C 5207 (Pages : 4) Name.......................................

Reg. No..................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

(CUCBCSS)

Mathematics

DMS 316—LINEAR INTEGRAL EQUATION

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. (a) Solve the integral equation :

$$\int_0^x \frac{\phi(t)}{\sqrt{x-t}}\,dt = x$$

(b) Show that if $\lambda_1 \neq \lambda_2$ and λ_1 and λ_2 are characteristic numbers of the Kernel $k(x,t)$, then the eigen functions of the equations :

$$\phi(x) - \lambda_1 \int_a^b k(x,t)\,\phi(t)\,dt = 0$$

$$\psi(x) - \lambda_2 \int_a^b k(t,x)\,\psi(t)\,dt = 0$$

are orthogonal.

(c) Find the solution of the following Volterra integral equation :

$$\phi(x) = \sin x + 2\int_0^x e^{(x-t)}\,\phi(t)\,dt.$$

(d) Solve the homogeneous Fredholm integral equation :

$$\phi(x) = \lambda \int_0^{2\pi} \sin(x+t)\,\phi(t)\,dt.$$

Turn over

(e) Find the Green's function for the Boundary value problem :

$$\frac{d^2y}{dx^2} = 0, \; y(0) = y(l) = 0.$$

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions with out omitting any unit.*

Unit I

II. (a) If the Kernel of an integral equation is quadratically summable and $|\lambda| < \frac{1}{B}$ where $B^2 = \int_a^b \int_a^b |k(x,t)|^2 \, dx dt$ then prove that the Neumann series for this equation converges in mean to a quadratically summable solution of $\phi(x) - \lambda \int_a^b k(x,t)\phi(t)\,dt = f(x)$. Is such a solution unique ? (10 marks)

(b) Solve the Fredholm integral equation $\phi(x) = 1 + \lambda \int_0^{\pi} \sin(x+t)\, \phi(t)\,dt$ and evaluate the resolvent Kernel. For what values of λ the Naumann series converges ? (10 marks)

III. (a) Discuss the solution of $\phi(x) - \lambda \int_a^b k(x,t)\phi(t)\,dt = f(x)$ when $k(x,t) = \sum_{k=1}^{n} a_k(x)\, b_k(t)$.

(6 marks)

(b) Solve the following Fredholm integral equation and discuss all possible cases

$$\phi(x) - \lambda \int_0^1 \phi(t)(1-3xt)\,dt = f(x).$$

(14 marks)

IV. (a) Find the resolvent Kernel for the Kernel :

$k(x,t) = (1+x)(1-t)\,;\, a = -1, b = 1.$ (10 marks)

(b) Describe the successive approximate solution of the integral equation

$\phi(x) = f(x) + \lambda \int_a^x K(x,t)\,\phi(t)\,dt.$ (10 marks)

Unit II

V. (a) State and prove Fredholm's first fundamental theorem. (12 marks)

(b) Using Fredholm determinant find the resolvent kernel of the integral equation

$$y(x) = e^{x} + \lambda \int_0^1 2e^{x} e^{t} y(t)\, dt.$$ (8 marks)

VI. (a) Prove that the characteristic constants of an integral equation with symmetric Kernel are real and a sequence of proper functions of such a Kernel can be orthonormalized. (10 marks)

(b) Find the characteristic constants and proper functions of the homogeneous integral equation whose kernel is :

$$k(x, t) = \begin{cases} t(x+1), & 0 \le x \le t \\ x(t+1), & t \le x \le 1. \end{cases}$$ (10 marks)

VII. (a) Prove that any bounded operator has a conjugate which is also bounded and has the same norm as the given operator. (10 marks)

(b) Solve the symmetric integral equation using Hilbert-Schmidt theorem

$$\phi(x) = 1 + \lambda \int_0^{\phi} \cos(x+t)\, \phi(t)\, dt.$$ (10 marks)

Unit III

VIII. (a) Show that the necessary and sufficient condition for symmetric square summable kernel to be separable is that it must possesses a finite number of eigen values. (10 marks)

(b) Solve the following non-homogeneous symmetric integral equation :

$$\phi(x) - 2\int_0^{\pi/2} k(x, t)\, \phi(t)\, dt = \cos 2x$$

$$k(x, t) = \begin{cases} \sin x \cos t, & 0 \le x \le t \\ \sin t \cos x, & t \le x \le \frac{\pi}{2}. \end{cases}$$ (10 marks)

IX. (a) Define Green's function for the boundary value problem :

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)\,y = -f(x) \text{ with } y(a) = y(b) = 0.$$

Show that the Green function $G(x, t)$ is symmetric. (10 marks)

(b) Find the Green function for the differential equation

$$xy'' + y' = 0$$

with the boundary condition $y(1) = \alpha y'(1), \alpha \neq 0$ and $y(x)$ is bounded as $x \to \infty$.

(10 marks)

X. (a) Find the characteristic constants and proper functions of the boundary value problem :

$$\frac{d}{dx}\left((2+x)^2\, y'\right) + \lambda y = 0, -1 \leq x \leq 1, y(-1) = 0, y(1) = 0.$$ (10 marks)

(b) Solve :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty, t > 0$$

$$u(x, 0) = e^{-x^2}, -\infty < x < \infty$$

$u(x, 0) = e^{-x^2}, -\infty < x < \infty$ by the Fourier method. (10 marks)

[4 × 20 = 80 marks]

C 5206 (Pages : 2) Name.......................................

Reg. No...................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

[PVT/SDE]

(CUCBCSS)

Mathematics

DMS 315—PROBABILITY THEORY

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

1. (a) Distinguish between a field and a sigma field, with suitable examples.

   (b) Define conditional probability. Show that it is a probability measure.

   (c) Define moment generating function of a random variable. Is it exists always ? Jusify your claim.

   (d) If variance of a random variable exists, then show that its mean also exists.

   (e) Prove or disprove : Every sequence of distribution functions need not converge.

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions without omiting any unit.*
Each question carries 20 *marks.*

Unit I

2. (a) Show by an example that pairwise independence need not imply mutual independence.

   (b) State and establish monotone property of probability measure.

3. (a) Show that the distribution function of a random variable is non-decreasing and right continuous.

   (b) Define (i) A random vector ; and (ii) Distribution function of a random vector. Discuss their important properties.

Turn over

4. (a) Explain the concept of discrete probabiliy space.

   (b) State and prove Jordan decomposition theorem on distribution functions.

UNIT II

5. (a) State and establish : (i) Liapounov inequality ; (ii) Markov inequality.

   (b) State and prove Fubini's theorem.

6. (a) Define convergence almost surely and convergence in probability regarding a sequence of random variables. Show that the former implies the latter. Is the converse true ? Justify.

   (b) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, then prove or disprove : $X_n + Y_n \xrightarrow{P} X + Y$.

7. (a) State and prove continuity theorem of characteristic functions.

   (b) State and prove Bochner's theorem.

UNIT III

8. (a) State and prove Borel-Cantelli Lemma. Is its converse true ? Justify.

   (b) State and establish Bernoulli's WLLN.

9. (a) State and prove classical CLT.

   (b) Examine whether conditions for classical CLT implies -Lindberg condition for CLT.

10. (a) State and establish Kolmogorov's SLLN for i.i.d. random variables.

   (b) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $P(X_n = 0) = 1 - n^{-2}$; $P(X_n = n) = P(X_n = -n) = 0.5 \ n^{-2}, n \geq 1$. Check whether the sequence obeys SLLNs or not.

(4 × 20 = 80 marks)

C 5205 (Pages : 4) Name..................................

Reg. No..............................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

[PVT/SDE]

(CUCBCSS)

Mathematics

DMS 315—OPERATIONS RESEARCH

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *the questions.*
Each question carries 4 *marks.*

I. (a) Prove that the sum of two convex functions is a convex function.

(b) What is degeneracy in an LP problem ?

(c) Explain integer linear programming ? Give an example of a mixed integer linear programming problem.

(d) Briefly explain backward recursion and forward recursion in a dynamic programming problem.

(e) Solve the game whose pay-off matrix is given by :

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}.$$

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 20 *marks.*

Unit I

II. (a) Find the point in the plane $x_1 + 2x_2 + 3x_3 = 1$ in E_3 nearest to the point $(-1, 0, 1)$.

(b) Let $f(x)$ be defined in a convex domain $K \subseteq E_n$ and be differentiable. Prove that $f(x)$ is a convex function iff $f(x_2) - f(x_1) \geq (x_2 - x_1)^1 \nabla f(x_1)$ for all x_1, x_2 in K.

Turn over

III. (a) Prove that a basic feasible solution of an LP problem is a vertex of the convex set of feasible solutions.

(b) Use revised simplex method to solve the problem :

$$\text{Maximize } 5x_1 + 3x_2$$

$$\text{subject to } 4x_1 + 5x_2 \le 10$$

$$5x_1 + 2x_2 \le 10$$

$$3x_1 + 8x_2 \le 12$$

$$x_1, x_2 \ge 0.$$

IV. (a) Describe the Caterer problem in Operations Research.

(b) Four operators A, B, C, D are to be assigned to four machines M_1, M_2, M_3, M_4 with the restriction that A and C cannot work on M_3 and M_2 respectively. The assignment costs are given below. Find the minimum assignment cost :

	M_1	M_2	M_3	M_4
A	5	2	–	5
B	7	3	2	4
C	9	–	5	3
D	7	7	6	2

Unit II

V. (a) Describe the generalized problem of maximum flow.

(b) Find the minimum path from v_1 to v_8 in the graph with arcs and arc lengths as given below :

Arc	(1, 2)	(1, 3)	(1, 4)	(2, 3)	(2, 6)	(2, 5)	(3, 5)
Length	1	4	11	2	8	7	3
Arc	(3, 4)	(4, 7)	(5, 6)	(5, 8)	(6, 3)	(6, 4)	
Length	7	3	1	12	4	2	
Arc	(6, 7)	(6, 8)	(7, 3)	(7, 8)			
Length	6	10	2	2			

VI. (a) Briefly explain the cutting plane method to solve an integer linear programming problem.

(b) Maximize $2x_1 + 5x_2$

subject to $0 \leq x_1 \leq 8$,

$0 \leq x_2 \leq 8$,

and either $4 - x_1 \geq 0$ or $4 - x_2 \geq 0$.

VII. (a) Let $f(x)$ be a real valued function in E_n, $G(x)$ a vector function consisting of real-valued functions $g_i(x), i = 1, 2, \ldots, m$ as components and $F(x, y) = f(x) + y' G(x)$ where y is a vector in E_m. If $F(x, y)$ has a point (x_0, y_0) for every $y \geq 0$, prove that $G(x_0) \leq 0$ and $y_0^1 G(x_0) = 0$.

(b) Show that the Kuhn-Tucker conditions fail to give max x_1, subject to $(1 - x_1)^3 - x_2 \geq 0$, $x_1 \geq 0$, $x_2 \geq 0$.

UNIT III

VIII. (a) Write the general form of a geometric programming problem.

(b) Solve the geometric programming problem :

Minimize $f(x) = \dfrac{c_1}{x_1 x_2 x_3} + c_2 x_2 x_3$

subject to $g_1(x) = c_3 x_1 x_3 + c_4 x_1 x_2 = 1$

$c_i > 0, x_j > 0$

where $i = 1, 2, 3, 4$ and $j = 1, 2, 3$.

IX. (a) Describe a method of dynamic programming to solve the problem :

Minimize $z = \sum_{j=1}^{n} f_j(u_j)$

subject to $\sum_{j=1}^{n} a_j u_j \geq b$

$a_j, b \in R, a_j \geq 0, b > 0$

$u_j \geq 0 ; j = 1, 2, \ldots, n.$

(b) A student has to take examination in three courses, A, B, C. He has three days available for study. He feels it would be best to devote a whole day to the study of the same course, so that he may study a course for one day, two days, or three days or not at all. His estimates of the grades he may get by study are as follows :

Study days \ Course	A	B	C
0	0	1	0
1	1	1	1
2	1	3	3
3	3	4	3

How should he study so that he maximizes the sum of his grades.

X. (a) State and prove the fundamental theorem of rectangular games.

(b) Write both the primal and dual LP problems corresponding to the rectangular game with the pay-off matrix is :

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}.$$

(4 × 20 = 80 marks)

C 5204 (Pages : 3) Name..................................

Reg. No...............................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

PVT/SDE

(CUCBCSS)

Mathematics

DMS 313—FLUID DYNAMICS

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 4 *marks.*

I. (a) Show that the magnitude of the vorticity multiplied by the cross sectional area is constant along the filament.

(b) Show that if the motion is irrotational then ψ and ϕ satisfy Laplace's equation.

(c) Write a short note an cavitation.

(d) What is doublet ? Explain.

(e) What is Rankine's solid ?

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 20 *marks.*

Unit I

II. (a) Obtain the equation of continuity for a liquid in irrotational motion.

(b) Show that $u = \frac{-2xyz}{\left(x^2+y^2\right)^2}$, $v = \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2}$, $w = \frac{y}{x^2+y^2}$,

are the velocity-components of a possible fluid motion. Is this motion irrotational ?

Turn over

III. (a) Show that the rate of change of total energy of any portion of the fluid as it moves about is equal to the rate of working of the pressures on the boundary.

(b) State and prove Kelvin's minimum energy theorem.

IV. (a) Obtain the pressure equation of motion in terms of the stream function.

(b) Show that the velocity potential $\phi = \frac{1}{2} \log \frac{(x+a)^2 + y^2}{(x-a)^2 + y^2}$ gives a possible motion, and determine the form of the streamlines.

UNIT II

V. (a) Discuss the streaming motion past a circular cylinder.

(b) If $w^2 = z^2 - 1$, prove that the streamline for which $\Psi = 1$ in $y^2\left(1+x^2\right) = x^2$.

VI. (a) Discuss the effect on a wall of a source parallel to the wall.

(b) Calculate the force on a wall due to a doublet of strength μ at distance a from the wall and inclined to it at an angle α.

VII. (a) Show that if we map the z-plane on the ξ-plane by a conformal transformation $\xi = f(z)$, a source in the z-plane will transform into a source at the corresponding point of the ξ-plane

(b) If there in a source at $(a, 0)$ and $(-a, 0)$ and sinks at $(0, -a)$ and $(0, -\alpha)$, all of equal strength, show that the circle through these four points in a streamline.

UNIT III

VIII. (a) Write a short note on 'aerofoil'.

(b) State and prove the theorem of Kulta and Joukowski on aerofoils.

IX. (a) What is Stokes' stream function ?

(b) Verify that :

$$\psi = \left(\frac{A}{r^2}\cos\theta + Br^2\right)\sin^2\theta.$$

is a possible form of stokes' stream function, and find the corresponding velocity potential.

X. (a) Consider a line source stretching along the axis from O to A, the strength at the distance ξ from the origin O being m/a per unit length where OA = a and m is a constant. Show that the streamlines are hyperbolas with foci at O and A.

(b) A and B are a simple source and sink of strength μ and μ' respectively in an infinite fluid. Show that the equation of the streamlines is :

$$\mu\cos\theta - \mu'\cos\theta' = \text{constant},$$

where θ, θ' are the angles which AP, BP make with AB, P being any point.

(4 × 20 = 80 marks)

C 5203 (Pages : 4) Name..................................

Reg. No..............................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

(PVT/SDE)

[CUCBCSS]

Mathematics

DMS 310—MEASURE AND INTEGRATION

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

I. (a) Let X be a measurable space and let $E \subset X$. Prove that E is a measurable set in X if and only if the function

$$\chi E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

is a measurable function.

(b) Let X be a measurable space and let f be a complex measurable function on X. Prove that there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.

(c) Define total variation of a complex measure. If μ is a positive measure, then prove that its total variation $|\mu| = \mu$.

(d) If λ_1 is absolutely continuous with respect to μ and λ_2 and μ are mutually singular then prove that $\lambda_1 + \lambda_2$ is absolutely continuous with respect to μ.

(e) Let (X, S) and (Y, $\mathcal{T}$) be measurable spaces and let $x \in X$. If $E \in S \times \mathcal{T}$, then prove that the x-section $E_x = \{y : (x, y) \in E\} \in \mathcal{T}$.

(5 × 4 = 20 marks)

Turn over

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 20 *marks.*

UNIT I

II. (i) Let $f : X \to [0, \infty]$ be a measurable function. Prove that there exist simple measurable functions s_n on X such that :

(a) $0 \le s_1 \le s_2 \le \ldots \le f$,

(b) $s_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

(ii) Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that :

(a) $0 \le f_1(x) \le f_2(x) \le \ldots. \le \infty$ for every $x \in X$,

(b) $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$.

Then prove that f is measurable and

$$\int_x f_n d\mu \to \int_x f\, d\mu \text{ as } n \to \infty.$$

III. (i) Let $f, g \in L^1(\mu)$ and α, β are complex numbers. Prove that $a\alpha + b\beta \in L^1(\mu)$ and

$$\int_X (a\alpha + b\beta)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

(ii) State and prove Lebesgue's dominated convergence theorem.

IV. (i) Let μ be a positive measure on a σ-algebra $\mathcal{M}$. in a set Ω so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$ and if φ is convex on (a, b), then prove that :

$$\varphi\left(\int_\Omega f\, d\mu\right) \le \int_\Omega (\varphi \circ f)\, d\mu.$$

(ii) Prove that $L^p(\mu)$ is a complete metric space for $1 \le p \le \infty$ and for every positive measure μ.

Unit II

V. (i) Let U be an open subset of a locally compact Hausdorff space X and let K be compact subset of U. Prove that there is an open set with compact closure such that $K \subset V \subset \bar{V} \subset U$.

(ii) State and prove Lusin's theorem.

VI. (i) If μ is a complex measure on a measurable space X, prove that $|\mu|(X) < \infty$.

(ii) Let μ be a real (signed) measure on a measurable space X. Prove that there exist positive measures μ^+ and μ^- on X such that $\mu = \mu^+ - \mu^-$.

VII. Let μ be a positive σ-finite measure on a σ-algebra $\mathcal{M}$ in a set X and let λ be a complex measure on $\mathcal{M}$. Prove that :

(a) There is a unique pair of complex measures λ_a and λ_s on $\mathcal{M}$ such that :

$$\lambda = \lambda_a + \lambda_s, \lambda_a \ll \mu, \ \lambda_s \perp \mu.$$

(b) There is a unique $h \in L^1(\mu)$ such that :

$$\lambda_a(E) = \int_E h \, d\mu$$

for every set $E \in \mathcal{M}$.

Unit III

VIII. (i) If μ, is a complex Borel measure on $\mathbb{R}^k$ and λ is a. positive number, then prove that :

$$m\left(\left\{x \in \mathbb{R}^k : (M\mu)(x) > \lambda\right\}\right) \leq 3^k \lambda^{-1} |\mu|\left(\mathbb{R}^k\right),$$

where M_μ denote the maximal function of a complex Borel measure μ.

(ii) If $f : [a, b] \to \mathbb{R}^1$ is differentiable at every point of $[a, b]$ and $f' \in L^1$, then prove that :

$$f(x) - f(a) = \int_a^x f'(t) \, dt.$$

IX. (i) Let $(X, \mathcal{S})$, $(Y, \mathcal{T})$ be measurable spaces and let f be an $(\mathcal{S} \times \mathcal{T})$-measurable function on $X \times Y$. Prove that f^y is a $\mathcal{S}$-measurable function for each $y \in Y$, where $f^y(x) = f(x, y)$.

(ii) Let $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \lambda)$ be σ-finite measure spaces and let f be an $(\mathcal{S} \times \mathcal{T})$-measurable function on $X \times Y$. If $0 \leq f \leq = \infty$ and if :

$$\varphi(x) = \int_X f_x \, d\lambda, \psi(y) = \int_Y f^y \, d\mu \; (x \in X, y \in Y),$$

then prove that φ is $\mathcal{S}$-measurable and ψ is T-measurable and

$$\int_X \varphi \, d\mu = \int_{X \times Y} f \, d(\mu \times \lambda) = \int_Y \psi \, d\lambda.$$

where $f_x(y) = f(x, y)$ for all $y \in Y$ and $f^y(x) = f(x, y)$ for all $x \in X$.

X. (i) Let (X, S) and $(Y, \mathcal{T})$ be measurable spaces and let $y \in Y$. If $E \in \mathcal{S} \times \mathcal{T}$, then prove that the y-section $E^y = \{x : (x, y) \in E\} \in \mathcal{S}$.

(ii) Let v be a positive measure on a σ-algebra $\mathcal{M}$, $\mathcal{M}^*$ be the completion of $\mathcal{M}$ relative to v and let f be a $\mathcal{M}^*$-measurable function. Prove that there exists an $\mathcal{M}$-measurable function g such that $f = g \; a.e.[v]$.

(4 × 20 = 80 marks)

C 5202 (Pages : 3) Name.......................................

Reg. No..................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

PVT/SDE

(CUCBCSS)

Mathematics

DMS 308—ALGEBRAIC GEOMETRY

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question is of 4 *marks.*

1. (a) Describe the variety $V\left(X^2+Y^2+Z^2-1, X\right)$ in $\mathbb{R}^3$.

   (b) Show that the variety $V\left(Y^2-X^3\right) \subseteq \mathbb{Q}^2$ is dense in the variety $V\left(Y^2-X^3\right) \subseteq \mathbb{R}^2$.

   (c) Verify whether the variety $V\left(ZY-X\left(X^2-Z^2\right)\right) \subseteq P^2(\mathbb{C})$ is connected.

   (d) Find the genus of the curve in $P^2(\mathbb{C})$ given by the polynomial X^2+Y^2.

   (e) Show that the varieties $V(Y)$ and $V\left(Y-X^2\right)$ are isomorphically oriented in $\mathbb{C}_{XY}$.

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question is of 20 *marks.*

Unit I

2. (a) Show that the rational circle $V\left(X^2+Y^2-1\right)$ in $\mathbb{Q}^2$ is dense in the corresponding circle in $\mathbb{R}^2$.

   (b) Verify whether the rational curve $V\left(X^4+Y^4-1\right)$ in $\mathbb{Q}^2$ is dense in the corresponding real curve in $\mathbb{R}^2$.

3. (a) Define dehomogenization of a variety $V \subseteq P^n(k)$.

   (b) Let $q_1, q_2, \ldots, q_r \in k\left[X_1, X_2, \ldots, X_{n+1}\right]$ be homogeneous. Let $V(q_1, q_2, \ldots, q_r) \subseteq P^n(k)$ be the projective variety defined by $q_1, q_2, \ldots, q_r$. Prove that with the usual notations $D_i(V(q_1, q_2, \ldots, q_r)) = V(D_i(q_1), D_i(q_2), \ldots, D_i(q_r))$.

4. (a) Show that every complex algebraic curve C in $P^2(\mathbb{C})$ is compact.

   (b) Let C be a complex algebraic curve and for $p \in C$ let U_p be a neighbourhood of p. Show that for each point p of C and for sufficiently small U_p either $C \cap U_p$ is topologically an open disk or $C \cap U_p$ is a one point union of finitely many open disks.

Unit II

5. (a) Let $C \in P^2(\mathbb{C})$ and $\{p_i\}$ be the finite set of non singular points of C. Show that $C \setminus \{p_i\}$ is orientable.

   (b) Show that if $C \in P^2(\mathbb{C})$ is non singular then C is orientable.

6. (a) With the usual notations prove that for ideals a_1, a_2 :

(i) $V(a_1 \cap a_2) = V(a_1) \cup V(a_2)$; and

(ii) $V(a_1 + a_2) = V(a_1) \cap V(a_2)$.

(b) Let $R = \mathbb{R}[X, Y]$, $a_1 = (Y)$ and $a_2 = (Y - X^2)$. Find $V(a_1 \cap a_2)$ and $V(a_1 + a_2)$.

7. (a) Define irreducible ideal.

(b) Show that every prime ideal is irreducible.

(c) Verify whether the ideal $(X^2Y - XY^2)$ is irreducible.

Unit III

8. (a) Define isomorphism between affine varieties in terms of polynomial maps.

(b) Show that if V and W are isomorphic affine varieties over $\mathbb{C}$ then their affine co-ordinate rings are $\mathbb{C}$-isomorphic.

9. (a) Let V be an irreducible variety in $\mathbb{C}^n$ and V' be a proper subvariety of V. Show that $V \setminus V'$ is dense in V.

(b) Let $V \subseteq \mathbb{C}^n$ and $V' \subseteq \mathbb{C}^m$ be varieties and $\pi : \mathbb{C}^m \to \mathbb{C}^n$ be the natural projection where $m \geq n$. Show that $\pi(V')$ is dense in V.

10. (a) Define hyper surface.

(b) Show that a variety $P \in \mathbb{C}^n$ is a hyper surface if and only if it is of pure dimension $n - 1$.

(4 × 20 = 80 marks)

C 5201 (Pages : 2) Name.......................................

Reg. No...................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

(PVT/SDE)

[CUCBCSS]

Mathematics

DMS 307—ALGEBRAIC NUMBER THEORY

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *marks.*

I. (a) Express $Q\left(\sqrt{2}, \sqrt[3]{5}\right)$ in the form $Q(\theta)$.

(b) Let $K = Q(\xi)$ where $\xi = e^{2\pi i/5}$. Calculate $N_K(\alpha)$ and $T_K(\alpha)$ for the value $\alpha = \xi + \xi^2$.

(c) Find all fractional ideals of Z.

(d) Show that if L is an n-dimensional lattice in R^n then $\frac{R^n}{L}$ is isomorphic to the n-dimensional torus T^n.

(e) Let D be the ring of integers of a number field K of degree n. Show that factorization in D is unique iff the class group H has order 1.

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 20 *marks.*

Unit I

II. (a) Show that every subgroup H of a free abelian group G of rank n is free of rank $s \leq n$.

(b) Find the order of the group G/H where G is free abelian with Z-basis, x, y, z, and H is generated by $x + 3y - 5z, 2x - 4y, 7x + 2y - 9z$.

Turn over

III. (a) Show that a complex number θ is an algebraic interger iff the additive group generated by all powers $1, \theta, \theta^2$..... is finitely generated.

(b) Show that the algebraic intergers form a subring of the field of algebraic numbers.

IV. (a) Show that the ring D of intergers of $Q(\xi)$ is $Z[\xi]$ where $\xi = e^{2\pi i/p}$, p an odd prime.

(b) Let D be the ring on intergers in a number field K and let $x \in D$. Show that x is a unit iff $N(x) = \pm 1$.

Unit II

V. (a) Show that factorization into irreducibles is not unique in the ring of integers of $Q(\sqrt{15})$.

(b) Show that for square free $d < -11$ the ring of integers of $Q(\sqrt{d})$ is not Euclidean.

VI. (a) Factorize the ideal $<18>$ in $Z[\sqrt{-17}]$ into prime ideals.

(b) Let D be the ring of integers of a number field K of degree n. Show that if a, b are non-zero ideals of D then there exists $\alpha \in a$ such that $\alpha a^{-1} + b = D$.

VII. (a) Show that an additive subgroup of R^n is a lattice iff it is discrete.

(b) State and prove two-square's theorem.

Unit III

VIII. (a) With usual notations, prove that Q-linearly independent elements of K map under σ to R-linearly independent element of L^{st}.

(b) Show that the class-group of a number field is a finite abelian group.

IX. (a) Show that the equation $x^4 + y^4 = z^2$ has no integer solutions with $x, y, z \neq 0$.

(b) Show that if $p(t) \in Z(t)$ is a monic polynomial, all of whose zeros in C have absolute value I, then every zero is a root of unity.

X. (a) Let K be a number field of degree $n = s + 2t$. Explain the logarithmic representation of K.

(b) State and prove Dirichlet units theorem.

C 5200

(Pages : 4)

Name.....................................

Reg. No.....................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

(PVT/SDE)

[CUCBCSS]

Mathematics

DMS 306—THEORY OF NUMBERS

(2017 to 2018 Admissions)

Time : Three Hours

Maximum : 100 Marks

Part A

Answer **all** *questions.*
Each question carries 4 marks.

1. (a) If a prime divides a product of integers, then prove that it divides atleast one of them :

   (b) Prove that for every odd integer n,

   $n^2 \equiv 1 \pmod 8$.

   (c) Prove that the Möbius function $\mu(n)$ is multiplicative.

   (d) Determine all units in the field $Q(\sqrt{3})$.

   (e) Prove that equivalence of ideals is an equivalence relation.

(5 × 4 = 20 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 20 marks.

Unit I

2. (a) Prove that a non-trivial module of rational integers consists precisely of the multiples of a fixed positive integer.

(7 marks)

   (b) Find the greatest common divisor and the least common multiple of 693 and 144.

(6 marks)

   (c) Prove that there exist infinitely many primes. (7 marks)

Turn over

3. (a) If $(a, m) = d$, then prove that the congruence

$$ax \equiv b \pmod{m}$$

has no solution if $d\ b$ and has a unique solution mod m/d if d/b.

(7 marks)

(b) Solve the system of congruences :

$$x \equiv 2 \pmod{3}$$

$$x \equiv 3 \pmod{5}$$

$$x \equiv 2 \pmod{7}.$$

(6 marks)

(c) If g is a primitive root mod m, them prove that $g, g^2, ..., g^{\phi(m)}$ are all incongruent modulo m and form a complete set of reduced residues.

(7 marks)

4. (a) Prove that there exist exactly $(p-1)/2$ quadratic residues and $(p-1)/2$ quadratic non-residues modulo p, where p is an odd prime.

(4 marks)

(b) If p is an odd prime, then prove that $(2/p) = (-1)^{(p^2-1)/8}$. (4 marks)

(c) State and prove Gauss's Lemma. (12 marks)

Unit II

5. (a) For any integer n, prove that $n! = \prod_{p \le n} p^{ep}$ where p is a prime and $e_p = \sum_{m \ge 1} \left[\frac{n}{p^m}\right]$.

(7 marks)

(b) For any integer m, prove that

$$\sum_{d/m} \phi(d) = m.$$ (7 marks)

(c) For $x > 1$, prove that :

$$\Psi(x) - \theta(x) = O\left(x^{1/2} \log^2 x\right).$$

(6 marks)

6. (a) If $s = \sigma + it$ and if $\sigma > 1$, then prove that the Riemann Zeta function G (s) is non-zero.

(10 marks)

(b) For $\sigma > 0$, prove that :

$$G(s) = \frac{1}{s-1} + \gamma + O_{(s-1)}$$

where $\gamma = \lim_{k \to \infty} \left(\sum_{n=1}^{k} n^{-1} - \log k \right)$.

(10 marks)

7. (a) For constant $\sigma = \text{Re}.\, s \geq 1$ and $1 \leq x < \infty$, prove that the series :

$$\sum_{m=1}^{\infty} m^{-1} \pi\left(x^{1/m}\right) x^{-s-1}$$

is uniformly convergent.

(10 marks)

(b) Let $f(x)$ be positive and non-decreasing.

If $g(x) = \int_1^x \frac{f(u)}{u} du \sim x$, then prove that $f(x) \sim x$.

(10 marks)

Unit III

8. (a) Prove that the set $Q(\sqrt{2}) = \{A + B\theta / \theta \text{ is a zero of } x^2 - 2\}$ is a field under ordinary addition and multiplication. Also prove that $Q(\sqrt{2})$ is isomorphic to $Q[x]/\{x^2 - 2\}$.

(8 marks)

(b) Prove that the Gaussian field $Q(i)$ is Euclidean. (6 marks)

(c) Prove that the polynomial $x^3 - x + 1$ is irreducible over Q. (6 marks)

9. (a) Prove that an ideal has only a finite number of factors. (6 marks)

(b) Prove that in every field of algebraic numbers, there exist infinitely many prime ideals.

(8 marks)

(c) If h is the class number of the algebraic number field k, then prove that for every ideal a of k, a^h is a principal ideal.

(6 marks)

10. (a) Prove that all primitive solutions of $x^2 + y^2 = z^2$ are of the form $x = 2ab, y = a^2 - b^2, z = a^2 + b^2$ or $x = a^2 - b^2, y = 2\,ab, z = a^2 + b^2$ with $(a, b) = 1$ and exactly one of a, b is even.

(10 marks)

(b) Prove that the Diophantine equation $x^4 + y^4 = z^2$ has no solution in integers.

(10 marks)

C 5199 (Pages : 3) Name.....................................

Reg. No.................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

[PVT/SDE]

(CUCBCSS)

Mathematics

DMS 304—TOPICS IN DISCRETE MATHEMATICS

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 150 Marks

Part A

Answer **all** *questions in this part.*
Each question carries 6 *marks.*

1. (a) Explain what is a phrase structure grammar that is used to specify a language.

   (b) If there is a $u-v$ walk in a graph G, then show that there is $u-v$ path in G.

   (c) With usual notations, prove that any simple graph G contains at least $\epsilon - \nu + \omega$ distinct cycles.

   (d) Define independent set of a graph. Prove that a set $S \subseteq V$ is an independent set of G if and only if V – S is a covering of G.

   (e) Define planar graphs. Illustrate it with an example. Is $K_{3,3} - \{e\}$ planar, for any edge e of $K_{3,3}$.

(5 × 6 = 30 marks)

Part B

Answer any **four** *questions not omitting any unit.*
Each question carries 30 *marks.*

Unit 1

2. (a) Construct a grammar that generates $L = \{a^i b^j : i \geq 0, j > i\}$. (15 marks)

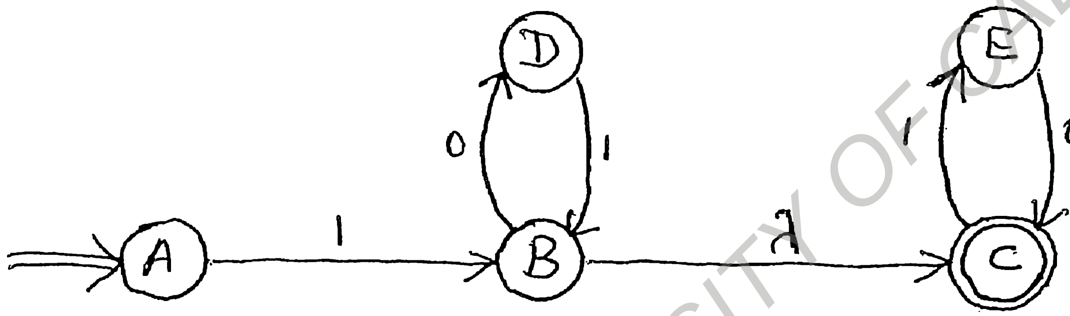
   (b) If L_1 and L_2 are type-2 languages and if $L_1 L_2 = \{\alpha\beta : \alpha \in L_1, \beta \in L_2\}$ prove that L_1L_2 is a type-2 language. (15 marks)

Turn over

3. (a) Show that $L = \{a^k : k = i^2, i \geq 1\}$ is not a finite state language. (15 marks)

(b) Find a finite state machine that accepts all binary sequences of the form any number of 0's followed by one or more 1's followed by one or more 0's followed by a 1, followed by any number of 0's, followed by a 1 and then followed by anything. (15 marks)

4. (a) For the finite state machine given by the transition diagram below, which of the sequences 101, 10101, 110, 11010, 10110, 11001 are accepted by the machine.



(15 marks)

(b) Describe the algorithm LARGEST and obtain its time complexity. (15 marks)

UNIT 2

5. (a) With usual notations, prove that $\sum_{v \in V} d(v) = 2\epsilon$. (10 marks)

(b) Show that $\omega(G) \leq \omega(G - e)$. (10 marks)

(c) Show that every induced subgraph of a complete graph is complete. (10 marks)

6. (a) Define centre of a graph. Illustrate it with an example. Show that the centre of a tree is a single vertex or two adjacent vertices. (15 marks)

(b) Find a simple graph G with $\delta = \nu - 3$ and $\kappa < \delta$. (15 marks)

7. (a) If G is eulerian, show that every block of G is eulerian. (7 marks)

(b) If G is a simple graph with $\nu \geq 3$ and $\delta \geq \frac{\nu}{2}$, prove that G is hamiltonian. (18 marks)

(c) Describe the Konigsberg bridge problem and its solution. (5 marks)

UNIT 3

8. (a) Let G be a bipartite graph with bipartition (X, Y). Prove that G contains a matching that saturates every vertex in X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$, where $N(S)$ denotes the neighbour set of S in G. (18 marks)

(b) Show that a tree has atmost one perfect matching. (12 marks)

9. (a) For any *two* integers $k \geq 2$ and $l \geq 2$,

prove that $r(k, l) \leq r(k, l-1) + r(k-1, l)$. (15 marks)

(b) Prove that $r(k, k) \geq 2^{\frac{k}{2}}$. (15 marks)

10. (a) In a critical graph, prove that no vertex cut is a clique. (10 marks)

(b) If G is a self dual plane graph, show that $\epsilon = 2\nu - 2$. (10 marks)

(c) Prove that $K_{3,3}$ is non-planar. (10 marks)

[4 × 30 = 120 marks]

C 5198 (Pages : 3) Name.....................................

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THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

[PVT/SDE]

(CUCBCSS)

Mathematics

DMS 303—DIFFERENTIAL GEOMETRY

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 150 Marks

Part A

Answer **all** *questions.*
Each question is of 6 *marks.*

1. (a) Let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A(x, y) = (x, 2x)$. Find all (x, y) such that $A(x, y) = 0$.

   (b) Define differential form of order k.

   (c) Sketch the vector field $X(p) = (p, X(p))$ where $X(x_1, x_2) = (x_2, x_1)$.

   (d) Describe the spherical image of the n-surface given by $f^{-1}(0)$ where $X(x_1, x_2) = -x_1^2 + x_2^2$.

   (e) Describe a parametrized 1-surface.

(5 × 6 = 30 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question is of 30 *marks.*

Unit I

2. (a) Let E be an open set in $\mathbb{R}^n$ and $f : E \to \mathbb{R}^m$. Show that f is continuously differentiable on E if and only if the partial derivatives $D_j f_i$ are continuous for all i and j.

   (b) Find all partial derivatives $D_j f_i$ at $(1, 1)$ for $f(x, y) = (x^2 + y^2, x + y)$.

Turn over

3. Let ω and λ be k-and m-forms respectively of class C' in E. Show that :

   (a) $d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda.$

   (b) If ω is of class C'' in E then $d^2\omega = 0.$

4. (a) Let $f : U \to \mathbb{R}$ be a smooth function where $U \subseteq \mathbb{R}^{n+1}$. Show that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p.

   (b) Show that if p is a regular point of f and $f(p) = c$, then the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $(\nabla f(p))^{\perp}$.

Unit II

5. (a) Describe the derivative $\dot{X}$ of a smooth vector field X along a parametrized curve α.

   (b) Show that :

   (i) $(\mathbf{X} + \mathbf{Y}) = \dot{\mathbf{X}} + \dot{\mathbf{Y}}.$

   (ii) $(\mathbf{X} \cdot \mathbf{Y}) = \dot{\mathbf{X}} \cdot \mathbf{Y} + \mathbf{X} \cdot \dot{\mathbf{Y}}$ where $\mathbf{X} \cdot \mathbf{Y}$ is the scvalar product.

6. (a) Describe Levi-Civita parallelism of smooth vector fields.

   (b) Show that :

   (i) If **X** is parallel along α then X has constant length.

   (ii) If **X** and **Y** are parallel vector fields along α then **X** + **Y** is also parallel along α.

   (iii) If **X** and **Y** are parallel vector fields along α then $\mathbf{X} \cdot \dot{\mathbf{Y}}$ is also parallel along α.

7. (a) Define parallel transport p_α along a piecewise smooth curve α.

   (b) Prove that :

   (i) p_α is a linear map.

   (ii) $p_\alpha(v) \cdot p_\alpha(w) = v \cdot w$ for all v and w.

Unit III

8. (a) Let $\phi(s,t) = (\cos s, \sin t, \cos t, \sin s)$. Show that ϕ is a 2-surface in $\mathbb{R}^4$.

   (b) Describe the image of the above ϕ as a cartesian product of two circles.

9. (a) Define local parametrization of an n-surface.

   (b) Show that every n-surface has a local parametrization.

10. (a) Define volume of a parametrized n-surface.

    (b) Let $\phi : U \to \mathbb{R}^{n+1}$ be a parametrized n-surface and $N : U \to S^n$ be the Gauss map. With the usual notations prove that $V(N) = \int_U |K| \left[\det\left(E_i^{\phi} \cdot E_j^{\phi}\right)\right]^{1/2}$.

(4 × 30 = 120 marks)

C 5197 (Pages : 4) Name.......................................

Reg. No..................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL/MAY 2021

[PVT/SDE]

(CUCBCSS)

Mathematics

DMS 302—FUNCTIONAL ANALYSIS

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 150 Marks

Part A

Answer **all** *questions.*
Each question carries 6 marks.

I. (a) Let Y be a finite dimensional subspace of a normed space X. Prove that Y is closed in X.

(b) Let X and Y be normed spaces with X finite dimensional. Prove that every bijective linear map from X onto Y is a homeomorphism.

(c) Let X be a Banach space. Prove that the set of all invertible operators is an open subset of the set of all bounded operators on $X, (BL(X))$.

(d) Let X be a nonzero Banach space over $\mathbb{C}$ and let $A \in BL(X)$. Prove that the spectrum of A is a non-empty subset of $\mathbb{C}$.

(e) Let H be a Hilbert space and $A \in BL(H)$ be a normal operator. If x_1 and x_2 are eigen vectors corresponding to distinct eigen values, then prove that $x_1 \perp x_2$.

(5 × 6 = 30 marks)

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 30 *marks.*

Unit I

II. (i) Let x be continuous K-valued function on $[-\pi, \pi]$ such that $x(\pi) = x(-\pi)$. Prove that the sequence of arithmetic means of the partial sums of the Fourier series of x converges to x uniformly on $[-\pi, \pi]$.

(ii) Let X be a normed space. If E_1 is open in X and $E_2 \subset X$, then prove that $E_1 + E_2$ is open in X.

(iii) Prove that the norms $\| \|_1$ and $\| \|_2$ on $\mathbb{R}^2$ are equivalent.

III. (i) Let X be a normed space. Prove that for every subspace Y of X and every $g \in Y'$ there is a unique Hahn-Banach extension of g to X if and only if X' is strictly convex.

(ii) Define Banach space and give an example of it.

(iii) Let X and Y be normed spaces and let $X \neq \{0\}$. Prove that BL (X, Y) is a Banach space in the operator norm if and only if Y is a Banach space.

IV. (i) State and prove uniform boundedness principle.

(ii) Let X and Y be normed spaces and let $F : X \to Y$ be linear. Prove that F is an open map if and only if there exists some $\gamma > 0$ such that for every $y \in Y$, there is some $x \in X$ with $F(x) = y$ and $\| x \| \leq \gamma \| y \|$.

Unit II

(i) Let X be a Banach space, $A \in BL(X)$ and $\| A^p \| < 1$ for some positive integer p. Prove that the bounded operator $I - A$ is invertible.

(ii) Let X be a Banach space. Prove that

$$\sigma(A) = \sigma_a(A) \bigcup \sigma_e(A') = \sigma(A'),$$

where A' denote the transpose of A

VI. (i) Let $1 \le p \le \infty$ and let $\frac{1}{p}+\frac{1}{q}=1$. Prove that the dual of C_{00} with the norm $\| \|_p$ is linearly isometric to l^q.

(ii) Let X and Y are Banach spaces and $F \in BL(X, Y)$. Prove that the range $R(F)$ is closed in Y if and only if $R(F')$ is closed in X'.

VII. (i) Prove that closed subspaces of reflexive normed spaces are reflexive.

(ii) Let X be a nonzero reflexive normed space. Prove that nonempty closed convex subset of X contains an element of minimal norm.

(iii) Let X be a Banach space. Prove that the set of all compact operators on $X, (CL(X))$ is a closed subspace of $BL(X)$.

UNIT III

VIII. (i) Let X be an inner product space with inner product $\langle\, ,\rangle$ and let $x, y \in X$. Prove that $\| x+y \|^2 = 2\left(\| x \|^2 + \| y \|^2 \right)$,

where $\| x \|^2 = \langle x, x\rangle$.

(ii) Let $\{u_\alpha\}$ be an orthonormal basis for a Hilbert space H. For every $x \in H$, prove that

$$x = \sum_{n=1}^{\infty} \langle x, u_n\rangle u_n,$$

where $\{u_1, u_2, ...\} = \{u_\alpha : \langle x, u_\alpha\rangle \neq 0\}$.

(iii) Let X be an inner product space, $E \subset X$ and let $x \in \bar{E}$. Prove that there exists a best approximation from E to x if and only if $x \in E$.

IX. (i) Let H be a Hilbert space and let F be a nonempty closed subspace of H. Prove that $H = F + F^{\perp}$.

(ii) Let $\{x_n\}$ be a bounded sequence in a Hilbert space H. Prove that $\{x_n\}$ has a weak convergent subsequence.

(iii) Let H be a Hilbert space and $A \in BL(H)$. Prove that there is a unique $B \in BL(H)$ such that for all $x, y \in H$,

$$\langle A(x), y \rangle = \langle x, B(y) \rangle.$$

X. (i) Let A be a self adjoint operator on a Hilbert space H. Prove that

$$\| A \| = \sup \left\{ \left| \langle A(x), x \rangle \right| : x \in H, \| x \| \leq 1 \right\}.$$

(ii) Let A be an operator onthe Hilbert space H. Prove that $\sigma(A)$ is contained in the closure of $\omega(A)$.

(iii) Prove that finite rank operators on a Hilbert space are compact operators.

(4 × 30 = 120 marks)

C 5196 (Pages : 4) Name.....................................

Reg. No.................................

THIRD AND FOURTH SEMESTER M.A./M.Sc./M.Com. DEGREE EXAMINATION, APRIL 2021

(CUCBCSS)

[PVT/SDE]

Mathematics

DMS 301—COMPLEX ANALYSIS

(2017 to 2018 Admissions)

Time : Three Hours Maximum : 150 Marks

Part A

Answer **all** *questions.*
Each question carries 6 *marks.*

1. (a) Compute $\int_{|z|=1} \frac{e^z}{z}\, dz.$

   (b) Show that the function e^z has an essential singularity at ∞.

   (c) Show that :

   $$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}.$$

   (d) Define the Euler's Gamma function $\Gamma(z)$. Prove that $\Gamma(z+1)=z\,\Gamma(z)$.

   (e) Find the natural boundary of :

   $$f(z)=\sum_{n=0}^{\infty} z^{2^n}.$$

(5 × 6 = 30 marks)

Turn over

Part B

Answer any **four** *questions without omitting any unit.*
Each question carries 30 *marks.*

Unit I

2. (a) Describe the Riemann surface associated with the mapping $w = z^3$. (12 marks)

(b) If the piecewise differential closed curve γ does not pass through the point 'a', then prove that the value of the integral

$$\int_{\gamma} \frac{dz}{z-a}$$

is a multiple of $2\pi i$.

(10 marks)

(c) Define the index $n(\gamma, a)$ of a point 'a' with respect to the closed curve γ not passing through the point a. Prove that $n(\gamma, a)$ is a constant in each of the regions determined by γ.

(8 marks)

3. (a) Suppose that $f(z)$ is analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point 'a' not on γ, prove that :

$$n(\gamma, a)\ f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz,$$

where $n(\gamma, a)$ is the index of 'a' with respect to γ.

(12 marks)

(b) Suppose that $\phi(\mathcal{G})$ is continuous on the arc γ. Prove that the function

$$F_n(z) = \int_{\gamma} \frac{\phi(\mathcal{G})\, d\mathcal{G}}{(\mathcal{G} - z)^n}$$

is analytic in each of the regions determined by γ and its derivative is $F_n'(z) = n\, F_{n+1}(z)$.

(12 marks)

(c) Prove that a non-constant analytic function maps open sets onto open sets.

(6 marks)

4. (a) If $f(z)$ is analytic in Ω, then prove that :

$$\int_{\gamma} f(z)\, dz = 0$$

for every cycle γ which is homologous to zero in Ω.

(16 marks)

(b) State and prove Rouche's theorem. (14 marks)

UNIT II

5. (a) If the functions $f_n(z)$ are analytic and non-zero in a region Ω and if $f_n(z)$ converges to $f(z)$, uniformly on every compact subset of Ω, then prove that $f(z)$ is either identically zero or never equal to zero in Ω.

(15 marks)

(b) Prove that :

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(15 marks)

6. (a) Let f be analytic in $|z| \leq \rho$ and suppose that $a_1, a_2, \ldots, a_n$ are the zeros of f in $|z| < \rho$ repeated according to multiplicity. If $f(0) \neq 0$, then prove that :

$$\log |f(0)| = -\sum_{i=1}^{n} \log \left(\frac{\rho}{|a_i|}\right) + \frac{1}{2\pi} \int_0^{2\pi} \log \left|f\left(\rho e^{i\theta}\right)\right| d\theta.$$

(15 marks)

(b) Prove that

$$\mathcal{G}(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \cdot \overline{|(1-s)}\, \mathcal{G}(1-s).$$

(15 marks)

7. State and prove Riemann Mapping theorem.

Unit III

8. (a) Prove that the sum of residues of an elliptic function is zero. (6 marks)

(b) Show that any elliptic function with periods w_1 and w_2 can be written as :

$$C \prod_{k=1} \frac{\sigma(z-a_k)}{\sigma(z-b_k)}$$

where C is a constant and $a_1, a_2, \ldots, a_n$ are the zeros and $b_1, b_2, \ldots, b_n$ are the poles of the elliptic function.

(12 marks)

(c) Briefly describe about the Weierstrass P-function $P(z)$. Prove that :

$$P(z) - P(u) = -\frac{\sigma(z-u)\cdot\sigma(z+u)}{(\sigma(z))^2\cdot(\sigma(u))^2}.$$

(12 marks)

9. (a) Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{2n+1}$ and $\sum_{n=0}^{\infty} \frac{(z-i)^n}{(2-i)^{n+1}}$ are analytic continuations of each other.

(14 marks)

(b) State and prove Schwarz Reflection principle. (16 marks)

10. (a) Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in $|z| < R$ and also at $z = r$. Prove that it is always possible to make direct-analytic extension of f to some points outside $|z| \le r$. Also if f is analytic in $|z| \le r$, then prove that $|z| \le r$ cannot be the disc of convergence of f.

(18 marks)

(b) If $a_n = 0$ except for $n = n_k$ where $\frac{n_{k+1}}{n_k} \ge \lambda > 1$, then show that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with a finite radius of convergence cannot be continued beyond the circle of convergence.