor k/2 \rfloor}(1 - (-1)^{k-1}i) H_{4Mc,\chi}(|D'|, |D|).
\end{aligned}$$

So, the second terms on both sides of equation (4.3) are also equal. This completes the proof. □

We now state a result which relates the Petersson scalar products in $J_{k,1}^{cusp}(M,\chi)$ and $S^{+}_{k-1/2}(4M,\chi)$ whose proof follows by using the same argument of the Proposition 5.1 in [15]. So we omit the details.

Lemma 4.2.3. *We have*

$$\langle \phi|\mathcal{Z}_1, \psi|\mathcal{Z}_1\rangle = c\langle \phi, \psi\rangle$$

for all $\phi, \psi \in J_{k,1}^{cusp}(M,\chi)$ *with*

$$c = 2\frac{i_M}{4^{k-3/2}}.$$

Hence, $\mathcal{Z}_1$ *is a canonical isomorphism which preserves the Hecke eigenforms and the Petersson scalar product structures.*

4.2.2 Decomposition of the space of Jacobi forms

We start with the following lemma.

Lemma 4.2.4. *Suppose* $4|D$*. Let* $D/4 \equiv 0, 1 \pmod 4$*. Then,*

$$P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}} = P^{+}_{k-1/2,4M,\chi_0;|D|}|U_4.$$

Proof. It is enough to show that m-th Fourier coefficients are equal for all integers $m \geq 1$. They are respectively given by

$$g^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}(m) = \delta_{\frac{|D|}{4},m} + \pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(4m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{Mc,\chi}(m,|D|/4) J_{k-3/2}\left(\frac{4\pi\sqrt{m\frac{|D|}{4}}}{Mc}\right)$$

and

$$g^{+}_{k-1/2,4M,\chi_0;|D|}(4m) = \delta_{|D|,4m} + \pi\sqrt{2}(-1)^{\lfloor\frac{k}{2}\rfloor}(1-(-1)^{k-1}i)(4m/|D|)^{k/2-3/4} \times \sum_{c\geq 1} H_{4Mc,\chi}(4m,|D|) J_{k-3/2}\left(\frac{4\pi\sqrt{4m|D|}}{4Mc}\right).$$

Since $4|D$, we have

$$\delta_{\frac{|D|}{4},m} = \delta_{|D|,4m}$$

and hence the first terms are equal. The second terms are also equal since

$$H_{Mc,\chi}(m,|D|/4) = H_{4Mc,\chi}(4m,|D|),$$

which follows by using their definitions. □

Lemma 4.2.5. *Let* $4|D,\ D/4 \equiv 0, 1(4)$. *Then,*

$$P_{k,1,M,\chi_0;D,r}|U_J(4)Z_1 = 2P^{+}_{k-1/2,M,\chi_0;\frac{|D|}{4}}$$

Proof. Follows from Proposition 4.2.2 and Lemma 4.2.4. $\square$

Lemma 4.2.6. *$S^+_{k-1/2}(M,\chi_0)$ is spanned by all the Poincaré series $P^+_{k-1/2,M,\chi_0;\frac{|D|}{4}}$, where D varies over all the discriminants with $4\big||D|$ and $D/4 \equiv 0,1 \pmod 4$.*

Proof. Let $f \in S^+_{k-1/2}(M,\chi_0)$ such that it is in the orthogonal complement of the sub space of $S^+_{k-1/2}(M,\chi_0)$spanned by the Poincaré series $P^+_{k-1/2,M,\chi_0;\frac{|D|}{4}}$, where D varies over all the discriminants with $4\big||D|$ and $D/4 \equiv 0,1 \pmod 4$. Then,

$$\langle f, P^+_{k-1/2,M,\chi_0;\frac{|D|}{4}}\rangle = 0,$$

for all $4\big||D|$ with $D/4 \equiv 0,1 \pmod 4$. This implies that

$$a_f\left(\frac{|D|}{4}\right) = 0,$$

for all $D \equiv 0,1 \pmod 4$, or equivalently

$$a_{f|B_4}(|D|) = 0,$$

for all $D \equiv 0,1 \pmod 4$. Therefore, $f|B_4 = 0$ (or) $f = 0$. Hence, the result follows. $\square$

Newforms in $J_{k,1}^{cusp}(M,\chi)$:

We now define the space of newforms in $J_{k,1}^{cusp}(M,\chi)$ by

$$J_{k,1}^{cusp;new}(M,\chi) = \text{the linear span of the set}\{P_{k,1,M,\chi;D,r}|U_J(4)\},$$

where D varies over all the discriminants with $4\big||D|$ and $D/4 \equiv 0, 1 \pmod 4$. A Hecke eigenform which is a simultaneous eigenform for Hecke operators $T_J(n), (n, M) = 1$ in $J_{k,1}^{cusp;new}(M,\chi)$ is called a newform. We have the image of Jacobi newforms under the Eichler-Zagier isomorphism:

Lemma 4.2.7. $J_{k,1}^{cusp;new}(M,\chi)|\mathcal{Z}_1 = S_{k-1/2}^{+}(M,\chi_0)$.

Proof. Follows from Lemma 4.2.5 and Lemma 4.2.6. □

We now state the following main theorem for the theory of newforms of Jacobi cusp forms:

Theorem 4.2.8.

$$J_{k,1}^{cusp}(M,\chi) = J_{k,1}^{cusp;new}(M,\chi) \bigoplus J_{k,1}^{cusp;new}(M,\chi)|B_J(4).$$

The space $J_{k,1}^{cusp;new}(M,\chi)$ is isomorphic to the space $S_{2k-2}(M/2,\chi^2)$ under a certain linear combination of Shimura lifts. Hence, The multiplicity one result holds good on $J_{k,1}^{cusp;new}(M,\chi)$.

Proof. (Theorem 4.2.8) From Theorem 3.2.12, we have the decomposition

$$S^+_{k-1/2}(4M,\chi_0) = S^+_{k-1/2}(M,\chi_0) \bigoplus S^+_{k-1/2}(M,\chi_0)|B_4.$$

Also, we have

$$\phi|B_J(4)\mathcal{Z}_1 = \phi|\mathcal{Z}_1 B_4,$$

where $\phi \in J^{cusp}_{k,1}(M,\chi)$. Using these we get

$$\begin{aligned}
J^{cusp}_{k,1}(M,\chi)|\mathcal{Z}_1 &= S^+_{k-1/2}(4M,\chi_0)\\
&= S^+_{k-1/2}(M,\chi_0) \bigoplus S^+_{k-1/2}(M,\chi_0)|B_4\\
&= J^{cusp;new}_{k,1}(M,\chi)|\mathcal{Z}_1 \bigoplus J^{cusp;new}_{k,1}(M,\chi)|B_J(4)|\mathcal{Z}_1\\
&= \left(J^{cusp;new}_{k,1}(M,\chi) \bigoplus J^{cusp;new}_{k,1}(M,\chi)|B_J(4)\right)|\mathcal{Z}_1.
\end{aligned}$$

Thus, by using $\mathcal{Z}_1$ defines an isomorphism from $J^{cusp}_{k,1}(M,\chi)$ into $S^+_{k-1/2}(4M,\chi_0)$, we have

$$J^{cusp}_{k,1}(M,\chi) = J^{cusp;new}_{k,1}(M,\chi) \bigoplus J^{cusp;new}_{k,1}(M,\chi)|B_J(4).$$

The multiplicity one result holds on $J^{cusp;new}_{k,1}(M,\chi)$, which follows from the Theorem 3.2.8 and the isomorphism $\mathcal{Z}_1$. $\square$

4.3 Theory of newforms of $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ and Saito-Kurokawa isomorphism

Let $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ be the space of Siegel cusp forms of weight k, level M, genus 2 and character χ. Let $V_{m,\chi}$ be the index shifting operator

$$V_{m,\chi} : J_{k,1}^{cusp}(M,\chi) \to J_{k,m}^{\text{cusp}}(M,\chi)$$

(as per [8]). If

$$\phi(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod 4}} c(D,r) e\left(\frac{r^2-D}{4}\tau + rz\right) \in J_{k,1}^{cusp}(M,\chi),$$

then

$$\phi|V_{m,\chi}(\tau,z) = \sum_{\substack{D<0, r\in\mathbb{Z},\\ D\equiv r^2 \pmod{4m}}} \left(\sum_{\substack{d|(r,m),\,(d,M)=1,\\ D\equiv r^2 \pmod{4md}}} \overline{\chi(d)} d^{k-1} c\left(\frac{D}{d^2},\frac{r}{d}\right) e\left(\frac{r^2-D}{4m}\tau + rz\right)\right).$$

Maass spezialschar in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$:

For $\phi \in J_{k,1}^{cusp}(M,\chi)$, we define the Maass embedding $\iota_{M,\chi}$ as follows (see [8]):

$$\phi|\iota_{M,\chi} = \sum_{m\geq 1} (\phi|_{k,1} V_{m,\chi})(\tau,z) e^{2\pi i m w}.$$

Denote the image of $J_{k,1}^{cusp}(M,\chi)$ under this embedding by $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$. Then we have the following result:

Proposition 4.3.1 ([8], Theorem 3.2)**.** *The map $\iota_{M,\chi}$ gives an embedding of $J_{k,1}^{cusp}(M,\chi)$ into $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$.*

In ([7], corollary 4.2) it was proved that the Fourier coefficients of the forms in $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ also satisfies certain relations analogous to the classical Maass relation. The converse part was also proved; i.e., if the coefficients of $F\in\mathcal{S}_k(\Gamma_0^2(M),\chi)$ satisfies the said relations, then $F\in\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$. Hence, let $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$ denote the Maass space in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$, defined as the image of $J_{k,1}^{cusp}(M,\chi)$ under the embedding $\iota_{M,\chi}$. Note that $\iota_{M,\chi}$ gives an isomorphism between the spaces $J_{k,1}^{cusp}(M,\chi)$ and $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$.

4.3.1 Newforms in $\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$

For $F\in\mathcal{S}_k^*(\Gamma_0^2(M),\chi)$, define the operator $B_S(4)$ by

$$F(\tau,z,\tau')|B_S(4)=F(4\tau,4z,4\tau').$$

For $\phi\in J_{k,1}^{cusp;new}(M,\chi)$, we have $\phi|\iota_{M,\chi}|B_S(4)=\phi|B_J(4)|\iota_{M,\chi}$.

Let

$$\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)=J_{k,1}^{cusp;new}(M,\chi)|\iota_{M,\chi}.$$

Thus, $\iota_{M,\chi}$ acts an isomorphism between the spaces $J_{k,1}^{cusp;new}(M,\chi)$ and $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$.

Now, using the thoery of newforms for the space $J_{k,1}^{cusp;new}(M,\chi)$ as in Theorem 4.2.8 we obtain the following:

Theorem 4.3.2. *Let $k \geq 2, M \geq 1$ be integers such that $32|M$. Let χ be a Dirichlet character modulo M such that $cond(\chi) = M$ and $cond(\chi^2) = M/2$. Also let $\chi(-1) = (-1)^k$. Then,*

$$\mathcal{S}_k^*(\Gamma_0^2(M),\chi) = \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi) \bigoplus \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)|B_S(4).$$

Also, the multiplicity one theorem is valid on $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$.

4.3.2 Saito-Kurokawa lift

Let N be an odd integer and $M = 2^{\alpha-2}N$, where $\alpha > 6$. Let χ be a primitive Dirichlet character modulo M. Let $T_S(p), T_S'(p)$ (for prime $p \nmid M$) and $U_S(p)$ (for prime $p|M$) denote standard Hecke operators in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ (see §4 of [8]). An eigenform in $\mathcal{S}_k(\Gamma_0^2(M),\chi)$ under the above operators is called a Hecke eigenform. A non-zero Hecke eigenform which belongs to $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is called a newform in the Maass space. For a newform $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$, let γ_p, ω_p and μ_p denote the corresponding eigenvalues with respect to $T_S(p), T_S'(p), U_S(p)$ respectively. Similarly, let $T_J(p)$ (for prime $p \nmid M$) and $U_J(p)$ (for prime $p|M$) denote standard Hecke operators in $J_{k,1}^{cusp}(M,\chi)$, and let $a_f(p)$ denote the corresponding eigenvalues for the newform $\phi \in J_{k,1}^{cusp;new}(M,\chi)$ which corresponds to the normalised Hecke eigenform

$f \in S_{2k-2}(M/2,\chi^2)$. Let $F = \phi|\iota_{M,\chi}$ (as in §4.3). Then, theorem 4.1 of [8] gives

$$\left.\begin{aligned}
&\gamma_p = a_f(p) + \chi(p)(p^{k-2}+p^{k-1}), \quad p \nmid M\\
&\omega_p = \chi(p)(p^{k-2}+p^{k-1})a_f(p) + \chi(p^2)(2p^{2k-3}+p^{2k-4}), \quad p \nmid M\\
&\mu_p = a_f(p), \quad p \mid M.
\end{aligned}\right\} \tag{4.4}$$

For a Hecke eigenform $F \in \mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$, the Andrianov zeta function $Z_F(s)$ has the Euler product expansion (see [8])

$$Z_F(s) = \prod_{p|M}(1-\mu_p p^{-s})^{-1}\prod_{p\nmid M} Q_p(p^{-s})^{-1},$$

where

$$Q_p(p^{-s}) = 1 - \gamma_p p^{-s} + (p\omega_p + (p^2+1)\chi(p^2)p^{2k-5})p^{-2s} - \gamma_p\chi(p^2)p^{2k-3-3s} + \chi(p^4)p^{4k-6-4s}.$$

This, combined with (4.4) give

$$\begin{aligned}
Z_F(s) &= L(s-k+1,\chi)L(s-k+2,\chi)\\
&\quad\times\prod_{p|M}(1-a_f(p)p^{-s})^{-1}\prod_{p\nmid M}(1-a_f(p)p^{-s}+\chi(p^2)p^{2k-3-2s})^{-1}\\
&= L(s-k+1,\chi)L(s-k+2,\chi)L(f,s),
\end{aligned}$$

where $L(f,s)$ is the L-function of f given by

$$L(f,s)=\sum_{n\geq 1}a_f(n)n^{-s}.$$

Now, using the isomorphisms ψ_k, $\mathcal{Z}_1$, $\iota_{M,\chi}$, and the Theorems 3.2.8, 3.2.12, 4.2.8, we get the following:

Theorem 4.3.3. *The space $\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is in one to one correspondence with $S_{2k-2}(M/2,\chi^2)$ under the Saito-Kurokawa isomorphism. A given normalised Hecke eigenform $f\in S_{2k-2}(M/2,\chi^2)$ is lifted into two equivalent Hecke eigenforms $F, F|B_S(4)$, where $F\in\mathcal{S}_k^{*;new}(\Gamma_0^2(M),\chi)$ is the newform satisfying*

$$Z_F(s)=L(s-k+1,\chi)L(s-k+2,\chi)L(f,s).$$

Finally, we make the following observation:

Remark 4.3.4. *The map $\mathcal{Z}_1$ is a canonical isomorphism from $J_{k,1}^{cusp}(M,\chi)$ into the space $S_{k-1/2}^+(4M,\chi_0)$, which has the decomposition:*

$$S_{k-1/2}^+(4M,\chi_0)=S_{k-1/2}^+(M,\chi_0)\oplus S_{k-1/2}^+(M,\chi_0)|B_4.$$

Hence, we have the corresponding decomposition for Jacobi forms as

$$J_{k,1}^{cusp}(M,\chi)=J_{k,1}^{cusp,new}(M,\chi)\oplus J_{k,1}^{cusp,new}(M,\chi)|B_J(4).$$

This gives, if $\phi \in J_{k,1}^{cusp,new}(M,\chi)$, then we have $\phi|\mathcal{Z}_1 \in S_{k-1/2}^{+}(M,\chi_0)$ and $\phi|B_J(4)\mathcal{Z}_1 = \phi|\mathcal{Z}_1 B_4 \in J_{k,1}^{cusp}(M,\chi)$. Now by using the space $S_{k-1/2}^{+}(M,\chi_0)$ is isomorphic to the space $S_{2k-2}(M/2,\chi^2)$ as module over Hecke algebra and using the embedding of $J_{k,1}^{cusp}(M,\chi)$ in the Maass space, we realise that a normalised Hecke eigenform $f \in S_{2k-2}(M/2,\chi^2)$ is lifted into two linearly independent Hecke eigenforms F and $F|B_S(4)$, where $F \in \mathcal{S}_k^{*,new}(\Gamma_0^2(M),\chi)$. We call the form $F \in \mathcal{S}_k^{*,new}(\Gamma_0^2(M),\chi)$ a newform of weight k, level M, character χ in the Maass space.

Chapter 5

A certain kernel function for L-values of half-integral weight Hecke eigenforms

In this chapter, we derive a non-cusp form of weight $k+1/2$ ($k \geq 2$, even) for $\Gamma_0(4)$ in the Kohnen plus space whose Petersson scalar product with a cuspidal Hecke eigenform g is equal to a constant times the L value $L(g, k-1/2)$. As a corollary, we obtain that for such a form g and the associated form f under the D^{th} Shimura-Kohnen lift the quantity $\frac{a_g(D)L(f,2k-1)}{\pi^{k-1}\langle g,g\rangle L(D,k)}$ is algebraic.

5.1 Preliminaries

Let $k \geq 2$ be an even integer. For N odd and square free integer, let $M^+_{k+1/2}(4N)$ denote the Kohnen 'plus' space containing the forms g in $M_{k+1/2}(4N)$ whose Fourier coefficients $a_g(n)$ are 0 unless $(-1)^k n \equiv 0, 1 \pmod 4$. We let $S^+_{k+1/2}(4N) = M^+_{k+1/2}(4N) \cap S_{k+1/2}(4N)$.

Let T_m be the m^{th} Hecke operator on $M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. These operators preserves the space of cusp forms. Suppose a non-zero form $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ satisfies the relation $f|T_m = a_f(m)f$ for every $m \geq 1$ and $a_f(1) = 1$. Then f is known as a normalized Hecke eigenform. The space $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ has an orthonormal basis of normalized eigenforms of all Hecke operators (for example, see theorem 6.15 in [9]); let $\mathcal{B}$ be such a basis. Similarly for the space of half integral weight forms in $S^+_{k+1/2}(4)$ we denote the Hecke operators by $T^+_{m^2}$ (as in [11]; the operators $T^+_{m^2}$ for $m \geq 1$ are generated by operators $T^+_{p^2}$ where p varies over all primes). Let $\mathcal{B}^+$ denotes an orthogonal basis of eigenforms for all Hecke operators $T^+_{m^2}$ ($m \geq 1$) on $S^+_{k+1/2}(4)$, (Theorem 1 of [11] gives the existence of such a basis). Note that elements of $\mathcal{B}^+$ can be chosen in such a way that their Fourier coefficients are real and algebraic numbers ([10], page 216, line 12).

For $k \geq 2$, Proposition 1 of [11] gives the decomposition:

$$M^+_{k+1/2}(4) = \mathbb{C}H_{k+1/2} \bigoplus S^+_{k+1/2}(4),$$

where $H_{k+1/2}$ is an Eisenstein series:

$$H_{k+1/2} := E^{i\infty}_{k+1/2} + 2^{-2k-1}(1-(-1)^k i)E^0_{k+1/2},$$

where $E^{i\infty}_{k+1/2}$ and $E^{0}_{k+1/2}$ are as defined in [11], page 254. $H_{k+1/2}$ takes the value 1 at infinity; this series was first studied by H. Cohen [4]. If $g \in M_{k+1/2}(4)$ with Fourier expansion $g = \sum_{n\geq 1} a_g(n)e^{2\pi i n z}$, define the standard operators U_4 and W_4 by

$$f|U_4 = \sum_{n\geq 1} a_f(4n)e^{2\pi i n z}, \text{ and } f|W_4 = \left(\frac{2z}{i}\right)^{-k-1/2} f\left(\frac{-1}{4z}\right)$$

Now, define the projection map Pr_+ (as in [14], page 185) from $M_{k+1/2}(4)$ into $M^{+}_{k+1/2}(4)$ by

$$Pr_+ = (-1)^{[\frac{k+1}{2}]}2^{-k}W_4U_4 + \frac{1}{3}.$$

The map Pr_+ satisfies $\langle g_1|Pr_+, g_2\rangle = \langle g_1, g_2|Pr_+\rangle$ for $g_1, g_2 \in M^{+}_{k+1/2}(4)$ where g_1 or g_2 is a cusp form (see [14], page 186, line 13). Thus, the projection map Pr_+ preserves the plus space of cusp forms $S^{+}_{k+1/2}(4)$ and it is hermitian on this space. It also preserves the space generated by the Cohen-Eisenstein series as it is the orthogonal complement of the space of cusp forms.

Let D be a positive fundamental discriminant and $s \in \mathbb{C}$ with $Re(s) >> 1$, define

$$L(D, s) := \sum_{n\geq 1} \left(\frac{D}{n}\right) n^{-s}.$$

Let $\Theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$ be the standard theta function, where $z \in \mathbb{H}$. It is a modular form of weight $1/2$ for $\Gamma_0(4)$ (we refer to chapter 15 of [5] or chapter 3, section 1 of [10] for the details). Let $k \geq 2$ be a positive integer and $D \equiv 0, 1 \pmod 4$

be a positive fundamental discriminant. Define

$$\Theta_D(z) := \Theta(Dz) = \sum_{n\in\mathbb{Z}} e^{2\pi i D n^2 z}.$$

It is a modular form of weight $1/2$ for $\Gamma_0(4D)$ with character $\left(\frac{D}{\cdot}\right)$ of conductor D or $4D$ according as $D \equiv 1(4)$ or $D/4 \equiv 2, 3(4)$ (refer [30], page 32, lines 16-18). We also define an Eisenstein series $E_{k,4D,\left(\frac{D}{\cdot}\right)}$ by

$$E_{k,4D,\left(\frac{D}{\cdot}\right)} = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ 4|c}} \left(\frac{D}{d}\right) (cDz + d)^{-k}.$$

It is a modular form of weight k, level $4D$ and character $\left(\frac{D}{\cdot}\right)$, and has rational Fourier coefficients (rationality of the Fourier coefficients can be proved by deriving the Fourier expansion explicitly, this is done towards the end of this chapter).

Let

$$g|Tr_4^{4D} = \frac{1}{[\Gamma_0(4) : \Gamma_0(4D)]} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4D)\backslash\Gamma_0(4)} g\big|_{k+1/2} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

denote the trace operator (adjoint to the inclusion map under Petersson scalar product) and it maps $S_{k+1/2}(4D)$ into $S_{k+1/2}(4)$. We take

$$\left\{ \begin{pmatrix} 1 & 0 \\ 4D_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \Big| D = D_1 D_2 \text{ and } \mu \pmod{D_2} \right\}$$

as a set of representatives to define the above trace operator (see page 196 of [14]).

5.2 Statement of results

Let $\lambda_{D,k}$ be a constant such that $\Theta_D E_{k,4D,(\frac{D}{\cdot})}\big|Tr_4^{4D}Pr_+ - \lambda_{D,k}H_{k+1/2} \in S^+_{k+1/2}(4)$. Also let $k \geq 2$ be even. Then we have,

Theorem 5.2.1.

$$\sum_{0<D \text{ fund. disc.}} \left(\Theta_D E_{k,4D,(\frac{D}{\cdot})}\big|Tr_4^{4D}Pr_+ - \lambda_{D,k}H_{k+1/2}\right) = \frac{\Gamma(k-1/2)}{(4\pi)^{k-1/2}} \sum_{g\in\mathcal{B}^+} L(g,k-1/2)\frac{g}{\langle g,g\rangle},$$

where the sum in left hand side is taken over all the fundamental discriminants $D > 0$ and the sum in right hand side is taken over an orthogonal basis of Hecke eigenforms of $S^+_{k+1/2}(4)$.

In the simplest case where $k = 6$, we use $S^+_{13/2}(4) = \mathbb{C}\delta$ and $S_{12} = \mathbb{C}\Delta$ (where δ and Δ are as defined in page 177 of [14]) :

$$\Delta(z) = 8000G_4(z)^3 - 147G_6(z)^2$$

and

$$\delta(z) = \frac{60}{2\pi i}\big(2G_4(4z)\theta'(z) - G_4'(4z)\theta(z)\big),$$

where

$$G_k(z) := \frac{1}{2}\zeta(1-k) + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$$

and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. We have $\delta|\mathcal{S}_1 = \Delta$ (this can be seen by comparing the Fourier coefficient of δ and Δ), where $\mathcal{S}_1$ is the first Shimura-Kohnen lift from $S^{+}_{k+1/2}(4)$ into S_{2k} (the definition of Shimura map defined by Kohnen is given in page 176 of [14]; see section §5.4 and equation (5.3)). We have,

Corollary 5.2.2.

$$\sum_{0<D \text{ fund. disc.}} \left(\Theta_D E_{6,4D,\left(\frac{D}{\cdot}\right)} | Tr_4^{4D} Pr_+ - \lambda_{D,6} H_{13/2}\right) = \frac{\Gamma(11/2) L(\delta, 11/2)}{(4\pi)^{11/2}} \frac{\delta}{< \delta, \delta >}.$$

5.3 Proof of the theorem

Observe that, for $\begin{pmatrix} a & b \\ Dc & d \end{pmatrix} \in \Gamma_0(4D)$, $4|c$ and $2|k$ (hence $\left(\frac{-4}{d}\right)^k = 1$) one has

$$\left(\Theta_D\Big|_{1/2}\begin{pmatrix} a & b \\ Dc & d \end{pmatrix}\right)(z) := \left(\frac{Dc}{d}\right)\left(\frac{-4}{d}\right)^{1/2}(Dcz+d)^{-1/2}\Theta_D\left(\frac{az+b}{Dcz+d}\right)$$
$$= \left(\frac{D}{d}\right)\Theta_D(z).$$

That is,

$$\Theta_D(z) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{1/2}(Dcz+d)^{-1/2}\sum_{n\in\mathbb{Z}} e^{2\pi i D n^2 \frac{az+b}{Dcz+d}}$$
$$= \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-1/2}\sum_{n\in\mathbb{Z}} e^{2\pi i D n^2 \frac{az+b}{Dcz+d}}.$$

Multiplying with $\frac{1}{2}(cDz+d)^{-k}\left(\frac{D}{d}\right)$ on both sides, we get

$$\frac{1}{2}(cDz+d)^{-k}\left(\frac{D}{d}\right)\Theta_D(z)$$
$$= \frac{1}{2}(cDz+d)^{-k}\left(\frac{D}{d}\right)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-1/2}\sum_{n\in\mathbb{Z}} e^{2\pi i D n^2 \frac{az+b}{Dcz+d}}$$

Sum over (c,d)'s (with $\gcd(c,d)=1$ and $4|c$)

$$\sum_{\substack{(c,d)=1\\4|c}} \frac{1}{2}(cDz+d)^{-k}\left(\frac{D}{d}\right)\Theta_D(z)$$
$$= \sum_{\substack{(c,d)=1\\4|c}} \frac{1}{2}(cDz+d)^{-k}\left(\frac{D}{d}\right)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-1/2}\sum_{n\in\mathbb{Z}} e^{2\pi i D n^2 \frac{az+b}{Dcz+d}}$$

We get

$$\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)} = \frac{1}{2}\sum_{n\in\mathbb{Z}}\sum_{\substack{(c,d)=1\\4|c}}\left(\frac{Dc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-k-1/2}e^{2\pi i D n^2 \frac{az+b}{Dcz+d}}$$
$$= \frac{1}{2}\sum_{\substack{(c,d)=1\\4|c}}\left(\frac{Dc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-k-1/2}$$

$$+\sum_{n\geq 1}\sum_{\substack{(c,d)=1\\4|c}}\left(\frac{Dc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-k-1/2}e^{2\pi iDn^2\frac{az+b}{Dcz+d}}$$

$$= E_{k+1/2,4D}+\sum_{n\geq 1}P_{k+1/2,4D;n^2D}.$$

In the above,

$$E_{k+1/2,4D}=\frac{1}{2}\sum_{\substack{(c,d)=1\\4|c}}\left(\frac{Dc}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(Dcz+d)^{-k-1/2}$$

is an Eisenstein series of weight $k+1/2$ for $\Gamma_0(4D)$. For each $n\geq 1$, the Poincarè series

$$P_{k+1/2,4D;n}=\sum_{\substack{(c,d)=1\\4D\big|c}}\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{k+1/2}(cz+d)^{-k-1/2}e^{2\pi in\frac{az+b}{cz+d}}$$

is a cusp form of weight $k+1/2$ for $\Gamma_0(4D)$ and characterized by

$$\langle g,P_{k+1/2,4D;n}\rangle=\frac{1}{[\mathrm{SL}_2(\mathbb{Z}):\Gamma_0(4D)]}\frac{\Gamma(k-1/2)}{(4\pi n)^{k-1/2}}a_g(n),$$

where $g=\sum_{n\geq 1}a_g(n)e^{2\pi inz}$ is an arbitrary cusp form of weight $k+1/2$, level $4D$. Using straightforward computations we have noticed that, Tr_4^{4D} maps Poincarè series to Poincarè series and Eisenstein series to Eisenstein series. We have

$$P_{k+1/2,4D;n^2D}\Big|Tr_4^{4D}=P_{k+1/2,4;n^2D}\ ,$$

and

$$E_{k+1/2,4D}\big|Tr_4^{4D}Pr_+ = H_{k+1/2}\,,$$

where $H_{k+1/2}$ is the Cohen-Eisenstein series defined in preliminaries. To get this, let $g \in S_{k+1/2}$. Now,

$$\Big\langle E_{k+1/2,4D}\big|Tr_4^{4D}, g\Big\rangle = \big\langle E_{k+1/2,4D}\,, g|\iota\big\rangle = 0,$$

since $g|\iota$ is cusp form (here, ι is the inclusion map, adjoint to the trace map Tr_4^{4D}). This proves Tr_4^{4D} preserves the space of Eisenstein series, which is orthogonal complement of the space of cusp forms with respect to the Petersson scalar product. So, we get a constant $\lambda_{D,k}$ such that

$$\begin{aligned}\left(\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)}\right)\big|Tr_4^{4D}Pr_+ &= \lambda_{D,k}H_{k+1/2} + \sum_{n\geq 1} P^+_{k+1/2,4;n^2D}\\ &= \lambda_{D,k}H_{k+1/2} + \sum_{n\geq 1}\frac{\Gamma(k-1/2)}{i_4(4\pi n^2 D)^{k-1/2}}\sum_{g\in\mathcal{B}^+}\overline{a_g(n^2D)}\frac{g}{\langle g,g\rangle}\\ &= \lambda_{D,k}H_{k+1/2} + \frac{\Gamma(k-1/2)}{i_4(4\pi)^{k-1/2}}\sum_{g\in\mathcal{B}^+}\left(\sum_{n\geq 1}\frac{a_g(n^2D)}{(n^2D)^{k-1/2}}\right)\frac{g}{\langle g,g\rangle},\end{aligned} \tag{5.1}$$

where $i_4 = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4)]$. Now, by summing both sides over all the fundamental discriminants $D > 0$, we get

$$\sum_{\substack{D \text{ fund. disc.}\\ D>0}}\left(\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)}\Big|Tr_4^{4D}Pr_+ - \lambda_{D,k}H_{k+1/2}\right)$$

$$= \sum_{\substack{D \text{ fund. disc.} \\ D>0}} \sum_{g\in\mathcal{B}^+} \left(\frac{\Gamma(k-1/2)}{i_4(4\pi)^{k-1/2}} \sum_{n\geq 1} \frac{a_g(n^2D)}{(n^2D)^{k-1/2}} \right) \frac{g}{\langle g, g\rangle}. \tag{5.2}$$

Now we state the following result:

<u>*Claim:*</u> As D varies over all the positive fundamental discriminants and n^2 $(n \geq 1)$ varies over all squares, then n^2D varies over all integers $m \geq 1$ such that $m \equiv 0, 1 \pmod 4$ where k is even.

<u>*Proof of claim:*</u> Let $m \geq 1$ be an integer such that $m \equiv 0, 1 \pmod 4$. If m is an odd integer, split $m = Dn^2$ where $D \equiv 1 \pmod 4$ and square-free with $n \geq 1$ since $m \equiv 1 \pmod 4$. If m is an even integer then $4|m$. We write $m/4 = dn^2$ where d is a square-free integer and $n \geq 1$. If $d \equiv 1 \pmod 4$, replace d by D so that $m = D(2n)^2$. If $d \equiv 2, 3 \pmod 4$, let $D = 4d$ so that $m = Dn^2$. Since d is square-free integer, in all the cases, D is a fundamental discriminant. This proves the claim. □

Thus the above equation (5.2) becomes

$$\sum_{\substack{D \text{ fund. disc.} \\ D>0}} \left(\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)} \middle| Tr_4^{4D} Pr_+ - \lambda_{D,k} H_{k+1/2} \right)$$
$$= \frac{\Gamma(k-1/2)}{i_4(4\pi)^{k-1/2}} \sum_{g\in\mathcal{B}^+} \left(\sum_{\substack{m\geq 1 \\ m\equiv 0,1 \pmod 4}} \frac{a_g(m)}{m^{k-1/2}} \right) \frac{g}{\langle g, g\rangle}$$
$$= \frac{\Gamma(k-1/2)}{i_4(4\pi)^{k-1/2}} \sum_{g\in\mathcal{B}^+} L(g, k-1/2) \frac{g}{\langle g, g\rangle}.$$

Note that in (5.2), before summing over all positive fundamental discriminants, right hand side is a finite linear combination of absolutely convergent series. Each of the series in the finite sum becomes an absolute convergent series $\sum\limits_{\substack{m\geq 1\\ m\equiv 0,1 \pmod 4}} \frac{a_g(m)}{m^{k-1/2}}$ after taking the sum over all fundamental discriminant $D > 0$. $\square$

5.4 Applications

We derive a certain algebraic nature involving some $L-$ series associated with a Hecke eigenform f and Fourier coefficient of g. We now use multiplicity result for $S^{+}_{k+1/2}(4)$ and its relation with S_{2k} via Shimura correspondence (as in [14], pages 176-177). Let $f \in S_{2k}$ be the normalized Hecke eigenform which corresponds to g via the identity: If D is a positive fundamental discriminant, then there exists a unique (upto a scalar multiple) non zero Hecke eigenform $g \in S^{+}_{k+1/2}(4)$ such that the following holds:

$$a_g(n^2D) = a_g(D)\sum_{d|n} d^{k-1}\mu(d)\left(\frac{D}{d}\right)a_f(n/d). \tag{5.3}$$

The multiplicity result for $S^{+}_{k+1/2}(4)$ states that: if $f \in S_{2k}$ is a Hecke eigenform and $g \in S^{+}_{k+1/2}(4)$ is a corresponding eigenform for all $T^{+}_{p^2}$ (with eigenvalue λ_p) via above equation (5.3) such that $f|T_p = \lambda_p g$ for all primes p, then the eigenspace generated by g has dimension one ([11]). The corresponding basis element g via (5.3) can be chosen in such a way that its Fourier coefficients are real and algebraic (refer [14], page 177, lines 4-6). Now, equation (5.3) gives,

$$\sum_{n\geq 1}\frac{a_g(n^2D)}{(n^2D)^{k-1/2}} = \frac{a_g(D)}{D^{k-1/2}}\sum_{n\geq 1}\frac{\sum_{d|n} d^{k-1}\mu(d)\left(\frac{D}{d}\right)a_f(n/d)}{n^{2k-1}}$$

$$= \frac{a_g(D)}{D^{k-1/2}} \sum_{n\geq 1} \frac{n^{k-1}\mu(n)\left(\frac{D}{n}\right)}{n^{2k-1}} \sum_{m\geq 1} \frac{a_f(m)}{m^{2k-1}}$$
$$= \frac{a_g(D)}{D^{k-1/2}} \frac{L(f,2k-1)}{L(D,k)}.$$

Substituting this in equation (5.1), we get

$$\begin{aligned} &\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)} \big| Tr_4^{4D} Pr_+ \\ &= \lambda_{D,k} H_{k+1/2} + \frac{\Gamma(k-1/2)}{i_4(4\pi)^{k-1/2}} \sum_{g\in\mathcal{B}^+} \frac{a_g(D)}{(D)^{k-1/2}} \frac{L(f,2k-1)}{L(D,k)} \frac{g}{\langle g,g\rangle} \\ &= \lambda_{D,k} H_{k+1/2} + \frac{\Gamma(k-1/2)D^{-k+1/2}}{i_4(4\pi)^{k-1/2}L(D,k)} \sum_{g\in\mathcal{B}^+} \frac{a_g(D)L(f,2k-1)g}{\langle g,g\rangle}. \end{aligned} \tag{5.4}$$

Claim: $\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)} \big| Tr_4^{4D} Pr_+$ has rational Fourier coefficients.

Proof of claim: Let $\mathcal{G} = \Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)} \in M_{k+1/2}(4D)$. Note that, since Θ_D and $E_{k,4D,\left(\frac{D}{\cdot}\right)}$ have rational Fourier coefficient, so has $\mathcal{G}$ (the rationality of the Fourier coefficients of $E_{k,4D,\left(\frac{D}{\cdot}\right)}$ is obtained by explicitly deriving its Fourier series expansion, which is given towards the end of this chapter). The projection operator Pr_+ picks up the coefficients such that $(-1)^k n \equiv 0, 1 \pmod 4$, so it does not change the nature of Fourier coefficients. It is enough to prove that $\mathcal{G}|Tr_4^{4D}$ has rational Fourier coefficients. We take

$$\left\{ \begin{pmatrix} 1 & 0 \\ 4D_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \middle| D = D_1 D_2 \text{ and } \mu \pmod{D_2} \right\}$$

as a set of representatives for the action of trace operator. Note that,

$$\begin{pmatrix} 1 & 0 \\ 4D_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1/4D \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 4D_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}.$$

By considering the action of the above representative matrices individually, one by one (in this order) and by proceeding along the standard arguments (see calculations in the appendix of [14]), we conclude that the Fourier coefficients of $\mathcal{G}|Tr_4^{4D}$ are rational. □

Finally, we have the following

Proposition 5.4.1. *If $g \in S^+_{k+1/2}(4)$ is a Hecke eigenform and $f \in S_{2k}$ is the corresponding eigenform via (5.3), then $\frac{a_g(D)L(f,2k-1)}{\pi^{k-1}\langle g,g\rangle L(D,k)}$ is algebraic over rationals.*

Proof: Let $d' = \dim(M^+_{k+1/2}(4))$, and $\{g_0 = H_{k+1/2}, g_1, \ldots, g_{d'-1}\}$ be a basis of Hecke eigenforms for $M^+_{k+1/2}(4)$, whose Fourier coefficients are real and algebraic (coefficients of $H_{k+1/2}$ are rationals, refer [4]). Let $j_r \equiv 0, 1 \pmod 4$ where $0 \leq r \leq d' - 1$. Let $j_0 = 0$ and $e'_{j_0} = H_{k+1/2}$ so that $a_{e'_{j_0}}(j_0) = 1$. Let us consider the j_1^{th} Poincarè series P_{j_1} in $S^+_{k+1/2}(4)$ which is not identically zero. Let $e'_{j_1} = \frac{P_{j_1}}{a_{P_{j_1}}(j_1)}$. Then $e'_{j_1} \in S^+_{k+1/2}(4)$ with $a_{e'_{j_1}}(j_1) = 1$ and $a_{e'_{j_1}}(j_0) = 0$. We pick e'_{j_2} from the orthogonal complement of span of $\{e'_{j_0}, e'_{j_1}\}$. To select this, we find a non-zero Poincarè series indexed by j_2 in this orthogonal complement and in the plus space. We consider the Poincarè series $P_{j_2} \in S^+_{k+1/2}(4)$ and take its projection inside the orthogonal complement of span of $\{e'_{j_0}, e'_{j_1}\}$. Denote this projection by P'_{j_2} ($P_{j_2} = P'_{j_2} \oplus P''_{j_2}$, where P''_{j_2} is in the linear span of $\{e'_{j_1}\}$). Let $e'_{j_2} = \frac{P'_{j_2}}{a_{P'_{j_2}}(j_2)}$. Then e'_{j_2} is orthogonal to both the forms e'_{j_0} and e'_{j_1} with $a_{e'_{j_2}}(j_2) = 1$ and $a_{e'_{j_2}}(j_0) = a_{e'_{j_2}}(j_1) = 0$. By proceeding in this way we have obtained a basis $\{e'_{j_0}, \ldots, e'_{j_r}, \ldots, e'_{j_{d'-1}}\}$ for the space $M^+_{k+1/2}(4)$, where $a_{e'_{j_r}}(j_r) = 1$ and $a_{e'_{j_r}}(j_i) = 0$ for $i = 0, 1, \ldots, r-1$.

Using this basis, we get another basis $\{e_{j_0}, e_{j_1}, \ldots, e_{j_{d'-1}}\}$ of $M^+_{k+1/2}(4)$ such that $a_{e_{j_r}}(j_i) = 1$ if $i = r$ and 0 otherwise, where $0 \le i, r \le d' - 1$. A direct computation gives this set of basis from the constructed basis $\{e'_{j_0}, \ldots, e_{j'_{d'-1}}\}$ (for a detailed one such proof, we refer to the proof of lemma 4.1 in [23]).

Now, consider the system of linear equations formed by writing $\{g_0, \ldots, g_{d'-1}\}$ in terms of the new basis $\{e_{j_0}, \ldots, e_{j_{d'-1}}\}$. We note that in this system of equations, both $\{g_0, \ldots, g_{d'-1}\}$ and $\{e_{j_0}, \ldots, e_{j_{d'-1}}\}$ are bases for $M^+_{k+1/2}(4)$. Hence the corresponding matrix $(a_{g_i}(j_r))_{d' \times d'}$ is invertible.

Now, equation (5.4) gives,

$$\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)}\big|Tr_4^{4D} Pr_+ = \lambda_{D,k} H_{k+1/2} + \frac{\Gamma(k-1/2)D^{-k+1/2}}{i_4(4\pi)^{k-1/2}L(D,k)} \sum_{g \in \mathcal{B}^+} \frac{a_g(D)L(f,2k-1)g}{\langle g, g\rangle}.$$

By comparing the consecutive $j_0, j_1, \ldots, j_{d'-1}$ Fourier coefficients from the above, we get a system of equations:

$$\big(a_{g_i}(j_r)\big)_{d' \times d'} X = Y.$$

Entries of Y are Fourier coefficients of $\Theta_D E_{k,4D,\left(\frac{D}{\cdot}\right)}\big|Tr_4^{4D} Pr_+$, which are rationals. Entries of X are

$$\left\{\lambda_{D,k}\,, \frac{\Gamma(k-1/2)D^{-k+1/2}}{i_4(4\pi)^{k-1/2}L(D,k)} \frac{a_{g_1}(D)L(f_1,2k-1)}{\langle g_1, g_1\rangle}, \ldots, \right.$$

$$\frac{\Gamma(k-1/2)D^{-k+1/2}}{i_4(4\pi)^{k-1/2}L(D,k)}\frac{a_{g_{d'-1}}(D)L(f_{d'-1},2k-1)}{\langle g_{d'-1},g_{d'-1}\rangle}\Bigg\},$$

where $\{f_1,\ldots,f_{d'-1}\}$ are normalized cuspidal Hecke eigenforms corresponding to the eigenforms $\{g_1,\ldots,g_{d'-1}\}$ respectively via Shimura correspondence as given by equation (5.3). If $K_g=\mathbb{Q}(a_{g_i}(j_r))_{0\le i,j\le d'-1}$, the number field, then entries of X are in K_g. Thus we get that the entries of X are algebraic numbers. Hence, $\frac{a_g(D)L(f,2k-1)}{\pi^{k-1}\langle g,g\rangle L(D,k)}$ is algebraic. □

Fourier expansion of the Eisenstein series $E_{k,4D,\left(\frac{D}{\cdot}\right)}$:

we derive the Fourier series expansion of the Eisenstein series $E_{k,4D,\left(\frac{D}{\cdot}\right)}$, and obtain that they are rational numbers. In the following computation, we have assumed that D is a positive fundamental discriminant. Also $k\ge 2$ is even, hence $\left(\frac{-4}{d}\right)^{k+1/2}=1$.

$$\begin{aligned}
&E_{k,4D,\left(\frac{D}{\cdot}\right)}\\
&\quad=\frac{1}{2}\sum_{\substack{(c,d)=1\\4|c}}\left(\frac{D}{d}\right)(cDz+d)^{-k}\\
&\quad=\frac{1}{2}\sum_{(4c,d)=1}\left(\frac{D}{d}\right)(4cDz+d)^{-k}\\
&\quad=\frac{1}{2}\sum_{d=1,-1}\left(\frac{D}{d}\right)(d)^{-k}+\frac{1}{2}\sum_{c\neq 0}\sum_{\substack{d\in\mathbb{Z}\\(4c,d)=1}}\left(\frac{D}{d}\right)(4cDz+d)^{-k}\\
&\quad=\frac{1}{2}\left(\left(\frac{D}{1}\right)(1)^{-k}+\left(\frac{D}{-1}\right)(-1)^{-k}\right)+\sum_{c\ge 1}\sum_{\substack{d\in\mathbb{Z}\\(4c,d)=1}}\left(\frac{D}{d}\right)(4cDz+d)^{-k}\\
&\quad=1+\sum_{c\ge 1}\sum_{d\in\mathbb{Z}}\left(\frac{D}{d}\right)\sum_{\substack{\delta|4c\\\delta|d}}\mu(\delta)(4cDz+d)^{-k}
\end{aligned}$$

$$= 1 + \sum_{c\geq 1}\sum_{\delta|4c}\mu(\delta)\sum_{d\in\mathbb{Z}}\left(\frac{D}{d\delta}\right)(4cDz + d\delta)^{-k}$$

$$= 1 + \sum_{c\geq 1}\sum_{\delta|4c}\frac{\mu(\delta)}{\delta^k}\sum_{d\in\mathbb{Z}}\left(\frac{D}{d\delta}\right)(\frac{4cD}{\delta}z + d)^{-k}$$

$d \mapsto Dn + d'$

$$= 1 + \frac{1}{D^k}\sum_{c\geq 1}\sum_{\delta|4c}\frac{\mu(\delta)}{\delta^k}\left(\frac{D}{\delta}\right)\sum_{d' \pmod D}\left(\frac{D}{d'}\right)\sum_{n\in\mathbb{Z}}(\frac{\frac{4cD}{\delta}z + d'}{D} + n)^{-k}$$

$$= 1 + \frac{(-2\pi i)^k}{D^k(k-1)!}\sum_{n\geq 1}n^{k-1}\sum_{c\geq 1}\sum_{\delta|4c}\frac{\mu(\delta)}{\delta^k}\left(\frac{D}{\delta}\right)\sum_{d \pmod D}\left(\frac{D}{d}\right)e^{2\pi in(\frac{\frac{4cD}{\delta}z+d'}{D}+n)}$$

$$= 1 + \frac{(-2\pi i)^k}{D^k(k-1)!}\sum_{n\geq 1}n^{k-1}\sum_{c\geq 1}\sum_{\delta|4c}\frac{\mu(\delta)}{\delta^k}\left(\frac{D}{\delta}\right)\sum_{d \pmod D}\left(\frac{D}{d}\right)e^{2\pi in\frac{d}{D}}e^{2\pi in\frac{4c}{\delta}z}$$

$$= 1 + \frac{(-2\pi i)^k\sqrt{D}}{D^k(k-1)!}\sum_{n\geq 1}n^{k-1}\left(\frac{D}{n}\right)\sum_{c\geq 1}\sum_{\delta|4c}\left(\frac{D}{\delta}\right)\frac{\mu(\delta)}{\delta^k}e^{2\pi in\frac{4c}{\delta}z}$$

$$= 1 + \frac{(-2\pi i)^k}{D^{k-1/2}(k-1)!}\sum_{\substack{n\geq 1\\ c\geq 1}}n^{k-1}\left(\frac{D}{n}\right)\sum_{\delta|4c}\frac{\mu(4c/\delta)}{(4c/\delta)^k}\left(\frac{D}{4c/\delta}\right)e^{2\pi in\delta z}$$

$$= 1 + \frac{(-2\pi i)^k}{D^{k-1/2}(k-1)!}\sum_{\substack{m\geq 1\\ c\geq 1}}\sum_{\substack{\delta|m\\ \delta|4c}}(m/\delta)^{k-1}\left(\frac{D}{m/\delta}\right)\frac{\mu(4c/\delta)}{(4c/\delta)^k}\left(\frac{D}{4c/\delta}\right)e^{2\pi imz}$$

$$= 1 + \left(\frac{(-2\pi i)^k}{D^{k-1/2}(k-1)!}\sum_{\substack{m\geq 1\\ c\geq 1}}\sum_{\substack{\delta|m/4\\ \delta|c}}(m/4\delta)^{k-1}\left(\frac{D}{m/4\delta}\right)\frac{\mu(c/\delta)}{(c/\delta)^k}\left(\frac{D}{c/\delta}\right)e^{2\pi imz}\right)$$

$$- \left(\frac{(-2\pi i)^k 2^{-k}\left(\frac{D}{2}\right)}{D^{k-1/2}(k-1)!}\sum_{\substack{m\geq 1\\ c\geq 1}}\sum_{\substack{\delta|m/2\\ \delta|c,\delta \text{ odd}}}(m/2\delta)^{k-1}\left(\frac{D}{m/2\delta}\right)\frac{\mu(c/\delta)}{(c/\delta)^k}\left(\frac{D}{c/\delta}\right)e^{2\pi imcz}\right)$$

$$= 1 + \left(\frac{(-2\pi i)^k}{D^{k-1/2}(k-1)!}\sum_{n\geq 1}\sum_{c\geq 1}\sum_{\delta|n}(n/\delta)^{k-1}\left(\frac{D}{n/\delta}\right)\frac{\mu(c)\left(\frac{D}{c/\delta}\right)}{c^k}e^{2\pi in4z}\right)$$

$$- \left(\frac{(-2\pi i)^k 2^{-k} \left(\frac{D}{2}\right)}{D^{k-1/2}(k-1)!} \sum_{n\geq 1} \sum_{c\geq 1} \sum_{\substack{\delta | n \\ \delta \text{ odd}}} (n/\delta)^{k-1} \left(\frac{D}{n/\delta}\right) \frac{\mu(c) \left(\frac{D}{c/\delta}\right)}{c^k} e^{2\pi i n 2z} \right)$$

$$= 1 + \left(\frac{(-2\pi i)^k}{D^{k-1/2}\Gamma(k)L(D,k)} \sum_{n\geq 1} \sum_{\delta | n} \delta^{k-1} \left(\frac{D}{\delta}\right) e^{2\pi i n 4z} \right)$$

$$- \left(\frac{(-2\pi i)^k 2^{-k} \left(\frac{D}{2}\right)}{D^{k-1/2}\Gamma(k)L(D,k)} \sum_{n\geq 1} \sum_{\substack{\delta | n \\ \delta \text{ odd}}} \delta^{k-1} \left(\frac{D}{\delta}\right) e^{2\pi i n 2z} \right)$$

$$= 1 + \frac{2}{L(D,1-k)} \sum_{n\geq 1} \sigma_{k-1,D}(n) e^{2\pi i n 4z} - \frac{2\left(\frac{D}{2}\right)}{L(D,1-k)2^k} \sum_{n\geq 1} \sigma^*_{k-1,D}(n) e^{2\pi i n 2z},$$

where $\sigma_{k-1,D}(n) = \sum_{d|n} \left(\frac{D}{d}\right) d^{k-1}$ and $\sigma^*_{k-1,D}(n) = \sum_{\substack{d|n \\ d \text{ odd}}} \left(\frac{D}{d}\right) d^{k-1}$. We have used functional equation for Dirichlet L function (notations as given in [2] theorem 12.11)):

$$L(D,1-k) = \frac{D^{k-1}\Gamma(k)}{(2\pi)^k} \left(e^{\frac{-\pi i k}{2}} + \left(\frac{D}{-1}\right) e^{\frac{\pi i k}{2}} \right) G(1,D) L(D,k)$$
$$= 2.\frac{D^{k-1}\Gamma(k)}{(-2\pi i)^k} \sqrt{D} L(D,k),$$

or,

$$\frac{(-2\pi i)^k}{D^{k-1/2}\Gamma(k)L(D,k)} = \frac{2}{L(D,1-k)}.$$

Since $L(D,1-k)/\zeta(1-2k) \in \mathbb{Q}$ and $\zeta(1-2k) \in \mathbb{Q}$, we have $L(D,1-k) \in \mathbb{Q}$ and hence the coefficients of the defined Eisenstein series are rational.

Chapter 6

Conclusions and Recommendations

In this thesis, we set up a generalized version of the Saito-Kurokawa isomorphism. Specifically, if f is a normalised newform of weight $2k-2$, level $M/2$ with character χ^2, we derived that the form f is lifted into two linearly independent Hecke eigenforms $F, F|B_S(4)$ in the Maass space of degree two Siegel modular forms of weight k, level M with character χ under the Saito-Kurokawa correspondence. Moreover, the relevant Saito-Kurokawa isomorphism maps the space of newforms $S^{new}_{2k-2}(M/2, \chi^2)$ into the space of newforms $S^{*,new}_k(\Gamma^2_0(M), \chi)$ in the Maass space. Here, $32|M$, $cond(\chi) = M$ and $cond(\chi^2) = M/2$.

In another problem, we derived a non-cusp form of half-integral weight $k + 1/2$ (k even) for $\Gamma_0(4)$ in the Kohnen plus space whose Petersson scalar product with a cuspidal Hecke eigenform g is equal to a constant times the L value $L(g, k - 1/2)$.

Future works:

We mention few problems for future work :

- Obtain a formula relating the Petersson inner products $\langle f, f\rangle$ and $\langle F, F\rangle$ where $f \in S_{2k-2}(M, \chi^2)$ and $F \in \mathcal{S}_k^*(\Gamma_0^2(M), \chi)$ where $32|M, cond(\chi) = M$ and $cond(\chi^2) = M/2$.
- Develop the theory of newforms for the spaces $S^+_{k-1/2}(8N)$ and $S^+_{k-1/2}(16N)$ where N is an odd and square-free integer.
- Obtain Saito-Kurokawa isomorphisms using the theory of newforms developed in the above cases.

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