C 23362 (Pages : 2) Name.....................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2022

(CBCSS)

Mathematics with Data Science

MTD 2C 10—REGRESSION TECHNIQUES AND TIME SERIES ANALYSIS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question carries 1 *weightage.*

1. State the assumptions in the general linear model.
2. Define a PRESS statistic. What is it used for ?
3. What is the effect of autocorrelated errors in a linear model and how would you approach them to rectify it ?
4. Define a stationary time series and a white noise process.
5. What is the difference between the Holt linear trend method and the Holt-Winters method ?
6. Explain the auto correlation function and its role in the analysis of time series.
7. When do you say that a linear time series is invertible ? Illustrate with an example.
8. Define an ARIMA (p, d, q) process and explain a method of removing its stationarity.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing two questions from each unit.*
Each question carries 2 weightage.

Unit I

9. Describe the role of residuals in statistical model checking.

10. Given the general linear model $y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_{p-1} x_{ip-1} + \epsilon_i,\ i = 1, 2, \ldots, n.$ obtain a test statistic for $H : \beta_j = c$.

11. Describe weighted least squares with known weights for the straight-line model.

Unit II

12. Prove or disprove the following process is covariance stationary : $X_t = U\sin(2\pi t) + V\cos(2\pi t)$, where U and V are independent random variables each with mean zero and variance one.

13. Explain the forecasting technique in the multiplicative seasonal model through Winters' method

14. Describe first order smoothing of a time series giving illustrative examples.

Unit III

15. Define an MA (q) process and obtain the conditions for its invertibility. Find the ACF of an MA (q) process

16. Explain how you forecast future observations for an AR (p) model.

17. Time series can be viewed as a realization of a stochastic process. Do you agree. Establish your claim.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries 5 weightage.

18. Let $Y = X\beta + \epsilon$ be a general linear model with $\epsilon \sim N(0, \sigma^2 I)$ and X be a matrix of full rank. Obtain the maximum likelihood estimate of β. Also discuss the equivalence of least square estimate of β with m.l.e of β.

19. (a) Define covariance stationary time series and give one example.

    (b) Describe second order smoothing of a time series giving illustrative examples.

20. Explain the methods of estimation of model parameters of an ARIMA.

21. Derive the Yule-Walker equations for an AR (p) model. Hence obtain the kth order auto correlation function of an AR (1) model.

(2 × 5 = 10 weightage)

C 23361 (Pages : 3) Name.......................................

Reg. No...................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2022

(CBCSS)

Mathematics with Data Science

MTD 2C 09—TOPOLOGY

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
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4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Define nearness relation on a set and give an example.
2. Let $\mathfrak{B} = \{[a,b) : a < b, a, b \in \mathbb{R}\} \cup \{(c,d] : a < b,\ a, b \in \mathbb{R}\}$. Is $\mathfrak{B}$ a sub basis for some topology on $\mathbb{R}$? If yes, what is that topology ? If no, why ?
3. Is boundedness a topological property ? Explain.
4. Show that the topological product of two connected spaces is connected.
5. Define totally disconnected space and illustrate with an example.
6. Define *dispersion point* and give an example.
7. Show that a space is T_1 if and only if singleton subsets are all closed.
8. Prove or disprove that the real line $\mathbb{R}$ with cofinite topology is a T_2 space.

(8 × 1 = 8 weightage)

Turn over

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Let $\{x_n\}$ be a sequence in a metric space (X, d). Show that $\{x_n\}$ converges to some $x \in X$ if and only if for every open set U containing x, there exists a positive integer N such that $x_n \in U \forall n \geq N$.

10. Consider the two semi-open interval topologies on $\mathbb{R}$. Prove that their meet is the usual topology while their join is the discrete topology on $\mathbb{R}$.

11. For a subset A of a topological space, show that $\bar{A} = A \cup A'$.

Module II

12. Show that the composite of two quotient maps is a quotient map.

13. Show that union of two connected sets with non-empty intersection is connected.

14. Show that every quotient space of a locally connected space is locally connected.

Module III

15. Show that regularity is a hereditary property.

16. Show that every compact T_2 space is a T_3 space.

17. Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then show that X is normal.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (a) Let $(X, \mathfrak{T})$) be a topological space the let $\mathcal{S}$ be a family of subsets of X. Then show that $\mathcal{S}$ is a sub base for $\mathfrak{T}$ if and only if $\mathcal{S}$ generates $\mathfrak{T}$.

    (b) For any subset A of a space X, show that $int(A) = X - \overline{(X - A)}$.

19. (a) Show that every continuous real valued function on a compact space is bounded and attains its extrema.

    (b) Define components. Also show that components are closed sets and any two components are mutually disjoint.

20. Let A be a closed subset of a normal space X and suppose $f : A \to [-1,1]$ is a continuous function. Then show that there exists a continuous function $F : X \to [-1,1]$ such that $F(x) = f(x)$ for all $x \in A$.

21. (a) Show that a topological space X is regular if and only if the family of all closed neighbourhoods of any point of X forms a local base at that point.

(b) Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta(X) = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$ in the product topology.

(2 × 5 = 10 weightage)

C 23360 (Pages : 5) Name.......................................

Reg. No....................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR/SUPPLEMENTARY) EXAMINATION, APRIL 2022

(CBCSS)

Mathematics with Data Science

MTD 2C 08—DIFFERENTIAL EQUATIONS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
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4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Find the singular points of the differential equation $(x-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\frac{1}{x}y=0$.

2. Define a solution of the differential equation $\frac{d^ny}{dx^n}=f\left[x, y, \frac{dy}{dx},\ldots\ldots, \frac{d^{n-1}y}{dx^{n-1}}\right]$,

   where f is a continuous real function defined in a domain D of real $(n+1)$-dimensional $\left(x, y, y',\ldots., y^{(n-1)}\right)$-space.

Turn over

3. Describe Bessel's equation of order zero.

4. Define Saddle point.

5. Verify whether E defined by $E(x, y) = -x^2 - y^2$ is negative definite in every domain D containing $(0, 0)$.

6. Define positive definite function.

7. Define singular integral of a first order partial differential equation $f(x, y, z, p, q) = 0$.

8. Find the type of the second order partial differential equation $u_{xx} - x^2 u_{yy} = 0$.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question has weightage 2.

Unit I

9. Locate and classify the singular points of the differential equation $\left(x^2 - 3x\right)\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + y = 0$.

10. Let the co-efficients a_{ij} and the functions $F_i \ (i, j = 1, 2, \ldots\ldots, n)$ in the linear system

$$\frac{dy_1}{dx} = a_{11}(x) y_1 + a_{12}(x) y_2 + \ldots\ldots + a_{1n}(x) y_n + F_1(x),$$
$$\frac{dy_2}{dx} = a_{21}(x) y_1 + a_{22}(x) y_2 + \ldots\ldots + a_{2n}(x) y_n + F_2(x),$$
$$\frac{dy_n}{dx} = a_{n1}(x) y_1 + a_{n2}(x) y_2 + \ldots\ldots + a_{nn}(x) y_n + F_n(x),$$

be continuous on the real interval $a \le x \le b$. Also, let x_0 be a point of the interval $a \le x \le b$, and let $c_1, c_2, \ldots, c_n$ be a set of real constants. Then prove that there exists a unique solution

$$\phi_1, \phi_2, \ldots\ldots, \phi_n$$

of the above system such that

$$\phi_1(x_0) = c_1, \phi_2(x_0) = c_2, \ldots\ldots, \phi_n(x_0) = c_n$$

and this solution is defined on the entire interval $a \le x \le b$.

11. Does there exist a unique solution ϕ of the third-order differential equation

$$\frac{d^3y}{dx^3} = x^2 + y\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2$$

such that

$$\phi(0) = 1, \phi'(0) = -3, \phi''(0) = 0 ?$$

Explain precisely why or why not.

UNIT II

12. Determine the nature of the critical point $(0, 0)$ of the system

$$\frac{dx}{dt} = 3x + 2y,$$

$$\frac{dy}{dt} = x + 2y$$

and determine whether or not the point is stable.

13. Let

$$m\frac{d^2x}{dt^2} = F(x),$$

where F is analytic for all values of x, be the differential equation of a conservative dynamical system. Consider the equivalent autonomous system

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = \frac{F(x)}{m},$$

and let $(x_c, 0)$ be a critical point of this system. Let V be the potential energy function of the dynamical system having differential equation $m\frac{d^2x}{dt^2} = F(x)$.

Then prove that if the potential energy function has a relative maximum at $x = x_c$, then the critical point $(x_c, 0)$ is a saddle point and is unstable.

14. Find the general solution of the non-linear system

$$\frac{dx}{dt} = x + 4y - x^2,$$

$$\frac{dy}{dt} = 6x - y + 2xy.$$

UNIT III

15. Eliminate the arbitrary function F from the following equations and find the corresponding partial differential equation :

(i) $F(xy, x + y - z) = 0$; and

(ii) $z = F\left(\frac{xy}{z}\right)$.

16. Write the general form of a second order semi-linear partial differential equation. Based on different conditions, explain the three classifications of second order semi-linear partial differential equation with one example to each.

17. Prove that the solution of the Neumann problem is unique upto the addition of a constant.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question has weightage 5.

18. Let f be a continuous function defined on a domain D of the xy-plane and ϕ be a continuous function defined on a real interval $\alpha \le x \le \beta$ and such that $[x, \phi(x)] \in D$ for all $x \in [\alpha, \beta]$. Also let x_0 be any real number such that $\alpha < x_0 < \beta$. Then prove that ϕ is a solution of the differential equation $\frac{dy}{dx} = f(x, y)$ on $[\alpha, \beta]$ and is such that $\phi(x_0) = y_0$ if and only if ϕ satisfies the integral equation

$$\phi(x) = y_0 + \int_{x_0}^{x} [f(t, \phi(t))]\, dt$$

for all $x \in [\alpha, \beta]$.

19. Find a power series solution of

$$\left(x^2-1\right)\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+xy=0$$

in powers of x.

20. Assume that the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2-(a+d)\lambda+(ad-bc)=0$$

are real and equal. Then prove that the critical point $(0,0)$ of the linear system

$$\frac{dx}{dt}=ax+by,$$

$$\frac{dy}{dt}=cx+dy,$$

where a, b, c and d are real constants, is a node.

21. (a) Prove that the general solution of the quasi-linear equation

$$P(x,y,z)\,p+Q(x,y,z)\,q=R(x,y,z),$$

where P, Q and R are continuously differentiable functions of x, y and z is

$$F(u,v)=0,$$

where F is an arbitrary differentiable function of u and v and

$u(x,y,z)=c_1$ and $v(x,y,z)=c_2$,

are two independent solutions of the system

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}.$$

(b) Reduce the equation

$$y^2u_{xx}-2xyu_{xy}+x^2u_{yy}=\frac{y^2}{x}u_x+\frac{x^2}{y}u_y$$

to a canonical form and solve it.

(2 × 5 = 10 weightage)

C 23359 (Pages : 3) Name..

Reg. No.....................................

SECOND SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY] EXAMINATION, APRIL 2022

(CBCSS)

Mathematics with Data Science

MTD 2C 07—NUMBER THEORY

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section / Part shall remain the same.*
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4. *There will be an overall ceiling for each Section / Part that is equivalent to the maximum weightage of the Section / Part.*

Section A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Prove that $\phi(n) > \frac{n}{6}$ for all n with atmost 8 distinct prime factors.
2. Prove that $d(n)$, the number of positive divisors of n, is odd if and only if n is a square.
3. State the Euler's summation formula.
4. For $x \geq 2, \theta(x)$ denote the Chebyshev's $\theta -$function and $\pi(x)$ denote the number of primes not exceeding x,then Prove that :

$$\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t} dt.$$

5. Prove that for every $n > 1$ there exist n consecutive composite numbers.

Turn over

6. Determine whether – 104 is a quadratic residue or non-residue of the prime 997.

7. Determine those odd primes p for which $(-3 \mid p) = 1$ and those for which $(-3 \mid p) = -1$.

8. How many different shift transformations are there with an N-letter alphabet.

(8 × 1 = 8 weightage)

Section B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. If f is an arithmetical function with $f(1) \neq 0$, show that there exists a unique arithmetical function f^{-1} such that $f * f^{-1} = f^{-1} * f = I$. Also obtain a recursive formula for f^{-1}.

10. Let f be a multiplicative function. Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n) f(n)$ for all n.

11. State and prove the Legendre's identity.

Unit II

12. State and prove Abel's identity.

13. For $n \geq 1$, prove that the n^{th} prime P_n satisfies the inequalities :

$$\frac{1}{6} n \log n < P_n < 12\left(n \log n + n \log \frac{12}{e}\right).$$

14. Prove that, there is a constant A such that :

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + A + O\left(\frac{1}{\log x}\right) \text{ for all } x \geq 2.$$

Unit III

15. State and prove Euler's criterion.

16. Prove that the Legendre's symbol is a completely multiplicative function of n.

17. Prove that the Diophantine equation $y^2 = x^3 + k$ has no solutions if k has the form $k = (4n-1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 \pmod 4$ divides m.

(6 × 2 = 12 weightage)

Section C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. If $x \geq 1$,

(a) Prove that $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O\left(x^{-s}\right), s > 0, s \neq 1,$ where $\zeta(s)$ denotes the Riemann zeta function and O denotes the big oh notation.

(b) $\sum_{n \leq x} n^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1} + O(x^{\alpha})$ if $\alpha \geq 0$.

19. For every integer $n \geq 2$, prove that :

$$\frac{1}{6} \frac{n}{\log n} < \pi(n) < 6 \frac{n}{\log n}$$

where $\pi(n)$ denote the number of primes not exceeding n.

20. (a) State and prove quadratic reciprocity law.

(b) Determine whether 219 is a quadratic residue or non-residue mod 383.

21. Suppose you know that your adversary is using an enciphering matrix A in the 26 letter alphabet. You intercept the ciphertext "WKNCCHSSJH" and you know that the first word is "GIVE". Find the deciphering matrix A^{-1} and read the message.

(2 × 5 = 10 weightage)

C 23358 (Pages : 3) Name.......................................

Reg. No.....................................

SECOND SEMESTER M.Sc. DEGREE [REGULAR/SUPPLEMENTARY] EXAMINATION, APRIL 2022

(CBCSS)

Mathematics with Data Science

MTD 2C 06—DISCRETE MATHEMATICS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
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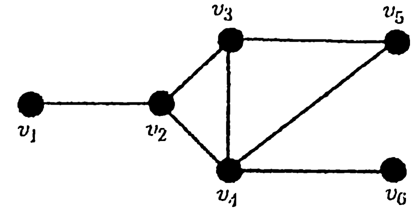
Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Give an example of a pair of non-isomorphic graphs having same degree sequence.
2. Show that similarity is an equivalence relation on the vertex set of a graph G.
3. Give any two examples of self-complementary graphs.
4. If $\delta(G) \geq 2$, then prove that G contains a cycle.
5. Does there exists an Euler graph with even number of vertices and odd number of edges ? Justify.
6. Find a minimal vertex cut of cardinality two for the following graph.



Turn over

7. If u and v are two strings, prove that $|uv| = |u| + |v|$.

8. Define grammar.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions by choosing two questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. How many non-isomorphic graphs of order 4 exist ? Sketch them.

10. Show that the rank over GF(2) of the incidence matrix of a graph G is $n - e$.

11. Sketch Peterson graph. Show that Peterson graph has girth five and circumference nine.

Unit II

12. State and prove Euler formula for planar graphs.

13. Prove that every connected graph contains a spanning tree.

14. Show that the complement of a simple planar graph with 11 vertices is non-planar.

Unit III

15. Let $\Sigma = \{a, b\}$ and let $n_a(w)$ and $n_b(w)$ denote the number of a's and b's in the string w, respectively. Find a grammar which generates the language $L = \{w: n_a(w) = n_b(w)\}$.

16. What do you mean by accepter and transducer ?

17. Define trap stage. Explain with an example.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5.

18. (i) What are self-complementary graphs ? Give an example.

    (ii) Prove that the order of a self-complementary graph is either a multiple of four or a multiple of four plus one.

19. State and prove Brooks' theorem.

20. Let G and G′ be simple connected graphs with isomorphic line graphs. Then G and G′ are isomorphic unless one of them is $K_{1,3}$ and the other is K_3.

21. Find grammars for $\Sigma = \{a, b\}$ that generates the sets of :

    (a) All strings with exactly one a

    (b) All strings with at least one a

    (c) All strings with no more than three a's.

    In each case, prove that the grammar obtained generates the indicate language.

(2 × 5 = 10 weightage)

C 5597 (Pages : 2) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. (CBCSS) REGULAR DEGREE EXAMINATION, APRIL 2021

Mathematics with Data Science

MTD 2C 10—REGRESSION TECHNIQUES AND TIME SERIES ANALYSIS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
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4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. What is variance inflation factor ? How is it used to detect multicollinearity ?
2. What are outliers ? How to detect outliers ?
3. What are studentized residuals ? What are their roles in regression analysis ?
4. Distinguish between stationary processes in the strict sense and weak sense.
5. Explain the technique of double exponential smoothing.
6. What are the roles of partial autocorrelation function in time series analysis ?
7. Describe a method of estimation of parameters of an AR(1) model.
8. Obtain auto correlation function of an MA (2) process.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Obtain the standard error of the least square estimators of parameters of the simple linear regression model.

10. Let $Y_i = \beta_0 + \beta_1 x_{i1} + \in_i, (i = 1, 2, \ldots, n)$ where the $\in_i$ are independence $N(0, \sigma^2)$. Obtain the confidence intervals on β_0 and β_1.

11. Describe the Cochrane-Orcutt iterative procedure for a simple linear regression model with first order auto correlated errors.

Unit II

12. Let $\{X_n\}$ be sequence of white noise with zero mean and constant variance σ^2. Let $S_n = \sum_{i=1}^{n} X_i$. Examine whether $\{S_n\}$ is covariance stationary.

13. Explain the forecasting technique in the additive seasonal model through Winters' method.

14. Obtain the estimate of variance of forecast errors for a linear trend model.

Unit III

15. Define a ARMA (p, q) process and obtain conditions for its stationarity in terms of the characteristic roots.

16. Derive the stationarity conditions for an AR(p) process. What is the condition for invertibility ?

17. Derivet he PACF of an ARIMA (p, q) process. How are they useful in the order determination of a stationary time series data ?

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. Describe the discounted least square method for a simple linear regression model.

19. Explain moving average smoothing and Holt-Winter smoothing. When will you use these methods ?

20. Define minimum mean square error forecast for a stationary time series. Show that an MMSE forecast for a stationary linear process is the conditional expectation.

21. Explain the duality between AR and MA processes. For the AR(1) and MA(1) processes, derive auto correlation and spectrum.

(2 × 5 = 10 weightage)

C 5596 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. DEGREE [REGULAR] EXAMINATION APRIL 2021

(CBCSS)

Mathematics with Data Science

MTD 2C 09—TOPOLOGY

(2020 Admission onwards)

Time : Three Hours Maximum: 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Define Sierpinski space. Is it metrizable ? Explain.
2. Give examples of hereditary, weakly hereditary and non-hereditary topological properties.
3. If X is connected, show that the only clopen subsets are X and ϕ.
4. Give an example of a Lindeloff space which is not compact. Explain.
5. Define mutually separated sets and explain with an example.
6. Define path connected space and explain with an example.
7. Show that compact subset of a Hausdorff space is closed.
8. Define Regular, completely regular and normal spaces.

($8 \times 1 = 8$ weightage)

Part B

Answer any **two** *questions from each module.*
Each question carries a weightage 2.

Module I

9. Define hereditary property and show that second countability is a hereditary property.

10. Let $\mathbb{R}_l$ be the real numbers with semi-open interval topology (lower limit topology) and let $X = \mathbb{R}_l \times \mathbb{R}_l$ be its topological product. Let $Y = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. What is the relative topology on Y ? Explain.

11. Let $\mathbb{R}$ be given in the cofinite topology. Let A be the set of all integers. Find the closure of A in $\mathbb{R}$.

Module II

12. Define Countable chain condition and show that every second countable space satisfies the countable chain condition.

13. Show that the product topology is the weak topology determined by the projection maps.

14. A subset C is a path component of a space X if and only if it is a maximal path connected subset of X.

Module III

15. Show that every Tychonoff space is T_3.

16. If X is a normal space show that for any closed set C and any open set G containing C, there exists an open set H such that $C \subset H$ and $\overline{H} \subset G$.

17. Show that the uniform limit of a sequence of continuous functions from a topological space into a metric space is continuous.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. (a) Let $\mathfrak{B}$ be the collection of all circular regions (excluding boundary) in $\mathbb{R}^2$ and $\mathfrak{B}'$ be the collection of all rectangular regions (excluding boundary) in $\mathbb{R}^2$. Show that these two collections are bases for topologies on $\mathbb{R}^2$. Also compare these two topologies.

(b) Prove of disprove semi open interval topology on $\mathbb{R}$ is second countable.

19. (a) Let D be dense subset of a topological space X, and let Y be a subspace of X. Prove of disprove $D \cap Y$ is dense in Y in the relative topology.

(b) For any subset A of a space X, show that $\overline{A} = A \cup A'$, where $\overline{A}$ is the closure and A′ is the derived set of A.

20. For a topological space X, show that the following statements are equivalent :

(i) X is locally connected.

(ii) Components of open subsets of X are open in X.

(iii) X has a base consisting of connected sets.

21. (a) Show that a regular Lindeloff space is normal.

(b) Let A and B be compact subsets of topological spaces X and Y respectively. Let W be an open subset of X × Y containing the rectangle A × B. Then show that there exist open sets U and V in X and Y respectively such that $A \subset U$, $B \subset V$ and $U \times V \subset W$.

(2 × 5 = 10 weightage)

6. Define negative semidefinite function.
7. Define complete integral of a first order partial differential equation $f(x, y, z, p, q) = 0$.
8. Fine the type of the second order partial differential equation $u_{xx} - x^2 u_{yy} = 0$.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Locate the classify the singular points of the differential equation :

$$\left(x^4 - 2x^3 + x^2\right) \frac{d^2y}{dx^2} + 2(x-1)\frac{dy}{dx} + x^2 y = 0.$$

10. Consider the differential equation :

$$\frac{d^n y}{dx^n} = f\left[x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right],$$

where the function f is continuous and satisfies a Lipschitz condition in a domain D of real $(n + 1)$ dimensional $\left(x, y, y', \ldots, y^{(n-1)}\right)$ space. Let $\left(x_0, c_0, c_1, \ldots, c_{n-1}\right)$ be a point of D. Then prove that there exists a unique solution ϕ of the nth order differential equation such that

$$\phi(x_0) = c_0, \phi'(x_0) = c_1, \ldots, \phi^{(n-1)}(x_0) = c_{n-1},$$

defined on some interval $|x - x_0| \le h$ about $x = x_0$.

11. Does there exist a unique solution ϕ of the third-order differential equation :

$$\frac{d^3y}{dx^3} = x^2 + y\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2$$

such that $\phi(0) = 1, \phi'(0) = -3, \phi''(0) = 0$?

Explain precisely why or why not.

Unit II

12. Determine the nature of the critical point (0, 0) of the system

$$\frac{dx}{dt} = 2x - 7y$$
$$\frac{dy}{dt} = 3x - 8y$$

and determine whether or not the point is stable.

13. Let

$$m\frac{d^2x}{dt^2} = F(x),$$

where F is analytic for all values of x, be the differential equation of a conservative dynamical system. Consider the equivalent autonomous system :

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = \frac{F(x)}{m},$$

and let $(x_c, 0)$ be a critical point of this system. Let V be the potential energy function of the dynamical system having differential equation $m\frac{d^2x}{dt^2} = F(x)$. Then prove that if the potential energy function has a horizontal inflection point at $x = x_c$, then the critical point $(x_c, 0)$ is of a degenerate type called a cusp and is unstable.

14. Find all the real critical points of the non-linear system :

$$\frac{dx}{dt} = 8x - y^2,$$
$$\frac{dy}{dt} = -6y + 6x^2.$$

Unit III

15. Find the general solution of $x(y^2 - z^2)p - y(z^2 + x^2)q = (x^2 + y^2)z.$

16. Reduce the equation $(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y$, where n is an integer, to a canonical form.

17. Suppose that u (x, y) is harmonic in a bounded domain D and continuous in $\overline{D} = D \cup B$. Then prove that u attains its maximum on the boundary B of D.

(6 × 2 = 12 weightage)

C 5595 (Pages : 4) Name..................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE [REGULAR] EXAMINATION APRIL 2021

(CBCSS)

Mathematics with Data Science

MTD 2C 08—DIFFERENTIAL EQUATIONS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Define ordinary point of a second order homogeneous differential equation.
2. Discuss the existence of a solution of the initial value problem :

$$|y'| + |y| = 0, \; y(0) = 1.$$

3. Describe Bessel's equation of order p.
4. Define critical point of the autonomous system :

$$\frac{dy}{dt} = P(x, y),$$
$$\frac{dy}{dt} = Q(x, y).$$

5. Verify whether E defined by $E(x, y) = x^2 + y^2$ is positive definite in every domain D containing (0, 0).

6. Define negative semidefinite function.
7. Define complete integral of a first order partial differential equation $f(x, y, z, p, q) = 0$.
8. Fine the type of the second order partial differential equation $u_{xx} - x^2 u_{yy} = 0$.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Locate the classify the singular points of the differential equation :

$$\left(x^4 - 2x^3 + x^2\right)\frac{d^2y}{dx^2} + 2(x-1)\frac{dy}{dx} + x^2 y = 0.$$

10. Consider the differential equation :

$$\frac{d^n y}{dx^n} = f\left[x, y, \frac{dy}{dx}, \ldots, \frac{d^{n-1}y}{dx^{n-1}}\right],$$

where the function f is continuous and satisfies a Lipschitz condition in a domain D of real $(n+1)$ dimensional $\left(x, y, y', \ldots, y^{(n-1)}\right)$ space. Let $\left(x_0, c_0, c_1, \ldots, c_{n-1}\right)$ be a point of D. Then prove that there exists a unique solution ϕ of the nth order differential equation such that

$$\phi(x_0) = c_0, \phi'(x_0) = c_1, \ldots, \phi^{(n-1)}(x_0) = c_{n-1},$$

defined on some interval $|x - x_0| \leq h$ about $x = x_0$.

11. Does there exist a unique solution ϕ of the third-order differential equation :

$$\frac{d^3y}{dx^3} = x^2 + y\frac{dy}{dx} + \left(\frac{d^2y}{dx^2}\right)^2$$

such that $\phi(0) = 1, \phi'(0) = -3, \phi''(0) = 0$?

Explain precisely why or why not.

Unit II

12. Determine the nature of the critical point (0, 0) of the system

$$\frac{dx}{dt} = 2x - 7y$$

$$\frac{dy}{dt} = 3x - 8y$$

and determine whether or not the point is stable.

13. Let

$$m\frac{d^2x}{dt^2} = F(x),$$

where F is analytic for all values of x, be the differential equation of a conservative dynamical system. Consider the equivalent autonomous system :

$$\frac{dx}{dt} = y,$$
$$\frac{dy}{dt} = \frac{F(x)}{m},$$

and let $(x_c, 0)$ be a critical point of this system. Let V be the potential energy function of the dynamical system having differential equation $m\frac{d^2x}{dt^2} = F(x)$. Then prove that if the potential energy function has a horizontal inflection point at $x = x_c$, then the critical point $(x_c, 0)$ is of a degenerate type called a cusp and is unstable.

14. Find all the real critical points of the non-linear system :

$$\frac{dx}{dt} = 8x - y^2,$$
$$\frac{dy}{dt} = -6y + 6x^2.$$

Unit III

15. Find the general solution of $x(y^2 - z^2)p - y(z^2 + x^2)q = (x^2 + y^2)z$.

16. Reduce the equation $(n-1)^2 u_{xx} - y^{2n} u_{yy} = ny^{2n-1} u_y$, where n is an integer, to a canonical form.

17. Suppose that $u\ (x, y)$ is harmonic in a bounded domain D and continuous in $\overline{D} = D \cup B$. Then prove that u attains its maximum on the boundary B of D.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. Let f be a continuous function defined on a domain D of the xy-plane and ϕ be a continuous function defined on a real interval $\alpha \le x \le \beta$ and such that $[x, \phi(x)] \in D$ for all $x \in [\alpha, \beta]$. Also let x_0 be any real number such that $\alpha < x_0 < \beta$. Then prove that ϕ is a solution of the differential equation $\frac{dy}{dx} = f(x, y)$ on $[\alpha, \beta]$ and is such that $\phi(x_0) = y_0$ if and only if ϕ satisfies the integral equation $\phi(x) = y_0 + \int_{x_0}^{x} [f(t, \phi(t))] dt$ for all $x \in [\alpha, \beta]$.

19. Find the power series solution of the differential equation $y'' + y = 0$ in powers of x.

20. Assume that the roots λ_1 and λ_2 of the characteristic equation

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

are real, unequal, and of the same sign. Then prove that the critical point (0, 0) of the linear system

$$\frac{dx}{dt} = ax + by,$$
$$\frac{dy}{dt} = cx + dy,$$

where a, b, c and d are real constants, is a nod

21. Prove that a necessary and sufficient condition that the Pfaffian differential equation

$$\vec{X} . d\vec{r} = P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz = 0$$

be integrable is that

$$\vec{X} . \text{curl } \vec{X} = 0.$$

(2 × 5 = 10 weightage)

C 5594 (Pages : 3) Name..................................

Reg. No..............................

SECOND SEMESTER M.Sc. [REGULAR] DEGREE EXAMINATION APRIL 2021

(CBCSS)

Mathematics with Data Science

MTD 2C 07—NUMBER THEORY

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Section A

Answer **all** *questions.*
Each question carries a weightage 1.

1. Prove that the Euler totient function $\phi(n)$ is multiplicative but not completely multiplicative.
2. Define the derivative of an arithmetical function. Find the derivative of $u(n)$, the unit function.
3. Define the big oh notation.
4. Prove that for every $n > 1$ there exist n consecutive composite numbers.
5. If $0 < a < b$, prove that there exist an x_0 such that $\pi(ax) < \pi(bx)$ if $x \geq x_0$, where $\pi(x)$ denote the number of primes not exceeding x.
6. Define the Legendre's symbol $(n|p)$ and calculate $(-1|p)$ for odd prime p.
7. Determine those odd primes p for which $(-3|p) = 1$ and those for which $(-3|p) = -1$.
8. How many possible affine enciphering transformations are there for digraphs in an N-letter alphabet ?

(8 × 1 = 8 weightage)

Turn over

Section B

Answer any **six** *questions, choosing* **two** *questions from each unit.*
Each question carries a weightage 2.

Unit I

9. Prove $n \geq 1$, prove that $\phi(n) = \prod_{p|n}\left(1 - \frac{1}{p}\right)$.

10. Prove that the Dirichlet multiplication is commutative and associative. Also prove that, if f and g are multiplicative, so is their Dirichlet product.

11. If $F(x) = \sum_{n \leq x} f(n)$, prove that $\sum_{n \leq x} \sum_{d|n} f(d) = \sum_{n \leq x} f(n)\left[\frac{x}{n}\right] = \sum_{n \leq x} F\left(\frac{x}{n}\right)$.

Unit II

12. For $n \geq 1$, prove that the nth prime P_n satisfies the inequalities $\frac{1}{6} n \log n < P_n < 12\left(n \log n + n \log \frac{12}{e}\right)$.

13. For $x \geq 2$, $\theta(x)$ denote the Chebyshev's θ-function and $\pi(x)$ denote the number of primes not exceeding x, then prove that

$$\theta(x) = \pi(x) \log x - \int_2^x \frac{\pi(t)}{t}\, dt$$

and

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t \log^2 t}\, dt.$$

14. For all $x \geq 1$, prove that $\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$.

Unit III

15. Prove that the Legendre's symbol $(n | p)$ is a completely multiplicative function of n.

16. For every odd prime p, prove that $(2 | p) = (-1)^{\frac{p^2-1}{8}}$.

17. State and prove the reciprocity law for Jacobi symbols.

(6 × 2 = 12 weightage)

Section C

Answer any **two** *questions.*
Each question carries a weightage 5.

18. If $x \geq 2$,

    (a) Prove that $\log[x]! = x\log x - x + O(\log x)$.

    (b) Prove that $\sum_{p\leq x}\left[\frac{x}{p}\right]\log p = x\log x + O(x)$, where the sum is extended over all primes $\leq x$ and O denotes the big oh notation.

19. Let P_n denote the nth prime. Then prove that the following asymptotic relations are logically equivalent :

    (a) $\lim\limits_{x\to\infty}\dfrac{\pi(x)\log x}{x} = 1$.

    (b) $\lim\limits_{x\to\infty}\dfrac{\pi(x)\log \pi(x)}{x} = 1$.

    (c) $\lim\limits_{n\to\infty}\dfrac{P_n}{n\log n} = 1$.

20. (a) Assume n is not congruent to 0 modulo p and consider the least positive residues mod p of the following $\dfrac{p-1}{2}$ multiples of n : $n, 2n, 3n, \ldots, \dfrac{p-1}{2}n$. If m denotes the number of these residues which exceed $\dfrac{p}{2}$, then prove that $m \equiv \sum_{t=1}^{\frac{p-1}{2}}\left[\dfrac{tn}{p}\right] + (n-1)\dfrac{p^2-1}{8} \pmod 2$.

    (b) If P is an odd positive integer, prove that the Jacobi symbol $(-1\,|\,P) = (-1)^{\frac{P-1}{2}}$.

21. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z/NZ)$ and D = $ad - bc$. Then prove that the following are equivalent ;

    (a) *g.c.d.* (D, N) = 1.

    (b) A has an inverse matrix.

    (c) If x and y are not both 0 in Z/nZ, then $A\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

    (d) A gives a one to one correspondence of $(Z/NZ)^2$ with itself.

(2 × 5 = 10 weightage)

C 5593 (Pages : 3) Name.......................................

Reg. No..................................

SECOND SEMESTER M.Sc. DEGREE (REGULAR) EXAMINATION APRIL 2021

(CBCSS)

Mathematics with Data Science

MTD 2C 06—DISCRETE MATHEMATICS

(2020 Admission onwards)

Time : Three Hours Maximum : 30 Weightage

General Instructions

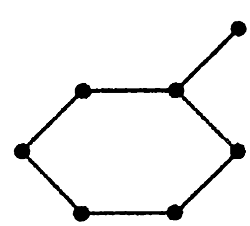
1. *In cases where choices are provided, students can attend* **all** *questions in each section.*
2. *The minimum number of questions to be attended from the Section/Part shall remain the same.*
3. *The instruction if any, to attend a minimum number of questions from each sub section/sub part/ sub division may be ignored.*
4. *There will be an overall ceiling for each Section/Part that is equivalent to the maximum weightage of the Section/Part.*

Part A

Answer **all** *questions.*
Each question has weightage 1.

1. Let n and m be the number of vertices and edges, respectively, of a simple graph G. Prove that $m \leq \binom{n}{2}$.

2. Find the complement of the following graph



3. Prove that the sum of degrees of all the vertices of a graph is equal to twice the number of edges.

4. Draw any two non-isomorphic trees on 4 vertices.

Turn over

5. Prove that every non-trivial tree has at least two leaves.

6. If G is k-critical, prove that $\delta \geq k-1$.

7. Find a grammar that generates $L = \{a^n b^{n+1} ; n \geq 0\}$.

8. Prove that $(L_1 L_2)^R = L_2^R L_1^R$.

(8 × 1 = 8 weightage)

Part B

Answer any **six** *questions by choosing two questions from each unit.*
Each question carries a weightage of 2.

Unit I

9. Show that, for any graph G, $\delta(G) \leq d(G) \leq \Delta(G)$. Show that in any group of two or more people, there are always two who have exactly same number of friends within the group.

10. Show that every nontrivial acyclic graph has at least two vertices of degree less than two. When it is connected, deduce that it has at least two vertices of degree one. When does equality hold ?

11. Does there exists an Eulerian graph G with n even and m odd ? Justify.

Unit II

12. Prove that a graph G is bipartite if and only if it does not contain any odd cycle.

13. Prove that an edge $e = xy$ of a connected graph G is a cut edge of G if and only if e belongs to no cycle of G.

14. Prove that a tree T on n vertices contains $n-1$ edges. Is the converse true ? Justify.

Unit III

15. Let the grammar $G = (\{S\}, \{a, b\}, S, P)$, with P given by S aSb and S λ and let the grammar $G_1 = (\{A, S\}, \{a, b\}, S, P_1)$ with P_1 given by S $a\text{Ab} \mid \lambda$ and A $a\text{Ab} \mid \lambda$. Prove that G and G_1 are equivalent.

16. Find a grammar that generates the following languages :

(i) $L_1 = \{a^n b^m : n \geq 0, m > n\}$; and

(ii) $L_2 = \{a^n b^{2n} : n \geq 0\}$

17. Show that the language $L = \{awa : w \{a, b\}^*\}$ is regular.

(6 × 2 = 12 weightage)

Part C

Answer any **two** *questions.*
Each question carries a weightage of 5

18. (i) Show that any *two* longest paths in a connected graph have a vertex in common.

(ii) Deduce that if P is a longest path in a connected graph G, then no path in $G - V(P)$ is as long as P.

19. Prove that for a nontrivial connected graph G, the following statements are equivalent:

(i) G is Eulerian.

(ii) The degree of each vertex of G is an even positive integer.

(iii) G is an edge-disjoint union of cycles.

20. Prove that K_5 and $K_{3,3}$ are non-planar graphs.

21. Let L be the language accepted by a nondeterministic finite accepter $M_N = (Q_N, \Sigma, \delta_N, q_0, F_N)$. Prove that there exists a deterministic finite accepter $M_D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ such that $L = L(M_D)$.

(2 × 5 = 10 weightage)